

ON NONCOMMUTATIVE WEIGHTED LOCAL ERGODIC THEOREMS ON L^p -SPACES

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ABSTRACT. In the present paper we consider a von Neumann algebra M with a faithful normal semi-finite trace τ , and $\{\alpha_t\}$ a strongly continuous extension to $L^p(M, \tau)$ of a semigroup of absolute contractions on $L^1(M, \tau)$. By means of a non-commutative Banach Principle we prove for a Besicovitch function b and $x \in L^p(M, \tau)$, the averages

$$\frac{1}{T} \int_0^T b(t) \alpha_t(x) dt$$

converge bilateral almost uniform in $L^p(M, \tau)$ as $T \rightarrow \infty$.

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1. INTRODUCTION

It is known (see for example [12]) that in the classical ergodic theory one of the powerful tools in dealing with the almost everywhere convergence of ergodic averages is the well-known Banach Principle, which can be formulated as follows:

Theorem 1.1. *Let (S, F, m) be a measurable space with a σ -finite measure and let X be a Banach space. Let $\{a_n\}$ be a sequence of continuous linear maps of X into the space of measurable functions on S . Assume that $\sup_n \{|a_n(x)(s)|\} < \infty$ for each $x \in X$ and almost all $s \in S$. If the sequence $a_n(x)$ converges almost everywhere for x in a dense subset of X , then this sequence converges for each $x \in X$.*

This principle is often applied in proofs concerning the almost everywhere convergence of weighted averages, moving averages, etc.

In a non-commutative setting the almost everywhere convergence of sequences of operators were applied to study of the individual ergodic theorems in von Neumann algebras by many authors [6],[7],[13],[17],[22] (see [10] for review). But in these investigations those ergodic theorems were obtained without using an analog of the Banach Principle. In [8] firstly a non-commutative analog of such principle was proved for quasi-uniform convergence. Using that result in [14] a uniform sequence weighted ergodic theorem was proved in the space of integrable operators affiliated with a von Neumann algebra. Recently, in [3] for the Banach Principle for bilateral uniform convergence has been adopted, and by means of it the Besicovitch weighted ergodic theorem has been proved.

In the present paper we are going to prove local and weighted local ergodic theorems on non-commutative L^p -spaces by means of the Banach principle. Note

that such kind of theorems in commutative settings were studied by many authors (see for example [1],[9],[12]). In a non-commutative setting we mention works [2], [4], [10], [11], [23].

Let us end this section with description of the organization of the paper. In Section 2, we recall some preliminary results and formulate the Banach Principle. There, to prove local ergodic theorem, we adopt the principle in a more convenient formulation. In the next Section 3 we prove the local ergodic theorem for semigroups of absolute contractions of L^p -spaces. Note that this section reviews the results of [2],[11]. Using the result of Sec. 3, in the last Section 4, we establish a weighted local ergodic theorem by means of the Banach principle.

2. PRELIMINARIES

Let M be a semifinite von Neumann algebra acting on a Hilbert space H , let τ be a faithful normal semifinite trace on M , let $P(M)$ be the complete lattice of all projections in M . A densely-defined closed operator x in H is said to be *affiliated* with M if $y'x \subset xy'$ for every $y' \in M'$, where M' is the commutant of the algebra M . An operator x , affiliated with M , is said to be τ -*measurable* if for each $\varepsilon > 0$ there exists $e \in P(M)$ with $\tau(e^\perp) \leq \varepsilon$ such that $eH \subset D_x$, where $e^\perp = \mathbf{1} - e$, $\mathbf{1}$ is the unit of M , D_x is the domain of definition of x . Let $S(M)$ be the set of all τ -measurable operators affiliated with M . Let $\|\cdot\|$ stand for the uniform norm in M . The *measure topology* in $S(M)$ is given by the system

$$V(\varepsilon, \delta) = \{x \in S(M) : \|xe\| \leq \delta \text{ for some } e \in P(M) \text{ with } \tau(e^\perp) \leq \varepsilon\},$$

$\varepsilon > 0$, $\delta > 0$, of neighborhoods of zero. Accordingly, a sequence $\{x_n\} \subset S(M)$ converges *in measure* to $x \in S(M)$, $x_n \rightarrow x$ (m), if, given $\varepsilon > 0, \delta > 0$, there is a number $N = N(\varepsilon, \delta)$ such that for any $n \geq N$ there exists a projection $e_n \in P(M)$ satisfying the conditions $\tau(e_n^\perp) < \varepsilon$ and $\|(x_n - x)e_n\| < \delta$.

