LOCATING-DOMINATING SETS IN HYPERGRAPHS

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ABSTRACT. A hypergraph is a generalization of a graph where edges can connect any number of vertices. In this paper, we extend the study of locating-dominating sets to hypergraphs. Along with some basic results, sharp bounds for the locationdomination number of hypergraphs in general and exact values with specified conditions are investigated. Moreover, locating-dominating sets in some specific hypergraphs are found.

1. INTRODUCTION

Let G be a graph with vertex set V(G) and edge set E(G). The number of elements in V(G) and E(G) is called the order and the size of G, respectively. The distance between two vertices u and v in G, denoted by d(u, v), is the length of a shortest u - v path in G. Let u be a vertex of a graph G, then the open neighborhood of u is $N(u) = \{v \in V(H) \mid uv \in E(G)\}$ and the closed neighborhood of u is $N[u] = N(u) \cup \{u\}$. A set D of vertices of G is a dominating set for G if every vertex v in V(G) - D has a neighbor in D, that is, for every $v \in V(G) - D, N(v) \cap D \neq \emptyset$.

A set $\mathfrak{L} = \{x_1, \ldots, x_k\}$ of vertices of a graph G is called a *locating set* if for every two distinct vertices u and v of G, $(d(u, x_1), \ldots, d(u, x_k)) \neq (d(v, x_1), \ldots, d(v, x_k))$. The *location number* (also called the *metric dimension* [11]) is the minimum cardinality of a locating set of G [19].

A set S of vertices of a graph G is called a *locating-dominating set* if it is both the locating and dominating set. An elaborate and more general definition is the following: A set S of vertices of G is called a locating-dominating set for G if for every two distinct elements $u, v \in V(G) - S$, we have $\emptyset \neq N(u) \cap S \neq N(v) \cap S \neq \emptyset$. The *location-domination number*, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set of G [17].

Locating-dominating sets in graphs were firstly studied by Slater [17]. The motivations of locating-dominating sets comes, for instance, from fault diagnosis in multiprocessor systems. Such a system can be modeled as a graph where vertices are processors and edges are links between processors. A considerable literature has been developed in this field (see [5, 7, 9, 10, 12, 14, 16, 18]). The decision problem for locating-dominating sets for directed graphs has been shown to be an

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NP-complete problem [8]. In [6], it was pointed out that each locating-dominating set is both the locating and dominating set. However, a set that is both the locating and dominating is not necessarily a locating-dominating set.

A hypergraph H is a pair (V(H), E(H)), where V(H) is a finite set of vertices and E(H) is a finite family of non-empty subsets of V(H), called hyperedges, with E = V(H). A subhypergraph K of a hypergraph H is a hypergraph with U $E \in E(H)$

vertex set $V(K) \subseteq V(H)$ and edge set $E(K) \subseteq E(H)$. The rank of H, denoted by rank(H), is the maximum cardinality of a hyperedge in E(H). H is linear if for distinct hyperedges E_i and E_j , $|E_i \cap E_j| \leq 1$, so in a linear hypergraph, there may be no repeated hyperedges of cardinality greater than one. A hypergraph H with no hyperedge is a subset of any other hyperedge is called *Sperner*.

A vertex $v \in V(H)$ is said to be *incident* with a hyperedge E of H if $v \in E$. If v is incident with exactly n hyperedges, then we say that the *degree* of v is n; if all the vertices $v \in V(H)$ have degree n, then H is n-regular. The maximum degree of any vertex in H is denoted by $\Delta(H)$. Similarly, if there are exactly n vertices incident with a hyperedge E, then we say that the size of E is n; if all the hyperedges $E \in E(H)$ have size n, then H is n-uniform. A simple graph is a 2-uniform hypergraph.

A path from a vertex v to another vertex u, in a hypergraph, is a finite sequence of the form $v, E_1, w_1, E_2, w_2, \ldots, E_{l-1}, w_{l-1}, E_l, u$, having length l such that $v \in E_1, w_i \in E_i \cap E_{i+1}$ for $i = 1, 2, \dots, l-1$ and $u \in E_l$. A hypergraph H is connected if there is a path between every two vertices of H. All the hypergraphs considered in this paper are connected Sperner hypergraphs.

A hypergraph H is said to be a hyperstar if $E_i \cap E_j = \mathcal{C} \neq \emptyset$, for any $E_i, E_j \in E(H)$. We will call \mathcal{C} , the *center* of the hyperstar. If there exists a sequence of hyperedges E_1, E_2, \ldots, E_k in a hypergraph H, then H is said to be (1) a hyperpath if $E_i \cap E_j \neq \emptyset$ if and only if |i-j| = 1; (2) a hypercycle if $E_i \cap E_j \neq \emptyset$ if and only if |i-j| = 1(mod k). A connected hypergraph H with no hypercycle is called a hypertree.

