Reasoning about permitted announcements

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Abstract

We formalize what it means to have permission to say something. We adapt the dynamic logic of permission by van der Meyden [22] to the case where atomic actions are public truthful announcements. We also add a notion of obligation. Our logic is an extension of the logic of public announcements introduced by Plaza [17] with dynamic modal operators for permission and for obligation. We axiomatize the logic and show that it is decidable.

Keywords Epistemic logic, deontic logic, public announcements, modal logic, axiomatisation, decidability, permission, obligation.

1 Introduction

Consider an art school examining works at an exhibition. A student is supposed to select one of the displayed works and is then permitted to make a number of intelligent observations about it, sufficient to impress the examiners with the breadth of her knowledge. Now in such cases it never hurts to be more informative than necessary, in order to pass the exam, but a certain minimum amount of intelligent information has to be passed on. This particular museum has both the Night Watch by Rembrandt and Guernica by Picasso on display in the same room! You pass the exam if you observe about the Night Watch that a big chunk of a meter or so is missing in the left corner,

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that was cut off in order to make the painting fit in the Amsterdam Townhall (a_1), and that the painter was Rembrandt van Rijn (a_2). Clearly, this is not a very difficult exam. You also pass the exam if you make two of the three following observations: that Guernica depicts the cruelties of the Spanish Civil war (b_1), that it is painted in black and white and not in colour (b_2), and that the painter was Pablo Picasso (b_3). It is not permitted to make observations about different paintings at the same time, so any conjunction of a_i's and b_j's is not permitted: it would amount to bad judgement if you cannot focus on a single painting. You are obliged to make two observations about the Rembrandt and in that case say nothing about the Picasso, or to make at least two of the three possible observations about the Picasso and in that case say nothing about the Rembrandt. We can treat the permissions and obligations in this setting in an extension of public announcement logic.

1.1 Dynamic epistemic logic

The logic of public announcements proposed by Plaza in [17] (reprinted in [18]) is an extension of multi-agent epistemic logic. It permits to express how agents update their knowledge after public announcements of true propositions. Let p be the proposition that there is a thunderstorm in Eindhoven, and a be the agent Pablo, whereas b is the agent Hans. The language of public announcement logic contains epistemic operators K_a and K_b such that $\neg K_a p$ stands for "Pablo does not know that there is a thunderstorm in Eindhoven". It also contains so-called dynamic epistemic operators $[\psi]$ such that $[K_b p]K_a p$ means "After Hans tells Pablo that there is a thunderstorm in Eindhoven, Pablo knows that there is a thunderstorm in Eindhoven. These operators $[\psi]$ are dynamic modal operators, of the necessity kind, that might be seen as labelled with the formula of the announcement, so we could see it as well as $\Box_\psi$ — that might appeal slightly more to a reader familiar with modal logic. The logic of public announcements is only *one* of many dynamic epistemic logics, for example, there are also logics for private announcements, logics combining factual and epistemic change, dynamic epistemic logics for belief revision, etc. A fair number of such extensions are presented in the monograph-type chapter [5]. Latest developments involve quantification over informative actions, as in [23, 1]. For a standard treatment of the logic of public announcements, see [24]. A number of paradoxes can be explained by their formalizations in dynamic epistemic logic, for example, announcement of $p \wedge \neg K_a p$ (the Moore-sentence [15]) makes p known to agent a: $K_a p$, and its weakening $K_a p \vee \neg p$ is the negation of $p \wedge \neg K_a p$. In public announcement logic, $[p \wedge \neg K_a p]\neg(p \wedge \neg K_a p)$ is a validity: this formula always becomes false when it is announced.

1.2 Deontic logic

Deontic logic is the logic formalizing notions such as ‘ought’, ‘might’, ‘should’ and ‘must’. Like many logics, it is rooted in Antiquity and the Middle Ages, e.g., in the Obligatio game/procedure [7]. Its modern roots are twofold, both non-modal and modal, namely Mally [13] and von Wright [25]. Without going into the details of Von Wright’s work, it is sufficient to say that this developed into a tradition of a propositional modal logic with modalities for obligation or permission, with the characteristic that such modalities bind formulas. Characterizable modal properties are considered such as $O\varphi \rightarrow \neg O\neg\varphi$, and also the possible equivalence of $P\varphi$ (it is permitted that φ) with $\neg O\neg\varphi$. Combining the former with the latter we should then get that if something is obligatory it should at least be permitted. Yet another property is $O(O\varphi \rightarrow \varphi)$, formalizing that it is required that obligations are fulfilled. Various interesting paradoxes have come out of such ‘standard’ deontic logic, we should mention at least Ross’s paradox: if you are permitted (or obliged) to do a or b, that seems to entail you are permitted to do a *and* you are permitted to do b. This suggest that in this standard modal setting $P\varphi \rightarrow P(\varphi \vee \psi)$ is invalid (because, clearly, $P\varphi$ does not imply $P\varphi$ *and* $P\psi$ for *any* ψ).

To dynamic epistemic logicians, spoilt by 25 years of thinking about actions and their execution, this seems a bewildering discussion: clearly obligations and permissions are about actions, how strange that people associate these with static observations, and ‘confuse’ the non-deterministic choice between two actions with the disjunction of two propositions. It is easy to forget that this really required a different frame of mind. For deontic logic this frame of mind was reset by John-Jules Meyer with his *A different approach to deontic logic: deontic logic viewed as a a variant of dynamic logic* [14], an approach that was later followed up by Van der Meyden in [22], the starting point for our proposal in this contribution.

1.3 Dynamic deontic logic

To formalize the concept of “having the permission to say” we extend Plaza’s public announcement logic with a modal operator P of permission, where $P\varphi$ expresses that it is permitted to say (i.e., announce) φ. To define the update of permissions after public announcements we employ a more general binary operator $P(\psi, \varphi)$ that expresses “*after the announcement of ψ it is permitted to announce φ*”—such that $P\varphi$ can then be defined by abbreviation as $P(\top, \varphi)$, where $\top$ is the true proposition.

Our proposal can be seen as an adaption of the dynamic logic of permission proposed by van der Meyden in [22]. Van der Meyden’s proposal was later elaborated on by Pucella et al. in [19]). In Van der Meyden’s work, $\Diamond(\alpha, \varphi)$ means “*there is a way to execute α which is permitted and after which φ is true.*” We treat the particular case where actions are public announcements. Thus, for α in van der Meyden’s $\Diamond(\alpha, \varphi)$ we take an announcement $\psi!$ such that $\Diamond(\psi!, \varphi)$ now means “*there is a way to execute the announcement ψ which is permitted* *and after which φ is true.*" The executability precondition for an announcement ('truthful public announcement') is the truth of the announcement formulas, therefore, the latter is equivalent to "*ψ is true and it is permitted to announce ψ, after which φ is true*". This suggests an equivalence of $\Diamond(\psi!, \varphi)$ with, in our setting, $P(\top, \psi) \wedge \langle\psi\rangle\varphi$, but our operator behaves slightly different. This is because we assume that if you have the permission to say something, you also have the permission to say something weaker, and because our binary permission operator allows update of permissions after an announcement.

In [22] van der Meyden also introduces a weak form of obligation. The meaning of $\mathcal{O}(\alpha, \varphi)$ is "*after any permitted execution of α, φ is true*". Similarly, we also introduce a binary obligation operator $O(\psi, \varphi)$, meaning "*After every announcement of ψ, the agents are obliged to announce φ.*"

Our work further relates to the extension of public announcement logic with protocols by [21, 26]. In their approach, one cannot just announce anything that is true, but one can only announce a true formula that is part of the *protocol*, i.e., that is the first formula in a sequence of formulas (standing for a sequence of successive announcements) that is a member of a set of such sequences called the protocol. In other words, one can only announce *permitted* formulas.

In the setting of informative actions like announcements we leave the beaten track for permission in one important aspect. Little Alice is given permission by her parents to invite uncle Charlie for her 8th birthday party with her children friends and for a delightful canoe trip on the river Thames, but not for the family dinner afterwards. When seeing uncle Charlie, she only mentions the canoe trip but not the children's party. She does not mention the family dinner. Has she transgressed the permissions given? Of course not. Permission to say $p \wedge q$ implies permission to say only q. She has also not transgressed the permission if she were not to invite him at all. Permission to say $p \wedge q$ implies permission to say nothing, i.e., to say the always true and therefore uninformative statement $\top$. Similarly, an obligation to say φ entails the obligation for anything entailed by φ. If you are obliged to say $p \wedge q$ you are also obliged to say q. Now saying q does not therefore mean you have fulfilled the original obligation of $p \wedge q$, you have only partially fulfilled the entailed weaker obligation of q. It may be worth to already point out as this stage that the weakening of announcement formulas is unrelated to Ross's Paradox [20]: this is about the obligation to do one of two possible *actions*—the alternative to that in public announcement logic would be the obligation to make one of two possible announcements (announcement of) φ and (announcement of) ψ, completely different from the obligation to make an announcement of (the disjuntive formula) $\varphi \vee \psi$. In dynamic epistemic logics, there is a clear distinction between actions and formulas.

1.4 Overview

We will first present the syntax and the semantics of our logic, continue with various validities and semantics observations, and conclude with the completeness of the axiomatisation and the decidability of the problem of satisfiability. After that we present an example in detail: the card game La Belote. We conclude with some observations relating to standard deontic logical topics, and a more detailed comparison of our proposal with the relevant dynamic logical literature, i.e. with [22, 19, 21].

2 The logic of permission and obligation to speak

2.1 Syntax

The logic $POPAL$ of permitted announcements is an extension of the multi-agent epistemic logic of public announcements [17].

Definition 1 (Language $\mathcal{L}_{popal}$) *The language $\mathcal{L}_{popal}$ over a countable set of agents N and a countable set of propositional atoms Θ is defined as follows:*

$$\varphi ::= p|\bot|\neg\varphi|\psi \vee \varphi|K_i\varphi|[\psi]\varphi|P(\psi,\varphi)|O(\psi,\varphi)$$

where $i \in N$ and $p \in \Theta$. The language $\mathcal{L}_{poel}$ is the fragment without announcement construct $[\psi]\varphi$, the language $\mathcal{L}_{pal}$ is the fragment without O and P, and the language $\mathcal{L}_{el}$ is the fragment restricted to the Boolean and epistemic operators.

The intuitive reading of $K_i\varphi$ is *"agent i knows that φ is true"* whereas $[\psi]\varphi$ is read as *"after announcing ψ, it is true that φ"*. We read $P(\psi,\varphi)$ as "*(ψ is true and) after announcing ψ, it is permitted to announce φ*". Similarly, $O(\psi,\varphi)$ stands for "*(ψ is true and) after announcing ψ, it is obligatory to announce φ*". Note that announcements are assumed to be public and truthful. Definitions by abbreviation of other Boolean operators are standard. Moreover, we define by abbreviation:

- $\langle\psi\rangle\varphi := \neg[\psi]\neg\varphi$;
- $P\varphi := P(\top,\varphi)$;
- $O\varphi := O(\top,\varphi)$.

Formula $P\varphi$ stands for "It is permitted to announce φ" and $O\varphi$ stands for "It is obligatory to announce φ" (the semantics also entails the *truth* of φ, in both cases); $\langle\psi\rangle\varphi$ stands for "ψ is true and after announcing ψ, φ is true." Note the difference with $[\psi]\varphi$: "*if* ψ is true, then after announcing it, φ is true." The latter is vacuously true if the announcement cannot be made.

The degree deg of a formula is a concept that will be used in the completeness proof, in Section 3.2. It keeps count of the number of P and O operators in a given formula.

Definition 2 (Degree) *The degree of a formula $\varphi \in \mathcal{L}_{popal}$ is defined inductively on the structure of φ as follows:*

$$\begin{array}{lcllcl}
deg(p) & = & 0 & deg(\psi_1 \vee \varphi_2) & = & max(deg(\psi_1), deg(\psi_2)) \\
deg(\bot) & = & 0 & deg([\psi]\varphi) & = & deg(\psi) + deg(\varphi) \\
deg(\neg\psi) & = & deg(\psi) & deg(P(\psi,\varphi)) & = & deg(\psi) + deg(\varphi) + 1 \\
deg(K_i\psi) & = & deg(\psi) & deg(O(\psi,\varphi)) & = & deg(\psi) + deg(\varphi) + 1
\end{array}$$

This is therefore not the usual *modal* degree function, that counts K_i operators. For all formulas $\varphi \in \mathcal{L}_{popal}$, $deg(\varphi) = 0$ iff φ does not contain any occurrence of P or O iff $\varphi \in \mathcal{L}_{pal}$.

