OPTIMAL SYSTEMS OF FUNDAMENTAL S-UNITS FOR LLL-REDUCTION

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Abstract. We show that a particular parameter plays a vital role in the resolution of S-unit equations, at the stage where LLL-reduction is applied. We define the notion of optimal system of fundamental S-units (with respect to this parameter), and prove that such a system exists and can be effectively constructed. Applying our results and methods, one can obtain much better bounds for the solutions of S-unit equations after the reduction step, than earlier. We briefly also discuss some effects of our results on the method of Wildanger and Smart for the resolution of S-unit equations.

1. Introduction

S-unit equations play a central role in the theory of Diophantine equations. On the one hand, there are many Diophantine problems which naturally give rise to S-unit equations. On the other hand, several types of classical Diophantine equations (such as e.g. norm form equations, discriminant form equations, index form equations) can be reduced to such equations. Here we only refer to the papers, survey papers and books [16], [23], [7], [6], [17], [26], [11] and the references given there.

It is well-known (see e.g. [23], [7], [17], [8] and the references there) that under general conditions the number of solutions of S-unit equations is finite. However, if the number of variables is more than two, then no bound is known for the solutions themselves. Moreover, a conjecture of Cerlienco, Mignotte and Piras [4] states that such equations are algorithmically unsolvable. In case of S-unit equations in two variables, by the help of Baker's method the solutions can also be bounded (see e.g. [15], [23], [7], [17], [18], [19] and the references given there).

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Further, based on LLL-type results of de Weger (cf. [29]) and an enumeration method of Wildanger [30], there exists an efficient algorithm for determining all solutions of any particular equations in this case; see [25].

The practical solution of S-unit equations in two unknowns (and also of some other types, e.g. of index form equations) consists of three main steps (see e.g. [21], [13], [14], [30], [25], [10], [1], and the references given there). First, applying Baker's method an initial upper bound C_{ini} is obtained for the unknowns (which are in the exponents). For the best known bounds see [20] in the complex and [31] in the p-adic case, respectively, and also the references given there. Roughly speaking, at this principal stage the "large" solutions are excluded. Though from the theoretical point of view this first step is the deepest and most important one, for practical purposes (i.e. for listing explicitly all solutions) the Baker-type bound is (typically) too large. Hence before going for the solutions, some further reduction is needed. In the second step applying some variants of the LLL-algorithm, one can reduce the bound C_{ini} considerably, to get a much smaller bound C_{red} for the solutions (see e.g. [25] and the references given there). This stage is frequently referred to as getting rid of the "medium" size solutions. However, if the number of variables (the rank of the S-unit group) is "large", even this reduced bound can be too high for getting all solutions, at least if one would like to simply apply some primitive listing algorithm. So to find the solutions effectively, some clever enumeration is needed. Such an algorithm has been worked out by Wildanger [30] in the complex case, and extended and adopted by Smart [25] for the S-unit case. The application of these algorithms as a third step (ideally) leads to the explicit solution of the original S-unit equation.

In this paper we focus on the second stage of this procedure. It turns out (in fact it is widely known already) that in pushing down the reduced bound C_{red} as much as possible, a certain parameter plays an important role. The aim of the present paper is to show that there is an optimal choice for this crucial parameter, and to give a method to actually find it. Hence ultimately we are able to get much better bounds C_{red} in case of particular equations than earlier. Having established our results, we briefly discuss their possible effects on the method of Wildanger [30] and Smart [25] for the practical resolution of S-unit equations. We note that the results presented in the paper lead to certain improvements of the method.

To formulate our results clearly, we need some preparation. For this purpose we introduce some (standard) notation.

1.1. **Valuations.** Let $\mathbb{K}$ be an algebraic number field of degree n, and let $M_{\mathbb{K}}$ be the set of places on $\mathbb{K}$. Choose from every place $v \in M_{\mathbb{K}}$ a valuation $|.|_v$ in the following way. If v is infinite and corresponds to an embedding $\sigma : \mathbb{K} \to \mathbb{C}$ then for every $\alpha \in \mathbb{K}$ let

$$|\alpha|_v = \begin{cases} |\sigma(\alpha)|, & \text{if } \sigma \text{ is real,} \\ |\sigma(\alpha)|^2, & \text{if } \sigma \text{ is complex.} \end{cases}$$

Further, if v is finite and corresponds to a prime ideal P of $\mathbb{K}$, then for every $\alpha \in \mathbb{K}$ put

$$|\alpha|_v = \begin{cases} 0, & \text{if } \alpha = 0, \\ \mathrm{Norm}(P)^{-\mathrm{ord}_P(\alpha)}, & \text{otherwise.} \end{cases}$$

By these choices we have the product formula, that is for every $\alpha \in \mathbb{K}$, $\alpha \neq 0$

$$\prod_{v \in M_{\mathbb{K}}} |\alpha|_v = 1 \tag{1}$$

holds. Let $S = \{v_1, \ldots, v_s\}$ be a finite subset of $M_{\mathbb{K}}$, containing all the infinite places, and write

$$U_S = \{\varepsilon : |\varepsilon|_v = 1 \text{ for all } v \in M_{\mathbb{K}} \setminus S\}$$

for the set of S-units. As is well-known, U_S is a finitely generated group of rank $s - 1$, containing the unit group of the ring of integers of $\mathbb{K}$.

1.2. S**-unit equations.** Let α_1, α_2 be fixed non-zero elements of $\mathbb{K}$. Consider the so-called S-unit equation

$$\alpha_1 x_1 + \alpha_2 x_2 = 1 \tag{2}$$

in two unknowns $x_1, x_2 \in U_S$. It is well-known (see e.g. [7]) that if x_1, x_2 is a solution to equation (2) then we have

$$x_i = \varepsilon_0^{b_{i,0}} \varepsilon_1^{b_{i,1}} \ldots \varepsilon_{s-1}^{b_{i,s-1}} \text{ for } i = 1, 2. \tag{3}$$

Here $\varepsilon_1, \ldots, \varepsilon_{s-1}$ is a fundamental system of S-units, and ε_0 is a root of unity in $\mathbb{K}$. Put $B_i = \max\limits_{1 \leq j \leq s-1} |b_{i,j}|$ for $i = 1, 2$ and $B = \max(B_1, B_2)$. Clearly, without loss of generality we may assume that $B = B_1$. From (3), for all $v \in M_{\mathbb{K}}$

$$\log |x_1|_v = \sum_{j=1}^{s-1} b_{1,j} \log |\varepsilon_j|_v$$

holds. In particular, we have

$$\begin{pmatrix} \log|\varepsilon_1|_{v_1} & \dots & \log|\varepsilon_{s-1}|_{v_1} \\ \vdots & \ddots & \vdots \\ \log|\varepsilon_1|_{v_s} & \dots & \log|\varepsilon_{s-1}|_{v_s} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} \\ \vdots \\ b_{1,s-1} \end{pmatrix} = \begin{pmatrix} \log|x_1|_{v_1} \\ \vdots \\ \log|x_1|_{v_s} \end{pmatrix}. \tag{4}$$

From the above equality (4) one easily gets that

$$B_1 \leq C^* \max_{1\leq j\leq s} |\log|x_1|_{v_j}| \tag{5}$$

with some constant C^*, depending only on our fundamental system of S-units. (The constant C^* is the crucial object from the point of view of the present paper; we shall come back to this point a bit later.) Then, by the help of the product formula (1), using standard arguments (see e.g. [7], [18]) we immediately get that

$$|x_1|_{v_t} \leq \exp\left(\frac{-B_1}{(s-1)C^*}\right) \tag{6}$$

holds for some $t \in \{1, \dots, s\}$. Hence from (2) we get

$$|1 - \alpha_2 x_2|_{v_t} \leq |\alpha_1|_{v_t} \exp\left(\frac{-B_1}{(s-1)C^*}\right). \tag{7}$$

Now using a Baker-type result (e.g. Matveev [20] if v_t is infinite and Yu [31] if v_t is finite, respectively) we get something like

$$|1 - \alpha_2 x_2|_{v_t} > \exp(-C_0 \log(B_2)). \tag{8}$$

Here C_0 is some constant depending only on $\alpha_1, \alpha_2, \mathbb{K}, S$. The inequalities (7) and (8) by our assumption $B = B_1$ yield an initial upper bound C_{ini} for B.