Theorem 2.1. [15] *Equipped with the measure topology, $S(M)$ is a complete topological $*$ -algebra.*

For a positive self-adjoint operator $x = \int_0^\infty \lambda de_\lambda$ affiliated with M one can define

$$\tau(x) = \sup_n \tau \left(\int_0^n \lambda de_\lambda \right) = \int_0^\infty d\tau(e_\lambda).$$

If $0 < p \leq \infty$, then

$$L^p = L^p(M, \tau) = \begin{cases} \{x \in S(M) : \|x\|_p = \tau(|x|^p)^{1/p} < \infty\}, & \text{for } p \neq \infty \\ (M, \|\cdot\|), & \text{for } p = \infty. \end{cases}$$

Here, $|x|$ is the *absolute value* of x , i.e. the square root of x^*x . By L_+^p (resp. L_{sa}^p) we denote the set of positive (resp. self-adjoint) elements of L^p . We refer a reader to [18] for more information about noncommutative integration and to [19, 21] for general terminology of von Neumann algebras.

There are several different types of convergences in $S(M)$ each of which, in the commutative case with finite measure, reduces to the almost everywhere convergence (see for example [16]). In the paper we deal with so called the *bilateral almost uniform* (b.a.u.) convergence in $S(M)$ for which $x_n \rightarrow x$ means that for every $\varepsilon > 0$ there exists $e \in P(M)$ with $\tau(e^\perp) \leq \varepsilon$ such that $\|e(x_n - x)e\| \rightarrow 0$. It is clear that b.a.u. implies convergence in measure. Now recall well known fact concerning b.a.u. convergence (see [15],[20]).

Lemma 2.2. *Let M be as above. If two sequences x_n and y_n converge b.a.u., then $x_n + y_n$ converges b.a.u.*

In [3] the following results has been proved.

Theorem 2.3. *Algebra $S(M)$ is complete with respect to the b.a.u. convergence.*

Lemma 2.4. *Let $0 \leq p < \infty$, and let $\{x_n\} \subset L^p$ be such that $\liminf_n \|x_n\|_p = s < \infty$. If $x_n \rightarrow x$ b.a.u., then $x \in L^p$ and $\|x\|_p \leq s$.*

Recall a non-commutative the Banach Principle (see [3]). Let $(X, \|\cdot\|, \geq)$ be an ordered real Banach space with the closed convex cone X_+ , $X = X_+ - X_+$. A subset $X_0 \subset X_+$ is said to be *minorantly dense* in X_+ if for every $x \in X_+$ there is a sequence $\{x_n\}$ in X_0 such that $x_n \leq x$ for each n , and $\|x - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. A linear map $a : X \rightarrow S(M)$ is called *positive* if $a(x) \geq 0$ whenever $x \in X_+$.

Theorem 2.5. *Let X be an ordered real Banach space with the closed convex cone X_+ . Let $a_n : X \rightarrow S(M)$ be a sequence of positive continuous (in the measure topology) linear maps satisfying the conditions*

- (i) *For every $x \in X_+$ and $\varepsilon > 0$ there is $b \in M$, $0 \neq b \leq I$, such that $\tau(I - b) < \varepsilon$, and*

$$\sup_n \|ba_n(x)b\| < \infty.$$

If, for every x from a minorantly dense subset $X_0 \subset X_+$,

- (ii) *$a_m(x) - a_n(x) \rightarrow 0$ b.a.u., $m, n \rightarrow \infty$,*

then (ii) holds on all of X .

Remark. According to Theorem 2.3 that the fundamental sequences in Theorem 2.5 indeed have their limits belonging to $S(M)$.