2.2 Semantics

The models of our logic are Kripke models with an additional *permission relation* $\mathcal{P}$ between states and pairs of sets of states, that represents, for each state, the announcements that are permitted to be done in this state.

Definition 3 (Permission Kripke Model) *Given a set of agents N and a set of atoms Θ,* permission Kripke models *have the form $\mathcal{M} = (S, \{\sim_i\}_{i \in N}, V, \mathcal{P})$ with S a non-empty set of states, for each $i \in N$, $\sim_i$ an equivalence relation between states of S, valuation function V mapping propositional atoms to subsets of S, and $\mathcal{P} \subseteq S \times 2^S \times 2^S$ such that if $(s, S', S'') \in \mathcal{P}$ then $s \in S'' \subseteq S'$.*

If the equivalence relation $\sim_i$ holds between states $s, t \in S$, this means that, as far as agent i is concerned, s and t are indiscernible. The membership of (s, S', S'') in $\mathcal{P}$ can be interpreted as follows: in state s, after an announcement that restricts the set of possible states to S', a further announcement in S' that restricts that set to S'' is permitted. We will explain this in more detail after giving the semantics.

We simultaneously define the restriction $\mathcal{M}_\psi$ of a model $\mathcal{M}$ after the public announcement of ψ, and the satisfiability relation $\models$. In the definitions we use the abbreviation $[\![\psi]\!]_\mathcal{M} = \{s \in S \mid \mathcal{M}, s \models \psi\}$. If no ambiguity results, we occasionally write $[\![\psi]\!]$ instead of $[\![\psi]\!]_\mathcal{M}$.

Definition 4 (Restricted model) *For any model $\mathcal{M}$ and any $\psi \in \mathcal{L}_{popal}$, we define the* restriction *$\mathcal{M}_\psi = (S_\psi, \sim_i^\psi, V_\psi, \mathcal{P}_\psi)$ where:*

- $S_\psi = [\![\psi]\!]_\mathcal{M}$
- *for all i, $\sim_i^\psi = \sim_i \cap (S_\psi \times S_\psi)$*
- *for all $p \in \Theta$, $V_\psi(p) = V(p) \cap S_\psi$*
- $\mathcal{P}_\psi = \{(s, S', S'') \in \mathcal{P} \mid s \in S_\psi,\ S' \subseteq S_\psi, S'' \subseteq S_\psi\}$

Definition 5 (Satisfiability relation) *Let $\mathcal{M}$ be a model and s be a state of S. The satisfiability relation $\models$ is defined inductively on the structure of φ:*

$\mathcal{M}, s \models p$ *iff* $s \in V(p)$

$\mathcal{M}, s \not\models \bot$

$\mathcal{M}, s \models \neg\psi$ *iff* $\mathcal{M}, s \not\models \psi$

$\mathcal{M}, s \models \psi_1 \vee \psi_2$ *iff* $(\mathcal{M}, s \models \psi_1$ *or* $\mathcal{M}, s \models \psi_2)$

$\mathcal{M}, s \models K_i\psi$ *iff for all* $t \sim_i s$, $\mathcal{M}, t \models \psi$

$\mathcal{M}, s \models [\psi]\chi$ *iff* ($\mathcal{M}, s \models \psi \Rightarrow \mathcal{M}_\psi, s \models \chi$)

$\mathcal{M}, s \models P(\psi, \chi)$ *iff for some* $(s, [\![\psi]\!]_\mathcal{M}, S'') \in \mathcal{P}$, $S'' \subseteq [\![\langle\psi\rangle\chi]\!]_\mathcal{M}$

$\mathcal{M}, s \models O(\psi, \chi)$ *iff for all* $(s, [\![\psi]\!]_\mathcal{M}, S'') \in \mathcal{P}$, $S'' \subseteq [\![\langle\psi\rangle\chi]\!]_\mathcal{M}$.

For all $\varphi \in \mathcal{L}_{popal}$, $\mathcal{M} \models \varphi$ *iff for all* $s \in S$, $\mathcal{M}, s \models \varphi$*; and* $\models \varphi$ *iff for all models* $\mathcal{M}$ *we have* $\mathcal{M} \models \varphi$.

We do not impose that S' and S'' are denotations of formulas in the language for (s, S', S'') to be in $\mathcal{P}$. This semantics is thus more general than the intuitive one for "having the permission to say". Indeed, if S' or S'' do not correspond to a restriction of S made by an announcement, then $(s, S', S'') \in \mathcal{P}$ does not correspond to some announcement being permitted.

The semantics of $P(\psi, \chi)$ expresses that after announcement of ψ it is permitted to announce a χ weaker than the restriction given in the relation $\mathcal{P}$. If the S'' in $(s, [\![\psi]\!], S'')$ is the denotation of some $[\![\langle\psi\rangle\varphi]\!]$, we get that after announcement of ψ it is permitted to announce a χ weaker than (implied by) φ.

2.3 Example: Art School

Consider the example in the introduction. In an art school examination you are asked to "describe precisely one (and only one) of the presented pictures". There are two distinct sets of intelligent observations to make (modelled as atomic propositional variables): $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$, with $A \cup B = \Theta$. The domain of discourse consists of all possible valuations $S = 2^\Theta$, in the actual state s all atoms are in fact true, and our student is in fact an omniscient agent g (i.e. $\sim_g = id_S$) that can announce anything she likes. The set $\mathcal{P}$ is given as $\mathcal{P} = \{(s, [\![\top]\!], [\![a_1 \wedge a_2]\!]), (s, [\![\top]\!], [\![b_1 \wedge b_2]\!]), (s, [\![\top]\!], [\![b_1 \wedge b_3]\!]), (s, [\![\top]\!], [\![b_2 \wedge b_3]\!]), (s, [\![\top]\!], [\![b_1 \wedge b_2 \wedge b_3]\!])\}$. Note that $[\![\top]\!] = S$. We now have that

- It is permitted to say a_1 ($\mathcal{M}, s \models P(\top, a_1)$), because $(s, [\![\top]\!], [\![a_1 \wedge a_2]\!]) \in \mathcal{P}$ and $[\![a_1 \wedge a_2]\!] \subseteq [\![\langle\top\rangle a_1]\!]_\mathcal{M}$ (where $[\![\langle\top\rangle a_1]\!]_\mathcal{M} = [\![a_1]\!]_\mathcal{M}$): it is permitted to say something weaker than $a_1 \wedge a_2$.

- It is not permitted to say $a_1 \wedge b_2$ ($\mathcal{M}, s \models \neg P(\top, a_1 \wedge b_2)$) because the denotation of that formula is not contained in either of the members of the set $\mathcal{P}$.

- It is not obligatory to say a_1 ($\mathcal{M}, s \not\models O(\top, a_1)$), because it is permitted to say $b_1 \wedge b_2$, and $[\![a_1]\!] \not\subseteq [\![b_1 \wedge b_2]\!]$.

- It is obligatory to say $ob := (a_1 \wedge a_2) \vee (b_1 \wedge b_2) \vee (b_2 \wedge b_3) \vee (b_1 \wedge b_3)$ as all members of $\mathcal{P}$ are stronger.

This last obligation is also the strongest obligation in this setting. It is, e.g., also obligatory to say $a_1 \vee b_1 \vee b_2$ ($\mathcal{M}, s \models O(\top, a_1 \vee b_1 \vee b_2)$) because this is weaker than *ob*. However, as already mentioned, this does not mean that a student has fulfilled her obligation when saying $a_1 \vee b_1 \vee b_2$ - she then only fulfills *part* of her obligation (and will therefore fail the exam!). We observe that our intuition of what an obligation is corresponds to the strongest obligation under our definition—reasons to prefer the current definition are technical, such as getting completeness right.

2.4 Valid principles and other semantic results

The O and P operators are not interdefinable. This is because the obligation to say φ means that anything not entailing φ may not be permitted to say, and not only that it is not permitted to say $\neg\varphi$. As an example, consider the following two models that have the same domain $S = \{s_1, s_2\}$, the same valuation $V(p) = \{s_1\}$, the same epistemic relation $\sim_i = S \times S$, but that differ on the permission relation: $\mathcal{M} = (S, V, \sim_i, \mathcal{P})$ and $\mathcal{M}' = (S, V, \sim_i, \mathcal{P}'$ where $\mathcal{P} = \{(s_1, S, \{s_1\}), (s_2, S, S)\}$ and $\mathcal{P}' = \{(s_1, S, \{s_1\}), (s_1, S, S), (s_2, S, S)\}$. Let $\mathcal{L}^-_{popal}$ be the language without the obligation operator O. The pointed models $(\mathcal{M}, s_1)$ and $(\mathcal{M}', s_1)$ have the same theory in that language: for all $\varphi \in \mathcal{L}_{popal}$, $(\mathcal{M}, s_1 \models \varphi$ iff $\mathcal{M}, s_1 \models \varphi)$. The proof is obvious for all inductive cases of φ except when φ takes shape $P(\psi, \varphi)$. In that case, observe from the semantics of P and the given relations $\mathcal{P}$ and $\mathcal{P}'$ that *only* formulas of type $P(\top, \varphi_2)$ can be true in these models, as the second argument of all triples in $\mathcal{P}$ and $\mathcal{P}'$ is the entire domain S. Further observe that in both models, anything that is true in s_1 is permitted to be said, formally for all $\varphi \in \mathcal{L}_{popal}$, $\mathcal{M}, s_1 \models \varphi \leftrightarrow P(\top, \varphi)$ and $\mathcal{M}', s_1 \models \varphi \leftrightarrow P(\top, \varphi)$. So $\mathcal{M}$ and $\mathcal{M}'$ are modally equivalent in $\mathcal{L}^-_{popal}$. On the other hand, as (s, S, S) is not in $\mathcal{P}$ we have that $\mathcal{M}, s_1 \models O(\top, p)$ but $\mathcal{M}', s_1 \models \neg O(\top, p)$, so the models are not modally equivalent in $\mathcal{L}_{popal}$. We conclude that:

Proposition 6 *$\mathcal{L}^-_{popal}$ is strictly less expressive than $\mathcal{L}_{popal}$.*

The standard validities for public announcement logic are preserved in this extension of the logic with permission and obligation (for details, see a standard introduction like [24]):

- $\models [\psi]p \leftrightarrow (\psi \rightarrow p)$
- $\models [\psi]\bot \leftrightarrow \neg\psi$
- $\models [\psi]\neg\varphi \leftrightarrow (\psi \rightarrow \neg[\psi]\varphi)$
- $\models [\psi](\varphi_1 \vee \varphi_2) \leftrightarrow ([\psi]\varphi_1 \vee [\psi]\varphi_2)$
- $\models [\psi]K_i\varphi \leftrightarrow (\psi \rightarrow K_i[\psi]\varphi)$
- $\models [\psi_1][\psi_2]\varphi \leftrightarrow [\langle\psi_1\rangle\psi_2]\varphi$

For example, $[\psi]p \leftrightarrow (\psi \rightarrow p)$ says that p is true after announcement of ψ iff ψ implies p (is true). As ψ is the condition to be able to make the announcement, this principle merely says that an announcement cannot change the valuation of atoms. Of course, for other formulas than atoms we cannot get rid of the announcement that way. A typical counterexample (the Moore-sentence) is that $(p \wedge \neg K_i p) \rightarrow (p \wedge \neg K_i p)$ is a trivial validity whereas $[p \wedge \neg K_i p](p \wedge \neg K_i p)$ is false, because whenever $p \wedge \neg K_i p$ can be announced, p is known afterwards: $K_i p$.

Additional to the principles for public announcement logic, two principles address how to treat a permission or obligation operator after an announced.