1.3. **The importance of C^*.** To get the reduced bound C_{red} by the LLL-algorithm, one starts from inequality (7) (or a variant of it) together with the already known information $B < C_{ini}$. Interestingly, the finally obtained reduced bound C_{red} depends very heavily on the constant C^*; the dependence is closely linear. This phenomenon should hopefully be (heuristically) clear from the examples presented in this paper, but already is well-known for experts for a long time; see e.g. the remarks of Tzanakis and de Weger [28] pp. 239-240, and Smart [24] p. 823. Hence it seems to be worth to try to keep C^* as small as possible. Apparently, so far this point remained more or less untouched, and the calculation of C^* is usually done in a rather casual way. Namely, in all occurrences in the literature the standard choice is to take something like

$$C^* := \min_{1\leq j\leq s} ||R_j^{-1}|| \tag{9}$$

(with certain refinements at some places). Here R_j is the (invertible) matrix obtained by deleting the j-th row of the matrix at the left hand side of (4), and $||A||$ stands for the row norm of a $k_1 \times k_2$ type real matrix $A = (a_{ij})_{\substack{1\le i\le k_1\\ 1\le j\le k_2}}$, i.e. $||A|| = \max_{1\le i\le k_1} \sum_{j=1}^{k_2} |a_{ij}|$. Using e.g. Cramer's rule, one can easily see that this choice of C^* is appropriate to have (5).

However, as it turns out, this choice of C^* can be rather far from being optimal, which results in a much worse value for C_{red} than possible. In the second section we show a very simple way to get an instant improvement upon the above choice. Further, we show that in fact the best C^* value exists, and depends only on the choice of the fundamental system of S-units. Finally, we prove that one can explicitly determine an optimal fundamental system of S-units (yielding the best value for C^*), and we give a (relatively) efficient algorithm for finding such a system. As the steps of our arguments and methods are connected in a rather organic way, we do not start with listing the main theorems, we prefer to formulate our results in a "linear" way. However, in order not to break the presentation, we give the proofs in a later section. In Section 3 we give an algorithm which finds an optimal system of fundamental S-units for LLL-reduction. In the fourth section of the paper we provide some numerical examples, including the bounds C_{red} obtained by the old method and by the new one. In the fifth section we give the proofs of our results. The sixth section is devoted to a brief discussion about the effects of our results on the method of Wildanger [30] and Smart [25] for the practical solution of S-unit equations. Finally, in the Appendix on the one hand we outline the reduction methods, and on the other hand we indicate how one can adjust the method developed in the paper if the valuations are not chosen in the "standard" way.

Note that our method can be adopted to the case where in (2) not a full system of fundamental S-units are involved, or the S-units form only an independent system.

Finally, we mention that a similar type investigation has been performed about Mordell-Weil bases of elliptic curves by Stroeker and Tzanakis [27], to reduce the final bound for the integral solutions of elliptic equations.

2. Optimizing C^*

We keep our notation from the previous section. Further, let $F = (\varepsilon_1, \ldots, \varepsilon_{s-1})$ be a system of fundamental S-units of $\mathbb{K}$, and define the

$s \times (s-1)$ matrix R_F by

$$R_F := \begin{pmatrix} \log|\varepsilon_1|_{v_1} & \dots & \log|\varepsilon_{s-1}|_{v_1} \\ \vdots & \ddots & \vdots \\ \log|\varepsilon_1|_{v_s} & \dots & \log|\varepsilon_{s-1}|_{v_s} \end{pmatrix}.$$

Let R'_F be any $(s-1) \times s$ matrix such that

$$R'_F \cdot R_F = E_{(s-1)\times(s-1)}, \tag{10}$$

with the identity matrix of size $(s-1) \times (s-1)$ on the right hand side. The importance of the matrices R'_F becomes clear in view of the following simple observation: starting from (4), by (10) we get

$$\begin{pmatrix} b_{1,1} \\ \vdots \\ b_{1,s-1} \end{pmatrix} = R'_F \cdot \begin{pmatrix} \log|x_1|_{v_1} \\ \vdots \\ \log|x_1|_{v_s} \end{pmatrix}.$$

Hence in (5) we can take C^* to be $||R'_F||$, with any matrix R'_F. So we define the norm $N(F)$ of the system F by

$$N(F) := \min_{R'_F} ||R'_F||,$$

where R'_F runs through the matrices for which (10) is valid. (We shall see later that the minimum does exist.) The system F is called optimal for LLL-reduction, if $N(F)$ is minimal among all choices of fundamental systems of S-units. As we shall also see, the minimum of $N(F)$ also exists. Hence we put

$$C^* := \min_F N(F), \tag{11}$$

where F runs through all systems of fundamental S-units. Then we have (5), of course using the optimal system F in (3). In order to compare this choice of C^* with the earlier one, we write $N_{old}(F)$ for the choice of C^* in (9). Note that (9) just means that we unnecessarily restrict ourselves to matrices R'_F having a constant zero column.

In the following proposition we describe the structure of the matrices R'_F, for fixed F.

Proposition 2.1. *Let R'_F be a matrix for which* (10) *is valid. Then for each $i \in \{1,\dots,s-1\}$ there exists a $u_i \in \mathbb{R}$ such that the i-th row of R'_F is of the form $\mathbf{w}_i - u_i \cdot \mathbf{1}$ with $\mathbf{w}_i = (w_{i,1},\dots,w_{i,s-1},0)$, where $\mathbf{1}$ is the s-tuple with all entries equal to* 1, *and*

$$\begin{pmatrix} w_{1,1} & \dots & w_{1,s-1} \\ \vdots & \ddots & \vdots \\ w_{s-1,1} & \dots & w_{s-1,s-1} \end{pmatrix} = \begin{pmatrix} \log|\varepsilon_1|_{v_1} & \dots & \log|\varepsilon_{s-1}|_{v_1} \\ \vdots & \ddots & \vdots \\ \log|\varepsilon_1|_{v_{s-1}} & \dots & \log|\varepsilon_{s-1}|_{v_{s-1}} \end{pmatrix}^{-1}.$$

The next result provides a simple tool to calculate $N(F)$ for a fixed system F of fundamental S-units. Before its formulation, we need a new notation. Define the central norm $|\mathbf{x}|_C$ of $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = (x_1, \ldots, x_n)$ in the following way. Let $y_1, \ldots, y_n$ be a rearrangement of $x_1, \ldots, x_n$ such that $y_1 \leq \cdots \leq y_n$. Then put

$$|\mathbf{x}|_C = \sum_{i=1}^{n} |y_l - y_i| \tag{12}$$

where $l = \lfloor (n+1)/2 \rfloor$. The number y_l is called the center of $\mathbf{x}$.

Proposition 2.2. *Using the previous notation, we have*

$$N(F) = \max_{1 \leq i \leq s-1} |\mathbf{w}_i|_C.$$

Note that the above result already gives a tool to improve upon the earlier choice of C^* in (9). Indeed, if one does not interested in finding the optimal system of fundamental S-units for LLL-reduction, but insists on his favorite system F, taking C^* to be $N(F)$ rather than $N_{old}(F)$ in (5), is already an improvement. By Proposition 2.2 the calculation of $N(F)$ takes only a fraction of time.

The next result (together with its proof) shows that there exists an optimal system of fundamental S-units indeed, and such a system can be effectively determined.

Theorem 2.1. *For any positive real constant c there are only finitely many systems F of fundamental S-units such that $N(F) \leq c$. Further, all such systems can be effectively determined.*

3. An algorithm to determine an optimal system F

In this section we outline an algorithm to determine an optimal system of fundamental S-units for LLL-reduction. The algorithm consists of two parts. First, by a heuristic method we (hopefully) obtain the best system F, then we check that our choice is best possible indeed.

In the first step we start from an arbitrary system F_0 of fundamental S-units. (Such a system can be found e.g. by Magma [3], but one can also use KASH [5] or PARI/GP [2]). Then we calculate the values of $w_{i,j}$ and choose u_i to be the center of $\mathbf{w}_i$ $(1 \leq i, j \leq s-1)$ in Proposition 2.1. Hence for the corresponding matrix R'_{F_0} we have that $||R'_{F_0}||$ is minimal (see the proof of Proposition 2.2). To find an optimal system F, in fact we need to find an unimodular $(s-1) \times (s-1)$ type matrix A_0 such that $N(F_0 A_0)$ is minimal.