In graphs, the theory of dominating sets and locating-dominating sets is extensively studied. Hypergraphs, in the context of domination, were firstly considered by Behr and Camarinopoulos in 1998 [4], and further considered by Acharya [1, 2] and Jose and Tuza [13]. In this paper, we consider hypergraphs in the context of location-domination. We give some sharp lower bounds for the location-domination number of hypergraphs. Also, we investigate the location-domination number of some well-known families of hypergraphs such as hyperpaths, hypercycles and kpartite hypergraphs.

2. Some Basic Results and Bounds

Two vertices u and v of a hypergraph incident with the same hyperedge are said to be coincident vertices. Let $E_i^{(d)} = \{E_{i_1}, E_{i_2}, \dots, E_{i_d}\}$ be a collection of hyperedges.

We denote the set of all the vertices, having degree d, incident with every hyperedge in $E_i^{(d)}$ by $S_i^{(d)}$, and we call it the *i*th *coincident set* of the vertices having degree d. It should be noted that for each set $E_i^{(d)} \subseteq E(H)$, there corresponds a coincident set $S_i^{(d)}$, which may be empty.

Coincident vertices have the same degree but two vertices having same degree may not be coincident as in the following example (illustrating the notion of the coincident set), the vertices v_6 and v_8 are not coincident although the degree of both the vertices is same.

Example 2.1. Let $V(H) = \{v_1, v_2, \dots, v_{10}\}$ and $E(H) = \{E_1, E_2, E_3\}$, where $E_1 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{v_1, v_4, v_5, v_6, v_7\}$ and $E_3 = \{v_2, v_3, v_4, v_5, v_8, v_9, v_{10}\}$. Then we can write, $E_1^{(1)} = \{E_1\}$, $E_2^{(1)} = \{E_2\}$, $E_3^{(1)} = \{E_3\}$, $E_1^{(2)} = \{E_1, E_2\}$, $E_2^{(2)} = \{E_1, E_3\}$, $E_3^{(2)} = \{E_2, E_3\}$, $E_1^{(3)} = \{E_1, E_2, E_3\}$ and the corresponding coincident sets are: $S_1^{(1)} = \emptyset$, $S_2^{(1)} = \{v_6, v_7\}$, $S_3^{(1)} = \{v_8, v_9, v_{10}\}$, $S_1^{(2)} = \{v_1\}$, $S_2^{(2)} = \{v_2, v_3\}$, $S_3^{(2)} = \{v_5\}$ and $S_1^{(3)} = \{v_4\}$.

Remark 2.2. Two vertices belonging to different coincident sets can have the same closed neighborhood. As in Example 2.1, the vertices v_4 and v_5 belong to different coincident sets, namely $S_1^{(3)}$ and $S_3^{(2)}$, respectively. However, $N[v_4] = N[v_5] = V(H)$.

Now, we discuss some properties of coincident sets in the following proposition:

Proposition 2.3. (1) The set of all non-empty coincident sets in a hypergraph H partitions V(H).

- (2) The number of non-empty coincident sets in a hypergraph H is bounded above $\sum_{h=1}^{\Delta(H)} {m \choose h} \text{ where } m \text{ is the size of } H$
- by $\sum_{d=1}^{\Delta(H)} {m \choose d}$, where *m* is the size of *H*.

Proof. (1) Note that $\cup S_i^{(d)} \subseteq V(H)$. Let $v \in V(H)$, then $v \in E_{i_l}$ for some hyperedge $E_{i_l} \in E_i^{(d)}$. Since for each set $E_i^{(d)}$ of hyperedges, there corresponds a coincident set $S_i^{(d)}$ yielding $v \in S_i^{(d)}$ for some i. Hence, $V(H) \subseteq \cup S_i^{(d)}$ implies $\cup S_i^{(d)} = V(H)$. Now, we show that $S_i^{(d)} \cap S_j^{(d)} = \emptyset$, or $S_i^{(d)} = S_j^{(d)}$ for $i \neq j$. Suppose that $S_i^{(d)} \cap S_j^{(d)} \neq \emptyset$ and let $v \in S_i^{(d)} \cap S_j^{(d)}$ $(i \neq j)$. Then by definition,

Suppose that $S_i^{(a)} \cap S_j^{(a)} \neq \emptyset$ and let $v \in S_i^{(a)} \cap S_j^{(a)}$ $(i \neq j)$. Then by definition, v will be incident with each hyperedge $E_{i_{l_1}}$ of $E_i^{(d)}$ and $E_{j_{l_2}}$ of $E_j^{(d)}$, respectively. This implies that $v \in E_{i_{l_1}} \cap E_{j_{l_2}}$. But, the considered graph is connected Sperner hypergraph so $v \in E_{i_{l_1}} \cap E_{j_{l_2}}$ holds only if $E_{i_{l_1}} = E_{j_{l_2}}$, which shows that $S_i^{(d)} = S_j^{(d)}$. It concludes the required result.

