On Linkedness of the Cartesian Product of Graphs

Gábor Mészáros*

January 9, 2015

Abstract

We study linkedness of the Cartesian product of graphs and prove that the product of an a-linked and a b-linked graphs is $(a+b-1)$-linked if the graphs are sufficiently large. Further bounds in terms of connectivity are shown. We determine linkedness of products of paths and products of cycles.

Introduction

Throughout this paper we use the notation of [1]. For the sake of completeness we recall definitions of the mainly used concepts. The *connectivity* of a simple graph $G = (V(G), E(G))$ (denoted by $\kappa(G)$) is the smallest number of vertices whose removal from G results in a disconnected graph or a graph of one vertex. In particular, a graph G is k-connected if $V(G) \geq k+1$ and the removal of any subset of at most $(k-1)$ vertices does not disconnect the graph. The *Cartesian product* of graphs G and H is the graph $G\Box H$ with vertices $V(G\Box H) = V(G) \times V(H)$, and $(x,u)(y,v)$ is an edge if $x = y$ and $uv \in E(H)$ or $xy \in E(G)$ and $u = v$. Product of graphs $G_1, \dots, G_t$ for $t \geq 3$ is defined recursively. Note that the Cartesian product is an associative operation. The graphs $G_1, \dots, G_t$ are called factors of $G_1 \Box \dots \Box G_t$. The Cartesian product is a well studied graph product and it gave rise to important classes of graphs; for example, the n-dimensional grid can be considered as the Cartesian product of lower dimensional grids. Hypercubes are well known members of this family with similar recursive structure: the Cartesian product of an m-dimensional hypercube and an n-dimensional hypercube is an $(m+n)$-dimensional one.

The study of graph products leads deep structural problems such as invariance and inheritance of graph parameters: connections between parameters of products and their factors have been extensively studied. Note that among the several graph products (see [6]) the Cartesian product is also known as *direct sum* referring to the fact that many of the classical graph parameters inherit additively. In case of *minimum, maximum* and *average degree* it can be shown easily that $\delta(G\Box H) = \delta(G) + \delta(H)$, $\Delta(G\Box H) = \Delta(G) + \Delta(H)$ and $\underline{d}(G\Box H) = \underline{d}(G) + \underline{d}(H)$. We present some further results with linear bounds. Chiue and Shieh [4] proved that the Cartesian product of a k-connected and an l-connected graph is $(k+l)$-connected. Later on, Špacapan [9] determined the connectivity number of $G\Box H$, namely $\kappa(G\Box H) = \min\big(\delta(G) + \delta(H), \kappa(G)\cdot|V(H)|, \kappa(H)\cdot|V(G)|\big)$. Győri and Plummer [5] proved that the Cartesian product of a k-extendable and an l-extendable graph is $(k+l+1)$-extendable (a graph G is k-extendable if G is connected, has a perfect matching and any matching of k edges in G can be extended to a perfect matching).

In this paper we study linkedness of the Cartesian product of graphs. Menger's theorem (see [1]) implies that a graph is k-connected if and only if for every (not necessarily disjoint) k-tuples $S = \{s_1, \dots, s_k\}$ and $T = \{t_1, \dots, t_k\}$ there exist disjoint paths $P_1, \dots, P_k$ joining every s_i to $t_{\pi(i)}$ for some permutation $\pi \in S_k$. Menger's theorem provides no control on the actual pairing of S and T via

*Research was supported by the Balassi Institute, the Fulbright Commission, and the Rosztoczy Foundation.

paths. A graph G is *k-linked* if, for every ordered set of $2k$ vertices $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$ there exist internally disjoint paths $P_1, \ldots, P_k$ such that each P_i is an $s_i - t_i$-path. It is a well-known but somewhat technical result that a graph G is k-linked if and only if the above condition holds for every choice of disjoint S and T sets. We use the proposition without proof and always assume that S and T are disjoint for the sake of simplicity. We use the notation $\text{link}(G)$ for the linkedness-number of a graph G, that is, the largest positive integer k for which G is k-linked.