As it has been pointed out that the Banach Principle is one of the basic tools to prove ergodic theorems. But the above formulated Principle is too complicated to apply, since it requires minorantly density of X_0 , which makes difficult to check the condition (ii). Basically, to obtain some ergodic theorems we really need the following theorem, which is an analog of the Banach Principle.

Theorem 2.6. *Let X be a Banach space and let $\mathbf{a}_n : X \rightarrow S(M)$ be a sequence of linear maps satisfying the conditions*

- (i) *For every $x \in X$ and $\varepsilon > 0$ there is $p \in P(M)$, with $\tau(p^\perp) < C(\varepsilon^{-1}\|x\|_X)^\alpha$, such that $\|p \mathbf{a}_n(x)p\| < \varepsilon$ for all $n \in \mathbb{N}$, here C and α are some positive constants.*

If, for every x from a dense subset $X_0 \subset X$,

- (ii) *$\mathbf{a}_m(x) - \mathbf{a}_n(x) \rightarrow 0$ b.a.u., $m, n \rightarrow \infty$,*

then (ii) holds on all of X .

Proof. Let $x \in X$. Due to density of X_0 in X , for given $\varepsilon > 0$ there is a sequence $\{x_n\} \subset X_0$ such that $\|x_n - x\|_X < (\varepsilon/2^{n+1})^{2/\alpha}$ for every $n \in \mathbb{N}$. Then from (i) for every $n \in \mathbb{N}$ there is a projection $p_n \in P(M)$ with $\tau(p_n^\perp) < C\varepsilon/2^{n+1}$ such that

$$\|p_n(\mathbf{a}_m(x_n - x))p_n\| < \varepsilon/2^{n+1} \quad \forall m \in \mathbb{N}. \quad (2.1)$$

Putting $p = \bigwedge_n p_n$, we have $\tau(p^\perp) < C\varepsilon/2$ and

$$\|p(\mathbf{a}_m(x_n - x))p\| \rightarrow 0, \quad n \rightarrow \infty \text{ uniformly in } m.$$

Therefore, from the last relation for given $\varepsilon > 0$ one finds $n_0 \in \mathbb{N}$ such that

$$\|p(\mathbf{a}_m(x_{n_0} - x))p\| \leq \frac{\varepsilon}{3} \quad (2.2)$$

for all $m \in \mathbb{N}$. Since $x_{n_0} \in X_0$ by condition (ii) there is a projection $q \in P(M)$ with $\tau(q^\perp) < \varepsilon/2$ and $N_0 \in \mathbb{N}$ such that

$$\|q(\mathbf{a}_m(x_{n_0}) - \mathbf{a}_n(x_{n_0}))q\| \leq \frac{\varepsilon}{3} \quad (2.3)$$

for all $m, n \geq N_0$. Letting $f = p \wedge q$ one gets $\tau(f^\perp) < \varepsilon(C + 1)/2$ and (2.2),(2.3) imply

$$\begin{aligned} \|f(\mathbf{a}_m(x) - \mathbf{a}_n(x))f\| &\leq \|p(\mathbf{a}_m(x_{n_0} - x))p\| + \|p(\mathbf{a}_n(x_{n_0} - x))p\| \\ &\quad + \|q(\mathbf{a}_m(x_{n_0}) - \mathbf{a}_n(x_{n_0}))q\| \leq \varepsilon \end{aligned}$$

This proves the assertion. \square

Remark. We should note that in the proved Theorem a Banach space X need not be ordered. Hence a condition of minorantly density of X_0 and positivity of \mathbf{a}_n are extra restrictions, which were important in Theorem 2.5. But the condition (i) in Theorem 2.6 is strong than one in Theorem 2.5. For example, it implies that each mapping \mathbf{a}_m ($m \in \mathbb{N}$) is continuous with respect to b.a.u. convergence, which can be seen from (2.1).