(2) By the definition of coincident set, for each $E_i^{(d)} \subseteq E(H)$, there corresponds a coincident set $S_i^{(d)}$. Also, since there are $\binom{m}{d}$ possible subsets of E(H) of cardinality d, so there will be at most as many coincident sets for each d.

If N[u] = N[v] for any two vertices u and v of a hypergraph H, then for each $S \subseteq V(H), N(u) \cap S = N(v) \cap S$, which implies that either u or v should belong to

every locating-dominating set of H. Hence, we have the following straightforward result:

Lemma 2.4. Let S be a locating-dominating set for a hypergraph H. If for any two distinct vertices u and v of H, N[u] = N[v], then at least one of u and v must belong to S.

By the definition of coincident set $S_i^{(d)}$, for every two distinct elements u, v of $S_i^{(d)}$, N[u] = N[v]. Hence, by Lemma 2.4, we have the following result:

Lemma 2.5. For any locating-dominating set S of a hypergraph and for a non-empty coincident set $S_i^{(d)}$, we have $|S| \ge |S \cap S_i^{(d)}| \ge |S_i^{(d)}| - 1$.

For a locating-dominating set S of a hypergraphs H and for a coincident set $S_i^{(d)}$, we let $C_i^{(d)} = S \cap S_i^{(d)}$ whenever $S_i^{(d)} \neq \emptyset$, and $C_i^{(d)} = \emptyset$ whenever $S_i^{(d)} = \emptyset$. Moreover, we let $C = \bigcup_{d=1}^{m} \bigcup_{i=1}^{\binom{m}{d}} C_i^{(d)}$. Since $S_i^{(d)} = \emptyset$ when $d > \Delta(H)$, therefore $C = \bigcup_{d=1}^{\Delta(H)} \bigcup_{i=1}^{\binom{m}{d}} C_i^{(d)}$.

A lower bound for the location-domination number of a hypergraph H is given in the following result:

Theorem 2.6. Let S be a minimum locating-dominating set for a hypergraph H with $m \ge 2$ hyperedges. Then

$$|S| \ge |C| = \sum_{d=1}^{\Delta(H)} \sum_{i=1}^{\binom{m}{d}} |C_i^{(d)}|.$$

Proof. Since non-empty coincident sets in a hypergraph H form a partition of V(H), by Proposition 2.3. So, every two coincident sets are either disjoint or equal, which yields that

$$|C| = \sum_{d=1}^{\Delta(H)} \sum_{i=1}^{\binom{m}{d}} |C_i^{(d)}|.$$

Further, Lemma 2.5 straightforwardly concludes that $|S| \ge |C|$.

The following result describes that the lower bound established above is sharp.

Theorem 2.7. Let H be a hypergraph in which every hyperedge contains at least two vertices of degree one. If S is a minimum locating-dominating set for H, then

$$|S| = |C| = \sum_{d=1}^{\Delta(H)} \sum_{i=1}^{\binom{m}{d}} |C_i^{(d)}|.$$

Proof. By hypothesis, $C_i^{(1)} \neq \emptyset$ for each i = 1, 2, ..., m. Let $u, v \in V(H) - C$. Since $|S_i^{(d)}| - |C_i^{(d)}| \leq 1$, there exist distinct coincident sets $S_i^{(d_1)}$ and $S_j^{(d_2)}$ such that $u \in S_i^{(d_1)}$ and $v \in S_j^{(d_2)}$. Assuming $d_1 \geq d_2$, there exists a hyperedge E_k such that $u \in E_k$ and $v \notin E_k$. Therefore $N(u) \cap C \neq N(v) \cap C$, and hence, together with Theorem 2.6, we have the required result.

We give two examples which show that the condition in Theorem 2.7 cannot be relaxed generally.

Example 2.8. Let H be a hypergraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E(H) = \{E_1 = \{v_1, v_2, v_3, v_4\}, E_2 = \{v_4, v_5\}\}$. Clearly, $C_2^{(1)} = \emptyset$ and the condition of Theorem 2.7 is not satisfied. Observe that $C = \{v_1, v_2\}$. But, C is not a locating-dominating set for H because $N(v_3) \cap C = N(v_4) \cap C$.

Example 2.9. Let *H* be a hypergraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E(H) = \{E_1 = \{v_1, v_2, v_3, v_4\}, E_2 = \{v_3, v_4, v_5, v_6\}, E_3 = \{v_1, v_2, v_5, v_6\}\}$. Clearly, $C_i^{(1)} = \emptyset$ but $C_i^{(2)} \neq \emptyset$, for all i = 1, 2, 3. Observe that $C = \{v_1, v_3, v_5\}$. But, *C* is not a locating-dominating set for *H* because $N(v_2) \cap C = N(v_4) \cap C$.

