CR-STRUCTURES OF CODIMENSION 2 ON TANGENT BUNDLES IN RIEMANN-FINSLER GEOMETRY

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Abstract. We determine a 2-codimensional CR-structure on the slit tangent bundle T_0M of a Finsler manifold (M, F) by imposing a condition regarding the almost complex structure Ψ associated to F when restricted to the structural distribution of a framed f-structure. This condition is satisfied when (M, F) is of scalar flag curvature (particularly flat) and in the Riemannian case (M, g) this last condition means that g is of constant curvature. This CR-structure is finally generalized by using one positive number but under more difficult conditions. ¹

INTRODUCTION

The Finsler geometry is very rich in remarkable tensor fields F of (1, 1)-type and associated structures. More precisely, there are: an (almost) tangent structure ($F^2 = 0$), an almost complex one ($F^2 = -I$) and also an almost product structure ($F^2 = I$). In [1] another wellknown type of structures, namely f-structure ($F^3 + F = 0$) is obtained in this geometry. In fact, this f-structure belongs to a very interesting particular case which is called *framed* fstructure and has, in addition to F, a set of vector fields and differential 1-forms interrelated. Moreover, a conformal deformation of the Sasaki type metric can be added in order to obtain a metric framed f-structure. This metric framed f-structure of M. Anastasiei was recently generalized in [7] and [14].

The present note is concerning with another kind of structures, namely the *CR-structures*, with an important rôle at the border between differential geometry and complex analysis, as it is pointed out in [6]. We restrict ourselves at the real case; more precisely, based on a relationship between framed *f*-structures and CR-structure established in [2, p. 130] we found a CR-structure on the slit tangent bundle T_0M of a Finsler manifold (M, F). This CR-structure is constructed with the above almost complex structure denoted Ψ_F in Section 2 and its existence is constrained by one condition expressing the vanishing of the Nijenhuis tensor of Ψ_F on the structural distribution of the framed *f*-structure from [1]. The above condition is expressed as a relation between the curvature of the Cartan nonlinear connection and the Jacobi endomorphism and is satisfied in dimension two or if (M, F) is of scalar flag curvature which in the particular case of Riemannian geometry (M, g) means that the metric *g* has a constant curvature. Several important classes of Finsler manifolds with scalar flag curvature are discussed in Chapter 7 of [5].

Inspired by [14] we generalize this CR-structure using a real number $\beta > \frac{1}{2}$ but with more difficult conditions. More precisely, we take into account the same vector fields and 1-forms as

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in the previous framed f-structure but deform the metric and the almost complex structure on both horizontal and vertical directions. At $\beta = 1$ we recover the previous CR-structure.

Finally, let us note that our CR-structures are of codimension 2 and the (complex) geometry of these structures was studied in [10]-[11] and recently in [8] and [9]. But for the Riemannian case the only studies until now are on hypersurfaces of Sasakian manifolds ([12]-[13]) and not on (slit) tangent bundle.

1. CR-structures from framed f-structures

Framed f-structures constitute a particular case of f-structures and a detailed study of this class of tensor fields of (1, 1)-type, especially from a local point of view, can be found in [15].

Let N be a smooth (2n+s)-dimensional manifold with $n, s \ge 1$ and fix D a distribution on N of dimension 2n. Considering D as a vector bundle over N let $\Gamma(D)$ be the module of its sections. Supposing D is endowed with a morphism $J: D \to D$ of vector bundles satisfying $J^2 = -I$ where I is the identity (Kronecker) morphism on D, the pair (D, J) is called *almost complex distribution*.

The first main notion is given by [2, p. 128]:

Definition 1.1 If for all $X, Y \in \Gamma(D)$ we have:

$$\begin{cases} [JX, JY] - [X, Y] \in \Gamma(D) \\ N_J(X, Y) := [JX, JY] - [X, Y] - J([X, JY] + [JX, Y]) = 0 \end{cases}$$
(1.1)

then (D, J) is a *CR-structure* on N and the triple (N, D, J) is a *CR-manifold*.

A second main notion is:

Definition 1.2 Let φ be a tensor field of (1, 1)-type and s pairs $(\xi_a, \eta^a), 1 \le a \le s$ of vector fields and 1-forms on N. If: i) $\varphi^3 + \varphi = 0$, $rank\varphi = 2n$,

ii) $\varphi^2 = -I + \sum_{a=1}^{s} \eta^a \otimes \xi_a, \ \varphi(\xi_a) = 0, \ \eta^a(\xi_b) = \delta_b^a, \ \eta^a \circ \varphi = 0,$ then the data (φ, ξ_a, η^a) is called a *framed f-structure*.