Proposition 7 *For all $p \in \Theta$, all $\psi, \varphi, \psi_1, \psi_2, \varphi_1, \varphi_2 \in \mathcal{L}_{popal}$*

1. $\models [\psi_1]P(\psi_2, \varphi) \leftrightarrow (\psi_1 \rightarrow P(\langle\psi_1\rangle\psi_2, \varphi))$
2. $\models [\psi_1]O(\psi_2, \varphi) \leftrightarrow (\psi_1 \rightarrow O(\langle\psi_1\rangle\psi_2, \varphi))$

Proof For all $\mathcal{M}$, all $s \in S$ and all $\psi_1, \psi_2, \varphi \in \mathcal{L}_{popal}$ we have:

1. ($\Rightarrow$) Suppose that $\mathcal{M}, s \models [\psi_1]P(\psi_2, \varphi)$ and $s \in [\![\psi_1]\!]_{\mathcal{M}}$. Then for some $S'' \subseteq [\![\langle\psi_2\rangle\varphi]\!]_{\mathcal{M}_{\psi_1}}, (s, [\![\psi_2]\!]_{\mathcal{M}_{\psi_1}}, S'') \in \mathcal{P}_{\psi_1}$. This implies that for some $S'' \subseteq [\![\langle\psi_1\rangle\langle\psi_2\rangle\varphi]\!]_{\mathcal{M}}$, $(s, [\![\langle\psi_1\rangle\psi_2]\!]_{\mathcal{M}}, S'') \in \mathcal{P}_{\psi_1}$, i.e. for some $S'' \subseteq [\![\langle\psi_1\rangle\langle\psi_2\rangle\varphi]\!]_{\mathcal{M}}, (s, [\![\langle\psi_1\rangle\psi_2]\!]_{\mathcal{M}}, S'') \in \mathcal{P}$. Finally $\mathcal{M}, s \models P(\langle\psi_1\rangle\psi_2, \varphi)$.

   ($\Leftarrow$) Suppose that $\mathcal{M}, s \models (\psi_1 \rightarrow P(\langle\psi_1\rangle\psi_2, \varphi))$. If $\mathcal{M}, s \not\models \psi_1$ then obviously $\mathcal{M}, s \models [\psi_1]P(\psi_2, \varphi)$. Otherwise $s \in [\![\psi_1]\!]_{\mathcal{M}}$ and $\mathcal{M}, s \models P(\langle\psi_1\rangle\psi_2, \varphi)$. Therefore, there exists $S'' \subseteq [\![\langle\langle\psi_1\rangle\psi_2\rangle\varphi]\!]_{\mathcal{M}}$ such that $(s, [\![\langle\psi_1\rangle\psi_2]\!]_{\mathcal{M}}, S'') \in \mathcal{P}$. Thus $s \in [\![\psi_1]\!]_{\mathcal{M}}$, $S'' \subseteq [\![\langle\psi_2\rangle\varphi]\!]_{\mathcal{M}_{\psi_1}}$ and $(s, [\![\psi_2]\!]_{\mathcal{M}_{\psi_1}}, S'') \in \mathcal{P}_{\psi_1}$. Finally $\mathcal{M}, s \models [\psi_1]P(\psi_2, \varphi)$.

2. ($\Rightarrow$) Suppose that $\mathcal{M}, s \models [\psi_1]O(\psi_2, \varphi)$ and $s \in [\![\psi_1]\!]_{\mathcal{M}}$. Then for all $(s, [\![\psi_2]\!]_{\mathcal{M}_{\psi_1}}, S'') \in \mathcal{P}_{\psi_1}$, $S'' \subseteq [\![\langle\psi_2\rangle\varphi]\!]_{\mathcal{M}_{\psi_1}}$. This implies that for all $(s, [\![\langle\psi_1\rangle\psi_2]\!]_{\mathcal{M}}, S'') \in \mathcal{P}_{\psi_1}$, $S'' \subseteq [\![\langle\psi_1\rangle\langle\psi_2\rangle\varphi]\!]_{\mathcal{M}}$ i.e. for all $(s, [\![\langle\psi_1\rangle\psi_2]\!]_{\mathcal{M}}, S'') \in \mathcal{P}$, $S'' \subseteq [\![\langle\psi_1\rangle\langle\psi_2\rangle\varphi]\!]_{\mathcal{M}}$. Finally $\mathcal{M}, s \models O(\langle\psi_1\rangle\psi_2, \varphi)$.

   ($\Leftarrow$) Suppose that $\mathcal{M}, s \models (\psi_1 \rightarrow O(\langle\psi_1\rangle\psi_2, \varphi))$. If $\mathcal{M}, s \not\models \psi_1$ then obviously $\mathcal{M}, s \models [\psi_1]O(\psi_2, \varphi)$. Otherwise $s \in [\![\psi_1]\!]_{\mathcal{M}}$ and $\mathcal{M}, s \models O(\langle\psi_1\rangle\psi_2, \varphi)$. Therefore, for all $(s, [\![\langle\psi_1\rangle\psi_2]\!]_{\mathcal{M}}, S'') \in \mathcal{P}$ we have $S'' \subseteq [\![\langle\langle\psi_1\rangle\psi_2\rangle\varphi]\!]_{\mathcal{M}}$. Thus $s \in [\![\psi_1]\!]_{\mathcal{M}}$ and for all $(s, [\![\psi_2]\!]_{\mathcal{M}_{\psi_1}}, S'') \in \mathcal{P}_{\psi_1}$ we have $S'' \subseteq [\![\langle\psi_2\rangle\varphi]\!]_{\mathcal{M}_{\psi_1}}$. Finally $\mathcal{M}, s \models [\psi_1]O(\psi_2, \varphi)$.

$\square$

For example, principle $[\psi_1]P(\psi_2, \varphi) \leftrightarrow (\psi_1 \rightarrow P(\langle\psi_1\rangle\psi_2, \varphi))$ of Proposition 7 says the following: *"(After announcing ψ_1 we have that (ψ_2 is true and after announcing ψ_2 it is permitted to announce φ)) iff (On condition that ψ_1 is true ($\langle\psi_1\rangle\psi_2$ is true and after announcing $\langle\psi_1\rangle\psi_2$ it is permitted to say φ))."* Using the meaning of the public announcement operator, the right part is the same as *"On condition that ψ_1 is true, after announcing ψ_1, ψ_2 is true and after then announcing ψ_2 it is permitted to say φ."* Which gets us back to the left part of the original equivalence.

Another validity of the logic spells out that equivalent announcements lead to equivalent permissions.

Proposition 8 *For all models $\mathcal{M}$ and all formulas $\psi, \psi', \varphi, \varphi' \in \mathcal{L}_{popal}$: If $\mathcal{M} \models (\psi \leftrightarrow \psi') \wedge ([\psi]\varphi \rightarrow [\psi']\varphi')$ then $\mathcal{M} \models P(\psi, \varphi) \rightarrow P(\psi', \varphi')$ and $\mathcal{M} \models O(\psi, \varphi) \rightarrow O(\psi', \varphi')$.*

Proof For all $\psi, \psi', \varphi, \varphi' \in \mathcal{L}_{popal}$, if $\mathcal{M} \models (\psi \leftrightarrow \psi')$ and $\mathcal{M} \models \langle\psi\rangle\varphi \rightarrow \langle\psi'\rangle\varphi'$, then $[\![\psi]\!]_{\mathcal{M}} = [\![\psi']\!]_{\mathcal{M}}$ and $[\![\langle\psi\rangle\varphi]\!]_{\mathcal{M}} \subseteq [\![\langle\psi'\rangle\varphi']\!]_{\mathcal{M}}$. It implies that for all $(s, [\![\psi]\!]_{\mathcal{M}}, S'') \in \mathcal{P}$, we have $(s, [\![\psi']\!]_{\mathcal{M}}, S'') \in \mathcal{P}$ and if $S'' \subseteq [\![\langle\psi\rangle\varphi]\!]$ then $S'' \subseteq [\![\langle\psi\rangle'\varphi']\!]$. $\square$

We continue with a proposition on allowed logical compositions of permitted and obliged announcements.

Proposition 9 *For all $\psi, \varphi, \varphi_1, \varphi_2 \in \mathcal{L}_{popal}$*

1. $\models (O(\psi, \varphi_1) \wedge O(\psi, \varphi_2)) \leftrightarrow O(\psi, \varphi_1 \wedge \varphi_2)$
2. $\models (P(\psi, \varphi_1) \wedge O(\psi, \varphi_2)) \rightarrow P(\psi, \varphi_1 \wedge \varphi_2)$
3. $\models (\psi \wedge O(\psi, \varphi) \wedge \neg P(\psi, \varphi)) \leftrightarrow (\psi \wedge \neg P(\psi, \top))$

Proof For all models $\mathcal{M}$ and all state $s \in S$ we have

1. $\mathcal{M}, s \models O(\psi, \varphi_1) \wedge O(\psi, \varphi_2)$ iff for all $(s, [\![\psi]\!], S'') \in \mathcal{P}$, $S'' \subseteq [\![\langle\psi\rangle\varphi_1]\!]$ and $S'' \subseteq [\![\langle\psi\rangle\varphi_2]\!]$ iff for all $(s, [\![\psi]\!], S'') \in \mathcal{P}$, $S'' \subseteq [\![\langle\psi\rangle\varphi_1]\!] \cap [\![\langle\psi\rangle\varphi_2]\!] = [\![\langle\psi\rangle(\varphi_1 \wedge \varphi_2)]\!]$ iff $\mathcal{M}, s \models O(\psi, \varphi_1 \wedge \varphi_2)$.
2. Suppose $\mathcal{M}, s \models P(\psi, \varphi_1) \wedge O(\psi, \varphi_2)$. Then for some $(s, [\![\psi]\!], S'') \in \mathcal{P}$, $S'' \subseteq [\![\langle\psi\rangle\varphi_1]\!]$ and for all $(s, [\![\psi]\!], S'') \in \mathcal{P}$, $S'' \subseteq [\![\langle\psi\rangle\varphi_2]\!]$. Thus, for some $(s, [\![\psi]\!], S'') \in \mathcal{P}$, $S'' \subseteq [\![\langle\psi\rangle\varphi_1]\!] \cap [\![\langle\psi\rangle\varphi_2]\!] = [\![\langle\psi\rangle(\varphi_1 \wedge \psi_2)]\!]$ which is equivalent to $\mathcal{M}, s \models P(\psi, \varphi_1 \wedge \varphi_2)$.
3. $\mathcal{M}, s \models \psi \wedge O(\psi, \varphi) \wedge \neg P(\psi, \varphi)$ if and only if $\mathcal{M}, s \models \psi$ and for all $(s, [\![\psi]\!], S'') \in \mathcal{P}$, $S'' \subseteq [\![\langle\psi\rangle\varphi]\!]$ and $S'' \not\subseteq [\![\langle\psi\rangle\varphi]\!]$. This is equivalent to $\mathcal{M}, s \models \psi$ and the fact that there is no S'' such that $(s, [\![\psi]\!], S'') \in \mathcal{P}$, which means that $\mathcal{M}, s \models \psi \wedge \neg P(\psi, \top)$.

$\square$

We also have that if φ is permitted, than any $\varphi \vee \psi$ is *also* permitted (namely anything weaker than φ is also permitted) and similarly, if φ is obligatory, than any $\varphi \vee \psi$ is *also* obligatory. In the example in the previous section we already illustrated that this notion of weakened obligation is not intuitive—one might rather see the announcement of $\varphi \vee \psi$ as something *towards* fulfilling an obligation. The weakened permission of $\varphi \vee \psi$ we find intuitive in the setting of permitted announcements. Unlike in the Ross Paradox [20], note that this is not choice between two different announcements, but the single announcement of a disjunction.