Recall a positive linear map $\alpha : L^1(M, \tau) \rightarrow L^1(M, \tau)$ will be called an *absolute contraction* if $\alpha(x) \leq \mathbf{1}$ and $\tau(\alpha(x)) \leq \tau(x)$ for every $x \in M \cap L^1$ with $0 \leq x \leq \mathbf{1}$. If α is a positive contraction in L^1 , then, as it is shown in [22], $\|\alpha(x)\|_p \leq \|x\|_p$ holds for each $x = x^* \in M \cap L^p$ and all $1 \leq p \leq \infty$. Besides, there exist unique continuous extensions $\alpha : L^p \rightarrow L^p$ for all $1 \leq p < \infty$ and a unique ultra-weakly continuous extension $\alpha : M \rightarrow M$ (see [11],[22]). This implies that, for every $x \in L^p$ and any positive integer k , one has $\|\alpha^k(x)\|_p \leq 2\|x\|_p$.

Let $\{\alpha_t\}_{t \geq 0}$ be semigroup of absolute contraction on L^1 . This means that each α_t is an absolute contraction on L^1 , $\alpha_0 = Id$ and $\alpha_t \alpha_s = \alpha_{t+s}$ for all $t, s \geq 0$. By the same symbol α_t we will denote its extension to L^p ($1 \leq p < \infty$). In the sequel we assume that the semigroup $\{a_t\}$ is strongly continuous in L^p , for fixed p , i.e. $\lim_{t \rightarrow s} \|\alpha_t f - \alpha_s f\|_p = 0$ for all $s \geq 0$ and $f \in L^p$.

For any $T > 0$ put

$$\beta_T(x) = \frac{1}{T} \int_0^T \alpha_t(x) dt \quad \text{for } x \in L^p(M, \tau).$$

It is clear that β_T is positive linear map, and maps L^p into itself. The following maximal theorem was proved in [22],[11].

Theorem 2.7. *Let $x \in L^p_{sa}$ then for any $\varepsilon > 0$, there exists projection $e \in P(M)$ such that $\tau(e^\perp) < C(\varepsilon^{-1}\|x\|_p)^p$ and*

$$\|e\beta_T(A)e\| \leq \varepsilon \quad \text{for all } T > 0.$$

3. LOCAL ERGODIC THEOREM

This section is devoted to the local ergodic theorem, which can be formulated as follows

Theorem 3.1. *Let $\{\alpha_t\}_{t \geq 0}$ be a strongly continuous extension to $L^p(M, \tau)$ of a semigroup of absolute contractions on $L^1(M, \tau)$. Then for every $x \in L^p(M, \tau)$ the averages $\beta_T(x)$ converge b.a.u. in $L^p(M, \tau)$ as $T \rightarrow 0$.*

Such kind of theorems were proved in [2], [4], [10], [11], [23]. Here we are going to provide a different proof based on the Banach Principle.

To prove the theorem, we need some auxiliary facts.

Lemma 3.2. *Let $x \in L^p_+$, then*

$$-\frac{1}{b} \int_0^a \alpha_s(x) ds \leq \beta_a(\beta_b(x)) - \beta_b(x) \leq \frac{1}{b} \int_b^{b+a} \alpha_s(x) ds \quad (3.1)$$

for every $a, b \in \mathbb{R}_+$.

Proof. Denote

$$y = \int_0^b \alpha_s(x) ds.$$

Then for a positive number $0 < h < a$ we have

$$\begin{aligned} \alpha_h(y) - y &= \int_h^{b+h} \alpha_s(x) ds - \int_0^b \alpha_s(x) ds \\ &= \int_b^{b+h} \alpha_s(x) ds - \int_0^a \alpha_s(x) ds \\ &\leq \int_b^{b+h} \alpha_s(x) ds \end{aligned}$$

here we have used that $\int_0^a \alpha_s(x) ds \geq 0$. Whence

$$\begin{aligned} \beta_a(y) - y &= \frac{1}{a} \int_0^a (\alpha_h(y) - y) dh \\ &\leq \frac{1}{a} \int_0^a \left(\int_b^{b+h} \alpha_s(x) ds \right) dh \\ &\leq \frac{1}{a} \int_0^a \left(\int_b^{b+a} \alpha_s(x) ds \right) dh \\ &= \int_b^{b+a} \alpha_s(x) ds. \end{aligned} \quad (3.2)$$