Linkedness is a natural strenghtening of connectivity. It is easy to see that k-linked graphs are $(2k-1)$-connected. Certainly, placing vertices $s_1, t_1, \ldots, s_{k-1}, t_{k-1}$ in a graph G to a cut D of size $(2k-2)$ makes impossible to join s_k to t_k if they are located in different components of $G - D$. It has been also known for some time that sufficient connectivity would imply linkedness. Bollobás and Thomason [2] gave the first linear upper bound proving that $22k$-connected graphs are k-linked. This bound has been improved to $10k$ by Thomas and Wollan [10] and it is also very likely that the connectivity needed to imply k-linkedness is significantly less than $10k$. When girth conditions are placed on the graph, then almost sharp results between connectivity and linkedness can be proven. Mader [8] proved that $2k$ connected graphs with sufficiently large girth are k-linked. The condition on the girth has been weakened by Kawarabayshi [7]. Note that neither Mader's nor Kawarabayshi's result can be applied for the Cartesian Product of graphs, as the girth of the Cartesian Product of two nonempty graphs is upperly bounded by four.

In this paper, we prove that the Cartesian product of an a-linked graph and a b-linked graph is $(a + b - 1)$ linked if the graphs are sufficiently large.

Theorem 1. *If G is an a-linked graph with $|V(G)| \geq 8a$ and H is a b-linked graph with $|V(H)| \geq 8b$ then $G \Box H$ is $(a + b - 1)$-linked.*

Remark that the bound in Theorem 1 is sharp. Let $n, k \in \mathbb{Z}^+$, $n \geq 2k - 1$ and construct a graph G as follows: take the complete graph K_n on n vertices and an additional vertex that is adjacent to $(2k-1)$ vertices in K_n. Easy to see that G is k-linked, $(2k-1)$-connected, while $G \Box G$ is $(4k-2)$-connected, hence it cannot be $2k$-linked as $2k$-linked graphs are $(4k-1)$-connected. As n does not depend on the choice of k (only $n \geq 2k - 1$ is required) it provides an infinite family of products where equality holds. Later on we prove that higher connectivity of G (with all other settings unchanged) yields better lower bound on the linkedness of the product graph.

It follows from Theorem 1 that the product $G \Box H$ of a k-linked graph G and graph H is also k-linked if H is connected, while disconnected H makes $G \Box H$ also disconnected. In the second part of the paper we find sufficient conditions for a graph H such that the product $G \Box H$ of H and a (sufficiently large) k-linked graph G is $(k+1)$-linked. Using that theorem, in the last section we determine linkedness of products of paths and products of cycles.

Theorem 2. *If G is a k-linked graph with $k \geq 2$ and $|V(G)| \geq \max(9, 4k)$ and H is a 2-connected graph then $G \Box H$ is $(k+1)$-linked.*

Before the proofs we fix further terminologies and notation. A *G-layer* G_x $(x \in V(H))$ of the Cartesian product $G \Box H$ is the subgraph induced by the set of vertices $\{(u, x) : u \in V(G)\}$. An H-layer is defined analogously. We call edges of $G \Box H$ lying in G-layers *horizontal* while edges lying in H-layers are called *vertical*. Unless misleading we also use the notation $G_z = G_x$ and $H_z = H_y$ for layers corresponding to $z = (x, y) \in V(G \Box H)$. The *projection* of vertex (u, v) to a horizontal layer G_x or a vertical layer H_y is (u, x) and (y, v), respectively. The set of neighbours of a vertex x in a graph G is denoted by $\Gamma_G(x)$. For a graph G we use the notation $G - x$ and $G - H$ for removing the vertex $x \in V(G)$ or the subgraph $H \subset G$ from G. The size of a set H is denoted by $|V(H)|$. The labelled vertices $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$ to be linked are sometimes called *terminals*, the sets $\{u_i, v_i\}$ are *pairs* or *matching terminals*. Finding a path for a pair is often called *joining* the pair.