Proposition 7 suggests the following translation $tr : \mathcal{L}_{popal} \rightarrow \mathcal{L}_{poel}$:

Definition 10 (the translation tr) *We define $tr(\varphi)$ by induction on the complexity of φ as follows:*

- $tr(p) = p$
- $tr(\bot) = \bot$
- $tr(\neg\varphi) = \neg tr(\varphi)$
- $tr(\psi \vee \varphi) = tr(\psi) \vee tr(\varphi)$
- $tr(K_i\varphi) = K_i tr(\varphi)$
- $tr(P(\psi, \varphi)) = P(tr(\psi), tr(\varphi))$
- $tr(O(\psi, \varphi)) = O(tr(\psi), tr(\varphi))$
- $tr([\psi]p) = tr(\psi) \to p$
- $tr([\psi]\bot) = \neg tr(\psi)$
- $tr([\psi]\neg\varphi) = tr(\psi) \to \neg tr([\psi]\varphi)$
- $tr([\psi](\varphi_1 \vee \varphi_2)) = tr([\psi]\varphi_1) \vee tr([\psi]\varphi_2)$
- $tr([\psi]K_i\varphi) = tr(\psi) \to K_i tr([\psi]\varphi)$
- $tr([\psi_1][\psi_2]\varphi) = tr([\langle\psi_1\rangle\psi_2]\varphi)$
- $tr([\psi_1]P(\psi_2, \varphi)) = tr(\psi_1) \to P(tr(\langle\psi_1\rangle\psi_2), tr(\varphi))$
- $tr([\psi_1]O(\psi_2, \varphi)) = tr(\psi_1) \to O(tr(\langle\psi_1\rangle\psi_2), tr(\varphi))$

An elementary proof by induction on the structure of φ, using Proposition 7, now delivers:

Proposition 11 *For all* $\varphi \in \mathcal{L}_{popal}$, $\models \varphi \leftrightarrow tr(\varphi)$.

In other words, adding public announcements to logical language with permitted and obligatory announcement does not increase the expressivity of the logic.

Finally, we need to show a property of the degree function. This property will be used in the completeness proof. Its proof is in the appendix.

Proposition 12 *For all* $\varphi \in \mathcal{L}_{popal}$, $deg(tr(\varphi)) = deg(\varphi)$.

3 Axiomatization

We define the axiomatization $POPAL$ and prove its soundness and completeness. Let $POPAL$ be the least set of formulas in our language that contains the axiom schemata and is closed under the inference rules in Table 1. We write $\vdash_{POPAL} \varphi$ for $\varphi \in POPAL$. We define the consistency and the maximality of a set x of formulas as usual: x is $POPAL$-consistent iff for all nonnegative integers n and for all formulas $\varphi_1, \ldots, \varphi_n \in x$, $\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \notin POPAL$ whereas x is maximal iff for all formulas φ, $\varphi \in x$ or $\neg\varphi \in x$.

all propositional tautologies	
$K_i\varphi \to \varphi$	truth
$K_i\varphi \to K_iK_i\varphi$	positive introspection
$\neg K_i\varphi \to K_i\neg K_i\varphi$	negative introspection
$[\psi]p \leftrightarrow (\psi \to p)$	atomic permanence
$[\psi]\bot \leftrightarrow \neg\psi$	ann. and false
$[\psi]\neg\varphi \leftrightarrow (\psi \to \neg[\psi]\varphi)$	ann. and negation
$[\psi](\varphi_1 \vee \varphi_2) \leftrightarrow ([\psi]\varphi_1 \vee [\psi]\varphi_2)$	ann. and disjunction
$[\psi]K_i\varphi \leftrightarrow (\psi \to K_i[\psi]\varphi)$	ann. and knowledge
$[\psi_1][\psi_2]\varphi \leftrightarrow [\langle\psi\rangle_1\psi_2]\varphi$	ann. composition
$[\psi]P(\psi', \varphi) \leftrightarrow (\psi \to P(\langle\psi\rangle\psi', \varphi))$	ann. and permission
$[\psi]O(\psi', \varphi) \leftrightarrow (\psi \to O(\langle\psi\rangle\psi', \varphi))$	ann. and obligation
$P(\psi, \varphi) \to \langle\psi\rangle\varphi$	permission and truth
$O(\top, \top)$	
$(O(\psi, \varphi_1) \wedge O(\psi, \varphi_2)) \leftrightarrow O(\psi, \varphi_1 \wedge \varphi_2)$	obligations composition
$(P(\psi, \varphi_1) \wedge O(\psi, \varphi_2)) \to P(\psi, \varphi_1 \wedge \varphi_2)$	obligation and permission comp.
$(\psi \wedge O(\psi, \varphi) \wedge \neg P(\psi, \varphi)) \leftrightarrow (\psi \wedge \neg P(\psi, \top))$	obligation and prohibition
From φ and $\varphi \to \psi$ infer ψ	modus ponens
From φ infer $K_i\varphi$	necessitation of K_i
From φ infer $[\psi]\varphi$	necessitation of announcement
From $(\psi \leftrightarrow \psi') \wedge (\langle\psi\rangle\varphi \to \langle\psi'\rangle\varphi')$ infer	
$(P(\psi, \varphi') \to P(\psi', \varphi))$ and $(O(\psi, \varphi) \to O(\psi', \varphi'))$	substitution

Table 1: Axiomatization of $POPAL$

3.1 Soundness

Proposition 13 *POPAL is sound on the class of all models.*

Proof By Propositions 7, 8 and 9. $\square$

Note that we have in particular that

Proposition 14 *For all* $\varphi \in \mathcal{L}_{popal}, \vdash_{POPAL} \varphi \leftrightarrow tr(\varphi)$.

3.2 Completeness

To prove the completeness result, let us define the canonical model for $POPAL$:

Definition 15 (Canonical Model)
The canonical model $\mathcal{M}^c = (S^c, \sim_i^c, V^c, \mathcal{P}^c)$ *is defined as follows:*

- S^c *is the set of all* $\vdash_{POPAL}$*-maximal consistent sets*
- *for any* $p \in \Theta$, $V^c(p) = \{x \in S^c \mid p \in x\}$
- $x \sim_i^c y$ *iff* $K_i x = K_i y$, *where* $K_i x = \{\varphi | K_i \varphi \in x\}$
- $\mathcal{P}^c = \{(x, S', S'') : \exists P(\psi, \varphi) \in x | \ S' = \{y \in S^c : \psi \in y\} \ , S'' = \{y \in S^c : \langle\psi\rangle\varphi \in y\} \cap A_\psi^x\} \bigcup \{(x, S', S'') : \exists(\psi \wedge \neg O(\psi, \varphi)) \in x | \ S' = \{y \in S^c : \psi \in y\} \ , S'' = A_\psi^x\}$

  where for all $\psi \in \mathcal{L}_{popal}$ *and all* $x \in S^c$ *we pose* $A_\psi^x = \{y \in S^c : \forall O(\psi, \chi) \in x, \langle\psi\rangle\chi \in y\}$.

For any set $x \in S^c$ and any formula $\psi \in \mathcal{L}_{popal}$, the set A_ψ^x is the set of all the $\mathcal{L}_{popal}$-maximal consistent sets that satisfy $\langle\psi\rangle\varphi$ for all announcements φ that are obligatory after the announcement of ψ.

Proposition 16 *The canonical model is a model.*

Proof The set of states and the valuation are clearly well defined, and as the equality is an equivalence relation between set of formulas, $\sim_i^c$ is an equivalence relation. $\mathcal{P}^c$ is a set of triplets of the expected form, the only thing we have to verify is that for every $(x, S', S'') \in \mathcal{P}^c$, we have $x \in S'' \subseteq S'$. Indeed, let $(x, S', S'') \in \mathcal{P}^c$, thus there are two possibilities:

1. First, there exists a $P(\psi, \varphi) \in x$ such that $S' = \{y \in S^c : \psi \in y\}$ and $S'' = \{y \in S^c : \langle\psi\rangle\varphi \in y\} \cap A_\psi^x$. In this case, clearly $S'' \subseteq S'$ because for all $y \in S^c$, $\langle\psi\rangle\varphi \in y$ only if $\psi \in y$. Now $x \in S''$ comes from the axiom "permission and truth".

2. Second, there exists a $(\psi \wedge \neg O(\psi, \varphi)) \in x$ such that $S' = \{y \in S^c : \psi \in y\}$ and $S'' = \{y \in S^c : \forall O(\psi, \chi) \in x, \langle\psi\rangle\chi \in y\}$. In this case, $S'' \subseteq S'$ comes from the fact that $\psi \in x$ implies $[\psi]O(\top, \top) \in x$, i.e. $O(\psi, \top) \in x$ and thus $y \in S''$ implies $\langle\psi\rangle\top \in y$. Now to show that $x \in S''$ let us consider

$O(\psi, \chi) \in x$ and let us show that $\langle\psi\rangle\chi \in x$. Indeed, by "obligation and prohibition", $\psi \wedge \neg O(\psi, \varphi) \in x$ implies $\psi \wedge P(\psi, \top) \in x$, now $P(\psi, \top) \wedge O(\psi, \chi) \in x$ implies $P(\psi, \chi) \in x$ by "obligation and permission comp.", and finally $\langle\psi\rangle\chi \in x$ by "permission and truth".

$\square$

In the canonical model, a state is a set of formulas. The link between the fact that a formula φ is in a set x and the fact that $\mathcal{M}^c, x \models \varphi$ is given by the Truth Lemma. In the proof of the Truth Lemma, we need the following

Lemma 17 *For any $x \in \mathcal{M}^c$ and any $\psi, \varphi, \alpha, \beta \in \mathcal{L}_{poel}$,*

1. *if $A_\psi^x \subseteq \{y : \langle\psi\rangle\varphi \in y\}$, then $O(\psi, \varphi) \in x$,*
2. *if $P(\alpha, \beta) \in x$ and $\{y : \langle\alpha\rangle\beta \in y\} \cap A_\alpha^x \subseteq \{y : \langle\alpha\rangle\varphi \in y\}$, then $P(\alpha, \varphi) \in x$*

Proof 1. By hypothesis, any maximal consistent set that contains $\langle\psi\rangle\chi$ for all $O(\psi, \chi) \in x$ contains also $\langle\psi\rangle\varphi$, thus $\{\langle\psi\rangle\chi : O(\psi, \chi) \in x\} \cup \{[\psi]\neg\varphi\}$ is inconsistent. By definition, it has a finite subset $\{\langle\psi\rangle\chi_1, \ldots, \langle\psi\rangle\chi_n, [\psi]\neg\varphi\}$ that is inconsistent. Thus $\vdash \langle\psi\rangle\chi_1 \wedge \ldots \wedge \langle\psi\rangle\chi_n \rightarrow \langle\psi\rangle\varphi$, i.e. $\vdash \langle\psi\rangle \bigwedge \chi_i \rightarrow \langle\psi\rangle\varphi$ and then $\vdash O(\psi, \bigwedge \chi_i) \rightarrow O(\psi, \varphi)$ by the inference rule (R). By axiom "obligation composition" $O(\psi, \bigwedge \chi_i) \in x$, and by modus ponens $O(\psi, \varphi) \in x$.

2. By hypothesis, any maximal consistent set that contains $\langle\psi\rangle\beta$ and $\langle\psi\rangle\chi$ for all $O(\psi, \chi) \in x$ contains also $\langle\psi\rangle\varphi$. Thus $\{\langle\psi\rangle\beta\} \cup \{\langle\psi\rangle\chi : O(\psi, \chi) \in x\} \cup \{[\psi]\neg\varphi\}$ is inconsistent. By definition, this set has a finite subset $\{\langle\psi\rangle\beta, \langle\psi\rangle\chi_1, \ldots, \langle\psi\rangle\chi_n, [\psi]\neg\varphi\}$ that is inconsistent. Thus $\vdash (\langle\psi\rangle\beta \wedge \langle\psi\rangle\chi_1 \wedge \ldots \wedge \langle\psi\rangle\chi_n) \rightarrow \langle\psi\rangle\varphi$, i.e. $\vdash \langle\psi\rangle(\beta \wedge \bigwedge \chi_i) \rightarrow \langle\psi\rangle\varphi$ and then $\vdash P(\psi, \beta \wedge \bigwedge \chi_i) \rightarrow P(\psi, \varphi)$. $O(\psi, \bigwedge \chi_i) \in x$ is true by axiom "obligation composition" and and $P(\psi, \beta) \in x$ by hypothesis. Thus $P(\psi, \beta \wedge \bigwedge \chi_i) \in x$ is true by axiom "obligation and permission comp.". Finally, $P(\psi, \varphi) \in x$ by modus ponens.