The last inequality (3.2) implies

$$\beta_a(\beta_b(x)) - \beta_b(x) \leq \frac{1}{b} \int_b^{b+a} \alpha_s(x) ds. \quad (3.3)$$

On the other hand, we have

$$\begin{aligned} \alpha_h(y) &= \int_h^{b+h} \alpha_s(x) ds \\ &\geq \int_h^b \alpha_s(x) ds \\ &\geq \int_a^b \alpha_s(x) ds \quad \text{for } 0 < h < a. \end{aligned}$$

Therefore,

$$\int_a^b \alpha_s(x) ds \leq \frac{1}{a} \int_0^a \alpha_h(y) dh$$

which yields

$$-\frac{1}{b} \int_0^a \alpha_s(x) ds \leq \beta_a(\beta_b(x)) - \beta_b(x).$$

This and (3.3) complete the proof. \square

Denote

$$X_0 = \text{span}\{\beta_T(x) : x \in L_+^p, T > 0\}. \quad (3.4)$$

Lemma 3.3. *The space X_0 is dense in L^p .*

Proof. Take $x \in L^1$, and show there is a sequence $\{x_k\}$ in X_0 which converges to x in norm of L^1 . Define a sequence $\{x_k\}$ by

$$x_k = k \int_0^{1/k} \alpha_s(x) ds. \quad (3.5)$$

Since any $x \in L^1$ can be represented by $x = \sum_{j=0}^3 i^k x_j$, where $x_j \in L_+^p$ ($j = 0, 1, 2, 3$), therefore x_k is a linear combination of $\beta_{1/k}(x_j)$, which implies that $\{x_k\} \subset X_0$. The strong continuity of α_s implies that for arbitrary $\varepsilon > 0$ there is $\delta > 0$ such that for every s with $|s| < \delta$ the inequality holds $\|\alpha_s(x) - x\|_p < \varepsilon$. Pick $k_0 \in \mathbb{N}$ such that $k_0 < \delta$, then

$$\|x_n - x\|_p \leq n \int_0^{1/n} \|\alpha_s(x) - x\|_p ds < \varepsilon \quad \forall n \geq k_0$$

which completes the proof. \square

Lemma 3.4. *Let $x \in L_+^p$, then*

$$\lim_{a \rightarrow 0} \beta_a(\beta_b(x)) = \beta_b(x) \quad \text{b.a.u.} \quad (3.6)$$

for every $b > 0$.

Proof. First denote

$$h(a) = \frac{1}{b} \int_0^a \alpha_s(x) ds, \quad g(a) = \frac{1}{b} \int_b^{b+a} \alpha_s(x) ds, \quad (3.7)$$

it is obvious that $h(a) \geq 0$, $g(a) \geq 0$ for all $a > 0$. Now due to the strong continuity of α_s we infer

$$\lim_{a \rightarrow 0} \|h(a)\|_p = 0, \quad \lim_{a \rightarrow 0} \|g(a)\|_p = 0.$$

From this we conclude that for any $\varepsilon > 0$ there is a sequence $\{a_k\} \subset \mathbb{R}_+$ such that $\tau(h^p(a_k)) < \varepsilon^2/2^{2k}$ for all $k \in \mathbb{N}$.

Let

$$h^p(a_k) = \int_0^\infty \lambda de_\lambda^{(k)}$$

be the spectral resolution of $h^p(a_k)$. Put $p_k = e_{\varepsilon/2^{k+1}}^{(k)}$, then $\tau(p_k^\perp) \leq \varepsilon/2^{k+1}$. From

$$p_k h^p(a_k) p_k = \int_0^{\varepsilon/2^{k+1}} \lambda^{1/p} de_\lambda^{(k)}$$

one sees that $p_k h(a_k) p_k \in M$, and with the inequality $h(a) \leq h(c)$ for $0 < a < c$ for sufficiently small a we have

$$\|p_k h(a) p_k\| \leq \|p_k h(a_k) p_k\| \leq \frac{\varepsilon}{2^{k+1}}. \quad (3.8)$$