A hypergraph H is said to be a *complete hypergraph* if for all $\{u, v\} \subseteq V(H)$, there is $E_i \in E(H)$ such that $\{u, v\} \subseteq E_i$. A *clique* of a hypergraph H, denoted by $\widetilde{H} = (V(\widetilde{H}), E(\widetilde{H}))$, is a complete subhypergraph of H.

A sharp upper bound for the location-domination number of a hypergraph H is given in the following lemma:

Lemma 2.10. If S is a locating-dominating set for a hypergraph H with n vertices, then $\lambda(H) \leq |S| \leq n-1$, and this bound is sharp.

Proof. It is easy to see that any n-1 vertices of H form a locating-dominating set S for H, which implies $\lambda(H) \leq |S| \leq n-1$.

For sharpness, consider a complete hypergraph H of order n. Since N[u] = N[v] for every two distinct vertices u and v of H, so it never be hold that a set S with |S| < n - 1 forms a locating-dominating set for H. For otherwise, there exist $x, y \in V(H) - S$ such that $N(x) \cap S = N(y) \cap S$.

Lemma 2.11. Let \widetilde{H} be a clique of hypergraph H and S be a locating-dominating set for H. If $S \subseteq V(\widetilde{H})$, then $|S| \geq |V(\widetilde{H})| - 1$.

Proof. Since all the vertices of \tilde{H} are mutually coincident, therefore they have the same closed neighborhoods. Hence, by Lemma 2.4, at least $|V(\tilde{H})| - 1$ elements from $V(\tilde{H})$ contained in S.

A vertex packing in a hypergraph H is a subset $\mathcal{P} \subseteq V(H)$ such that no two elements of \mathcal{P} belong to the same hyperedge of H. The packing number is the maximum cardinality of such a set \mathcal{P} , and we denote it by π [15].

Theorem 2.12. Let H be a linear hypergraph of order n with $S_i^{(1)} \ge 2$ for all i. Then $\lambda(H) \le n - \pi$ and this bound is sharp. Proof. By the definition, for any two distinct $u, v \in \mathcal{P}$, we have $N(u) \neq N(v)$. So, $V(H) - \mathcal{P}$ is a locating-dominating set for H. Hence $\lambda(H) \leq n - \pi$ since π is the largest size of packing \mathcal{P} . Further, the bound is sharp if H is a complete hypergraph. \Box

3. LOCATION-DOMINATION IN SOME SPECIFIC HYPERGRAPHS

Since in a uniform linear hypergraph, we have $C_i^{(2)} = \emptyset$ and $C_i^{(1)} \neq \emptyset$ for all *i*. So, Theorem 2.7 yields the following consequence:

Corollary 3.1. Let *H* be a *k*-uniform linear hypergraph with *m* hyperedges. Then for $k \ge 4$, $\lambda(H) = \sum_{i=1}^{m} |C_i^{(1)}|$.

Corollary 3.2. Let H be a k-uniform linear hyperpath with m hyperedges. Then for $k \ge 4$, $\lambda(H) = m(k-3) + 2$.

Proof. Observe that $|C_1^{(1)}| = |C_m^{(1)}| = k-2$, whereas for $2 \le i \le m-1$, $|C_i^{(1)}| = k-3$ hence, we have $\lambda(H) = m(k-3) + 2$.

Corollary 3.3. Let H be a k-uniform linear hypercycle with m hyperedges. Then for $k \ge 4$, $\lambda(H) = m(k-3)$.

Proof. Note that, for every k-uniform linear hypercycle $|C_i^{(1)}| = k - 3$, for all $1 \le i \le m$. Therefore $\lambda(H) = m(k - 3)$.

Theorem 3.4. In a k-uniform hypercycle $C_{m,k}$ $(k \ge 4)$ with $m \ge 3$ hyperedges, if $C_i^{(1)} = \emptyset$ for all *i* and the order of each non-empty coincident set $S_i^{(2)}$ is same, then $\lambda(C_{m,k}) = m(\lfloor \frac{k}{2} \rfloor - 1).$

Proof. Since $C_i^{(1)} = \emptyset$ for all *i* and each non-empty coincident set $S_i^{(2)}$ has the same order, so each $C_i^{(2)}$ has exactly $\lfloor \frac{k}{2} \rfloor - 1$ elements. Since there are *m* hyperedges, therefore $\lambda(C_{m,k}) = m(\lfloor \frac{k}{2} \rfloor - 1)$.

The following result for hyperpaths also shows that the lower bound established in Theorem 2.6 is sharp.

Theorem 3.5. Let H be a hyperpath with m hyperedges. If $S_i^{(1)}$, for i = 1, m, and each non-empty coincident set $S_i^{(2)}$ has at least two elements, then

$$\lambda(H) = \sum_{d=1}^{\Delta(H)} \sum_{i=1}^{\binom{m}{d}} |C_i^{(d)}|.$$

Proof. If $C_i^{(1)} \neq \emptyset$, for all *i*. Then, by Theorem 2.7, we have the required result. If $C_i^{(1)} = \emptyset$ for $2 \le i \le m - 1$, then we have the following three cases for $S_i^{(1)}$.