$\square$

Proposition 18 (Truth Lemma for $\mathcal{L}_{poel}$) *For all $\varphi \in \mathcal{L}_{poel}$ we have:*

$$\Pi(\varphi): \text{ for all } x \in S^c, \mathcal{M}^c, x \models \varphi \text{ iff } \varphi \in x$$

Proof The proof is by induction on the degree of φ.

Base case If $deg(\varphi) = 0$ then $\varphi \in \mathcal{L}_{el}$ and $\Pi(\varphi)$ is a known result (See [6] or [9] for details). Note that $(S^c, \sim_i^c, V^c)$ is the classical canonical model for $\mathcal{L}_{el}$.

Induction steps Let $k \in \mathbb{N}$, let us suppose that $\Pi(\psi)$ is true for all $\psi \in \mathcal{L}_{poel}$ such that $deg(\psi) \leqslant k$.

Note that it follows that $\Pi(\psi)$ is true for all $\psi \in \mathcal{L}_{popal}$ such that $deg(\psi) \leqslant k$. Indeed, for all such ψ, for all $x \in S^c$, $\mathcal{M}^c, x \models \psi$ iff $\mathcal{M}^c, x \models tr(\psi)$ iff $tr(\psi) \in x$ iff $\psi \in x$.

Let φ be such that $deg(\varphi) \leq k+1$ and let us reason by induction on the structure of φ.

- **$\varphi = p; \bot; \neg\psi; \varphi_1 \vee \varphi_2; K_i\psi$:** See the proof of the truth lemma for $\mathcal{L}_{el}$ in [6] or [9].

- **$\varphi = P(\psi, \chi)$:**

  ($\Rightarrow$) Suppose that $\mathcal{M}^c, x \models P(\psi, \chi)$ and let $S'' \subseteq [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}^c}$ be such that $(x, [\![\psi]\!]_{\mathcal{M}^c}, S'') \in \mathcal{P}^c$.

  Two possibilities:

  - First, there exists $P(\alpha, \beta) \in x$ s.t. $(*)[\![\psi]\!]_{\mathcal{M}^c} = \{y \in S^c : \alpha \in y\}$ and $S'' = \{y \in S^c : \langle\alpha\rangle\beta \in y\} \cap A^x_\alpha$. Now we know, by hypothesis, that $S'' \subseteq [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}^c} = [\![\langle\alpha\rangle\chi]\!]_{\mathcal{M}^c}$ (by $(*)$), this implies by lemma 17.2, that $P(\alpha, \chi) \in x$. By $(*)$ again and the inference rule (R) we obtain that $P(\psi, \chi) \in x$.

  - Second, there exists $\neg O(\alpha, \beta) \in x$ s.t. $[\![\psi]\!]_{\mathcal{M}^c} = \{y \in S^c : \alpha \in y\}$ and $S'' = A^x_\alpha$. On one hand, this implies that $\vdash \psi \leftrightarrow \alpha$ and then $\neg O(\psi, \beta) \in x$. On the other hand, with the fact that $S'' \subseteq [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}^c}$ we obtain, by lemma 17.1, that $O(\psi, \chi) \in x$. Now, if we suppose $P(\psi, \chi) \notin x$ then $\psi \wedge O(\psi, \chi) \wedge \neg P(\psi, \chi) \in x$ and thus $\psi \wedge \neg P(\psi, \top) \in x$ by "obligation and prohibition". Therefore, by the same axiom, $O(\psi, \beta) \in x$, which leads to a contradiction. This shows that $P(\psi, \chi) \in x$.

  ($\Leftarrow$) If $P(\psi, \chi) \in x$ then let us define $S' = [\![\psi]\!]_{\mathcal{M}^c}$ and $S'' = [\![\langle\psi\rangle\chi]\!] \cap A^x_\psi$. We then obtain, by definition of $\mathcal{P}^c$, $(x, S', S'') \in \mathcal{P}^c$, and then, as $S'' \subseteq [\![\langle\psi\rangle\chi]\!] = [\![tr(\langle\psi\rangle\chi)]\!]$, $\mathcal{M}_c, x \models P(\psi, \chi)$.

- **$\varphi = O(\psi, \chi)$:**

  ($\Leftarrow$) Suppose that $O(\psi, \chi) \in x$ and $\mathcal{M}_c, x \not\models O(\psi, \chi)$. Thus $\mathcal{M}_c, x \models \psi$ otherwise we would have $\mathcal{M}_c, x \models O(\psi, \chi)$. Now $\mathcal{M}_c, x \not\models O(\psi, \chi)$ implies that there exists $(x, [\![\psi]\!], S'') \in \mathcal{P}^c$ such that $S'' \not\subseteq [\![\langle\psi\rangle\chi]\!]$. That is impossible, because by definition $S'' \subseteq A^x_\psi \subseteq [\![\langle\psi\rangle\chi]\!]$.

  ($\Rightarrow$) Suppose that $O(\psi, \chi) \notin x$ and $\mathcal{M}_c, x \models O(\psi, \chi)$. Then $\neg O(\psi, \chi) \in x$ and, by definition of $\mathcal{P}^c$, $(x, [\![\psi]\!], A^x_\psi) \in \mathcal{P}^c$. But then $\mathcal{M}_c, x \models O(\psi, \chi)$ leads to $A^x_\psi \subseteq [\![\langle\psi\rangle\chi]\!] = [\![tr(\langle\psi\rangle\chi)]\!]$. By Proposition 11, IH (with $deg(tr(\langle\psi\rangle\chi)) \leq k$) and Lemma 17.1, $O(\psi, \chi) \in x$. Contradiction.

$\square$

Proposition 19 *POPAL is sound and complete with respect to the class of all models.*

Proof The soundness has been shown in Proposition 13. By Proposition 18 we can show the completeness with respect to the class of all models. Indeed, for all $\varphi \in \mathcal{L}_{popal}$: $\models \varphi \Rightarrow \models tr(\varphi) \Rightarrow \mathcal{M}^c \models tr(\varphi) \Rightarrow \vdash tr(\varphi) \Rightarrow \vdash \varphi$. $\square$

4 Decidability

We prove in this section that *POPAL* is decidable by proving a *small model property*. To do so, we will use the filtration method.

Definition 20 (Closed set) *Let $X \subseteq \mathcal{L}_{poel}$. We shall say that X is* closed *if the following properties are satisfied:*

- *it is closed under subformulas*
- *for all $P(\psi, \varphi) \in X$, $tr(\langle\psi\rangle\varphi) \in X$*
- *for all $O(\psi, \varphi) \in X$, $tr(\langle\psi\rangle\varphi) \in X$*

Definition 21 (P-filtration) *Let $\mathcal{M} = (S, \sim_i, V, \mathcal{P})$ be a model and Γ be a closed set of formulas. Let $\leftrightsquigarrow_\Gamma$ be the relation on S defined, for all $s, t \in S$, by:*

$$s \leftrightsquigarrow_\Gamma t \quad \textit{iff} \quad \forall\, \varphi \in \Gamma : (\mathcal{M}, s \models \varphi \textit{ iff } \mathcal{M}, t \models \varphi)$$

We call the filtration of $\mathcal{M}$ through Γ *(or simply* the filtration of $\mathcal{M}$*) the model $\mathcal{M}^\Gamma = (S^\Gamma, \sim_i^\Gamma, V^\Gamma, \mathcal{P}^\Gamma)$ where:*

- $S^\Gamma = S/_{\leftrightsquigarrow_\Gamma}$
- *$|s| \sim_i^\Gamma |t|$ iff for all $K_i\varphi \in \Gamma$, $(\mathcal{M}, s \models K_i\varphi$ iff $\mathcal{M}, t \models K_i\varphi)$*
- $V^\Gamma(p) = \begin{cases} \emptyset \; if \; p \notin \Gamma \\ V(p)/_{\leftrightsquigarrow_\Gamma} \; if \; p \in \Gamma) \end{cases}$
- *$\mathcal{P}^\Gamma = \{(|s|, S^1, S^2)$: there exists $t \in |s|$ and $S'' \subseteq S$ s.t. $S''/_{\leftrightsquigarrow_\Gamma} = S^2$ and $(t, \bigcup(S^1), S'') \in \mathcal{P}\}$*

In this definition, S^1 is a set of equivalence classes, and $\bigcup S^1$ is the set of all states that are represented by an element of S^1. Note that $\leftrightsquigarrow_\Gamma$ is an equivalence relation. For all $s \in S$, let us denote $|s|_\Gamma$ (or simply $|s|$) the equivalence class of s with respect to $\leftrightsquigarrow_\Gamma$; and for $S' \subseteq S''$, we write $\leftrightsquigarrow_\Gamma(S')$ for $\{t \in S \mid \exists s \in S' : s \leftrightsquigarrow_\Gamma t\}$. Here is a useful lemma:

Lemma 22 *Let $\Gamma \subset \mathcal{L}_{poel}$ be a finite closed set. For any model $\mathcal{M}$, its filtration $\mathcal{M}^\Gamma$ contains at most 2^m nodes, where $m = Card(\Gamma)$.*

Proof Let $\mathcal{M}$ be a model. Let $g : S^\Gamma \to 2^\Gamma$ defined by $g(|s|) = \{\psi \in \Gamma : \mathcal{M}, s \models \psi\}$. It follows from the definition of $\leftrightsquigarrow_\Gamma$ that g is well-defined and injective. Thus the size of S^Γ is at most 2^m. $\square$

The epistemic relations of a model and their filtrations over a set Γ are linked by the following property:

Proposition 23 *Let $\mathcal{M}$ be a model and Γ be a closed set of formulas. Then for all $s, t \in S$, for all $\varphi \in \Gamma$:*

1. $s \sim_i t \Rightarrow |s| \sim_i^\Gamma |t|$.
2. $|s| \sim_i^\Gamma |t|$ *and* $K_i\varphi \in \Gamma$ *and* $\mathcal{M}, s \models K_i\varphi \Rightarrow \mathcal{M}, t \models \varphi$.

Proof

1. *Let $s, t \in S$ such that $s \sim_i t$, and let $K_i\varphi \in \Gamma$. Then we have $\mathcal{M}, s \models K_i\varphi$ iff for all $u \sim_i s, \mathcal{M}, u \models \varphi$ iff for all $u \sim_i t$, $\mathcal{M}, u \models \varphi$ iff $\mathcal{M}, t \models K_i\varphi$. Then by definition of $\sim_i^\Gamma$ we obtain $|s| \sim_i^\Gamma |t|$.*
2. *Let us suppose the first part of the implication. Since $|s| \sim_i^\Gamma |t|$, $K_i\varphi \in \Gamma$ and $\mathcal{M}, s \models K_i\varphi$ then $\mathcal{M}, t \models K_i\varphi$. Since $\sim_i$ is reflexive $\mathcal{M}, t \models \varphi$.*

$\square$

Proposition 23 is sufficient to prove the following:

Proposition 24 (Filtration lemma) *Let $\mathcal{M}$ be a model and Γ be a closed set of formulas. For all $\varphi \in \Gamma$ we have:*

$$(F^o\varphi) \quad \forall s \in S, (\mathcal{M}, s \models \varphi \textit{ iff } \mathcal{M}^\Gamma, |s| \models \varphi).$$

Proof By induction on the degree of φ.

base case If $deg(\varphi) = 0$ then $\varphi \in \mathcal{L}_{el}$ and the proof of $(F^o\varphi)$ comes by induction on the complexity of φ (see [6] or [9] for details, note that Γ is in particular closed under subformulas).

induction steps Let $k \in \mathbb{N}$. Suppose that $(F^o\psi)$ is true for all $\psi \in \mathcal{L}_{poel}$ such that $deg(\psi) \leqslant k$. Let φ be such that $deg(\varphi) \leq k+1$ and let us reason on the structure of φ.