Letting $p = \bigwedge p_k$, one finds $\tau(p^\perp) < \varepsilon/2$. It follows from (3.8) that

$$\|p h(a) p\| \leq \|p_k h(a) p_k\| \leq \frac{\varepsilon}{2^{k+1}} \quad \text{for all } k \in \mathbb{N}. \quad (3.9)$$

By the same argument one finds $q \in P(M)$ with $\tau(q^\perp) < \varepsilon/2$ such that

$$\|q g(a) q\| \rightarrow 0 \quad \text{as } a \rightarrow 0. \quad (3.10)$$

Put $e = p \wedge q$, then $\tau(e^\perp) < \varepsilon$. Now Lemma 3.2 implies that

$$e h(a) e \leq e(\beta_a(\beta_b(x)) - \beta_b(x))e \leq e g(a) e$$

whence from (3.9)-(3.10) one gets

$$\|e(\beta_a(\beta_b(x)) - \beta_b(x))e\| \leq \max\{\|e h(a) e\|, \|e g(a) e\|\} \rightarrow 0 \quad \text{as } a \rightarrow 0.$$

This completes the proof. \square

The proved lemma and Lemma 2.2 yields the following

Corollary 3.5. *For any $x \in X_0$, we have*

$$\lim_{a \rightarrow 0} \beta_a(x) = x \quad \text{b.a.u.}$$

Now we are ready to prove the formulated Theorem 3.1.

Proof. Take $X = L^p$ in Theorem 2.6. Then due to Theorem 2.7 the condition (i) of Theorem 2.6 is satisfied. Now take an arbitrary sequence of positive numbers $\{a_n\}$ such that $a_n \rightarrow 0$. Then according to Lemma 3.5 one sees that $\beta_{a_n}(x)$ converges b.a.u. for every $x \in X_0$. From Lemma 3.3 we already knew that X_0 is dense in L^p . Hence, all the conditions of Theorem 2.6 are satisfied, which implies the assertion of the theorem. \square

Remark. Note that similar results were proved in [2] and [11], respectively in L^1 and L^p spaces. But our approach uses the Banach principle.

4. A WEIGHTED LOCAL ERGODIC THEOREM

In this section by means of Theorem 3.1 and the Banach principle we are going to prove a weight local ergodic theorem.

Recall that a function $P : \mathbb{R}_+ \rightarrow \mathbb{C}$ is called *trigonometric polynomial* if it has the following form

$$P(t) = \sum_{j=1}^n \kappa_j e^{2\pi i \theta_j t}, \quad t \in \mathbb{R}_+ \quad (4.1)$$

for some $\{\kappa_j\} \subset \mathbb{C}$, and $\{\theta_j\} \subset \mathbb{R}$. By $\mathbb{P}(\mathbb{R}_+)$ we denote the set of all trigonometric polynomials defined on \mathbb{R}_+ . We say that a measurable function $b : \mathbb{R}_+ \rightarrow \mathbb{C}$ is a *Besicovitch function* if

- (i) $b \in L^\infty(\mathbb{R}_+)$;

(ii) Given any $\varepsilon > 0$ there is $P \in \mathbb{P}(\mathbb{R}_+)$ such that

$$\limsup_{T \rightarrow 0} \frac{1}{T} \int_0^T |b(t) - P(t)| dt < \varepsilon. \quad (4.2)$$

Remark. A similar notion of Besicovitch weights was introduced, for example, in [5].

The next simple lemma will be used in the proof of main result which was proved in [14].

Lemma 4.1. *If a sequence $\{\tilde{a}_n\}$ in M is such that for every $\varepsilon > 0$ there exist a b.a.u. convergent sequence $\{a_n\} \subset M$ and a positive integer n_0 satisfying $\|\tilde{a}_n - a_n\| < \varepsilon$ for all $n \geq n_0$, then $\{\tilde{a}_n\}$ also converges b.a.u.*

The main result of this section is the following

Theorem 4.2. *Let M be a von Neumann algebra with a faithful normal semi-finite trace τ , and $\{\alpha_t\}_{t \geq 0}$ be a strongly continuous extension to $L^p(M, \tau)$ of a semigroup of absolute contractions on $L^1(M, \tau)$. If b is a Besicovitch function and $x \in L^p(M, \tau)$, then the averages*

$$\tilde{\beta}_T(x) = \frac{1}{T} \int_0^T b(t) \alpha_t(x) dt \quad (4.3)$$

converge b.a.u. in $L^p(M, \tau)$.