To a framed f-structure we associate [2, p. 130]: 1) the torsion tensor field S of (1, 2)-type:

$$S = N_{\varphi} + 2\sum_{a=1}^{s} d\eta^{a} \otimes \xi_{a}.$$
(1.2)

2) the structural distribution D:

$$D = \{X \in \Gamma(TM); \eta^{1}(X) = \dots = \eta^{s}(X) = 0\} = \bigcap_{a=1}^{s} \ker \eta^{a}.$$
 (1.3)

For a 1-form η we use the differential:

$$2d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]).$$
(1.4)

These notions lead to:

Definition 1.3 The framed f-structure is called *D*-normal if S vanishes on D i.e. S(X, Y) = 0 for all $X, Y \in \Gamma(D)$.

The relationship between the above structures was pointed out by A. Bejancu in Proposition 1.1 of [2, p. 130]:

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Proposition 1.4 If (φ, ξ_a, η^a) is a *D*-normal framed f-structure then $(D, J = \varphi|_D)$ is a CR-structure.

Proof The restriction J of φ to D is obviously an almost complex structure. The conditions (1.1) result from the fact that for $X, Y \in \Gamma(D)$ we have:

$$S(X,Y) = 0 = [JX,JY] + \varphi^2([X,Y]) - \varphi([X,JY] + [JX,Y]) - \sum_{a=1}^s \eta^a([X,Y])\xi_a.$$
(1.5)

For other details see the cited reference. \Box

2. A metric framed f-structure on the tangent bundle of a Finsler manifold

Let M be now a smooth m-dimensional manifold with $m \geq 2$ and $\pi : TM \to M$ its tangent bundle. Let $x = (x^i) = (x^1, ..., x^m)$ be the local coordinates on M and $(x, y) = (x^i, y^i) = (x^1, ..., x^m, y^1, ..., y^m)$ the induced local coordinates on TM. Denote by O the null-section of π .

Recall after [5] that a Finsler fundamental function on M is a map $F: TM \to \mathbb{R}_+$ with the following properties:

F1) F is smooth on the slit tangent bundle $T_0M := TM \setminus O$ and continuous on O,

F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,

F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is positive definite.

The tensor field $g = \{g_{ij}(x, y); 1 \le i, j \le \}$ is called the Finsler metric and the homogeneity of F implies:

$$F^{2}(x,y) = g_{ij}y^{i}y^{j} = y_{i}y^{i}$$
(2.1)

where $y_i = g_{ij}y^j$. The pair (M, F) is called *Finsler manifold*.

On T_0M we have two distributions:

i) $V(TM) := \ker \pi_*$, called the vertical distribution and not depending of F. It is integrable and has the basis $\{\frac{\partial}{\partial y^i}; 1 \le i \le m\}$. A remarkable section of it is the Liouville vector field $\Gamma = y^i \frac{\partial}{\partial u^i}$.

ii) H(TM) with the basis $\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial y^i} - N_i^j \frac{\partial}{\partial y^j}\}$ where:

$$N_{j}^{i} = \frac{1}{2} \frac{\gamma_{00}^{i}}{\partial y^{j}}$$
(2.2)

with $\gamma_{00}^i = \gamma_{jk}^i y^j y^k$ built from the usual Christoffel symbols:

$$\gamma_{jk}^{i} = \frac{1}{2}g^{ia} \left(\frac{\partial g_{ak}}{\partial x^{j}} + \frac{\partial g_{ja}}{\partial x^{k}} - \frac{\partial g_{jk}}{\partial x^{a}}\right).$$
(2.3)

H(TM) is often called the *Cartan* (or canonical) *nonlinear connection* of the geometry (M, F) and a remarkable section of it is *the geodesic spray*:

$$S_F = y^i \frac{\delta}{\delta x^i}.\tag{2.4}$$

In particular, if g does not depend on y we recover the Riemannian geometry.

The dual basis of the above local basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ of $\Gamma(T_0M)$ is $(dx^i, \delta y^i = dy^i + N^i_j dx^j)$. On T_0M we have a Riemannian metric of Sasaki type:

$$G_F = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j.$$
(2.5)

Another Finslerian object is the tensor field of (1, 1)-type $\Psi_F : \Gamma(T_0M) \to \Gamma(T_0M)$:

$$\Psi_F\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \quad \Psi_F\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}.$$
(2.6)

It results that Ψ_F is an almost complex structure and the pair (Ψ_F, G_F) is an almost Kähler structure on T_0M .

In order to obtain a framed f-structure on T_0M associated to the Finslerian function F, the following objects are considered in [1]:

$$\begin{cases} \xi_1 = S_F, \xi_2 = \Gamma \\ \eta^1 = \frac{1}{F^2} y_i dx^i, \eta^2 = \frac{1}{F^2} y_i \delta y^i \\ \varphi = \Psi_F + \eta^1 \otimes \xi_2 - \eta^2 \otimes \xi_1 \\ G = \frac{1}{F^2} G_F. \end{cases}$$
(2.7)

Then the main result of [1] is that the data $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2)$ is a framed *f*-structure on T_0M with η^a the *G*-dual of ξ_a , $1 \le a \le 2$ and, moreover:

$$G(\varphi, \varphi) = G - \eta^1 \otimes \eta^1 - \eta^2 \otimes \eta^2.$$
(2.8)

Also, ξ_a are unitary vector fields with respect to G and $(G, \varphi, \xi_a, \eta^a)$ is a *metric framed* f-structure.