Proof of Theorem 1

Recall a straightforward corollary of Menger's Theorem:

Lemma 1. *If G is k-connected, $m, n \in \mathbb{Z}^+$, $m+n \leq k$, then for every disjoint tuples $D = \{d_1, \dots, d_m\}$, $S = \{s_1, \dots, s_n\}$ and $T = \{t_1, \dots, t_n\}$ there exist disjoint paths $P_1, \dots, P_n$ in $G - D$ joining every s_i to $t_{\pi(i)}$ for some $\pi \in S_n$.*

Proof. Use Menger's Theorem on sets S and T in the graph $G - D$ which is $(k - m)$- connected. □

We first settle the case when a or b is equal to 1. Note that being 1-linked is equivalent to connectivity.

Lemma 2. *Let G be k-linked, H be connected. Then $G \Box H$ is k-linked as well.*

Proof. Let M denote the set of $2k$ (arbitrarily chosen and paired) terminals in $G \Box H$. Take a G-layer G_x $(x \in H)$ with terminals $u_1, \dots, u_t$ $(1 \leq t \leq 2k)$. If $t = 2k$, use the condition that G_x is k-linked and find the necessary paths within the layer. Otherwise, let $D = \{u_1, \dots, u_t\}$, $S = M - D$ and let T consist of $(2k - t)$ non-terminal vertices in G_x.

We use Lemma 1 for the graph $G \Box H$ which is $(2k - 1)$-connected as G is k-linked and H is connected. Using Lemma 1 one can find $(2k - t)$ paths $P_1, \dots, P_{2k-t}$ from S to T in $G \Box H - D$. For each P_i path let p_i denote its terminal endpoint in S and let p_i' denote the first vertex of P_i in G_x (the vertex where P_i first "enters" G_x). Truncate P to a $p_i - p_i'$ path. Using the condition that G_x is k-linked, one can find k paths $Q_1, \dots, Q_k$ that join the $2k$ vertices of the set $D \cup \{p_i' : 1 \leq i \leq (2k-t)\}$, with the obvious matching (p_i' is paired with the original pair of p_i). Note that the truncation is performed in order to get the above P_i and Q_i paths disjoint. Also, observe that the path system $Q_1, \dots, Q_k$ extended by paths $P_1, \dots, P_{2k-t}$ at the p_i' vertices forms an appropritate path system for the initial matching. That completes the proof. □

From now on, we may assume $a \geq b \geq 2$. We prove a more general form of Theorem 1:

Theorem 3. *If G is an a-linked graph with $|V(G)| \geq 8a$ and H is a $(2b - 1)$-connected graph with $a \geq b$ then $G \Box H$ is $(a + b - 1)$-linked.*

Proof. Our main goal in the proof is to carry out one of the following tasks.

i) Join one terminal to its pair within a layer and proceed by induction on an appropriate subgraph.

ii) For every pair (x, y) find paths P_x, P_y with other endvertices x' and y', such that x' and y' share the same horizontal layer. Following that we will find a path Q joining x' and y' and join x and y by the concatenation $P_x - Q - P_y$.

For the latter task, observe that, as the total number of terminals is $(2a + 2b - 2)$ and $a \geq b$, two approriate horizontal G-layers will be sufficient to contain and join all the x'-s and y'-s. The bottleneck of the idea is that all the P_x, P_y paths have to be disjoint. We also want to make sure that these paths enter only one of the above distinguished horizontal layers containing the x'-s and y'-s. We will use Lemma 1 to guarantee such conditions. We call a G-layer *crowded* if it contains more than $(2a - 1)$ terminals. Observe that crowded G-layers necessarily contain at least one pair of matching terminals.

If there exists a crowded G-layer G_x $(x \in H)$ in $G \Box H$, take a pair $u_1, v_1 \in G_x$. As $|\Gamma_H(x)| \geq (2b-1)$, there exists $y \in \Gamma_H(x)$ such that G_y contains no terminal. The appropriate neighbours of u_1 and v_1 in G_y can be joined by a path within G_y. We can join u_1 and v_1 by extending that path on both ends by the vertical edges from u_1 and from v_1 to G_y. For every remaining terminal u of G_x we find a vertical neighbour not belonging to G_y as follows (note that case i) and case ii) do not exclude each other).

i) Link u to its pair if they are adjacent by a vertical edge.

ii) If the terminal u has a vertical neighbour u' that is neither a terminal nor has it been previously assigned as a vertical neighbour to another terminal in G_x, choose u'.

iii) If neither of the previous cases applies, then H_u contains all terminals lying outside of G_x and its pair v lies in G_x. Switch (u_1, v_1) to (u, v) and start the procedure again with joining u and v. The second round terminates without encountering the same problem.