Proof. Let \mathbb{B} be the unit circle in \mathbb{C} , i.e. $\mathbb{B} = \{z \in \mathbb{C} : |z| = 1\}$. By μ we denote the normalized Lebesgue measure on \mathbb{B} . Let $\tilde{M} = M \otimes L^\infty(\mathbb{B}, \mu)$ with $\tilde{\tau} = \tau \otimes \mu$. Let $\tilde{L}^q = L^q(\tilde{M}, \tilde{\tau})$ where $q \geq 1$.

Let us fix $\lambda \in \mathbb{B}$ and define a map $\tilde{\alpha}_t^{(\lambda)}$ on \tilde{L}^1 by

$$(\tilde{\alpha}_t^{(\lambda)}(f))(z) = \alpha_t(f(\lambda^t z)), \quad f \in \tilde{L}^1, \quad z \in \mathbb{B}, \quad t > 0. \quad (4.4)$$

One can see that for $f \in \tilde{L}_+^1$

$$\begin{aligned} (\tilde{\alpha}_t^{(\lambda)}(f)) &= \int_{\mathbb{B}} \tau(\alpha_t(f(\lambda^t z))) d\mu(z) \\ &\leq \int_{\mathbb{B}} \tau(f(\lambda^t z)) d\mu(z) = \tilde{\tau}(f) \end{aligned}$$

and $\tilde{\alpha}_t^{(\lambda)}(\mathbf{1}) \leq \mathbf{1}$. These mean that $\{\tilde{\alpha}_t^{(\lambda)}\}$ is a semigroup of absolute contractions of \tilde{L}^1 . By the same symbol denote its extension to \tilde{L}^p . Strong continuity of α_t on L^p implies that $\tilde{\alpha}_t$ is so on \tilde{L}^p . Therefore, according to Theorem 3.1 for every $f \in \tilde{L}^p$ the averages

$$\frac{1}{T} \int_0^T \tilde{\alpha}_t^{(\lambda)}(f) dt$$

converge b.a.u. in \tilde{L}^p as $T \rightarrow 0$. By Lemma 4.1 [3] we infer that the averages

$$\frac{1}{T} \int_0^T (\tilde{\alpha}_t^{(\lambda)}(f))(z) dt = \frac{1}{T} \int_0^T \alpha_t(f(\lambda^t z)) dt$$

converge b.a.u. in $L^p(M, \tau)$ for almost all $z \in \mathbb{B}$. Applying this to the function $f(z) = zx$, here $x \in L_+^p(M, \tau) \cap M$ we obtain b.a.u. convergence of

$$z \frac{1}{T} \int_0^T \lambda^t \alpha_t(x) dt \quad \text{for almost all } z \in \mathbb{B}.$$

This implies that the averages

$$\frac{1}{T} \int_0^T \lambda^t \alpha_t(x) dt \quad \text{converge b.a.u. as } T \rightarrow 0 \text{ for every } \lambda \in \mathbb{B}. \quad (4.5)$$

Now pick an arbitrary $\varepsilon > 0$. Since b is a Besicovitch function, then there exists $P_\varepsilon \in \mathbb{P}(\mathbb{R}_+)$ such that $P_\varepsilon(t) = \sum_{j=1}^n \kappa_j \lambda_j^t$ and (4.2) is satisfied, where $\{k_j\}_{j=1}^n \subset \mathbb{C}$, $\{\lambda_j\} \subset \mathbb{B}$. Consequently, from (4.5) and Lemma 2.2 we obtain that

$$\frac{1}{T} \int_0^T P_\varepsilon(t) \alpha_t(x) dt \quad (4.6)$$

converge b.a.u. as $T \rightarrow 0$.