Case 1: $(S_i^{(1)} \neq \emptyset \text{ for all } 2 \le i \le m-1).$

Let $v, v' \in V(H) - C$. If $v \in E_1$ and $v' \in E_m$. Then, clearly, $N(v) \cap C \neq N(v') \cap C$ because $(N(v) \cap C) \subset E_1$ and $(N(v') \cap C) \subset E_m$.

If v and v' are common vertices, then again $N(v) \cap C \neq N(v') \cap C$ because H is Sperner and each $S_i^{(2)}$ is disjoint.

If $v \in E_i$ and $v' \in E_j$, for $2 \le i, j \le m-1$ and $i \ne j$. Then $N(v) = \{E_{i-1} \cap E_i\} \cup \{E_i \cap E_{i+1}\} \ne \{E_{j-1} \cap E_j\} \cup \{E_j \cap E_{j+1}\} = N(v')$, and hence $N(v) \cap C \ne N(v') \cap C$.

If $v \in E_i$ and v' is a common vertex. Then by above discussion, we note that both the vertices have their distinct open neighborhoods in C.

The other two cases when $S_i^{(1)} \neq \emptyset$ for some $2 \le i \le m-1$ and when $S_i^{(1)} = \emptyset$ for all $2 \le i \le m-1$ follows from Case 1.

We know that a 2-uniform linear hyperpath (hypercycle) is a simple path (cycle). The exact value for $\lambda(H)$ of a simple path (cycle) is already determined by Slater.

Theorem 3.6. [17] Let $P_{n\geq 2}$ be a simple path and $C_{n\geq 3}$ be a simple cycle. Then $\lambda(P_n) = \lambda(C_n) = \lceil \frac{2n}{5} \rceil$.

Observe that, in the case of 3-uniform linear hyperpath H with two hyperedges, $\lambda(H) = 2$. We also observe that, the 3-uniform linear hyperpath with three and four hyperedges, respectively, has the location-domination number 3 and 4, respectively. In the next result, we determine the explicit value for the location-domination number for 3-uniform linear hyperpaths with more than four hyperedges.

Theorem 3.7. Let $P_{m,3}$ be a 3-uniform linear hyperpath with $m \ge 5$ hyperedges. Let m = 3a + b + 2, where $a \ge 1$ and $0 \le b \le 2$. Then $\lambda(P_{m,3}) = 2a + b + 2$.

Proof. Let v_i represents a vertex of degree one in the hyperedge E_i and $v_{i,i+1} \in E_i \cap E_{i+1}$ is common vertex of degree two. Since E_1 and E_m contains two vertices of degree one, we denote them as v_1 , v'_1 for E_1 and v_m , v'_m for E_m .

Our claim is that $\lambda(P_{m,3}) = 2a + b + 2$. We prove it by showing that a set $S \subseteq V(H)$ with |S| = 2a + b + 2 is a minimum locating-dominating set for $P_{m,3}$. We consider the following three cases:

Case 1: (b = 0) Let $S = \{v'_1, v'_m\} \cup \{v_{3i+2,3(i+1)}, v_{3(i+1),3(i+1)+1} \mid 0 \le i \le a - 1\}$. This set S is a minimum locating-dominating set of order 2a + 2 because of the following unequal S-neighborhoods: $N(v_1) \cap S = \{v'_1\}$; $N(v_m) \cap S = \{v'_m\}$; $N(v_{3i+1,3i+2}) \cap S = \{v_{3i,3i+1}, v_{3i+2,3(i+1)}\}$ with $v_{0,1} = v'_1$; $N(v_{3i+2}) \cap S = \{v_{3i+2,3(i+1)}, v_{3(i+1),3(i+1)+1}\}$; $N(v_{3(i+1)+1}) \cap S = \{v_{3(i+1),3(i+1)+1}\}$ and $N(v_{m-1,m}) \cap S = \{v_{m-2,m-1}, v'_m\}$.

Case 2: (b = 1) The set $S = \{v'_1, v'_m\} \cup \{v_{3i+2,3(i+1)}, v_{3(i+1),3(i+1)+1} \mid 0 \le i \le a-1\} \cup \{v_{m-1,m}\}$ is a required minimum locating-dominating set of order 2a + 3 because of the following unequal S-neighborhoods of all the elements in $V(P_{m,3}) - S$: $N(v_m) \cap S = \{v_{m-1,m}, v'_m\}; N(v_{m-2,m-1}) \cap S = \{v_{m-3,m-2}, v_{m-1,m}\}; N(v_{m-1}) \cap S = \{v_{m-1,m}\}$ and those listed in Case 1. **Case 3:** (b = 2) Let $S = \{v'_1, v'_m\} \cup \{v_{3i+2,3(i+1)}, v_{3(i+1),3(i+1)+1} \mid 0 \le i \le a-1\} \cup \{v_{m-2,m-1}, v_{m-1,m}\}$. Together with the unequal S-neighborhoods of all the elements in $V(P_{m,3}) - S$ listed in the Cases 1 and 2, and the S-neighborhood $N(v_{m-1}) \cap S = \{v_{m-2,m-1}, v_{m-1,m}\}$, one can conclude that S is a minimum locating-dominating set of order 2a + 4.