Define a new pairing of the remaining $(a + b - 2)$ pairs of terminals by substituting every u by u'. Observe that G and $H - x - y$ are a-linked and $(b-1)$-linked and have at least $8a$ and $(8b-2)$ vertices, respectively. By inductional hypothesis, $G\Box(H-x-y)$ is $(a+b-2)$-linked and so there exist $(a+b-2)$ paths joining the newly defined $(a + b - 2)$ pairs. The extension of these paths by the appropriate $\overline{uu'}$ edges results in a path system that joins the original pairing.

Assume now that $G\Box H$ contains no crowded G-layer. For a terminal u our first goal is to find a path with horizontal edges to a vertex $u' \in G_u$ such that $H_{u'}$ is devoid of terminals and endvertices of previously routed paths of the same kind. We carry out this task in several rounds, defining a u' vertex and a corresponding $u - u'$ path for every u terminal of a given G-layer within a round. As long as the number of terminals on layers being or having been processed does not exceed $(2a - 1)$, Lemma 1 provides an easy way of assignment. We will frequently use the following *truncation* operation during our proof. Assume we are given a path P of horizontal edges with a terminal end u and a non-terminal endvertex $\hat{u}$, whose $H_{\hat{u}}$ layer does not contain terminals or vertices of previously defined paths. Starting with u, we read the vertices of P in precedence order until we find the first vertex u' that has the same properties as $\hat{u}$. We stop and truncate P to an $u - u'$ path. Observe that the main importance of the truncation operation is gaining control of the length of the joining paths that is not automatically guaranteed by Menger's theorem. The above tuncation of the paths makes sure that we can find at every step an appropriate canditate for the role of $\hat{u}$.

Consider all G-layers $G_1, \dots, G_n$ containing $0 < s_1 \le \dots \le s_n < 2a$ terminals. Choose $1 \le t \le n$ such that $\sum_{i=1}^{t-1} s_i \le (2a - 1)$ and $\sum_{i=1}^{t} s_i > (2a - 1)$. We design our algorithm as follows:

i) In round 1, choose a set of s_1 vertices in G_1 whose corrensponding H-layers do not contain any terminal. Use Menger's theorem to find s_1 disjoint paths between the terminals of G_1 and the set. Truncate these paths and define the set D_1 as the set of the non-terminal endpoints of the truncated paths.

ii) In round i for $2 \le i \le (t-1)$, let T denote the set of terminals in G_i and let D_i be the projection of D_{i-1} to G_i. Choose a set S of s_i vertices in G_i whose corrensponding H-layers do not contain any terminal or vertex of D_i. Easy to see that $|D_i| = \sum_{j=1}^{i} s_j \le (2a - 1)$, hence the conditions of Lemma 1 hold. Take s_i paths joining (in some order) S and T. Truncate the paths and update D_i by adding the set of the paths's non-terminal endpoints.

iii) In the remaining $(n - t + 1)$ rounds $(t \le i \le n)$, choose a set of s_i vertices in G_i whose corrensponding H-layers do not contain any terminal. Use Menger's Theorem to find s_i disjoint paths joining (in some order) the terminals and the newly chosen vertices.

We refer to the previous phase as a *global horizontal shift*. Observe that each terminal u was given a non-terminal vertex $u' \in G_u$ and an uu' path $P_{uu'}$ of horizontal edges, such that:

A) $P_{uu'}$ does not intersect with other paths defined in the phase.

B) $H_{u'}$ consist of at most $(n - t + 1)$ vertices belonging to other paths defined in the phase (at most one at each layer during the last $(n - t + 1)$ steps).

