A Hardy-Littlewood Integral Inequality on Finite Intervals with a Concave Weight

(*To appear in: Periodica Mathematica Hungarica*)

HORST ALZER[a] and MAN KAM KWONG[b1]

[a] *Morsbacher Str. 10, 51545 Waldbröl, Germany*
H.Alzer@gmx.de

[b] *Department of Applied Mathematics, The Hong Kong Polytechnic University, Hunghom, Hong Kong*
mankwong@polyu.edu.hk

Abstract. We prove: For all concave functions $w : [a, b] \to [0, \infty)$ and for all functions $f \in C^2[a, b]$ with $f(a) = f(b) = 0$ we have

$$\left(\int_a^b w(x)f'(x)^2\,dx\right)^2 \le \left(\int_a^b w(x)f(x)^2\,dx\right)\left(\int_a^b w(x)f''(x)^2\,dx\right).$$

Moreover, we determine all cases of equality.

Keywords and phrases. Integral inequality, HELP-type inequality, concave weight function.

2010 Mathematics Subject Classification. 26D10, 26D15.

[1] The research of this author is fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. PolyU 5012/10P).

1. Introduction

In 1932, Hardy and Littlewood [6] established integral inequalities of the form

$$\left(\int_J f'(x)^2\,dx\right)^2 \le k(J)\left(\int_J f(x)^2\,dx\right)\left(\int_J f''(x)^2\,dx\right), \tag{1.1}$$

where J is either $\mathbf{R} = (-\infty, \infty)$ or $\mathbf{R}^+ = (0, \infty)$, and f is a twice-differentiable function such that $f, f'' \in L^2(J)$, with the best possible constants

$$k(\mathbf{R}) = 1, \quad k(\mathbf{R}^+) = 2. \tag{1.2}$$

The square integrability of f' and the existence of a finite constant $k(J)$ independent of f are implicitly parts of the conclusion.

The significance of the result is in providing an estimate of the "size" of the derivative of a function when bounds on the "sizes" of the function and its second derivative are known, with potential applications in the study of differential equations.

The above results of Hardy and Littlewood are reproduced in Sections 7.9 and 7.8 of the classical work "Inequalities" by Hardy, Littlewood, and Pólya [7]. The case $J = \mathbf{R}$ is an immediate consequence of integration by parts and the Cauchy-Schwarz inequality. The $J = \mathbf{R}^+$ case has a lengthy proof using calculus of variations. Shorter proofs are found later; see, for example, [13].

Inequalities of the form (1.1) are special cases of a more general result, called the Landau inequalities, obtained by replacing the L^2 norms (of f' on the lefthand side and of f and f'' on the righthand side) by other L^p norms, with $1 \le p < \infty$. We note that now the best constant $k(p, J)$ depends both on J and p. It is known that $k(p, \mathbf{R}) < k(p, \mathbf{R}^+)$ for all p. The exact values of $k(p, J)$ are known only for $p = 1$ (due to Berdyshev [1]) and $p = \infty$ (due to Hadamard and Landau, respectively), besides the Hardy-Littlewood result (1.2) for $p = 2$. We only mention here that $k(1, \mathbf{R}) = 2$. For details on the other results and further readings, we refer the readers to the monograph [12].

Many further extensions have been obtained. Gabushin used three (possibly) different norms, L^p, L^q and L^r for f, f' and f'', respectively, and also studied higher derivative analogues. Kato [8] replaced $L^2(J)$ with a Hilbert space, and the derivatives of f with Af and $A^2 f$, respectively, where A is an m-dissipative operator. Everitt [5] substituted the differentiation operator with a general self-adjoint second-order differential operator, and showed that the corresponding best constant can be determined using the m_λ function that Titchmarsh introduced in the Weyl theory of limit-point/limit-circle classification of second-order differential operators. The resulting inequality has been dubbed the HELP (Hardy-Everitt-Littlewood-Pólya) inequality. See [4] for a survey of this topic.