In the next result, we determine the explicit value for the location-domination number for 3-uniform linear hypercycles having more than five hyperedges.

Theorem 3.8. Let $C_{m,3}$ be a 3-uniform linear hypercycle with $m \ge 6$ hyperedges. Let m = 3a + b, where $a \ge 2$ and $0 \le b \le 2$. Then $\lambda(C_{m,3}) = 2a + b$.

Proof. In $C_{m,3}$, each $v_i \in E_i$ represents a vertex of degree one and $v_{i,i+1} \in E_i \cap E_{i+1}$ with $v_{m,m+1} = v_{m,1}$. We prove that $\lambda(C_{m,3}) = 2a+b$ by showing that a set $S \subseteq V(H)$ with |S| = 2a + b is a minimum locating-dominating set for $C_{m,3}$. We discuss the following three cases for b:

Case 1: (b = 0) Let $S = \{v_{3i+1,3i+2}, v_{3i+2,3i+3} \mid 0 \le i \le a-1\}$. Then |S| = 2a. One can see that S is a minimum locating-dominating set because of the following unequal S-neighborhoods: $N(v_{3i+1}) \cap S = \{v_{3i+1,3i+2}\}; N(v_{3i+2}) \cap S = \{v_{3i+1,3i+2}, v_{3i+2,3i+3}\}; N(v_{3i+3}) \cap S = \{v_{3i+2,3i+3}\}; N(v_{3i+3,3(i+1)+1}) \cap S = \{v_{3i+2,3i+3}, v_{3(i+1)+1,3(i+1)+2}\}.$

Case 2: (b = 1) The set $S = \{v_{3i+1,3i+2}, v_{3i+2,3i+3} \mid 0 \le i \le a-1\} \cup \{v_{m,1}\}$ is a minimum locating-dominating set of order 2a + 1 because all the elements in $V(C_{m,3}) - S$ have unequal S-neighborhoods $N(v_1) \cap S = \{v_{m,1}, v_{1,2}\}; N(v_m) \cap S = \{v_{m,1}\}$ and those listed in Case 1.

Case 3: (b = 2) Let $S = \{v_{3i+1,3i+2}, v_{3i+2,3i+3} \mid 0 \le i \le a-1\} \cup \{v_{m-1,m}, v_{m,1}\}$ be the set of order 2a + 2. Then the S-neighborhoods $N(v_1) \cap S = \{v_{m,1}, v_{1,2}\}$; $N(v_{m-1}) \cap S = \{v_{m-1}, v_m\}$; $N(v_m) \cap S = \{v_{m-1,m}, v_{m,1}\}$ and those listed in Case 1 all are unequal, which implies that S is a minimum locating-dominating set. \Box

Proposition 3.9. Let H be a hyperstar with n vertices and $m \ge 2$ hyperedges. If for all $1 \le i \le m$, $|S_i^{(1)}| = 1$ and $|\mathcal{C}| > 1$, then $\lambda(H) = n - 2$, where \mathcal{C} is the center of the hyperstar.

Proof. Suppose $A = \mathcal{C}$ and $B = \bigcup_{i=1}^{m} S_i^{(1)}$. Since any two vertices of A have their same closed neighborhood so, by Lemma 2.4, |A| - 1 vertices of A must belong to a minimum locating-dominating set S for H. Further, for any two elements u_1 and u_2 of B, we have $N(u_1) = N(u_2)$. It follows that if there exists $\{u_1, u_2\} \subseteq B - S$, then $N(u_1) \cap S = N(u_2) \cap S$, which implies that S contains exactly |B| - 1 vertices of B. Since $A \cup B = V(H)$. Hence, $\lambda(H) = |A| + |B| - 2$.

If each coincident set $S_i^{(1)}$ $(1 \le i \le m)$ in a hyperstar with *m* hyperedges has at least two elements, then we have the following straightforward proposition:

Proposition 3.10. Let *H* be a hyperstar with $m \ge 2$ hyperedges and for all *i*, $|S_i^{(1)}| > 1$, then $\lambda(H) = \sum_{i=1}^m |C_i^{(1)}| + |\mathcal{C}| - 1$, where \mathcal{C} is the center of the hyperstar.

In a k-uniform linear hyperstar, $|C_i^{(1)}| = k - 2$, for all $1 \le i \le m$, therefore we have the following consequence:

Corollary 3.11. Let H be a k-uniform linear hyperstar with $m \ge 3$ hyperedges. Then for $k \ge 3$, $\lambda(H) = m(k-2)$.