Note that the condition $V(G) \geq 8a$ guarantees that every step of the horizontal shift can be carried out without running out of space; we have at most $4a$ terminals in the graph, each of which requires at most one new H layer during that phase. Our next goal is to carry out a *global vertical shift*. We take two G-layers that contain neither terminals nor vertices belonging to paths of the previous phase and call them G_α and G_β. For each u' of the previous phase we define a vertex u'' and a $u' - u''$ path in $H_{u'}$ such that:

i) $u'' \in G_\alpha$ or $u'' \in G_\beta$,

ii) if (u, v) are a pair, then u'' and v'' belong to the same G-layer,

iii) G_α and G_β both have at most a pairs of (u'', v'') vertices,

iv) the path $P_{u'u''}$ does not intersect other paths of the recent or the previous phase (with the exception of $P_{uu'}$). In addition, if $u'' \in G_\alpha$, then $P_{u'u''} \cap G_\beta = \emptyset$, if $u'' \in G_\beta$, then $P_{u'u''} \cap G_\alpha = \emptyset$.

Clearly, G_α and G_β will provide room for the final step of joining the terminals. As both layers are a-linked, all (u'', v'') pairs can be joined by disjoint paths. Our initial pair (u, v) will be joined by an $u - u' - u'' - v'' - v' - v$ path. It remains to show that the $P_{u'u''}$ can be found with the above conditions. Distribute the (u, v) terminal pairs among G_α and G_β an arbitrary, balanced way (the layers receive $\lfloor \frac{a+b-1}{2} \rfloor$ and $\lceil \frac{a+b-1}{2} \rceil$ terminal pairs). For given u' an u'' vertices, we may assume, without loss of generality, that $u'' \in G_\alpha$. The underlying $H_{u'}$-layer is $(2b-1)$-connected. It contains at most $(n-t+1)$ vertices of horizontal paths and the projection of u' to G_β. If $(n-t+2) \leq (2b-2)$, we can find a $P_{u'u''}$ path that contains none of the listed vertices.

If $(n-t+2) > (2b-2)$, then $s_1 = \cdots = s_n = 1$ or $s_1 = \cdots = s_{n-1} = 1, s_n = 2$. These rather simple cases can be handled by very simple case-by case analysis. Choose an empty H layer for every pair of terminals. As each G-layer is $(2a-1)$-connected, and there are $(a+b-1)$ pairs of terminals, we can set a path between a terminal u and the assigned u' endpoint within G_u without entering the other assigned H-layers. We join (u, v) by an $(u - u' - v' - v)$ path. We leave the detailed analysis as an exercise for the reader.

There are a few more cases to consider. If there is only one G-layer that is devoid of terminals, label it G_α and choose an arbitrary G_β layer with at most a terminals. As the average terminal load of a G layer is $\frac{2a+2b-2}{8b-1}$, such layer can be easily found. In that case, G_β skips phase one and we proceed in phase two, saving all the horizontal edges for the final joining.

If there is no available G-layer, we need a more elaborate work to proceed. Let G_x denote a G-layer with the fewest number of terminals. By simple averaging, one can easily show (just as we did above) that G_x contains at most a terminals. We, in fact, prove that x has a neighbour $y \in \Gamma_H(x)$ such that G_y contains at most a terminals. Recall that H is $(2b-1)$-connected, hence $|\Gamma_H(x)| \geq (2b-1)$. Assuming that every G-layer corresponding to a neighbour of x consists of at least $(a+1)$ terminals would yield at least $(2b-1)(a+1) > (2a+2b-2)$ terminals in total that is not possible. Let $y \in \Gamma_H(x)$ be chosen as described above. We distinguish two cases:

1. If G_x and G_y do not share a pair of terminals, that is, every pair has terminals in at most one of them, simply label $G_\alpha = G_x$, $G_\beta = G_y$ and use the shifting techniques without applying horizontal shift on these two layers. The two layers will collect all the terminals and perform the final joining.

2. If there is a pair of terminals (u, v) such that $u \in G_x$ and $v \in G_y$, we use an inductional step and reduce the problem to a smaller graph. Apply horizontal shift on the terminals of G_x and G_y such that every terminal t gets a pseudopair, u' and v' share an H layer and no other pair of pseudopairs share their H layers. Recall that x and y are adjacent and so are u' and v'. Join the pseudopairs and so u and v by the union of the three paths. For a remaining pseudopair t'

(that is neither equal to u' nor to v'), choose an arbitrary vertical edge with other endpoint t'' such that $t'' \notin V(G_x) \cup V(G_y)$. Apply induction on G and $H - x - y$ just as in the main proof. Note that the base of our induction is the case $b = 1$, which is covered in Lemma 2.