In this paper we are concerned with a weighted form of (1.1) which holds for functions defined on finite intervals and subjected to suitable boundary conditions. More precisely, we determine the best possible constant κ such that we have for all concave functions $w : [a, b] \to [0, \infty)$ and for all $f \in C^2[a, b]$ with $f(a) = f(b) = 0$:

$$\left(\int_a^b w(x) f'(x)^2\,dx\right)^2 \le \kappa \cdot \left(\int_a^b w(x) f(x)^2\,dx\right)\left(\int_a^b w(x) f''(x)^2\,dx\right). \tag{1.3}$$

Two previous works of the second author pertain to the current paper. First, we note that if J is replaced by a finite interval $[a, b]$, an inequality of the form (1.1), as well as its more general Landau analogue, cannot hold, as shown by the counterexample $f(x) = x$. However, if one of the three sets of boundary conditions

$$f(a) = f(b) = 0, \quad f(a) = f'(b) = 0, \quad \text{or} \quad f'(a) = f(b) = 0,$$

is imposed, then (1.1) (or its Landau analogue) is preserved with the same best constant for $J = \mathbf{R}$. Indeed, it was shown by Kwong and Zettl [10] that any one of these finite-interval results is equivalent to the original result for $J = \mathbf{R}$. The case $J = \mathbf{R}^+$ also has a finite-interval equivalent associated with the boundary condition $f'(a)/f(a) = f'(b)/f(b)$. For details, see [12].

In [13], it was proved that a Landau-type inequality holds when a weight function w is added in defining the norms of functions, provided that w is an increasing function:

$$\|f'\|^2 \leq k(p, \mathbf{R}^+)\, \|f\| \cdot \|f''\|, \tag{1.4}$$

where

$$\|f\| = \left(\int_J w(x) |f(x)|^p \, dx \right)^{1/p}.$$

We are not claiming that the constant $k(p, \mathbf{R}^+)$ that appears on the righthand side of (1.4) is the best possible constant for a given weight w; it is, however, best possible over all choices of increasing w. Another important point to note is that, unlike the classical Landau inequalities, the same constant is required in (1.4), irrespective of whether $J = \mathbf{R}$ or $\mathbf{R}^+$.

2. Lemmas

In this section, we collect a few basic facts on concave functions, which we need for our purposes. Proofs for the first three lemmas given below as well as more information on this subject can be found, for example, in the monographs Niculescu and Persson [14], Roberts and Varberg [15], and Royden and Fitzpatrick [16, Section 6.6].

We recall that a function $f : I \to \mathbf{R}$, where $I \subset \mathbf{R}$ is an interval, is said to be concave if

$$\lambda f(x) + (1 - \lambda) f(y) \leq f(\lambda x + (1 - \lambda) y)$$

for all $x, y \in I$ and $\lambda \in (0, 1)$. If $-f$ is concave, then f is called convex.

Lemma 1. *If $f : [a, b] \to \mathbf{R}$ is concave, then f is continuous on (a, b) and the limits*

$$\lim_{x \to a+} f(x) \quad \text{and} \quad \lim_{x \to b-} f(x)$$

exist.

From now on, we always assume that (if necessary) f has been modified such that $f(a) = \lim_{x \to a+} f(x)$ and $f(b) = \lim_{x \to b-} f(x)$ exist, so that f is continuous on $[a, b]$.

The left and right derivatives of a function are defined by

$$f'_-(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c}, \quad f'_+(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c}.$$

Lemma 2. *If $f : [a, b] \to \mathbf{R}$ is concave, then f'_- and f'_+ exist and are decreasing on (a, b).*

Lemma 3. *If $f : (a, b) \to \mathbf{R}$ is concave, then we have for $c, x \in (a, b)$:*

$$f(x) - f(c) = \int_c^x f'_-(t) dt = \int_c^x f'_+(t) dt.$$

The next result is nothing more than integration by parts for integrable functions. It is given, for example, in [17, p. 32] and in [16, p. 128, Problem 52].

Lemma 4. *Let $w : [a,b] \to \mathbf{R}$ be concave. If $f \in C^1[a,b]$ with $f(a) = f(b) = 0$, then*

$$\int_a^b w(t)f'(t)dt = -\int_a^b w'_-(t)f(t)dt.$$

Lemma 5. *Let $w : [a,b] \to [0,\infty)$ be concave. Then there exists a sequence of non-negative concave functions (w_n) $(n = 1,2,...)$ with $w_n \in C^2[a,b]$ such that w_n converges uniformly to w on $[a,b]$.*

Proof. A result of Bremermann [2] states that if f is convex on $[a,b]$, then for every $k \in \mathbf{N}$ there exists a sequence of k-times differentiable convex functions (f_n) with $f_n(x) \geq f_{n+1}(x) \geq f(x)$ and f_n converges to f on $[a,b]$. Applying Dini's theorem we conclude that the convergence is uniform. This implies that there exists a sequence of concave functions (w_n) with $w_n \in C^2[a,b]$ such that w_n converges uniformly to w on $[a,b]$. If it happens that $w_n \geq 0$ on $[a,b]$ for all n, then (w_n) is our desired sequence.