- $\varphi = p; \bot; \neg\psi; \varphi_1 \vee \varphi_2, K_i\varphi$**:** See the proof of the filtration lemma in [6] or [9].
- $\varphi = P(\psi, \chi)$**:** Let $s \in S$. By construction of Γ we know that

$$\begin{cases} \leftrightsquigarrow_\Gamma([\![\psi]\!]_\mathcal{M}) = [\![\psi]\!]_\mathcal{M} \text{ and} \\ (*) \leftrightsquigarrow_\Gamma([\![\langle\psi\rangle\chi]\!]_\mathcal{M}) = \leftrightsquigarrow_\Gamma([\![tr(\langle\psi\rangle\chi)]\!]_\mathcal{M}) = [\![tr(\langle\psi\rangle\chi)]\!]_\mathcal{M} = [\![\langle\psi\rangle\chi]\!]_\mathcal{M}. \end{cases}$$

($\Rightarrow$) Suppose $\mathcal{M}, s \models P(\psi, \chi)$. Let $S'' \subseteq [\![\langle\psi\rangle\chi]\!]_\mathcal{M} = [\![tr(\langle\psi\rangle\chi)]\!]_\mathcal{M}$ be such that $(s, [\![\psi]\!], S'') \in \mathcal{P}$, and let $S^{oo} = S''/_{\leftrightsquigarrow_\Gamma}$. We have (by IH) that $S^{oo} \subseteq [\![tr(\langle\psi\rangle\chi)]\!]_{\mathcal{M}^\Gamma}$ and we obtain that $(|s|, [\![\psi]\!]_{\mathcal{M}^\Gamma}, S^{oo}) \in \mathcal{P}^\Gamma$ by definition of the filtration and $(*)$. Finally, $\mathcal{M}^\Gamma, |s| \models P(\psi, \chi)$

($\Leftarrow$) Suppose $\mathcal{M}^\Gamma, |s| \models P(\psi, \chi)$. Let $S^{oo} \subseteq [\![tr(\langle\psi\rangle\chi)]\!]_{\mathcal{M}^\Gamma}$ be such that $(|s|, [\![\psi]\!]_{\mathcal{M}^\Gamma}, S^{oo}) \in \mathcal{P}^\Gamma$. Then by definition of $\mathcal{P}^\Gamma$, there exists $t \in |s|$ and S'' such that $S''/_{\leftrightsquigarrow_\Gamma} = S^{oo}$ and $(t, [\![\psi]\!], S'') \in \mathcal{P}$. By IH, $S^{oo} \subseteq [\![tr(\langle\psi\rangle\chi)]\!]_{\mathcal{M}^\Gamma}$ implies that $S'' \subseteq [\![tr(\langle\psi\rangle\chi)]\!]_\mathcal{M}$. Therefore, $\mathcal{M}, t \models P(\psi, \chi)$. Finally, as $s \leftrightsquigarrow_\Gamma t$, $\mathcal{M}, s \models P(\psi, \chi)$.

- $\varphi = O(\psi, \chi)$**:** Let $s \in S$,

  ($\Rightarrow$) Suppose $\mathcal{M}, s \models O(\psi, \chi)$ and let S^{oo} be such that $(|s|, [\![\psi]\!]_{\mathcal{M}^\Gamma}, S^{oo}) \in \mathcal{P}^\Gamma$, we want to show that $S^{oo} \subseteq [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}^\Gamma}$. By definition of the filtration, we can construct S'' such that $S''/_{\leftrightsquigarrow_\Gamma} = S^{oo}$ and $(t, [\![\psi]\!], S'') \in \mathcal{P}$ for some $t \in |s|$. Thus $S'' \subseteq [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}}$, because $\mathcal{M}, t \models O(\psi, \chi)$ (as $s \leftrightsquigarrow_\Gamma t$). Finally $S^{oo} \subseteq [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}^\Gamma}$ by $(*)$ and IH.

  ($\Leftarrow$) Suppose $\mathcal{M}^\Gamma, |s| \models O(\psi, \chi)$ and let S'' be such that $(s, [\![\psi]\!], S'') \in \mathcal{P}$. We show that $S'' \subseteq [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}}$. Let $S^{oo} = S''/_{\leftrightsquigarrow_\Gamma}$, then by definition of the filtration, $(|s|, [\![\psi]\!]_{\mathcal{M}^\Gamma}, S^{oo}) \in \mathcal{P}^\Gamma$. Thus $S^{oo} \subseteq [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}^\Gamma}$ and then $S'' \subseteq [\![\langle\psi\rangle\chi]\!]_{\mathcal{M}^\Gamma}$ by $(*)$ and IH.

$\square$

Definition 25 (Closure) *For all $\varphi \in \mathcal{L}_{poel}$, we construct the P-Closure of φ, noted $Cl(\varphi)$, inductively on the structure of φ:*

- $Cl(p) = \{p\}$
- $Cl(\bot) = \{\bot\}$
- $Cl(\neg\varphi) = \{\neg\varphi\} \cup Cl(\varphi)$
- $Cl(\psi \vee \varphi) = \{\psi \vee \varphi\} \cup Cl(\psi) \cup Cl(\varphi)$
- $Cl(K_i\varphi) = \{K_i\varphi\} \cup Cl(\varphi)$
- $Cl(P(\psi, \varphi)) = \{P(\psi, \varphi)\} \cup Cl(\psi) \cup Cl(\varphi) \cup Cl(tr(\langle\psi\rangle\varphi))$.
- $Cl(O(\psi, \varphi)) = \{O(\psi, \varphi)\} \cup Cl(\psi) \cup Cl(\varphi) \cup Cl(tr(\langle\psi\rangle\varphi))$.

Proposition 26 *For all $\varphi \in \mathcal{L}_{poel}, Cl(\varphi)$ is well-defined and it is a finite closed set.*

Proof The proof is by induction on the degree of φ.

[base case] If $deg(\varphi) = 0$ then $\varphi \in \mathcal{L}_{el}$ and we only need to prove that $Cl(\varphi)$ is a well-defined finite set closed under subformulas, which is straightforward.

[inductive cases] Let $k \in \mathbb{N}$, let us suppose that $Cl(\psi)$ is a well-defined finite closed set for any ψ such that $deg(\psi) \leq k$. Let φ be such that $deg(\varphi) \leq k + 1$ and let us reason inductively on the structure of φ.

- $\varphi = p; \bot; \neg\psi; \varphi_1 \vee \varphi_2; K_i\psi$**:** Trivial.
- $\varphi = P(\psi, \chi)$ **or** $O(\psi, \chi)$**:** By IH, $Cl(\psi), Cl(\chi)$ and $Cl(tr(\langle\psi\rangle\chi))$ are well-defined finite closed sets, so $Cl(P(\psi, \chi))$ and $Cl(O(\psi, \chi))$ are well-defined finite sets. We only need to prove that they are closed, which is straightforward.

$\square$

Proposition 27 (Finite model property)
Let $\varphi \in \mathcal{L}_{poel}$, if φ is satisfiable then φ is satisfiable in a model containing at most 2^m nodes, where $m = Card(Cl(\varphi))$.

Proof Suppose that $\mathcal{M}$ and s are such that $\mathcal{M}, s \models \varphi$. Let $\Gamma = Cl(\varphi)$. Then by Proposition 24, $\mathcal{M}^\Gamma, |s| \models \varphi$. By Lemma 22, $\mathcal{M}^\Gamma$ contains at most 2^m states. □

Theorem 28 *POPAL is decidable.*

Proof Let $\varphi \in \mathcal{L}_{popal}$ be a formula, the following procedure decides whether φ is satisfiable or not:

1. Compute $\Phi = tr(\varphi)$
2. Compute $\Gamma = Cl(\Phi)$
3. For all models $\mathcal{M}$ of size $\leq 2^{Card(\Gamma)}$ check if there exists $s \in \mathcal{M}$ such that $\mathcal{M}, s \models \Phi$.

□

5 Extended example: La Belote

We now consider the French card game "la Belote". For a full description of the game, see `http://en.wikipedia.org/wiki/Belote`. The game is played with four players, who form two teams, and with 32 cards of a regular full deck of cards (the ranks 2 to 6 are eliminated). The name of the game, "belote", is also used in the game to designate a pair of a King and a Queen of a trump suit.

After the deal, and after the choice of a trump suit, the first person to play chooses a card of her hand, followed by the other players in clockwise order. The player who dealt the highest trump card or the highest of the same color as the first player's card wins the round and starts the next round. Except for the first player of a round, each player has to follow suit or, if she cannot, to play trump. Moreover, when a trump is played, it is forbidden to play a lower trump.

The act of playing a card is the public announcement that the corresponding card belonged to the corresponding player. We model the game with the set of propositional atoms Θ expressing card ownership, namely $\{RC_i \mid R \in \{7, 8, 9, 10, J, Q, K, A\}, C \in \{\clubsuit, \heartsuit, \diamondsuit, \spadesuit\}, i \in \{1, 2, 3, 4\}\}$. An atom RC_i stands for 'player i holds a card with rank R of suit C'. For any suit C and player i, we introduce the abbreviations $C_i = \bigvee_R RC_i$, and $C_i^{>R} = \bigvee_{R'>R} R'C_i$.

A model $\mathcal{M} = (S, \{\sim_i\}, V, \mathcal{P})$ is called a "model of La Belote" if

1. for each state s, for any R and C, there is exactly one i such that $s \in V(RC_i)$ (i.e. the states of $\mathcal{M}$ are deals of cards);
2. for any $s, t \in S$ and any i, $s \sim_i t$ implies that for all R, C: $s \in V(RC_i)$ iff $t \in V(RC_i)$ (i.e. each player can distinguish different cards);

3. $\mathcal{P}$ is constructed from $(S, \{\sim_i\}, V)$ according to the rules of the game.

The last item means that in a given deal s, for all the cards p held by an agent i that are permitted by the rules to be played, $(s, S, S_{p_i}) \in \mathcal{P}$. If after p has been played by player i it is permitted for player j to play q, then we also need that $(s, S_{p_i}, S_{\langle p_i \rangle q_j}) \in \mathcal{P}$. And so on, for all possible moves.

Let $\mathcal{M}$ be a model of La Belote. The trump suit has been selected before the game starts, we will suppose that it is clubs. The set of atoms is partially ordered as follows ($*$ can be one of the players $1, 2, 3, 4$). First, any trump is higher than any non-trump. But the card are also ordered in the following way: for non-trumps (i.e. for any $C \neq \clubsuit$):

$$7C_* < 8C_* < 9C_* < JC_* < QC_* < KC_* < 10C_* < AC_*$$

For trumps:

$$7\clubsuit_* < 8\clubsuit_* < Q\clubsuit_* < K\clubsuit_* < 10\clubsuit_* < A\clubsuit_* < 9\clubsuit_* < J\clubsuit_*$$

For more details, see the mentioned website. We now list a number of model validities of Belote. These formulas are valid at the beginning of each round of the game, in other words, the models $\mathcal{M}$ considered below result from any iteration of a sequence of four permitted announcements. We will call 1 the player that opens the round, followed by 2, etc.

1-One player at once:

For all $\psi \in \mathcal{L}_{popal}$, all $i \neq j$, all $p_i, q_j \in \Theta$, $\mathcal{M} \models P(\psi, p_i) \rightarrow \neg P(\psi, q_j)$.
Two different players are not allowed to play simultaneously.

2-Each card is played once:

For all $p \in \Theta$, all $\psi \in \mathcal{L}_{popal}$, $\mathcal{M} \models \neg P(\langle p \rangle \psi, p)$.
If a card has been played once, it cannot be played again.

3-Obligation to follow suit:

For all ranks R and all suits C, $\mathcal{M} \models C_2 \rightarrow O(RC_1, C_2)$.
If the player 2 can follow the suit asked by 1, he is obliged to do so.

4-Obligation to play trump:

For all ranks R, all suit $C \neq \clubsuit$, $\mathcal{M} \models \clubsuit_2 \rightarrow O(RC_1, C_2 \vee \clubsuit_2)$.
If the player 2 can follow the suit asked by 1 or play trump, he is obliged to do so.

5-Permission to say "belote et rebelote":

For all players i, $\mathcal{M} \models K\clubsuit_i \wedge Q\clubsuit_i \wedge (P(\psi, Q\clubsuit_i) \vee P(\psi, K\clubsuit_i)) \rightarrow P(\psi, Q\clubsuit_i \wedge K\clubsuit_i)$.
If a player one is allowed to play the queen of the trump suit, he is allowed to announce that he has the royal couple (called the "belote"). This does not mean that she is allowed to play both cards, but playing one of them she is allowed to announce that she also has the other one.