□

We briefly mention that our method with somewhat rougher estimates yields the following variant of Theorem 1. This bound is sharp apart from a small (≤ 6) constant term for in infinite class of graphs.

Theorem 4. *Suppose G is a-linked, k-connected graph, H is a h-connected graph ($h \leq k$, G and H are sufficiently large) then $G \square H$ is $\frac{a}{2a+1}(k+h)$-linked.*

Proof. We copy the proof of Theorem 1. Let us denote $L := \frac{a}{2a+1}(k+h)$. If there exists a crowded G-layer (containing at least $(k+1)$ terminals), find a matching pair of terminals (which exists by piegon-hole principal), join them, empty the layer as before and proceed by induction. Otherwise, global shift horizontally, allocate $t := \lceil \frac{L}{a} \rceil$ empty G-layers $G_{\alpha_1}, \ldots, G_{\alpha_t}$, distribute the terminal pairs among them via vertical paths and reduce the problem to linking within horizontal layers. □

We believe that the statement of Theorem 1 is true even without the indicated condition on the minimal size of the graphs (we only assume the condition to guarantee enough room for the shifting techniques). Nevertheless, in the case when $\frac{\text{link}(G)}{v(G)} > \frac{1}{8}$ the shifting techniques presented in the main proof fail to work as one has to deal with an aboundance of terminals congested on the layers. Linking of the terminals in that case is likely to lead a rather lenghty and tedious case-by-case analysis involving ac hoc solutions which we do not find particularly interesting.

Proof of Theorem 2

Assume we are given the pairing of $(2k+2)$ terminals in $G \square H$. We use the technique of the proof of Theorem 1 and follow a case-by-case analysis.

1. If there exist a G-layer G_i with $3 \leq s_i \leq k$ elements, then no G-layer is crowded (no G-layer contains $2k$ or more terminals). Choose $G_\alpha = G_i$ and apply the horizontal and vertical shift techniques on the remaining $(2k+2) - s_i \leq (2k-1)$ terminals. Observe that $n = (t-1)$, that is, one can use Lemma 1 in every G-layer during the horizontal shift. In the vertical phase the H-layer is 2-connected and the path joining u' and u'' only has one vertex to avoid (corresponding to G_α or G_β).

2. If there exist G-layers G_i and G_j such that $s_i = 1$, $s_j = 2$ or $s_i = s_j = 2$, choose $G_\alpha = G_i$ and $G_\alpha = G_j$. Solution for the previous case works here as well.

3. If $s_1 = \cdots = s_{2k+2} = 1$, use the separate technique presented for small cases at the end of proof of Theorem 1.

4. If $s_1 = \cdots = s_{n-1} = 1$, $k+1 \leq s_n \leq 2k-1$, choose $\{G_\alpha, G_\beta\} = \{G_1, G_2\}$ and apply the shifting technique. Lemma 1 handles every G-layer just as in Case 1.

5. If $s_1 = s_2 = 1$, $s_3 = 2k$, join a pair u_1, v_1 within G_2 using Lemma 1 and shift the remaining terminals vertically. If a terminal u_2 has no available neighbour, then $v_2 \in G_2$ and we can switch pair (u_1, v_1) to (u_2, v_2) and repeat the argument, just as in the crowded layer case of the proof of Theorem 1.

6. If $s_1 = 2$, $s_2 = 2k$, similar technique works as in Case 5.

7. If $s_n \geq 2k+1$, we have all terminals (or all with one exc eption) on the same G_x-layer for some $x \in H$. Let $y, z \in \Gamma_H(x)$. We can distribute the pairs of terminals between G_y and G_z by using appropriate vertical $\overline{xy}$ and $\overline{xz}$ edges and join u', v' endpoints within the horizontal layer. If $s_n = 2k+1$, the missing terminal can be routed to the appropriate layer. We leave the details as an exercise.