For any row vector $\mathbf{b} = (b_1, \ldots, b_n)$ write $\mathbf{b}^* = (b_1 - u, \ldots, b_n - u)$, where u is the center of $\mathbf{b}$. We call $\mathbf{b}^*$ the centralization of $\mathbf{b}$. If B is a matrix then let B^* denote the matrix obtained from B by replacing all its rows by their centralizations. Observe that for any matrices A, B (of appropriate sizes) we have $(A \cdot B)^* = (A \cdot B^*)^*$. Hence our problem reduces to finding the following minimum:

$$\min_A ||(AR'_{F_0})^*|| \tag{13}$$

where A runs through all the unimodular matrices of type $(s-1) \times (s-1)$. If this minimum is taken at some matrix A', we simply have $A_0 = (A')^{-1}$.

We determine the above minimum with a heuristic algorithm, which seems to work well. The algorithm produces a sequence of systems $F_0, F_1, F_2, \dots$ with $N(F_0) > N(F_1) > N(F_2) > \dots$, and terminates at some point. Suppose that we have already made i steps, and currently we are working with F_i. Let

$$R'_{F_i} = \begin{pmatrix} w_{1,1}^{(i)} & \dots & w_{1,s}^{(i)} \\ \vdots & \ddots & \vdots \\ w_{s-1,1}^{(i)} & \dots & w_{s-1,s}^{(i)} \end{pmatrix}$$

belong to the optimal choice, i.e. $||R'_{F_i}|| = N(F_i)$. Write $\mathbf{w}_t^{(i)} = (w_{t,1}^{(i)}, \dots, w_{t,s}^{(i)})$ for the t-th row of R'_{F_i} $(t = 1, \dots, s-1)$. Suppose that $||R'_{F_i}|| = \sum\limits_{l=1}^{s} |w_{j,l}^{(i)}|$ (that is the j-th row of R'_{F_i} yields the row norm $||R'_{F_i}||$). To improve upon $||R'_{F_i}||$ we need to find an unimodular matrix A such that for some row $\mathbf{a} = (a_1, \dots, a_{s-1})$ of A we have $a_j \neq 0$, and further

$$\left| \sum_{t=1}^{s-1} a_t \mathbf{w}_t^{(i)} \right|_C < ||R'_{F_i}||.$$

Indeed, otherwise we cannot "replace" the vector $\mathbf{w}_j^{(i)}$ by any "shorter" one, and hence we cannot improve upon $||R'_{F_i}||$. In the first part of the algorithm we try to find such a row vector $\mathbf{a}$ of a simple shape, and then we iterate the procedure. More precisely, we consider the minimum (13), however, only for A running through the unimodular matrices which are different from the identity matrix only in their j-th row; namely, the (j, j)-th entry of A equals 1, and all the other entries in its j-th row may assume the values $-1, 0, 1$ only. Having the minimum at A', we define F_{i+1} by $F_{i+1} = F_i(A')^{-1}$, and then repeat the procedure. By Theorem 2.1 we know that this algorithm terminates, and produces a system F_k as output.

Now we should check that the final system is optimal. (Note that this was the case for every example we tested the algorithm for, so we think that this is the typical phenomenon.) For this we need some preparation; in fact we need to establish two simple properties of the central norm.

Lemma 3.1. *The central norm is homogeneous, that is for any* $\mathbf{x} \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$ *we have* $|t\mathbf{x}|_C = |t||\mathbf{x}|_C$.

Lemma 3.2. *Let* $e_i \in \mathbb{R}^n$ *denote the vector with all coordinates* 0, *except for the* i*-th one which is* 1 $(i = 1, \ldots, n-1)$, *and put*

$$T := \{\mathbf{e}_i : i = 1, \ldots, n-1\} \cup \{-\mathbf{e}_i : i = 1, \ldots, n-1\} \cup \{\mathbf{e}_0, -\mathbf{e}_0\}$$

where $\mathbf{e}_0 = \mathbf{e}_1 + \cdots + \mathbf{e}_{n-1}$. *Further, set*

$$H := \{\mathbf{x} \in \mathbb{R}^n : x_n = 0 \text{ and } |\mathbf{x}|_C \leq 1\}$$

where x_n *is the last entry of* $\mathbf{x}$. *Then* H *is the convex hull of* T.

As a corollary of the above two lemmas we get the following statement, which will be needed in the last step of our algorithm.

Lemma 3.3. *Let* F *be a system of fundamental* S*-units, and choose an* R'_F *as before. Then for any (integral) unimodular matrix* A *of type* $(s-1) \times (s-1)$ *with* $||(AR'_F)^*|| < N(F)$ *we have that the row vectors of* A *belong to the convex hull of the set*

$$\{\pm N(F)\mathbf{b}_1, \ldots, \pm N(F)\mathbf{b}_s\},$$

where $\mathbf{b}_i$ *is the* i*-th row of* R_F $(i = 1, \ldots, s)$.

Now we can outline the final part of our algorithm. Having the output F_k by the first stage of the procedure, we calculate $N(F_k)$ (by the help of Proposition 2.2). Then using Lemma 3.3, we get an upper bound c_0 for each entry of the possible unimodular matrices A. Typically, this bound is very small (around at most 2-3). We keep the notation introduced above Lemma 3.1. In particular, we assume again that the value of $||R'_{F_k}||$ belongs to the j-th row of R'_{F_k}. Then we check all row vectors $\mathbf{a} = (a_1, \ldots, a_{s-1})$ such that $a_j > 0$ and all the entries of $\mathbf{a}$ are at most c_0 in absolute value. If for all such vectors $\mathbf{a}$ we have $\left|\mathbf{a}R'_{F_k}\right|_C \geq N(F_k)$, then the value of $N(F_k)$ cannot be improved and F_k is an optimal system. Otherwise, for every appropriate row vectors $\mathbf{a}$ we check all unimodular matrices with entries at most c_0 in absolute value, containing $\mathbf{a}$ as a row. In case we can improve upon $N(F_k)$, we switch back to the first part of the algorithm (with the improved system), then return to this second part later on, and so on. By Theorem 2.1 we eventually get an optimal system of fundamental S-units. We mention that the second part is much more time consuming than the first one, at least if s is "large". (It is not surprising in the light of the observation that the number of row vectors $\mathbf{a}$ to be checked is $c_0(2c_0+1)^{s-2}$ at this stage.) However, note that in all cases we encountered the second part has never been necessary in the sense that the system F_k obtained by the first part of the algorithm has already been optimal.

We conclude this section by two remarks. First we note that our simple heuristic algorithm works so well is very probably due to the fact that the systems of fundamental units provided e.g. by Magma are already LLL-reduced, hence the initial system is "closely" optimal. The other thing we mention is that our algorithm is not optimized, it probably can be improved. Further, maybe there are other (possibly more efficient) ways to get the optimal system. For example, after having an original system F_0 one can search for the minimizing unimodular matrix A in (13) in the following way. Let $\mathcal{L}$ denote the lattice spanned by the vectors $\mathbf{w}_t^{(0)}$ $(t=1,\dots,s-1)$. If F_0 is not optimal then there is some unimodular matrix A such that for all row vectors $\mathbf{a}$ of A we have $\sum_{t=1}^{s}|a'_t| < N(F_0)$ where $\mathbf{w}_a := (a'_1,\dots,a'_s) = (\mathbf{a}R'_{F_0})^*$. This implies that $\mathbf{w}_a$ is a vector of the lattice $\mathcal{L}$ of Euclidean length less than $\sqrt{s}N(F_0)$. Hence all the appropriate row vectors $\mathbf{a}$ can be efficiently determined by the algorithm of Fincke and Pohst [9]. Having the set of possible vectors $\mathbf{a}$, we can build up the appropriate unimodular matrices A, and we can find the minimum (13). However, as we mentioned, the original algorithm was efficient enough for our present purposes, so here we do not take up the problem of optimizing the method.

4. Examples

In this section we present some examples to illustrate our method and also to give some numerical evidence why the parameter C^* is so important. In fact we have worked out several examples, but we give here only four, out of which three can be found in the literature. We exhibit a "proper" S-unit equation (i.e. with a finite valuation involved) and three pure unit equations. Among the latter ones the first two correspond to totally real fields, while the last one to a totally complex field. These examples are of larger size. We mention that in case of smaller examples, our algorithm worked with the same efficiency, but much faster. However, naturally in the "small" cases the gain using the new values of C^* is certainly smaller.