Let N_1, N_2, \ldots, N_t be t disjoint finite sets with $|N_i| = n_i$. A complete t-partite r-uniform hypergraph is $H = K_{n_1, n_2, \ldots, n_t}^r$ has the vertex set $V(H) = \bigcup_{i=1}^t N_i$ and for each subset $E_j \subseteq V(H), E_j \in E(H)$ if $|E_j| = r$ and $|E_j \cap N_i| \le 1$ [3].

Lemma 3.12. For each $t \in \mathbb{Z}^+$ and $t \ge 2$, if S is a locating-dominating set for $K^r_{n_1,n_2,\ldots,n_t}$, then $|S \cap N_i| \ge n_i - 1$ for all $1 \le i \le t$.

Proof. Suppose that $A = N_{t_1}, 1 \le t_1 \le t$. and $B = \bigcup_{\substack{i=1\\i \ne t_1}}^t N_i$. Then for each $u \in A$

and for each $v \in B$, we have $N(u) \neq N(v)$ whereas, for any two elements u_1, u_2 of A, we have $N(u_1) = N(u_2)$. Hence, if there exists $\{u_1, u_2\} \subseteq A - S$, then we have $N(u_1) \cap S = N(u_2) \cap S$, which implies $|S \cap N_i| \geq n_i - 1$.

Theorem 3.13. Let $K_{n_1,n_2,...,n_t}^r$ be a complete t-partite r-uniform hypergraph. Then for $t \in \mathbb{Z}^+$ and for all $i, v_i \in N_i, S = \bigcup_{i=1}^{t} (N_i - \{v_i\})$ is a locating-dominating set for $K_{n_1,n_2,...,n_t}^r$ if and only if there is at most one partite set of cardinality 1.

Proof. Suppose that there are two partite sets $N_{i_{t_1}}$ and $N_{i_{t_2}}$ such that $n_{i_{t_1}} = n_{i_{t_2}} = 1$. Let $u \in N_{i_{t_1}}$ and $v \in N_{i_{t_2}}$. Then, by the definition of $K^r_{n_1,n_2,\dots,n_t}$, both the vertices have their same open neighborhoods in S, which implies that S is not a locating-dominating set, a contradiction.

Conversely, let $R = \bigcup_{i=1}^{t} (N_i - S)$ and N_{i_t} is the unique partite set such that $n_{i_t} = 1$. Then, by the definition of S, the set R has exactly t vertices of K_{n_1,n_2,\ldots,n_t}^r , that is, R has exactly one vertex from each N_i . Since every two partite sets are disjoint, so every vertex in R has its different open neighborhood in S.

Remark 3.14. In $K_{n_1,n_2,...,n_t}^r$, note that, if there are p partite sets of cardinality 1, then all the vertices in these partite sets except one will belong to every locating-dominating set for $K_{n_1,n_2,...,n_t}^r$.

Let H be a hypergraph and let $k \in \mathbb{Z}^+$. The *k*-section of H is $H_k = (V(H), E_k)$, where for a set $X \subseteq V(H)$, X belongs to E_k if any of the following conditions holds: (1) $|X| \leq k$ and $X \in E(H)$,

(2) |X| = k and there exists $E_j \in E(H)$ such that $X \subseteq E_j$ [3].

Lemma 3.15. For a hypergraph H and for any $k \in \mathbb{Z}^+$, a set $S \subset V(H)$ is locatingdominating in H if and only if it is locating-dominating in H_k . Moreover, $\lambda(H) = \lambda(H_k)$.

Proof. Let S_1 and S_2 be two locating-dominating sets for H and H_k , respectively. Let N[u] and N[v] be closed neighborhoods of $u, v \in V(H)$. By the definition of k-section hypergraph, we note that N[u] = N[v] in H if and only if N[u] = N[v] in H_k . Hence, by Lemma 2.4, S_1 and S_2 are the same sets. \Box

For any vertex $v \in V(H)$, let $E(v) = \{E_i \in E(H) \mid v \in E_i\}$. Then the edge degree of v is $d_e(v) = |E(v)|$. If $d_e(v) = 1$, then we say that v is a pendant vertex.

The natural partition in a hypergraph H is a partition $P = \{P_1, P_2, \ldots, P_t\}$ of V(H) such that for every pair $u, v \in V(H), u, v \in P_i$ if and only if E(u) = E(v). The elements $P_i \in P$ are called the *levels* of H. A hypergraph $H_L = (V(H_L), E(H_L))$ is called the *level hypergraph* of H if we delete every vertex except one from each level of H.