6-Obligation to go up at trump:

> For all $\psi \in \mathcal{L}_{popal}$, all player i and all R, $\mathcal{M} \models \clubsuit_i^{>R} \to O(\langle\psi\rangle R\clubsuit_{i-1}, \clubsuit_i^{>R})$. It says that if the previous player played trump and if you have higher cards than the played trump, then you are obliged to play one of them.

We apply these conditional rules about the permission to speak to the following state (deal) s, where each player has 2 cards.

$$
\begin{array}{ccccc}
 & & Bill(B) & & \\
 & & \boxed{Q\heartsuit}\boxed{K\spadesuit} & & \\
Anne & \boxed{7\diamondsuit} & & \boxed{Q\clubsuit} & Charles \\
(A) & \boxed{8\heartsuit} & & \boxed{K\clubsuit} & (C) \\
 & & \boxed{8\clubsuit}\boxed{A\clubsuit} & & \\
 & & Diane(D) & &
\end{array}
$$

Anne starts the game. According to the rule, our model validates the following formulas:

- $\mathcal{M}, s \models P(8\heartsuit_A) \wedge P(7\diamondsuit_A) \wedge \neg P(8\heartsuit_A \wedge 7\diamondsuit_A)$:

  Anne has the permission to play one of her cards, but not both.

- $\mathcal{M}, s \models O(8\heartsuit_A, Q\heartsuit_B)$:

  If Anne plays the $8\heartsuit$ card, Bill is obliged to play a card of the same suit, he cannot play his $K\spadesuit$ card (rule (3)).

- $\mathcal{M}, s \models P(\langle 8\heartsuit_A\rangle Q\heartsuit_B, Q\clubsuit_C \wedge K\clubsuit_C)$:

  Charles has the permission to announce that he has both cards of the "belote" (rule (5)).

- $\mathcal{M}, s \models O(\langle\langle 8\heartsuit_A\rangle Q\heartsuit_B\rangle Q\clubsuit_C, A\clubsuit_D)$:

  The 'go up at trump" applies if Charles plays the queen of clubs. As Diane has a unique higher trump, she has the obligation to play it.

6 Comparison to the literature

6.1 Classic deontic principles and paradoxes

As we reviewed before, deontic logic started out with Von Wright's operators P and O binding formulas in expression $P\varphi$ and $O\varphi$, then came Meyer's and van der Meyden's mind-frame switch to operators P and O binding actions, and finally we treat communicative actions that are represented by the announced formulas. We introduce by abbreviation the permission or obligation to speak φ as $P\varphi$ and $O\varphi$. Well, if we end up with such expressions, how do its validities relate to the standard and historical von Wright approach? In this subsection, we summarily treat that matter.

First, a disappointment: the P and O operators we have introduced are *not normal modal operators* (the triples in the $\mathcal{P}$ relation rather suggest a modality with a neighbourhood-type of semantics). They do not satisfy necessitation! A formula may be valid, but that does not make it an obligation, or permitted; if you are not allowed to announce p nor $\neg p$, it does not help you a great deal that $p \vee \neg p$ is a validity! Something else has to be underlined: our formalism allows to consider situations in which nothing is permitted to be said, which is equivalent to the fact that everything is obligatory. To avoid such borderline cases, we will often consider the class of models in which for all ψ, $\psi \to P(\psi, \top)$ is valid, called the "permissive models". In the class of permissive models the obligation and prohibition axiom becomes the classical one: $O\varphi \to P\varphi$: something obligatory is permitted. Obligation distributes over conjunction (and implication), as $O(\varphi \wedge \psi) \leftrightarrow (O\varphi \wedge O\psi)$ is a special case (in the case where the first argument is $\top$) of Proposition 9.1. Permission does not distribute over conjunction: we may have that p and q are both permissible announcements, such that Pp and Pq are true, but not at the same time, $P(p \wedge q)$ may be false. This reflects that for a given Kripke model with domain S and actual state s the relation $\mathcal{P}$ may contain $(s, S, [\![p]\!])$ and $(s, S, [\![q]\!])$ but not $(s, S, [\![p \wedge q]\!])$. However, given weakening of permitted announcements, a valid principle indeed is $P(\varphi \wedge \psi) \to (P\varphi \wedge P\psi)$.

Permitted announcements are true, obligatory ones also in the permissive models: $P\varphi \to \varphi$ and $O\varphi \to \varphi$. A principle obviously false in classic deontic logic. But one has to realize the special reading of such implications in our setting! $P\varphi \to \varphi$ is valid because a precondition for a permitted announcement is the truth of the announcement formula. It does not formalize that all permitted actions always take place. A similar slip of the deontic mind occurs when observing that $P\varphi \to P(\varphi \vee \psi)$ is valid. Doesn't this conflict with Ross's Paradox [20]? We addressed this matter in the introduction, let us go over the details. Ross's Paradox is about the reading (for permission and for obligation) that 'to be permitted to do a or b' entails 'to be permitted to do a' *and* 'to be permitted to do b'. In the setting of permitted announcements we have to clearly distinguish the action of announcing from the formula being announced. Permission to announce a or b indeed entails permission to perform either announcement, and choose between them. This is a nondeterministic action. This is different from the permission to make an announcement weaker than the announcement of a, such as $a \vee b$. In other words, permission to announce a or b is not the same as permission to announce $a \vee b$. Possibly, "permission to announce a or b" might be called *ambiguous*, as the 'or' may mean logical disjunction of formulas or non-deterministic choice between programs. But once the reading has been chosen, the course is clear.

We already observed that obligation and permission are not interdefinable. In Proposition 6 we showed that obligation adds to the expressivity of the logic. So $O\varphi \leftrightarrow \neg P \neg \varphi$ is not valid. Now, Clearly, $O\varphi \to \neg O \neg \varphi$ is valid in the class of permissive models. But then again, even in the permissive cases, $P\varphi \vee P\neg\varphi$ is not valid: there is nothing against both p and $\neg p$ being forbidden announcements at the same time! For yet another example, consider the schema $O(O\varphi \to \varphi)$, formalizing the requirement that obligations are fulfilled. In our setting, either we are in a non-permissive case and thus this obligation is satisfied, or it is a permissive one and thus as $O\varphi \to \varphi$ is valid, this is equivalent to the validity of $O\top$, which indeed is a validity (note that $\top$ is weaker than any obligatory announcement, and that weakening holds for obligation).

A more recent development in deontic logic is the interaction between obligations and permissions and explicit agency [8, 11]. The well-known Meinong-Chisholm reduction of "The agent is obliged to do a" to "It is obligatory that the agent does a" seems to have an interesting parallel in the logic of permitted announcements. In the logic of public announcements, the announcement by agent a is typically reduced to 'the (public) announcement of 'agent a knows φ'. It is relevant to recall at this stage that public announcements are supposedly made by outsiders of the system, not by agents modelled explicitly in the logical language. This observation can be applied in the logic of permitted and obligatory announcements! A Meinongian turn to permitted announcements seems to interpret $OK_i\varphi$—"It is obligatory that agent i announces φ" (announcements of φ by an agent i in the system are known to be true by that agent, so in fact have form $K_i\varphi$)—as an indirect form of agency in our logic, namely, we can let it stand for "Agent i is obliged to announce φ."

6.2 Deontic action logics

For the purpose of comparing our work with the existing literature we present the version of the semantics for permission that was presented in the precursor [3] to our work. Our current understanding of $P(\psi, \varphi)$ is that "*after the announcement of ψ it is permitted to give* at most *the information φ*". Any weakening of φ is also permitted. Instead, in [3] it is ' '*after the announcement of ψ it is permitted to give* exactly *the information φ*". We will write $P^=$ for that modality. It has the semantics: for all $\mathcal{M}$ and s in the domain of $\mathcal{M}$:

$$\mathcal{M}, s \models P^=(\psi, \varphi) \text{ iff } (s, [\![\psi]\!], [\![\langle\psi\rangle\varphi]\!]) \in \mathcal{P}.$$

Our current logic subsumes this somewhat different logic of permission. Let us expand a given relation $\mathcal{P}$ with all supersets for the third argument of a triple in that relation: for all subsets S''' of the domain of a given model $\mathcal{M}$, if $(s, S', S'') \in \mathcal{P}$ and $S'' \subseteq S'''$, then add (s, S', S''') to $\mathcal{P}$. Call the resulting relation $\mathcal{P}^=$ and let $\mathcal{M}^=$ be the model with $\mathcal{P}^=$ instead of $\mathcal{P}$. On the language without obligation, inductively define a translation $\bullet^=$ that replaces all occurrences of P by $P^=$. We now have that $\mathcal{M}, s \models P(\psi, \varphi)$ iff $\mathcal{M}^=, s \models P^=(\psi, \varphi)$.

The Dynamic Logic of Permission by Van der Meyden To formalize the permission to act van der Meyden proposed a dynamic logic of permission [22], later followed up by Pucella and Weismann's [19]. Van der Meyden's logic is an adaptation of propositional dynamic logic (PDL) [10] in which the models contain a set $\mathcal{P} \subseteq S \times \mathbf{Act} \times S$ that links every action in **Act** to the set of all transitions between states in S that are permitted for it. In other words, $\mathcal{P}$ is a set of permitted transitions. The syntax of this language contains the following constructs $\Diamond(\alpha, \varphi)$ that means "*there is a way to execute action α which is permitted and after which φ is true*". Our semantics for $P^{=}(\psi, \varphi)$ consists of the particular case where actions are public announcements. Thus, for α in van der Meyden's $\Diamond(\alpha, \varphi)$ we take an announcement $\psi!$ such that $\Diamond(\psi!, \varphi)$ now means 'it is permitted to announce ψ, after which φ is true'. The precise correspondence is:

Proposition 29 *$\Diamond(\varphi!, \theta)$ is equivalent to $P^{=}(\top, \varphi) \wedge \langle\varphi\rangle\theta$*

Proof Given a model $\mathcal{M}$ with domain S, we can see the announcement $\varphi!$ as an atomic action which links each state $s \in [\![\varphi]\!]_{\mathcal{M}}$ to the same state $s \in S_{\varphi}$. This is a permitted action in van der Meyden's semantics if and only if $(s, S, S_{\varphi}) \in \mathcal{P}$. By definition, $\mathcal{M}, s \models P^{=}(\top, \varphi)$ iff $(s, S, S_{\varphi}) \in \mathcal{P}$. The formula θ should then hold after the permitted announcement of φ. □

Van der Meyden's $\Diamond(\varphi!, \theta)$ is found in a syntactic variant $Perm(\varphi)\theta$ in [19]. Now, we have that $P^{=}(\top, \varphi)$ is equivalent to $Perm(\varphi)$. Given the abbreviation $P(\varphi)$ in our language for $P^{=}(\top, \varphi)$, the correspondence is therefore very close.

Merging Frameworks for Interaction by van Benthem et al. In "Merging Frameworks for Interaction" [21] (see also [12]), a logic for protocols in dynamic epistemic logic is proposed that can be interpreted as a logic for permitted events – and in particular permitted announcements. A protocol is a set of event sequences, and an announcement is an example of such an event; "*being in the protocol*" can therefore be understood as "*being permitted to be said*". An objective of this publication was to merge epistemic temporal logic [16] with dynamic epistemic logic [4, 24], and the axiomatization of the language with added protocols is facilitated by the translation of the latter into the former.