In each example we consider an S-unit equation of the shape

$$x_1 + x_2 = 1 \tag{14}$$

in $x_1, x_2 \in U_S$, for some particular choice of $\mathbb{K}$ and S. Just as in (3), we write

$$x_i = \varepsilon_0^{b_{i,0}} \varepsilon_1^{b_{i,1}} \dots \varepsilon_{s-1}^{b_{i,s-1}} \text{ for } i = 1, 2. \tag{15}$$

Note that here the system $\varepsilon_1, \dots, \varepsilon_{s-1}$ is certainly not fixed during the examples, in fact we use two different systems in each example (corresponding to different values of C^*).

In every example, starting from an initial system F_0 of fundamental S-units, using our algorithm we determine an optimal system F_k. (Note that at the examples from the literature we use the system F_0 from the corresponding papers, otherwise we used Magma [3] to get an F_0.) Remark that the first stage of our algorithm typically takes a few seconds (up to a few minutes when s is larger) to terminate and provide an optimal system F_k. However, to check that F_k is optimal indeed by the second part of the algorithm takes much more time (around 30 minutes for "large" s). An optimized and more sophisticated version of the algorithm would probably work better, but we do not take up this question here.

Description of the tables. We provide a table for each example, containing several data. We give the values of the reduced bounds C_{red} corresponding to the choices $C^* = N_{old}(F_0)$, $N(F_0)$ and $N(F_k)$, respectively. Naturally, in the first two cases we use the system F_0 in (15), while in case of $C^* = N(F_k)$ the system F_k is used. To execute the reduction steps, we used Lemmas 7.1 and 7.2 given in the Appendix. Note that these reduction lemmas have very many variants in the literature. We chose these ones because they are relatively simply formulated, and they are appropriate for our present purposes. To make the reduction, when there was no available initial upper bound for B, as it is not an important point from the point of view of the present paper, instead of going through Baker's method we just started with the (plausible) bound $B < 10000 =: C_{ini}$.

In the tables we provide the ratios of the C^* values, as well (more precisely the ratios corresponding to the actual C^* and $N_{old}(F_0)$). We also indicate the ratios of the C_{red} values (that is, the ratios corresponding to the actual C_{red} and the reduced bound corresponding to the choice $C^* = N_{old}(F_0)$). Note that these ratios are remarkably close to each other, which shows that the dependence of C_{red} is very nearly linear in C^*. This phenomenon is not surprising in view of Lemmas 7.1 and 7.2: the new lower bound for B in each iteration is almost linear in $1/C_1$ in both cases - and $1/C_1$ is linear in C^*. (For the details cf. subsection 7.1 of the Appendix.)

Finally, we introduce an indicator called "domain ratio" to compare the remaining domains to be checked after the reduction. This indicator is defined in the natural way, i.e. as

$$\left(\frac{2C_{red}^{(1)}+1}{2C_{red}^{(0)}+1}\right)^{2s-2}.$$

Here the constants C_{red} are the reduced constants ($C_{red}^{(0)}$ corresponds to the choice $C^* = N_{old}(F_0)$ and $C_{red}^{(1)}$ to the actual choice), and the exponent $2s-2$ by (15) is just the number of variables (as $b_{1,0}$ and $b_{2,0}$ do not really count). Note that in our examples this indicator shows that the size of the domain to be checked using the new method is a tiny fraction of the size of the domain remaining by the old method.

Example 1. This example is from [25]. Let $\mathbb{K} = \mathbb{Q}(\vartheta)$ where $\vartheta^8 + 1 = 0$. Let S be the set containing the infinite valuations of $\mathbb{K}$, and a finite valuation corresponding to the prime ideal $P = (\pi)$ with $\pi = 1 - \vartheta$. We have $N_{\mathbb{K}/\mathbb{Q}}(\pi) = 2$, and a fundamental system of S-units is given by

$$\varepsilon_1 = \vartheta^2 + \vartheta^4 + \vartheta^6, \varepsilon_2 = -\vartheta^2 - \vartheta^3 - \vartheta^4, \varepsilon_3 = 1 + \vartheta^3 - \vartheta^5, \varepsilon_4 = \pi = 1 - \vartheta$$

(see [25]). Note that the element $\varepsilon_0 = -\vartheta^7$ generates the sixteen roots of unity of $\mathbb{K}$. Put

$$F_0 = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4).$$

A simple calculation shows that $N_{old}(F_0) = 1.442695\ldots$. Further, we also have $N(F_0) = 1.442695\ldots$. Then, executing the above algorithm starting with F_0, in three steps we get a new system of fundamental S-units F_3 given by the transformation

$$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&1&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&1&1&0\\ 0&1&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0\\ 1&1&-1&-1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ 1&1&-1&-1\\ 1&1&0&-1\\ 2&2&-1&-1 \end{pmatrix}.$$

Actually, we have

$$F_3 = (\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4^2, \varepsilon_2\varepsilon_3\varepsilon_4^2, \varepsilon_2^{-1}\varepsilon_4^{-1}, \varepsilon_2^{-1}\varepsilon_3^{-1}\varepsilon_4^{-1}).$$

A simple calculation gives that $N(F_3) = 0.931871\ldots$. The second part of our algorithm reveals that F_3 is an optimal system of fundamental S-units.

Then, using Lemmas 7.1 and 7.2 from the Appendix, we get the reduced bounds in the table below. Note that we need to perform the reduction for each choice of the valuations, and that the worst case belongs to the finite valuation in S. We also mention that working with F_0, we used the initial upper bound $B \leq 1066$, from [25]. In case of F_3, using the inverse of the basis transformation matrix, from this we have $B \le 3198$. We started our reduction with these bounds. We summarize the results of our computations in Table 1. For the definitions of the entries of the table (and also in case of the other tables) see the preceding subsection.

	using F_0 and $N_{old}(F_0)$	using F_0 and $N(F_0)$	using F_3 and $N(F_3)$
C^*	1.442695...	1.442695...	0.931871...
C_{red}	1031	1031	651
C^* ratio	1	1	0.645923...
C_{red} ratio	1	1	0.631425...
domain ratio	1	1	0.025325...

Table 1

Example 2. The data of this example is from [10] (the authors considered a different equation). Let $\mathbb{K} = \mathbb{Q}(\vartheta)$ where $\vartheta^{10} - 15\vartheta^8 + \vartheta^7 + 66\vartheta^6 + \vartheta^5 - 96\vartheta^4 - 7\vartheta^3 + 37\vartheta^2 + 12\vartheta + 1 = 0$. Let S be the set of infinite valuations of $\mathbb{K}$. An integral basis of $\mathbb{K}$ is given by $\omega_1, \dots, \omega_{10}$ with $\omega_i = \vartheta^{i-1}$ $(i = 1, \dots, 9)$ and

$$\omega_{10} = \frac{9 + 27\vartheta + 43\vartheta^2 + 20\vartheta^3 + 37\vartheta^4 + 5\vartheta^5 + 32\vartheta^6 + 3\vartheta^7 + 26\vartheta^8 + \vartheta^9}{47}.$$

The coordinates of a fundamental system $\varepsilon_1, \dots, \varepsilon_9$ of S-units with respect to this integral basis is given by

$$[21, 107, 192, -5, -120, -40, 84, 20, 30, -60],$$

$$[16, 99, 139, -56, -113, -7, 56, 9, 14, -30],$$

$$[10, 4, 65, 197, 85, -110, 56, 34, 50, -90],$$

$$[21, 35, 196, 346, 94, -206, 129, 66, 97, -177],$$

$$[0, -53, -31, 200, 145, -90, 14, 24, 35, -60],$$

$$[8, 24, 40, 33, -1, -27, 25, 10, 15, -28],$$

$$[15, 13, 118, 248, 78, -143, 84, 45, 66, -120],$$

$$[0, 1, 0, 0, 0, 0, 0, 0, 0, 0],$$

$$[4, 19, 42, 0, -26, -8, 17, 4, 6, -12],$$

respectively (see [10]). Put

$$F_0 = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9).$$

We have $N_{old}(F_0) = 2.285921\ldots$ and $N(F_0) = 1.564168\ldots$. Then by the above algorithm starting with F_0, in three steps we get a new system of fundamental S-units F_3 given by the transformation

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&1&-1&0&-1\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1\\ 0&0&0&1&0&1&-1&0&-1\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

Actually, we have

$$F_3 = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_1\varepsilon_5, \varepsilon_4\varepsilon_6, \varepsilon_4^{-1}\varepsilon_6, \varepsilon_7, \varepsilon_3\varepsilon_4^{-1}\varepsilon_8).$$

By a simple calculation we get that $N(F_3) = 1.209236\ldots$. Using the second part of our algorithm we obtain that F_3 is an optimal system of fundamental S-units. In this case we performed the reduction starting with the initial bound $B < 10000 =: C_{ini}$ both with F_0 and with F_3. The results of our calculations are summarized in Table 2.

	using F_0 and $N_{old}(F_0)$	using F_0 and $N(F_0)$	using F_3 and $N(F_3)$
C^*	2.285921...	1.564168...	1.209236...
C_{red}	2079	1416	1011
C^* ratio	1	0.684261...	0.528993...
C_{red} ratio	1	0.681096...	0.486291...
domain ratio	1	0.000996...	0.000002...