On the other hand, from (4.2) one gets

$$\left\| \frac{1}{T} \int_0^T b(t) \alpha_t(x) dt - \frac{1}{T} \int_0^T P_\varepsilon(t) \alpha_t(x) dt \right\| = 2 \left(\frac{1}{T} \int_0^T |P_\varepsilon(t) - b(t)| dt \right) \|x\| < 2\varepsilon \|x\| \quad (4.7)$$

Now Lemma 4.1 implies that the average (4.3) converges b.a.u. in $L_+^p \cap M$ as $T \rightarrow 0$. This means that b.a.u. convergence of

$$(\tilde{\beta}_T(x))^* = \frac{1}{T} \int_0^T \overline{b(t)} \alpha_t(x) dt. \quad (4.8)$$

The last relation with (4.3) yields that both

$$\tilde{\beta}_T^{(r)}(x) = \frac{1}{T} \int_0^T \Re(b(t)) \alpha_t(x) dt \quad \text{and} \quad \tilde{\beta}_T^{(i)}(x) = \frac{1}{T} \int_0^T \Im(b(t)) \alpha_t(x) dt$$

averages converge b.a.u. too.

Put

$$\tilde{\beta}_T^{(R)}(x) = \tilde{\beta}_T^{(r)}(x) + \beta_T(x), \quad \tilde{\beta}_T^{(I)}(x) = \tilde{\beta}_T^{(i)}(x) + \beta_T(x),$$

here as before

$$\beta_T(x) = \frac{1}{T} \int_0^T \alpha_t(x) dt.$$

Now according to Theorem 2.7 given $\varepsilon > 0$ there exists a projection $e \in P(M)$ with $\tau(e^\perp) < C(\varepsilon^{-1} \|x\|_p)^p$ such that

$$\sup_T \|e \beta_T(x) e\| < \varepsilon$$

Note that, since b from $L^\infty(\mathbb{R}_+)$ without loss of generality we may assume that $|b(t)| \leq 1$ for almost every $t \in \mathbb{R}_+$. Therefore, one finds $0 \leq \Re(b) + 1 \leq 2$ which implies that

$$e \tilde{\beta}_T^{(R)}(x) e \leq 2e \beta_T(x) e$$

for every $T \in \mathbb{R}_+$. This immediately yields

$$\|e \tilde{\beta}_T^{(R)}(x) e\| \leq 2\varepsilon$$

Since $\tilde{\beta}_T^{(R)} : X = L_+^p \rightarrow S(M)$ is a positive linear continuous maps, and the set $X_0 := L_+^p \cap M$ is dense in $X = L_+^p$, by Theorem 2.6 we obtain the b.a.u. convergence of $\tilde{\beta}_T^{(R)}(x)$ for all $x \in L_+^p$. Remembering that the averages $\beta_T(x)$ also converge b.a.u. one gets the convergence of $\tilde{\beta}_T^{(r)}(x)$, $x \in L_+^p$. Analogously, $\tilde{\beta}_T^{(i)}(x)$ converges b.a.u. for all $x \in L_+^p$. Therefore, by Lemma 2.2 the averages

$$\tilde{\beta}_T(x) = \tilde{\beta}_T^{(r)}(x) + i \tilde{\beta}_T^{(i)}(x)$$

converge b.a.u. for every $x \in L^p_+$, hence for every $x \in L^p$.

It remains to show that the limits of these averages belong to L^p . Taking into account that $\|a_t(x)\|_p \leq 2\|x\|_p$ for each t , we get $\|\tilde{\beta}_T(x)\|_p \leq 2\|x\|_p$ for all $x \in L^p$. This finishes proof due to Lemma 2.4. \square

Remark. In the proof we could use Theorem 2.5 instead of Theorem 2.6, since in that case we may take $X = L^p_{sa}$, $X_0 = L^p_+ \cap M$. Indeed, X is an ordered Banach space with closed cone $X_+ = L^p_+$, and X_0 is a minorantly dense subset of L^p_+ (see [3]).

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