In the contrary case, we modify w_n as follows. First we note that, for every $\epsilon > 0$ there is a natural number n_0 such that for all $n \geq n_0$ and for all $x \in [a,b]$:

$$|w(x) - w_n(x)| < \epsilon/2. \tag{2.1}$$

We assume that w_n is negative somewhere in $[a,b]$ and set

$$w_n(x_0) = \min_{a \leq x \leq b} w_n(x).$$

Then, $w_n(x_0) < 0$. From (2.1) we obtain

$$w(x_0) - w_n(x_0) < \epsilon/2.$$

Since $w(x_0) \geq 0$, it follows that

$$w_n(x_0) > w(x_0) - \epsilon/2 \geq -\epsilon/2.$$

Let $w_n^*(x) = w_n(x) - w_n(x_0)$. Then, w_n^* is non-negative and concave on $[a,b]$. Moreover, we get for $n \geq n_0$ and $x \in [a,b]$:

$$|w(x) - w_n^*(x)| \leq |w(x) - w_n(x)| + |w_n(x_0)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This implies that w_n^* converges uniformly to w on $[a,b]$. $\square$

3. Main Result

We are now in a position to present our main result. The following theorem reveals that the best possible constant factor in (1.3) is given by $\kappa = 1$.

Theorem. *For all concave functions $w : [a,b] \to [0,\infty)$ and for all functions $f \in C^2[a,b]$ with $f(a) = f(b) = 0$ we have*

$$\Big(\int_a^b w(x)f'(x)^2\,dx\Big)^2 \leq \Big(\int_a^b w(x)f(x)^2\,dx\Big)\Big(\int_a^b w(x)f''(x)^2\,dx\Big). \tag{3.1}$$

Let $w \not\equiv 0$ and $f \not\equiv 0$. Then, the sign of equality holds in (3.1) if and only if

$$f(x) = \lambda \sin\left(\frac{n\pi(x-a)}{b-a}\right), \tag{3.2}$$

where λ is a real number and n is a natural number, and w is linear in each of the subintervals

$$J_k = \Big[a + \frac{(k-1)(b-a)}{n}, a + \frac{k(b-a)}{n}\Big], \quad k = 1, ..., n. \tag{3.3}$$

Proof. From Lemma 5, we conclude that it suffices to prove inequality (3.1) under the additional assumption that the weight function $w \in C^2[a,b]$.

Integration by parts gives

$$\int_a^b w''(x) f(x)^2 dx = \Big[w'(x) f(x)^2\Big]_a^b - 2\int_a^b w'(x) f(x) f'(x)^2 dx \tag{3.4}$$

and

$$\int_a^b w'(x) f(x) f'(x) dx = \Big[w(x) f(x) f'(x)\Big]_a^b - \int_a^b w(x)[f(x) f''(x) + f'(x)^2] dx. \tag{3.5}$$

Since $f(a) = f(b) = 0$, we obtain from (3.4) and (3.5):

$$\int_a^b w''(x) f(x)^2 dx = 2\int_a^b w(x)[f(x) f''(x) + f'(x)^2] dx. \tag{3.6}$$

Using $-w''(x) \geq 0$ and (3.6), we find

(3.7)

$$\int_a^b w(x) f'(x)^2 dx \leq \int_a^b w(x) f'(x)^2 dx - \frac{1}{2}\int_a^b w''(x) f(x)^2 dx = \int_a^b \big(\sqrt{w(x)} f(x)\big) \cdot \big(-\sqrt{w(x)} f''(x)\big) dx.$$

The Cauchy-Schwarz inequality yields

$$\int_a^b \big(\sqrt{w(x)} f(x)\big) \cdot \big(-\sqrt{w(x)} f''(x)\big) dx \leq \Big(\int_a^b w(x) f(x)^2 dx\Big)^{1/2} \Big(\int_a^b w(x) f''(x)^2 dx\Big)^{1/2}. \tag{3.8}$$

Combining (3.7) and (3.8) leads to (3.1).

Next, we discuss the case of equality. A short calculation reveals that equality holds in (3.1) if f is given by (3.2) and if w is linear in each of the subintervals J_k given in (3.3).