Table 2

Example 3. This example is from [30]. Let $\mathbb{K} = \mathbb{Q}(\vartheta)$ where

$$\vartheta^9 + \vartheta^8 - 8\vartheta^7 - 7\vartheta^6 + 21\vartheta^5 + 15\vartheta^4 - 20\vartheta^3 - 10\vartheta^2 + 5\vartheta + 1 = 0.$$

Note that $\mathbb{K}$ is the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_{19})$. Let S be the set of infinite valuations of $\mathbb{K}$. An integral basis of $\mathbb{K}$ is given by $1, \vartheta, \ldots, \vartheta^8$. The coordinates of a fundamental system $\varepsilon_1, \ldots, \varepsilon_8$ of S-units with respect to this integral basis is given by

$$[1, -4, -10, 10, 15, -6, -7, 1, 1], [0, 3, 0, -1, 0, 0, 0, 0, 0],$$

$$[1, -2, -3, 1, 1, 0, 0, 0, 0], [2, 0, -9, 0, 6, 0, -1, 0, 0],$$

$$[0, 1, 0, 0, 0, 0, 0, 0, 0], [2, 0, -1, 0, 0, 0, 0, 0, 0],$$

$$[2, 0, -4, 0, 1, 0, 0, 0, 0], [0, -5, 5, 10, -5, -6, 1, 1, 0],$$

respectively; see [30]. (Note that in [30] $\varepsilon_2 = 3\vartheta - \vartheta^4$ is written, which is a typo.) Put

$$F_0 = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8).$$

A simple calculation shows that $N_{old}(F_0) = 2.561675\ldots$. Further, we also have $N(F_0) = 1.872827\ldots$. Then our algorithm in six steps yields a new system of fundamental S-units F_6 given by the transformation

$$\begin{pmatrix} 1&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&-1&0&0&0&1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ -1&0&1&1&1&1&1&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} = \begin{pmatrix} 0&0&0&1&1&1&1&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&-1&-1&-1&-1&0\\ -1&0&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&0\\ 0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1&0\\ 1&0&-1&-1&-1&-1&-1&0 \end{pmatrix}.$$

That is, we have

$$F_6 = (\varepsilon_3\varepsilon_4^{-1}\varepsilon_8, \varepsilon_2, \varepsilon_4\varepsilon_8^{-1}, \varepsilon_1\varepsilon_3^{-1}\varepsilon_4\varepsilon_8^{-1}, \varepsilon_1\varepsilon_3^{-1}\varepsilon_4\varepsilon_5\varepsilon_8^{-1},$$
$$\varepsilon_1\varepsilon_3^{-1}\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_8^{-1}, \varepsilon_1\varepsilon_3^{-1}\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8^{-1}, \varepsilon_4)$$

with $N(F_6) = 1.343979\ldots$. The second part of our algorithm verifies that F_6 is optimal. We started the reduction with the initial bound $B \le 2076$ for F_0 (see [30]), while using the inverse of the basis reduction matrix we could start with $B \le 4152$ in case of F_6. The data and the information derived from them are given in Table 3.

	using F_0 and $N_{old}(F_0)$	using F_0 and $N(F_0)$	using F_6 and $N(F_6)$
C^*	2.561675...	1.872827...	1.343979...
C_{red}	1664	1210	824
C^* ratio	1	0.731094...	0.524648...
C_{red} ratio	1	0.727163...	0.495192...
domain ratio	1	0.006122...	0.000013...

Table 3

Example 4. This example is new. Let $\mathbb{K} = \mathbb{Q}(\vartheta)$ where

$$\vartheta^{18} + \vartheta^{17} + \vartheta^{16} + \vartheta^{15} + \vartheta^{14} + \vartheta^{13} + \vartheta^{12} + \vartheta^{11} + \vartheta^{10} + \vartheta^9 +$$
$$+\vartheta^8 + \vartheta^7 + \vartheta^6 + \vartheta^5 + \vartheta^4 + \vartheta^3 + \vartheta^2 + \vartheta + 1 = 0,$$

so $\mathbb{K}$ is the cyclotomic field $\mathbb{Q}(\zeta_{19})$ (with $\zeta_{19} = \vartheta$). Let S be the set of infinite valuations of $\mathbb{K}$. An integral basis of $\mathbb{K}$ is given by $1, \vartheta, \ldots, \vartheta^{17}$.

The coordinates of a fundamental system $\varepsilon_1, \dots, \varepsilon_8$ of S-units with respect to this integral basis (obtained by Magma [3]) is given by

$$[1,1,1,1,1,1,1,1,1,0,1,1,1,1,1,1,1,1],$$
$$[0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0],$$
$$[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,1,1],$$
$$[1,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1],$$
$$[1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1],$$
$$[1,1,1,1,1,1,1,1,1,1,1,1,1,0,1,1,1,1],$$
$$[1,1,1,1,1,1,1,1,1,1,0,1,1,1,1,1,1,1],$$
$$[0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0],$$

respectively. Put

$$F_0 = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8).$$

A simple calculation shows that $N_{old}(F_0) = 1.280834\dots$ and $N(F_0) = 0.936410\dots$. Then the above algorithm in seven steps provides a new system of fundamental S-units F_7 given by the transformation

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&-1&0&0&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&-1&1&0&0&1&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&-1&1&-1&1&1&-1&-1\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 1&0&0&0&1&1&0&0\\ 1&-1&1&0&1&1&0&0\\ 1&-1&1&0&1&1&-1&-1\\ 1&-1&1&-1&1&1&-1&-1\\ 1&0&0&0&0&1&0&0\\ 1&-1&1&0&1&1&0&-1\\ 1&-1&0&0&1&1&0&0 \end{pmatrix}.$$

Actually, we have

$$F_7 = (\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8, \varepsilon_3^{-1}\varepsilon_4^{-1}\varepsilon_5^{-1}\varepsilon_7^{-1}\varepsilon_8^{-1}, \varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_7, \varepsilon_5^{-1},$$
$$\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_7\varepsilon_8, \varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8, \varepsilon_4^{-1}\varepsilon_5^{-1}, \varepsilon_4^{-1}\varepsilon_5^{-1}\varepsilon_7^{-1})$$

with $N(F_7) = 0.67198843\dots$. By the second part of our algorithm we get that F_7 is an optimal system of fundamental S-units. To execute the reduction, we used the initial bound $B < 10000$ both with F_0 and with F_7. The outcome of our calculations is contained in Table 4.

	using F_0 and $N_{old}(F_0)$	using F_0 and $N(F_0)$	using F_7 and $N(F_7)$
C^*	1.280834...	0.936410...	0.671988...
C_{red}	792	550	386
C^* ratio	1	0.731094...	0.524649...
C_{red} ratio	1	0.694444...	0.487373...
domain ratio	1	0.002938...	0.000010...

Table 4

5. Proofs

In this section we give the proofs of our results.