Example 3.16. Let *H* be the hypergraph with $V(H) = \{v_1, v_2, ..., v_{10}\}$ and $E(H) = \{E_1, E_2, E_3\}$, where $E_1 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{v_4, v_5, v_6\}$ and $E_3 = \{v_6, v_7, v_8, v_9, v_{10}\}$. Then the collection $P = \{P_1, P_2, P_3, P_4, P_5\}$ with $P_1 = \{v_1, v_2, v_3\}$, $P_2 = \{v_4\}$, $P_3 = \{v_5\}$, $P_4 = \{v_6\}$, and $P_5 = \{v_7, v_8, v_9, v_{10}\}$ is called the natural partition of *H*. The elements $P_i \in P$ are called the levels of *H*. The level hypergraph of *H* is the graph with vertex set $V(H_L) = \{v_1, v_4, v_5, v_6, v_7\}$ and the edge set $E(H_L) = \{E'_1, E'_2, E'_3\}$ with $E'_1 = \{v_1, v_4\}$, $E'_2 = \{v_4, v_5, v_6\}$ and $E'_3 = \{v_6, v_7\}$.

Theorem 3.17. If H_L is the level hypergraph of a hypertree H, then the set

$$S = \{v \in V(H_L) \mid E'(v) = \{E'_i\} \text{ in } H_L, \text{ where } E'_i \in E(H_L)\}$$

is a locating-dominating set for H_L .

Proof. Let $u, v \in V(H_L) - S$. Then, by the definitions of the level hypergraph H_L and the set S, u and v are not pendant vertices. Note that, $N(u) \cap S = \{u' \in \bigcup_{i=1}^{l_1} E'_{i_r} \mid u'$ is a pendant vertex $\}$ and $N(v) \cap S = \{v' \in \bigcup_{s=1}^{l_2} E'_{i_s} \mid v'$ is a pendant vertex $\}$, where $E'_{i_1}, E'_{i_2}, \ldots, E'_{i_{l_1}}$ and $E'_{i_1}, E'_{i_2}, \ldots, E'_{i_{l_2}}$ with $2 \leq l_1, l_2 \leq m$ are the hyperedges such that $u \in \bigcap_{r=1}^{l_1} E'_{i_r}$ and $v \in \bigcap_{s=1}^{l_2} E'_{i_s}$. Since H_L is the level hypergraph, therefore there is at least one hyperedge $E'_i \in E(H_L)$ such that $u \in E'_i$ but $v \notin E'_i$, which proves the theorem. \Box

The primal graph, prim(H), of a hypergraph H is a graph with vertex set V(H)and vertices x and y of prim(H) are adjacent if and only if x and y are contained in a hyperedge. The middle graph, M(H), of H is a subgraph of prim(H) formed by deleting all loops and parallel edges. The dual of $H = (\{v_1, v_2, \ldots, v_m\}, \{E_1, E_2, \ldots,$ E_k), denoted by H^* , is the hypergraph whose vertices are $\{e_1, e_2, \ldots, e_k\}$ corresponding to the hyperedges of H and with hyperedges $V_i = \{e_j : v_i \in E_j \text{ in } H\}$, where $i = 1, 2, \ldots, m$. In other words, the dual H^* swaps the vertices and hyperedges of H.

From the definition of the primal graph of a hypergraph H, note that, closed neighborhood of any vertex in H is same as in prim(H). Thus, from Lemma 2.4, we have the following straightforward result:

Theorem 3.18. Let H be a hypergraph and prim(H) be the primal graph of H. Then $\lambda(H) = \lambda(prim(H)) = \lambda(M(H))$.

The primal graph of the dual H^* of a hypergraph H is not a simple graph. In this case, the middle graph of H^* is a simple graph. By using the same argument as above, we have the following result:

Theorem 3.19. Let H^* be the dual of a hypergraph H and $M(H^*)$ be the middle graph of H^* . Then $\lambda(H^*) = \lambda(M(H^*))$.

Theorem 3.20. [6] Let G be a graph of order $n \ge 2$. Then $\lambda(G) = 1$ if and only if $G \cong P_2$.

Theorem 3.21. Let H be a hypergraph. Then $\lambda(H) = 1$ if and only if $H \cong P_2$, where P_2 is a 2-uniform linear hyperpath with one hyperedge.

Proof. Suppose that $H \cong P_2$. Then, by Theorem 3.20, $\lambda(H) = 1$.

Conversely, suppose that $\lambda(H) = 1$ and $S = \{v\}$ is a locating-dominating set for H. Then $H \cong P_2$ otherwise, there exist $v_1, v_2 \in V(H) - S$ such that $N(v_1) \cap S = N(v_2) \cap S$.

Remark 3.22. Given a hypergraph H, the hypergraph resulting from deleting the repeated and non-maximal edges is denoted by H_{nm}^r . Then for any hypergraph H, $\lambda(H) = \lambda(H_{nm}^r)$ since the considered graphs are Sperner.

From Theorem 3.21 and Remark 3.22, it follows that for every hypergraph H, $\lambda(H_{nm}^r) = 1$ if and only if $\lambda(H_{nm}^r) \cong P_2$, where P_2 is a 2-uniform linear hyperpath with one hyperedge.

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