Now, we assume that the sign of equality is valid in (3.1). Let $w \not\equiv 0$ and $f \not\equiv 0$. Applying Lemma 5, (3.7), and (3.8) yields

$$\int_a^b w_n(x) f'(x)^2 dx \leq -\int_a^b w_n(x) f(x) f''(x) dx \leq \Big(\int_a^b w_n(x) f(x)^2 dx\Big)^{1/2} \Big(\int_a^b w_n(x) f''(x)^2 dx\Big)^{1/2}.$$

Letting $n \to \infty$ gives

$$\int_a^b w(x) f'(x)^2 dx \leq -\int_a^b w(x) f(x) f''(x) dx \leq \Big(\int_a^b w(x) f(x)^2 dx\Big)^{1/2} \Big(\int_a^b w(x) f''(x)^2 dx\Big)^{1/2}. \tag{3.9}$$

By assumption, the expressions on the left-hand side and right-hand side are equal. This implies that the sign of equality holds in the second inequality of (3.9). It follows that

$$\sqrt{w(x)} f(x) = \lambda_0 \sqrt{w(x)} f''(x)$$

for some constant λ_0. Since w is non-negative and concave with $w \not\equiv 0$, we conclude that w is positive on (a,b). Hence,

$$f(x) = \lambda_0 f''(x)$$

with $\lambda_0 \neq 0$. Together with the boundary conditions $f(a) = f(b) = 0$, we see that f must be of the form given in (3.2).

Since f is equal to 0 at the end-points of each subinterval J_k, we conclude that the hypothesis of the Theorem is satisfied for each k. Hence, for all $k \in \{1, 2, ..., n\}$ we have

$$\left(\int_{J_k} w(x) f'(x)^2\,dx\right)^2 \le \left(\int_{J_k} w(x) f(x)^2\,dx\right)\left(\int_{J_k} w(x) f''(x)^2\,dx\right). \tag{3.10}$$

We claim that equality must hold in (3.10) for each k.

The following equivalence is easy to verify:

If A, B, and C are positive numbers, then

$$B^2 \le AC \iff B \le \epsilon A + \frac{1}{4\epsilon} C \quad \textit{for all} \quad \epsilon > 0. \tag{3.11}$$

Moreover, "$<$" holds on the left-hand side if and only if it holds on the right-hand side.

Applying (3.11) reveals that (3.10) is equivalent to

$$\int_{J_k} w(x) f'(x)^2\,dx \le \epsilon \int_{J_k} w(x) f(x)^2\,dx + \frac{1}{4\epsilon}\int_{J_k} w(x) f''(x)^2\,dx \quad \text{for all} \quad \epsilon > 0. \tag{3.12}$$

If strict inequality in (3.10) holds for at least one k, then the corresponding inequality (3.12) must also be strict. After adding up (3.12) for all k, we obtain

$$\int_a^b w(x) f'(x)^2\,dx < \epsilon \int_a^b w(x) f(x)^2\,dx + \frac{1}{4\epsilon}\int_a^b w(x) f''(x)^2\,dx \quad \text{for all} \quad \epsilon > 0.$$

This implies that the sign of equality does not hold in (3.1), a contradiction.

It remains to show that w is linear in each J_k under the assumptions that f has the representation (3.2) and that equality is valid in (3.10) for each k. We insert (3.2) in (3.10) (with "$=$" instead of "$\le$") and substitute

$$x = a + \frac{(k-1)(b-a)}{n} + \frac{b-a}{n\pi} t.$$

Then we get

$$\int_0^\pi W(t)\cos^2(t)dt = \int_0^\pi W(t)\sin^2(t)dt \tag{3.13}$$

with

$$W(t) = w\Big(a + \frac{(k-1)(b-a)}{n} + \frac{b-a}{n\pi} t\Big).$$

The function W is concave. We show that W is linear on $[0, \pi]$, which implies that w is linear on J_k.