Proof of Proposition 2.1. Noting that the rank of R_F is $s-1$ and that by the product formula (1) we have $\sum_{j=1}^{s} \log|\varepsilon_i|_{v_j} = 0$ for each $i \in \{1, \dots, s-1\}$, the statement follows from the elementary theory of systems of linear equations. $\square$

Proof of Proposition 2.2. In view of Proposition 2.1, the only freedom we have in the choice of R'_F is to choose $u_1, \dots, u_{s-1}$. Take any $i \in \{1, \dots, s-1\}$, and let $z_1, \dots, z_s$ be the rearrangement of the entries of $\mathbf{w}_i$ such that $z_1 \leq \dots \leq z_s$. Writing $w_{i,s} = 0$, it is obvious that

$$\min_{u_i \in \mathbb{R}} \sum_{j=1}^{s} |w_{i,j} - u_i|$$

is achieved by the choice $u_i = z_l$, where $l = \lfloor (s+1)/2 \rfloor$. Hence the statement follows. $\square$

Proof of Theorem 2.1. Let $T = (\eta_1, \dots, \eta_{s-1})$ be an arbitrary system of fundamental S-units, and suppose that for the system F we have $N(F) \leq c$. Then we have $R_F = R_T A$ with some integral unimodular matrix A of size $(s-1)\times(s-1)$. Obviously, it is sufficient to "bound" A. Observe that in view of Proposition 2.2, $N(T)$ and $N(F)$ are exclusively ruled by

$$W_T := \begin{pmatrix} \mathbf{w}_1^{(T)} \\ \vdots \\ \mathbf{w}_{s-1}^{(T)} \end{pmatrix} \quad \text{and} \quad W_F := \begin{pmatrix} \mathbf{w}_1^{(F)} \\ \vdots \\ \mathbf{w}_{s-1}^{(F)} \end{pmatrix},$$

respectively (with the obvious notation). Since $W_F = A^{-1} W_T$, W_T and W_F are bases of the same lattice. As the last entry of each $\mathbf{w}_i^{(F)}$ is zero, the restriction $|\mathbf{w}_i^{(F)}|_C \leq c$ yields that the absolute values of the components of $\mathbf{w}_i^{(F)}$ are all at most c. That is, W_F is a basis of a fixed lattice, within a bounded region, and our statement follows. $\square$

Proof of Lemma 3.1. The statement is a trivial consequence of the definition (12) and of the fact that (using the notation above (12)) by symmetry

$$|\mathbf{x}|_C = \sum_{i=1}^{n} |y_{l'} - y_i|$$

holds, where $l' = \lfloor (n+2)/2 \rfloor$. □ □

Proof of Lemma 3.2. First we prove that any convex linear combination of the vectors in T belongs to H. For this purpose, choose arbitrary non-negative real numbers $\lambda_0^+, \lambda_0^-, \lambda_1^+, \lambda_1^-, \ldots, \lambda_{n-1}^+, \lambda_{n-1}^-$ such that $\sum_{i=0}^{n-1} (\lambda_i^+ + \lambda_i^-) = 1$. Put

$$\mathbf{x} = \sum_{i=0}^{n-1} (\lambda_i^+ \mathbf{e}_i + \lambda_i^- (-\mathbf{e}_i)).$$

Then for each $i \in \{1, \ldots, n-1\}$ the i-th entry x_i of $\mathbf{x}$ is given by $x_i = \lambda_0^+ - \lambda_0^- + \lambda_i^+ - \lambda_i^-$, while $x_n = 0$. Hence clearly,

$$|\mathbf{x}|_C = |(\lambda_1^+ - \lambda_1^-, \ldots, \lambda_{n-1}^+ - \lambda_{n-1}^-, \lambda_0^- - \lambda_0^+)|_C.$$

Thus

$$|\mathbf{x}|_C \le \sum_{i=0}^{n-1} |\lambda_i^+ - \lambda_i^-| \le \sum_{i=0}^{n-1} (\lambda_i^+ + \lambda_i^-) = 1,$$

and the statement follows.

Let now $\mathbf{x} \in H$ be arbitrary, and write $\mathbf{x} = (x_1, \ldots, x_n)$. Let $y_1, \ldots, y_n$ be a rearrangement of $x_1, \ldots, x_n$ such that $y_1 \le \cdots \le y_n$ and write $x_i = y_{\nu(i)}$, where ν is the underlying permutation of the indices $1, \ldots, n$. (Note that $y_{\nu(n)} = x_n = 0$.) As usual, put $l = \lfloor (n+1)/2 \rfloor$, for $i = 1, \ldots, n-1$ write $\lambda_i^+ = \max\{y_{\nu(i)} - y_l, 0\}$ and $\lambda_i^- = \max\{y_l - y_{\nu(i)}, 0\}$, and set $\lambda_0^+ = \max\{y_l, 0\}$ and $\lambda_0^- = \max\{-y_l, 0\}$. Then on the one hand, $x_i = (y_{\nu(i)} - y_l) + y_l$ implies

$$\mathbf{x} = \sum_{i=0}^{n-1} (\lambda_i^+ \mathbf{e}_i + \lambda_i^- (-\mathbf{e}_i)).$$

Further, on the other hand

$$\sum_{i=0}^{n-1} (\lambda_i^+ + \lambda_i^-) = \sum_{i=1}^{n} |y_l - y_{\nu(i)}| = |\mathbf{x}|_C \le 1$$

as $\mathbf{x} \in H$, and the lemma follows. □

Proof of Lemma 3.3. Let $\mathbf{a}_i$ be the i-th row of A $(i = 1, \dots, s-1)$. Using Proposition 2.1 without loss of generality we may assume that all entries of the last column of R'_F are zero. Hence by $||(AR'_F)^*|| < N(F)$ Lemmas 3.1 and 3.2 imply that $\mathbf{a}_i R'_F$ belongs to the convex hull of the set

$$\{\pm N(F)\mathbf{e}_0, \pm N(F)\mathbf{e}_1, \dots, \pm N(F)\mathbf{e}_{s-1}\}$$

(where the $\mathbf{e}_i$ are defined in Lemma 3.2). However, then using $R'_F \cdot R_F = E_{(s-1)\times(s-1)}$ we get that $\mathbf{a}_i$ belongs to the convex hull of the set

$$\{\pm N(F)\mathbf{b}_0, \pm N(F)\mathbf{b}_1, \dots, \pm N(F)\mathbf{b}_{s-1}\},$$

where $\mathbf{b}_0 = \mathbf{b}_1 + \dots + \mathbf{b}_{s-1}$. Since by the product formula (1) we have $\mathbf{b}_s = -\mathbf{b}_0$, the lemma follows. □

6. Effects on the method of Wildanger and Smart

In this section we discuss about the effects of the results in the paper on the method of Wildanger [30] and Smart [25] for the resolution of S-unit equations. For this purpose, first we briefly and schematically outline the main steps of the method. After that we show that our results yield certain improvements of the method. At this point the author would like to express his deep thanks to Attila Pethő for the many fruitful and stimulating discussions and advice about the contents of this section.

6.1. **The method of Wildanger and Smart.** Now we briefly sketch the method worked out by Wildanger [30] and Smart [25] for the resolution of S-unit equations. We follow the presentation in [25], with certain simplifications. First we need to introduce some notation.

Let $\mathbb{K}$ and S be as before. For $K \in \mathbb{R}$ with $K > 1$ write

$$\langle\langle K, S\rangle\rangle = \{\alpha \in \mathbb{K} : 1/K \leq |\alpha|_v \leq K \text{ for all } v \in S\}.$$

Let $\mathcal{L}$ denote the set of solutions of the S-unit equation (2), that is

$$\mathcal{L} = \{(x_1, x_2) \in U_S \times U_S : \alpha_1 x_1 + \alpha_2 x_2 = 1\}.$$

Further, put

$$\mathcal{L}_{H_i} = \{(x_1, x_2) \in \mathcal{L} : B \leq H_i\}$$

where B is defined after (3), and set

$$\mathcal{L}_{H_i}(K) = \{(x_1, x_2) \in \mathcal{L}_{H_i} : x_1 \in \langle\langle K, S\rangle\rangle\}.$$

Starting from (2) after making the standard steps (see section 1.2) we arrive at (4). Write $\mathcal{E}_1, \dots, \mathcal{E}_{s-1}$ for the columns of the matrix at the left hand side of (4), and $\mathcal{X}$ for the vector on the right hand side. Then (4) can be rewritten as

$$b_{1,1}\mathcal{E}_1 + \cdots + b_{1,s-1}\mathcal{E}_{s-1} = \mathcal{X}. \tag{16}$$

As earlier, denote by C_{red} the reduced bound obtained for B (defined after (3)) after executing Baker's method and the LLL-algorithm. Put

$$K_0 = \max_{v\in S} \exp(C_{red}|\log|\varepsilon_1|_v| + \cdots + C_{red}|\log|\varepsilon_{s-1}|_v|). \tag{17}$$

Then by the help of (16) one can easily get that

$$\mathcal{L} = \mathcal{L}_{H_0}(K_0) \tag{18}$$

with $H_0 = C_{red}$ (see Lemma 1 of [25]).