8. If $s_1 = s_2 = k+1$, we may assume none of the layers contain a pair, otherwise we can proceed by matching a pair within a layer, allocating new terminal vertex u' instead of the original terminal u on the layer, shifting and using induction as previously. Let $G_1 = G_x$, $G_2 = G_y$ for some $x, y \in H$ and let $z \in \Gamma_H(x) - \{y\}$ (as H is 2-connected, such z has to exist). Shift terminals horizontally within G_x if necessarily in order to get for every terminal u a uu' path with endpoint u', such that $H_{u'}$ contains neither a terminal nor a vertex belonging to the shifting paths (in case there was no shift necessary, let $u' = u$). We pick a single terminal $u \in G_x$ and take a path $u - u' - u''$ where u'' denotes the projection of u' to G_y. We connect u'' with the pair of u in G_y using Lemma 1. For the remaining $2k$ pairs, we set a vertical paths for each terminal in G_x and G_y to G_z. For a terminal $w \in G_x$ there is no obstacle in H_w to find a path to its projection to G_z. If $w \in G_y$, we use the fact that H is 2-connected and that it contains at most one vertex of G_x we might have used previously. Having set the vertical paths, we join the projections in G_z.

□

Assuming that the product $G \Box H$ is k-linked yields no essential lower bound neither on linkedness nor on connectivity of G or H. The theorem of Bollobás and Thomason [2] together with the result of Špacapan [9] show that large degree is sufficient to imply high linkedness while the component graphs are connected but might not even be 2-connected. That is, there exist a function f such that $\delta(G) + \delta(H) \geq f(k)$ implies $G \Box H$ is k-linked. Using the improved bound presented in [10] we know that $f(k) \leq 10k$ and detailed analysis might yield even better bounds.

Linkedness of Hypercubes, Products of Paths and Products of Cycles

We determine the linkedness-number of the n-dimensional grid, that is, the Cartesian product of n paths, and the linkedness of product of n cycles. We use a straightforward corollary of Theorem 2:

Corollary 1. *If G is a k-linked graph, $k \geq 2$, $|V(G)| \geq \max(9, 4k)$ then $G \Box C_m$ is $(k+1)$-linked, where C_m denotes the cycle of length m.*

Proposition 1. *For cycles of length $m_1, \ldots, m_t$ ($m_i \geq 3$, $t \geq 2$) $\operatorname{link}(C_{m_1} \Box \ldots \Box C_{m_t}) = t$.*

Proof of Proposition 1. We first investigate the product of two cycles.

Lemma 3. *For cycles of length m and n ($m, n \geq 3$) $\operatorname{link}(C_m \Box C_n) = 2$.*

Proof. It can be shown by a simple but rather lengthy case-by-case analysis that $C_3 \Box C_3$, $C_3 \Box C_4$ and $C_4 \Box C_4$ are 2-linked. If $\max(m, n) \geq 5$, one of the cycles can be shortened by substituting an empty layer with vertical / horizontal edges joining its neighbours and proceed by induction. □

Now the general statement in Proposition 1 follows directly from Corollary 1 and Lemma 3. □

Proposition 2. *Let Q_n denote the n-dimensional hypercube.* $\operatorname{link}(Q_n) = \lceil \frac{n}{2} \rceil$ *if $n \neq 3$.*

Proof of Proposition 2. As Q_n is n-connected, the linkedness number of Q_n is at most $\lceil \frac{n}{2} \rceil$. Equality holds for $n = 1$ and 2. Q_3 is not 2-linked as being a planar graph with non-triangle faces. Q_4 is 2-linked.

Lemma 4. *For the five dimensional hypercube* $\text{link}(Q_5) = 3$.

Proof. We distinguish two cases:

Case 1 Assume there exist terminals x_1, y_1 satisfying $d(x_1, y_1) \leq 4$ where $d(x, y)$ denotes the distance of vertices x and y. In other words, our current assumption is that x_1 and y_1 are not "opposite" vertices of Q_5. Because of symmetries of the graph Q_5, we may assume without loss of generality that $x_1 = (0,0,0,0,0)$ and $y_1 = (\underline{v}, 0)$, $\underline{v} \in Q_4$. Also, let us denote $Q_5 = Q_4^0 \cup Q_4^1$, the decomposition of Q_5 into affine hyperplanes being isomorphic to Q_4 (with respect to the last coordinate). Certainly, $x_1, y_1 \in Q_4^0$ and we may assume that $Q_4^0 - x_2 - y_2 - x_3 - y_3$ is connected (otherwise switch to pair (x_2, y_2)). Join x_1 to y_1 in Q_4 by any path of length 4 encountering no other terminal. We want to join the remaining two pairs in Q_4^1. If a terminals $u \in \{x_2, y_2, x_3, y_3\}$ lies in Q_4^0, we define a crossing path that ends at $u' \in Q_4^1$. If the projection of u to Q_4^1 is not a terminal vertex (or if it happens to be the pair of u), we take that very edge as the required path. In every other case there is a $v \in \Gamma_{Q_4^0}(u)$ such that the projection of v to Q_4^1 is available, yielding an appropriate path of length 2.

Case 2 If $d(x_1, y_1) = d(x_2, y_2) = d(x_3, y_3) = 5$, there exist - up to isomorphism - 5 possible arrangements of the terminals. We leave the easy case-by-case analysis to the reader.

□

As $Q_n = Q_{n-2} \square C_4$, Corollary 1 applies (for $n \geq 4$) and so the proof is complete. □

Proposition 3. *Let P_m denote the path of m vertices and let $G = P_{m_1} \square \ldots \square P_{m_t}$. Then we have*

i) $\text{link}(G) = 1$ *if* $t = 3$, $m_1 = m_2 = 2$ *and*

ii) $\text{link}(G) = \lceil \frac{t}{2} \rceil$ *if* $t \neq 3$, $m_i \geq 2$ *or* $t = 3$, $m_3 \geq m_2 \geq 3$.

Proof of Proposition 3. The first statement is obvious as G is a planar graph. For $t \neq 3$, let Q_n be an induced subgraph of G containing terminals $x_1, \ldots, x_p$, $p \geq 1$. As $G - x_2 - \cdots - x_p$ is $(t-p)$-connected, the set of remaining terminals can be routed to Q_n and linking can be performed. The case $t = 3$, $m_3 \geq m_2 \geq 2$ can be solved by the previous idea using the fact that $P_2 \square P_3 \square P_3$ is 2-linked. □

Acknowledgement

The author wishes to thank Professor Ervin Győri for his helpful suggestions, interest and valuable guidance. The author is also grateful for the Balassi Institute, the Fulbright Commission and the Rosztoczy Foundation for their kind and generous support.

References

[1] B. Bollobás, *Modern Graph Theory*, Springer Graduate Texts in Mathematics, Volume 184 (1998)

[2] B. Bollobás, C. Thomason, *Highly Linked Graphs*, Combinatorics, Probability, and Computing, Volume 16 (1993), 1-7.

[3] B. Brešar, P. Dorbec, W. Goddard, B. L. Hartnell, M. A. Henning, S. Klavžar, D. F. Rall, *Vizing's Conjecture: a Survey and Recent Results*, Journal of Graph Theory, Volume 69, 46-76.

[4] W. S. Chiue, B. S. Shieh, *On Connectivity of the Cartesian Product of Two Graphs*, Applied Mathematical Computation, Volume 102 (1999), 129-137.

[5] E. Győri, M. Plummer, *The Cartesian Product of a k-Extendable and an l-Extendable Graph is $(k+l+1)$-Extendable*, Discrete Mathematics, Volume 101 (1992), 87-96.

[6] W. Imrich, S. Klavžar, *Product Graphs: Structure and Recognition*, J. Wiley& Sons, New York, (2000)

[7] K. Kawarabayshi, *k-Linked Graphs with Girth Condition*, Journal of Graph Theory, Volume 45 (2004), 48-50.

[8] W. Mader, *Topological Subgraphs in Graphs of Large Girth*, Combinatorica, Volume 18 (1998), 405-412.

[9] S. Špacapan, *Connectivity of Cartesian Products of Graphs*, Applied Mathematics Letters, Volume 21 (2008), 682-685.

[10] R. Thomas, P. Wollan, *An Improved Linear Edge Bound for Graph Linkages*, European Journal of Combinatorics, Volume 26 (2005), 309-324.