The identity $\cos^2(A) - \sin^2(A) = \cos(2A)$ and (3.13) imply

$$\int_0^\pi W(t)\cos(2t)dt = 0. \tag{3.14}$$

Using Lemma 4 and (3.14) gives

$$\int_0^\pi W'_-(t)\sin(2t)dt = \int_0^{\pi/2} W'_-(t)\sin(2t)dt + \int_{\pi/2}^\pi W'_-(t)\sin(2t)dt = 0. \tag{3.15}$$

We have

$$\int_{\pi/2}^{\pi} W'_-(t)\sin(2t)dt = -\int_0^{\pi/2} W'_-(\pi - t)\sin(2t)dt, \tag{3.16}$$

so that (3.15) and (3.16) lead to

$$\int_0^{\pi/2} \Delta(t)\sin(2t)dt = 0, \tag{3.17}$$

where

$$\Delta(t) = W'_-(t) - W'_-(\pi - t).$$

Since W'_- is decreasing on $(0,\pi)$, it follows that $\Delta \geq 0$ on $(0,\pi/2)$. Moreover, the function $t \mapsto -W'_-(\pi - t)$ is decreasing on $(0,\pi/2)$, which implies that Δ is also decreasing on $(0,\pi/2)$. Since $\sin(2t) > 0$ for $t \in (0,\pi/2)$, we conclude from (3.17) that $\Delta(t) = 0$ for all $t \in (0,\pi/2)$. Hence,

$$W'_-(t) = W'_-(\pi - t) \quad \text{for all} \quad t \in (0,\pi/2]. \tag{3.18}$$

Let ϵ be any small positive number. Since W'_- is decreasing we conclude from (3.18) that W'_- must be a constant on $(\epsilon, \pi - \epsilon)$. This is valid for any small $\epsilon > 0$, which reveals that W'_- is a constant on $(0,\pi)$. Applying Lemma 3 reveals that W is linear. □

Corollary. *For all concave and increasing functions $w : [a,b] \to [0,\infty)$ and for all functions $f \in C^2[a,b]$ with $f(a) = f'(b) = 0$ we have*

$$\left(\int_a^b w(x)f'(x)^2\,dx\right)^2 \leq \left(\int_a^b w(x)f(x)^2\,dx\right)\left(\int_a^b w(x)f''(x)^2\,dx\right). \tag{3.19}$$

Let $w \not\equiv 0$ and $f \not\equiv 0$. Then, the sign of equality holds in (3.19) if and only if

$$f(x) = \lambda \sin\left(\frac{(2n-1)\pi(x-a)}{2(b-a)}\right),$$

where λ is a real number and n is a natural number, and w is linear in each of the subintervals

$$I_k = \left[a + \frac{2(k-1)(b-a)}{2n-1}, a + \frac{2k(b-a)}{2n-1}\right], \quad k = 1, ..., n-1$$

and is constant in the subinterval

$$I = \left[a + \frac{2(n-1)(b-a)}{2n-1}, b\right].$$

Proof. We extend w and f to $[a, 2b-a]$ so that each function is even with respect to $x = b$, that is,

$$w(b+\sigma) = w(b-\sigma) \quad \text{and} \quad f(b+\sigma) = f(b-\sigma) \quad \text{for } \sigma \in (0, b-a].$$

Then, w is non-negative and concave on $[a, 2b-a]$ and $f \in C^2[a, 2b-a]$ with $f(a) = f(2b-a) = 0$. So, w and f satisfy the hypotheses of the Theorem on the extended domain $[a, 2b-a]$ and the conclusion follows from the Theorem. □

An obvious analogous result holds for functions f that satisfies $f'(a) = f(b) = 0$.

4. Concluding Remarks

In view of the Theorem two questions arise naturally.

(i) Does the Theorem remain valid if we remove the "concavity" requirement?

(ii) Is there a corresponding result for convex weight functions? More precisely, given a convex function $w : [a,b] \to [0,\infty)$, does there exist a constant C such that for all $f \in C^2[a,b]$ with $f(a) = f(b) = 0$:

$$\kappa(w,f) = \frac{\left(\int_a^b w(x) f'(x)^2\,dx\right)^2}{\left(\int_a^b w(x) f(x)^2\,dx\right)\left(\int_a^b w(x) f''(x)^2\,dx\right)} \le C\,?$$

An example, constructed with the help of the MAPLE software, reveals that in both cases the answer is "no".