Put

$$s_i = \max_{v\in S} \max(|\alpha_i|_v, |\alpha_i^{-1}|_v) \text{ for } i = 1, 2$$

and

$$s_3 = \max_{v\in S} \min(|\alpha_2^{-1}|_v).$$

Now let K_i, K_{i+1} be real numbers such that $\max(s_1, s_2, s_3, (s_3-1)/s_1) < K_{i+1} < K_i$ and let $H_i \in \mathbb{Z}$. Note that we have $K_{i+1} > 1$. For $v \in S$ define the sets

$$T_{1,v}(H_i, K_i, K_{i+1}) = \left\{(x_1, x_2) \in \mathcal{L}_{H_i}(K_i) : |-\alpha_1 x_1 - 1|_v < \frac{1}{1 + s_1 K_{i+1}}\right\},$$

$$T_{2,v}(H_i, K_i, K_{i+1}) = \left\{(x_1, x_2) \in \mathcal{L}_{H_i}(K_i) : \left|-\frac{1}{\alpha_1 x_1} - 1\right|_v < \frac{1}{1 + s_1 K_{i+1}}\right\},$$

$$\begin{aligned} T_{3,v}(H_i, K_i, K_{i+1}) = \{(x_1, x_2) \in \mathcal{L}_{H_i}(K_i) : |-\alpha_2 x_2 - 1|_v < \frac{s_1}{K_{i+1}}, \\ \alpha_2 x_2 \in \langle\langle 1 + s_1 K_i, S\rangle\rangle\}, \end{aligned}$$

$$\begin{aligned} T_{4,v}(H_i, K_i, K_{i+1}) = \{(x_1, x_2) \in \mathcal{L}_{H_i}(K_i) : \left|-\frac{\alpha_2 x_2}{\alpha_1 x_1} - 1\right|_v < \frac{s_1}{K_{i+1}}, \\ \frac{\alpha_2 x_2}{\alpha_1 x_1} \in \langle\langle 1 + s_1 K_i, S\rangle\rangle\}. \end{aligned}$$

Further, for $j = 1, 2, 3, 4$ let

$$T_j(H_i, K_i, K_{i+1}) = \bigcup_{v\in S} T_{j,v}(H_i, K_i, K_{i+1}).$$

Then by Lemma 2 of [25] we have the decomposition

$$\mathcal{L}_{H_i}(K_i) = \mathcal{L}_{H_{i+1}}(K_{i+1}) \bigcup_{j=1}^{4} T_j(H_i, K_i, K_{i+1}), \tag{19}$$

with

$$H_{i+1} = C^* \cdot C^+ \tag{20}$$

where C^* is the crucial constant investigated in the paper (see (5)) and

$$C^+ = \max\left(\log\left(\frac{s_1 K_{i+1}+1}{s_2}\right), \log\left(\frac{s_1 K_{i+1}+1}{s_3}\right), \log(K_{i+1})\right). \tag{21}$$

For more details and explanation see [30] and [25]. Now what happens, is that under certain conditions (which usually hold in the first part of the algorithm; see Lemma 3 of [25]) the set $\bigcup_{j=1}^{4} T_j(H_i, K_i, K_{i+1})$ occurring in (19) proves to be empty. This is not the case in the later part of the algorithm. However, at that stage one can rather efficiently use the method of Fincke and Pohst [9] to completely enumerate this set. Hence starting from $i = 0$, by the repeated application of (19) the whole procedure can be iterated. Finally we are left only with a set $\mathcal{L}_{H_j}(K_j)$ for some small values of K_j and H_j, which can also be enumerated without any trouble. Hence in this way we can get all solutions of the original S-unit equation (2).

6.2. **Some improvements.** In this subsection we indicate at which points our results may improve upon the above described method of Wildanger and Smart.

By our method presented in the paper we are able to minimize the value of C_{red}. Hence the initial value of K_0 in (17) can be taken much smaller than previously. In particular, one may even use his/her original fundamental system F_0 of units. Working with the value $N(F_0)$ defined in the paper instead of $N_{old}(F_0)$ used earlier, we get a better C_{red} than previously. As in this case all the other parameters in (17) are unchanged, we already obtain some improvement.

Furthermore, in the definition (20) of H_{i+1} the constant C^* is used. As by our results one can take a much smaller value for this parameter than earlier, in each iteration of the algorithm we get a smaller value for H_{i+1}, and hence the procedure can be made to "converge" faster. Here the above remark applies again: using the original non-optimized system F_0, but working with the new norm $N(F_0)$, one already gets some improvement.

7. Appendix

This final section has two distinct parts. In the first subsection we provide the reduction lemmas we used to obtain the reduced bounds C_{red} for B, both in the finite and in the infinite case. In the second subsection we show how one can adjust the method for other choices of the valuations. (We give our motivation for doing so, as well.)

7.1. **Reduction.** There are very many variants of reduction lemmas, both in the finite and in the infinite case; we chose a lemma of Smart [24] in the p-adic case, and a result of Gaál and Pohst [12] in the complex case. Note that these lemmas have to be applied for each possible choice of the valuation $v \in S$, and the final reduced bound will be the maximum of the bounds obtained for each separate v.

7.1.1. *The p-adic case.* To execute the reduction, in the p-adic case we use a lemma of Smart [24], which is based upon results of de Weger [29]. For its formulation we need some preparation, in which we follow the presentation in [24] with slight modifications. (For more details see [24].)

Let P be a prime ideal in $\mathbb{K}$, corresponding to a finite valuation v in S. Suppose that P lies above the rational prime p, and suppose that $\mathrm{ord}_p(\Lambda) \geq C_1 B - C_2$ with some constants $C_1 > 0$ and C_2, where $\Lambda = \alpha_1 x_1 = 1 - \alpha_2 x_2$ in (2). Now assuming that B is not too small (otherwise we can derive a much better bound for B than with the reduction), we can find $\mu_i \in \mathbb{K}$ $(i = 0, 1, \dots, s'-1)$ with $s' = s$ or $s-1$, such that

$$\alpha_2 x_2 = \mu_0 \prod_{i=1}^{s'} \mu_i^{k_i}$$

where $k_i \in \mathbb{Z}$ with $|k_i| \leq B$ $(i = 1, \dots, s')$. Note that these μ_i can actually be found (see [24]).

Let $\mathbb{Q}_p$ and $\mathbb{K}_P$ denote the p-adic closure of $\mathbb{Q}$ and the P-adic closure of $\mathbb{K}$, respectively. Then we can write $\mathbb{K}_P = \mathbb{Q}_p(\phi)$ with some $\phi \in \mathbb{K}_p$; put $n_0 := [\mathbb{K}_P : \mathbb{Q}_p]$. Further, set

$$\Delta = \log_p \mu_0 + \sum_{i=1}^{s'} k_i \log_p \mu_i \in \mathbb{K}_P.$$

Then we can write

$$\Delta = \sum_{i=0}^{n_0-1} \Delta_i \phi^i$$

where

$$\Delta_i = \beta_{0,i} + \sum_{j=1}^{s'} k_j \beta_{j,i}$$

with the corresponding $\beta_{ji} \in \mathbb{Q}_p$ $(i = 0, 1, \ldots, n_0 - 1)$. Let $\lambda \in \mathbb{Q}_p$ such that

$$\mathrm{ord}_p(\lambda) = \min_{1 \le j \le s'} \left(\min_{0 \le i \le n_0 - 1} (\mathrm{ord}_p(\beta_{j,i})) \right).$$

Then (assuming again without loss of generality that B is "not too small") by a simple calculation, including Evertse's trick (see [28] and [24]) we get that

$$\mathrm{ord}_p(\Delta_i/\lambda) \ge C_1 B - C_3 \quad (i = 0, 1, \ldots, n_0 - 1).$$

Here C_3 is a constant which can be explicitly given in terms of C_2, λ, ϕ. Write

$$\Delta_i/\lambda = \kappa_{0,i} + \sum_{j=1}^{s'} k_j \kappa_{j,i}, \quad \kappa_{j,i} \in \mathbb{Z}_p,\ 0 \le i \le n_0 - 1$$

with the obvious notation.