For the purpose of our comparison we present what is known as the *forest* generated by a pointed epistemic model $(\mathcal{M}, s)$ and a number of pointed events models. A prefix-closed set of such pointed events sequences can be seen as the *protocol* Π, and the set consisting of such sequences preceded by a state in the model wherein they are executed as the *history* H pertaining to the model $\mathcal{M}$. (Given an initial state s, and say a sequence of first $\psi!$ and then $\varphi!$ as allowed according to protocol, we write $s\psi\varphi$ for that history: the announcements in sequence are simply written one after the other.) Relative to this protocol we can construct a temporal epistemic model $\mathcal{M}_{\Pi}$ (details omitted). We now have that $\mathcal{M}_{\Pi}, h \models \langle\psi\rangle\varphi$ iff:

- $\mathcal{M}_{\Pi}, h \models \psi$
- $h' = h\psi \in \Pi$
- $\mathcal{M}_{\Pi}, h' \models \varphi$

This *suggests* to translate $P^{=}(\psi, \varphi)$ in $\mathcal{L}_{popal}$ by $[\psi]\langle\varphi\rangle\top$ in the logical language $\mathcal{L}_{tpal}$ of van Benthem et al.'s protocol logic $TPAL$. (For convenience, we write announcements $\psi!$ and $\varphi!$ instead of singleton event models with precondition ψ and φ, respectively.) Unfortunately, this translation is imprecise. Consider executing these two announcements in a state s of an initial model $\mathcal{M}$. If $s\psi \notin H$ then $\mathcal{M}_\Pi, s \models [\psi]\langle\varphi\rangle\top$: after a non-permitted announcement, anything is permitted to be said, because anything holds after a necessity-type modal operator that cannot be executed. But $\mathcal{M}, s \not\models P^{=}(\psi, \varphi)$, because $(s, [\![\psi]\!], [\![\langle\psi\rangle\varphi]\!])$ is not in the $\mathcal{P}$ relation to validate it. In other words, in our logic we get the *full* forest produced by the protocol of all truthful public announcements, but some branches are coloured with *permitted* and others are coloured with not-permitted. The Van Benthem et al. approach produces a forest restricted to the protocol (i.e., restricted to permitted announcements only).

A more serious problem with such a translation is as follows. Our semantics allows that if something is later permitted to be said, we are already permitted to say something now in a different way, a consequence of the axiom "announcement and permission" $[\psi]P(\psi', \varphi) \leftrightarrow (\psi \rightarrow P(\langle\psi\rangle\psi', \varphi))$. (This axiom holds for $P^{=}$ as well.) In TPAL this would amount to requiring that (announcement) protocols are *postfix*-closed in the restricted sense that if $\pi'\pi'' = \pi \in \Pi$, then there is a single announcement ξ (combining all the announcements in the initial π' part in one complex announcement) such that $\xi\pi'' \in \Pi$.

Our logic with P instead of $P^{=}$ and with obligation O as well makes the comparison even more problematic. As we now know, the notion of "*obligation to say* φ" cannot be captured only by the negation of permission to say anything else than φ (except in a very radical dictatorship), but much more by the fact that all that does not say *at least* φ is not permitted. This notion of obligation we consider a strong point of our logic $POPAL$, in which it differs from known other proposals.

An interesting recent appearance on dynamic deontic logic, with the difference that 'the right to say' is derived from 'the right to know', is [2].

7 Conclusions and further research

We proposed a logic for the permission and obligation to say something. We axiomatized the logic and have shown that it is decidable. The analysis of the card game 'La Belote' illustrates the logic.

Various issues are left for further research. Our tentative observations on the relation between $P\varphi$ and $O\varphi$ and 'classic' deontic principles, in Section 6.1, seems worthwhile to pursue more systematically. The generalization to more complex dynamics than (public) announcements is obvious. This would allow to model permission for an individual to say something in the presence of some but not all other agents. Such a logic of permitted announcement with more explicit agency is under development: in that logic the primitive is that agent i is permitted to say something, and to some agent j only but not to all agents; instead of permission relation $\mathcal{P}$ we then have an individual permission relation

$\mathcal{P}_i$. We also consider to expand the framework with changing permissions, as in Pucella et al. [19].

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Appendix

Proposition 12 says that for all $\varphi \in \mathcal{L}_{popal}$, the property $\Pi(\varphi) : deg(\varphi) = deg(tr(\varphi))$ is true. To prove it, we will first introduce two preliminary lemmas.

Lemma 30 *For all formula $\varphi \in \mathcal{L}_{poel}, \Pi(\varphi)$.*

Proof Trivial, because in that case $tr(\varphi) = \varphi$. □

Lemma 31 *For all $\psi \in \mathcal{L}_{popal}$, if $\Pi(\psi)$ then for all $\theta \in \mathcal{L}_{poel}$ such that $\Pi(\chi)$ is true for any χ subformula of θ, $\Pi([\psi]\theta)$ is true.*

Proof Let us prove it by induction on the structure of θ. We denote $=_{IH}$ the equalities that come from induction hypothesis and $=^*$ the ones that come from $\Pi(\psi)$.

Base cases:

- $\theta = p$:

$$\begin{aligned} deg(tr([\psi]p)) &= deg(tr(\psi) \to p) \\ &= max(deg(tr(\psi)), deg(p)) \\ &= deg(tr(\psi)) \\ &=^* deg(\psi) \\ &= deg([\psi]p) \end{aligned}$$

Inductive cases: let us suppose $\Pi([\psi]\chi)$ is true for all χ subformula of θ.

- $\theta = \bot$:

$$\begin{aligned} deg(tr([\psi]\bot)) &= deg(\neg tr(\psi)) \\ &= deg(tr(\psi)) \\ &=^* deg(\psi) \\ &= deg([\psi]\bot). \end{aligned}$$

- $\theta = \neg\chi$:

$$\begin{aligned} deg(tr([\psi]\neg\chi)) &= deg(tr(\psi) \to \neg tr([\psi]\chi)) \\ &= max(deg(tr(\psi)), deg(tr([\psi]\chi))) \\ &=^*_{IH} max(deg(\psi), deg([\psi]\chi)) \\ &= deg(\psi) + deg(\chi) \\ &= deg([\psi]\neg\chi) \end{aligned}$$

- $\theta = \chi_1 \vee \chi_2$:

$$\begin{aligned} deg(tr([\psi](\chi_1 \vee \chi_2))) &= deg(tr([\psi]\chi_1) \vee tr([\psi]\chi_2)) \\ &= max(deg(tr([\psi]\chi_1)), deg(tr([\psi]\chi_2))) \\ &=_{IH} max(deg([\psi]\chi_1), deg([\psi]\chi_2)) \\ &= max(deg(\psi) + deg(\chi_1), deg(\psi) + deg(\chi_2)) \\ &= deg(\psi) + max(deg(\chi_1), deg(\chi_2)) \\ &= deg([\psi](\chi_1 \vee \chi_2)) \end{aligned}$$

- $\theta = K_i\chi$:

$$\begin{aligned} deg(tr([\psi]K_i\chi)) &= deg(tr(\psi) \to K_i tr([\psi]\chi)) \\ &= max(deg(tr(\psi)), deg(tr([\psi]\chi))) \\ &=^*_{IH} max(deg(\psi), deg[\psi]\chi)) \\ &= deg([\psi]\chi) = deg(\psi) + deg(\chi) \\ &= deg([\psi]K_i\chi) \end{aligned}$$

- $\theta = P(\psi', \chi)$:

$$\begin{aligned} deg(tr([\psi]P(\psi',\chi))) &= deg(tr(\psi) \to P(tr(\langle\psi\rangle\psi'), tr(\chi))) \\ &= max(deg(tr(\psi)), deg(P(tr(\langle\psi\rangle\psi'), tr(\chi)))) \\ &=^* max(deg(\psi), deg(tr(\langle\psi\rangle\psi')) + deg(tr(\chi)) + 1) \\ &=_{IH} max(deg(\psi), deg(\langle\psi\rangle\psi') + deg(tr(\chi)) + 1) \\ &= deg(\langle\psi\rangle\psi') + deg(tr(\chi)) + 1 \\ &=_{IH} deg(\psi) + deg(\psi') + deg(\chi) + 1 \\ &= deg([\psi]P(\psi',\chi)) \end{aligned}$$

- $\theta = O(\psi', \chi)$: Idem

□

Lemma 32 *For all $\psi \in \mathcal{L}_{popal}$, if $\Pi(\psi)$ then for all $\theta \in \mathcal{L}_{popal}$ such that $\Pi(\chi)$ is true for any χ subformula of θ, $\Pi([\psi]\theta)$ is true.*

To prove it, we introduce the [.]-degree of a formula:

Definition 33 *The [.]-degree is defined inductively as follows:*

$$\begin{array}{ll} deg_{[.]}(p) = deg_{[.]}(\bot) = 0; & deg_{[.]}(\psi_1 \vee \varphi_2) = max(deg_{[.]}(\psi_1), deg_{[.]}(\psi_2)) \\ deg_{[.]}(\neg\psi) = deg_{[.]}(\psi); & deg_{[.]}([\psi]\varphi) = deg_{[.]}(\psi) + deg_{[.]}(\varphi) + 1 \\ deg_{[.]}(K_i\psi) = deg_{[.]}(\psi); & deg_{[.]}(P(\psi,\varphi)) = deg_{[.]}(\psi) + deg_{[.]}(\varphi). \end{array}$$

Proof Let us prove it by induction of the [.]-degree of θ. The previous lemma gives us the result in the base case where $deg_{[.]}\theta = 0$. Let us suppose $\Pi([\psi]\chi)$ is true for all $\chi \in \mathcal{L}_{popal}$ such that $deg_{[.]}\chi \leqslant n$. Let ν be such that $deg_{[.]}(\nu) \leqslant n+1$ and let us prove $\Pi([\psi]\nu)$ by induction on the structure of ν

- **$\nu = p, \bot, \neg\mu, \mu_1 \vee \mu_2, K_i\mu, P(\mu_1, \mu_2), O(\mu_1, \mu_2)$:** already done.
- **$\nu = [\psi']\chi$:**

$$\begin{aligned} deg(tr([\psi][\psi']\chi)) &= deg(tr([\langle\psi\rangle\psi']\chi)) \text{ by definition} \\ &= deg([\langle\psi\rangle\psi']\chi) \\ &\text{(because } \Pi(\psi') \text{ (by IH) and } deg_{[.]}(\chi) \leqslant n) \\ &= deg(\langle\psi\rangle\psi') + \mathrm{deg}(\chi) \\ &= deg(\psi) + deg(\psi') + deg(\chi) \\ &= deg([\psi][\psi']\chi) \end{aligned}$$

□

Proof (of lemma 12) Let us prove it by induction on the structure of φ.

Base case: $\varphi = p, \bot$**:** in this case, $tr(\varphi) = \varphi$ and the result is trivial.

Inductive steps: let us suppose that $\Pi(\theta)$ is true for any subformula θ of φ, and let us prove $\Pi(\varphi)$ in the following cases:

- **$\varphi = \neg\psi$:**

$$\begin{aligned} deg(tr(\neg\psi)) &= deg(\neg tr(\psi)) \\ &= deg(tr(\psi)) \\ &= deg(\psi) \\ &= deg(\neg\psi) \end{aligned}$$

- $\varphi = \chi \vee \psi$:

$$\begin{aligned} deg(tr(\chi \vee \psi)) &= deg(tr(\chi) \vee tr(\psi)) \\ &= max(deg(tr(\chi)), deg(tr(\psi))) \\ &= max(deg(\chi), deg(\psi)) \\ &= deg(\chi \vee \psi) \end{aligned}$$

- $\varphi = K_i\psi$:

$$\begin{aligned} deg(tr(K_i\psi)) &= deg(K_i tr(\psi)) \\ &= deg(tr(\psi)) \\ &= deg(\psi) \\ &= deg(K_i\psi) \end{aligned}$$

- $\varphi = P(\chi, \psi)$:

$$\begin{aligned} deg(tr(P(\chi, \psi))) &= deg(P(tr(\chi), tr(\psi))) \\ &= deg(tr(\chi)) + deg(tr(\psi)) + 1 \\ &= deg(\chi) + deg(\psi) + 1 \\ &= deg(P(\chi, \psi)) \end{aligned}$$

- $\varphi = O(\chi, \psi)$:

$$\begin{aligned} deg(tr(O(\chi, \psi))) &= deg(O(tr(\chi), tr(\psi))) \\ &= deg(tr(\chi)) + deg(tr(\psi)) + 1 \\ &= deg(\chi) + deg(\psi) + 1 \\ &= deg(O(\chi, \psi)) \end{aligned}$$

- $\varphi = [\psi]\chi$:

  Proved by Lemma 32.

$\square$