Let $[a,b] = [0,1]$, $w(x) = x^4$, and $\delta \in (0,1/2)$. The piecewise polynomial function

$$f(x) = \frac{x}{\delta} \quad (0 \le x \le \delta), \quad f(x) = \sum_{k=0}^{5} a_k x^k \quad (\delta < x \le 2\delta), \quad f(x) = \frac{1-x}{1-2\delta} \quad (2\delta < x \le 1)$$

is determined by the interpolation conditions

$$f(0) = f(1) = 0, \quad f(\delta) = f(2\delta) = 1, \quad f'(\delta) = \frac{1}{\delta}, \quad f'(2\delta) = -\frac{1}{1-2\delta}, \quad f''(\delta) = f''(2\delta) = 0.$$

These conditions guarantee that $f \in C^2[0,1]$. MAPLE is used to compute the coefficients:

$$a_0 = \frac{48\,\delta - 17}{2\,\delta - 1}, \qquad a_1 = -\frac{183\,\delta - 64}{\delta\,(2\,\delta - 1)}, \quad a_2 = \frac{12(23\,\delta - 8)}{\delta^2\,(2\,\delta - 1)},$$

$$a_3 = -\frac{2(99\,\delta - 34)}{\delta^3\,(2\,\delta - 1)}, \quad a_4 = \frac{68\,\delta - 23}{\delta^4\,(2\,\delta - 1)}, \quad a_5 = -\frac{3(3\,\delta - 1)}{\delta^5\,(2\,\delta - 1)}.$$

Straightforward computation gives

$$\kappa(w,f) = \frac{(3003 + 14474\,\delta^3 - 53525\,\delta^4 - 12344\,\delta^5)^2}{26\,\delta(858 + 72450\,\delta^5 - 531793\,\delta^6 + 674178\,\delta^7)(2042 - 11999\,\delta + 20182\,\delta^2)}.$$

Since δ appears as a factor in the denominator, we obtain $\lim_{\delta \to 0} \kappa(w,f) = \infty$.

(iii) The following example shows that the "monotonicity" requirement on w in the Corollary cannot be eliminated.

Let $[a,b] = [0,1]$, $w(x) = 1 - x$ and $f(x) = \sin(\pi x/2)$. Straightforward computation gives

$$\kappa(w,f) = \left(\frac{\pi^2 + 4}{\pi^2 - 4}\right)^2 \approx 5.5835.$$

References

[1] V.I. Berdyshev, "The best approximation in $L(0,\infty)$ to the differentiation operator," *Math. Zametki*, **5** (1971), 477-481.

[2] H.J. Bremermann, "Complex convexity", *Trans. Amer. Math. Soc.*, **82** (1956), 17-51.

[3] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Math., vol. 19, Amer. Math. Soc., 1998.

[4] W.D. Evans and W.N. Everitt, "HELP inequalities for limit-circle and regular problems," *Proceedings: Mathematical and Physical Sciences*, **432** (1991), 367-390

[5] W.N. Everitt, "On an extension to an integro-differential inequality of Hardy, Littlewood and Pólya," *Proc. Roy. Soc. Edinburgh*, **69** (1971/72), 295-333.

[6] G.H. Hardy and J.E. Littlewood, "Some integral inequalities connected with the calculus of variations," *Quart. J. Math.*, Oxford Ser. 2, **3** (1932), 241-252.

[7] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[8] T. Kato, "On an inequality of Hardy, Littlewood and Pólya," *Adv. Math.*, **7** (1971), 217-218.

[9] M. K. Kwong and A. Zettl, "An extension of the Hardy-Littlewood inequality," *Proc. Amer. Math. Soc.*, **77** (1979), 117-118.

[10] M. K. Kwong and A. Zettl, "Remarks on best constants for norm inequalities among powers of an operator," *J. Approx. Theory*, **26** (1979), no. 3, 249-258.

[11] M. K. Kwong and A. Zettl, "Norm inequalities of product form in weighted L^p spaces," *Proc. Roy. Soc. Edinburgh.*, **89A** (1981), 293-307.

[12] M. K. Kwong and A. Zettl, *Norm inequalities for derivatives and differences*, Lecture Notes in Mathematics 1536, Springer Verlag, 1992.

[13] M. K. Kwong and A. Zettl, "An alternate proof of Kato's inequality." *Evolution Equations, Lecture Notes in Pure and Appl. Math.*, **234**, 275-279, Dekker, New York, 2003.

[14] C. Niculescu and L.-E. Persson, *Convex Functions and Their Applications*, CMS Books in Mathematics, vol. 23, Springer, 2006.

[15] A.W. Roberts and D.E. Varberg, *Convex Functions*, Academic Press, New York, 1973.

[16] H.L. Royden and P.M. Fitzpatrick, *Real Analysis, 4th Edition*, Boston, Prentice Hall, 2010.

[17] E.C. Titchmarsh, *The Theory of Functions*, Oxford University Press, London, 1939.