For $\gamma \in \mathbb{Z}_p$ and $u \in \mathbb{Z}$ let $\gamma^{(u)}$ denote the unique rational integer such that $0 \le \gamma^{(u)} \le p^u - 1$ and $\gamma \equiv \gamma^{(u)} \pmod{p^u}$. Further, for $u \in \mathbb{Z}$ set

$$L = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ 0 & & 1 & & & \\ \kappa_{1,0}^{(u)} & \ldots & \kappa_{s',0}^{(u)} & p^u & & 0 \\ \vdots & & \vdots & & \ddots & \\ \kappa_{1,n_0-1}^{(u)} & \ldots & \kappa_{s',n_0-1}^{(u)} & 0 & & p^u \end{pmatrix} \in \mathbb{Z}^{(s'+n_0)\times(s'+n_0)}$$

and

$$\underline{y} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\kappa_{0,0}^{(u)} \\ \vdots \\ -\kappa_{0,n_0-1}^{(u)} \end{pmatrix} \in \mathbb{Z}^{s'+n_0}.$$

Let $\mathcal{L}$ denote the lattice generated by the column vectors of L over $\mathbb{Z}$, and set

$$\ell(\mathcal{L}, \underline{y}) = \begin{cases} \min_{\underline{x} \in \mathcal{L}, \underline{x} \ne \underline{0}} ||\underline{x}|| & \text{if } \underline{y} = \underline{0}, \\ \min_{\underline{x} \in \mathcal{L}} ||\underline{x}|| & \text{otherwise.} \end{cases}$$

The following result is Lemma 5 in [24]. Note that the statement is in fact a consequence of Lemmas 3.4, 3.5 and 3.6 of [29].

Lemma 7.1. *Using the previous notation, suppose that* $ord_p(\Lambda) \geq C_1B - C_2$ *with* $B \leq X_0$*. Then* $\ell(\mathcal{L}, \underline{y}) > \sqrt{s'}X_0$ *implies that* $B < (u + C_3)/C_1$*.*

To use this lemma, recall that by (6) we have

$$|\alpha_1 x_1|_v \leq |\alpha_1|_v \exp\left(\frac{-B}{(s-1)C^*}\right)$$

for some $v \in S$. Assuming that v is the valuation occurring in the above argument, this yields

$$p^{-f_p e_p \mathrm{ord}_p(\alpha_1 x_1)} \leq |\alpha_1|_v \exp\left(\frac{-B}{(s-1)C^*}\right),$$

where $\mathrm{Norm}(P) = p^{f_p}$ and e_p is the ramification index of P. Hence a simple calculation gives

$$\mathrm{ord}_p(\Lambda) \geq \left(\frac{1}{f_p e_p \log p(s-1)C^*}\right) B - \left(\frac{\log|\alpha_1|_v}{f_p e_p \log p}\right).$$

Now we can apply Lemma 7.1 with the above inequality, using that $B \leq X_0$ holds for some constant X_0.

7.1.2. *The complex case.* To execute the reduction in the complex case we use the following result, which is an immediate consequence of Lemma 1 of Gaál and Pohst [12].

Lemma 7.2. *Let* $\xi_1, \dots, \xi_k$ *be non-zero real numbers, and let* $x_1, \dots, x_k$ *be integers. Put* $X = \max(|x_1|, \dots, |x_k|)$*. Suppose that*

$$|x_1\xi_1 + \cdots + x_k\xi_k| < C_2 \exp(-C_1 X) \text{ and } X < X_0$$

hold with some positive constants C_1, C_2 *and* X_0*. Further, let* b_1 *be the first vector of an LLL-reduced basis of the lattice spanned by the columns of the* $k \times (k+1)$ *type matrix*

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ H\xi_1 & H\xi_2 & \dots & H\xi_k \end{pmatrix}$$

where H *is some positive constant. Then*

$$|b_1| \geq \sqrt{(k+1)2^{k-1}} \cdot X_0$$

implies that

$$X \leq \frac{\log H + \log C_2 - \log X_0}{C_1}.$$

Now we briefly explain how to apply this lemma. For this purpose, let $v \in S$ be an infinite valuation such that $|1-\alpha_2 x_2|_v \le C_2 \exp(-C_1 B)$ (see (7)). Then using the inequality $|\log x| \le 2|x-1|$ which holds for $|x-1| < 0.795$, we get

$$|b_{2,1} \log |\varepsilon_1|_v + \cdots + b_{2,s-1} \log |\varepsilon_{s-1}|_v| \le 2C_2 \exp(-C_1 B).$$

Then we can apply Lemma 7.2 with the previous inequality, knowing that $B < X_0$ for some constant X_0.

7.2. **Adjusting the method for other choices of the valuations.** In this subsection we indicate how one can adjust our results for other choices of the valuations. The motivation is that if S contains only the infinite places (i.e. we are interested in pure unit equations) then there is an alternative, also natural choice for the valuations: we simply take the absolute value of the real conjugates and the absolute value of one from each pair of complex conjugates of elements $\alpha \in \mathbb{K}$, without squaring in the complex cases. To be more general, keep the previous notation, and take arbitrary non-zero rational numbers $r_1, \ldots, r_s$. Choose now valuations $|.|_{v_i'}$ such that $|\alpha|_{v_i'} := |\alpha|_{v_i}^{r_i}$ for all $\alpha \in \mathbb{K}$, where $|\alpha|_{v_i}$ is the previously defined ("standard") valuation corresponding to $v_i \in S$ $(i = 1, \ldots, s)$. For simplicity, we do not introduce new notation but use the previous one, with the convention that everything is adopted for this new choice of the valuations. We have the following variants of Propositions 2.1 and 2.2.

Proposition 7.1. *Let R_F' be a matrix for which* (10) *is valid. Then for each $i \in \{1, \ldots, s-1\}$ there exists a $u_i \in \mathbb{R}$ such that the i-th row of R_F' is of the form $\mathbf{w}_i - u_i \cdot (1/r_1, \ldots, 1/r_s)$ with $\mathbf{w}_i = (w_{i,1}, \ldots, w_{i,s-1}, 0)$, where*

$$\begin{pmatrix} w_{1,1} & \ldots & w_{1,s-1} \\ \vdots & \ddots & \vdots \\ w_{s-1,1} & \ldots & w_{s-1,s-1} \end{pmatrix} = \begin{pmatrix} \log |\varepsilon_1|_{v_1'} & \ldots & \log |\varepsilon_{s-1}|_{v_1'} \\ \vdots & \ddots & \vdots \\ \log |\varepsilon_1|_{v_{s-1}'} & \ldots & \log |\varepsilon_{s-1}|_{v_{s-1}'} \end{pmatrix}^{-1}.$$

Note that using the assertion $\sum_{j=1}^{s} \frac{1}{r_j} \log |\varepsilon_i|_{v_j'} = 0$ $(i = 1, \ldots, s-1)$, the statement can be proved in a similar manner as Proposition 2.1. We suppress the details.

To formulate our last statement, write $r_i = q_i/p_i$ with $p_i, q_i \in \mathbb{Z}$, $q_i > 0$, $\gcd(p_i, q_i) = 1$ and put $t_i = q/r_i$ where $q = \gcd(q_1, \ldots, q_s)$. Finally, let $\mathbf{w}_i'$ be the $\left(\sum_{i=1}^{s} |t_i|\right)$-tuple such that the first $|t_1|$ entries of $\mathbf{w}_i'$ equal $r_1 w_{i,1}$, the next $|t_2|$ entries of $\mathbf{w}_i'$ equal $r_2 w_{i,2}$, etc. $(i = 1, \ldots, s)$, with the convention $w_{i,s} = 0$.

Proposition 7.2. *Using the above notation, we have*

$$N(F) = \frac{\max_{1\le i\le s-1} |\mathbf{w}_i'|_C}{q}.$$

Observe that for $i = 1, \ldots, s$ we have $\sum_{j=1}^{s} |w_{i,j} - \frac{1}{r_j} u_i| = \frac{1}{q} \sum_{j=1}^{s} |t_j||r_j w_{i,j} - u_i|$. Using this assertion, Proposition 7.2 can be verified similarly to Proposition 2.2. We omit the details once again.

Finally, we note that as one can easily see, using Propositions 7.1 and 7.2 in place of Propositions 2.1 and 2.2, respectively, all the arguments of the paper remain valid also under this general choice of the valuations - after making the necessary (but rather obvious) alternations.

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