3. Putting all together

The last paragraph of the previous Section provides the ingredients of the first Section with $N = T_0 M$, s = 2 and n = m - 1 which motivates our choice $m \ge 2$. The structural distribution is then:

$$D_F = \ker \eta^1 \cap \ker \eta^2 = \{\xi_1\}^{\perp G} \cap \{\xi_2\}^{\perp G} = \{\xi_1\}^{\perp G_F} \cap \{\xi_2\}^{\perp G_F}$$
(3.1)

where $\{X\}^{\perp G}$ is the *G*-orthogonal complement of $span\{X\}$. We have $D_F = (span\{\xi_1, \xi_2\})^{\perp G_F}$ and this implies that D_F has the dimension 2m - 2. For a geometrical meaning of the distribution $span\{\xi_1, \xi_2\}$ in [1] is defined the differential 2-form ω_F , naturally associated to the metric framed *f*-structure:

$$\omega_F = G(\cdot, \varphi \cdot) \tag{3.2}$$

and it follows that $span{\xi_1, \xi_2}$ is the kernel of ω_F . Also, the homogeneity of F implies the homogeneity of $S_F = \xi_1$ which means:

$$[\Gamma, S_F] = [\xi_2, \xi_1] = \xi_1 \tag{3.3}$$

and thus $span\{\xi_1,\xi_2\}$ is an integrable distribution; see also Theorem 3.15 of [3, p. 236].

A concrete expression of D_F appears in [4, p. 11]. More precisely, consider after the cited paper:

i) the horizontal vector fields:

$$h_i = \frac{\delta}{\delta x^i} - \frac{1}{F^2} y_i S_F \tag{3.4}$$

and the corresponding (m-1)-distribution $\mathcal{H}_{m-1} = span\{h_i; 1 \leq i \leq m\}$, ii) the vertical vector fields:

$$v_i = \frac{\partial}{\partial y^i} - \frac{1}{F^2} y_i \Gamma \tag{3.5}$$

and also the corresponding (m-1)-distribution $\mathcal{V}_{m-1} = span\{v_i; 1 \leq i \leq m\}$. We have:

$$D_F = \mathcal{H}_{m-1} \otimes \mathcal{V}_{m-1} \tag{3.6}$$

and the same Theorem 3.15 of [3, p. 236] proves the integrability of \mathcal{V}_{m-1} ; see also [4, p. 12]. Regarding the integrability of the nonlinear connection H(TM) we have:

$$\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^i_{jk} \frac{\partial}{\partial y^i} \tag{3.7}$$

where:

$$R_{jk}^{i} = \frac{\delta N_{j}^{i}}{\delta x^{k}} - \frac{\delta N_{k}^{i}}{\delta x^{j}}.$$
(3.8)

The tensor field $R = \{R_{jk}^i(x, y); 1 \le i, j, k \le m\}$ is called *the curvature* of the Cartan nonlinear connection and:

$$R_j^i := R_{kj}^i y^k \tag{3.9}$$

are the components of the Jacobi endomorphism $\Phi = R_j^i \frac{\partial}{\partial y^i} \otimes dx^j$, [4, p. 5]. We are ready now for the first main result:

Theorem 3.1 If the curvature tensor of (M, F) has the form:

$$R^i_{jk} = \lambda (X^i_k y_j - X^i_j y_k) \tag{3.10}$$

with λ a smooth function on T_0M and the tensor field $\{X_j^i(x,y); 1 \leq i, j \leq m\}$ satisfying:

$$y_i X_j^i = y_j \tag{3.11}$$

for all $i, j \in \{1, ..., m\}$ then the pair $(D_F, J_F = \Psi_F|_{D_F})$ is a CR-structure on T_0M .

Proof We express the Nijenhuis tensor field of Ψ_F as:

$$N_{\Psi_F}(X,Y) = [\Psi_F X, \Psi_F Y] - [X,Y] - \Psi_F(A(X,Y)) = B(X,Y) - \Psi_F(A(X,Y))$$
(3.12)

with $A(X,Y) := [X, \Psi_F Y] + [\Psi_F X, Y]$ and $B(X,Y) = [\Psi_F X, \Psi_F Y] - [X,Y]$. It follows that $B(X,Y) = A(\Psi_F X,Y)$ and then:

$$N_{\Psi_F}(X,Y) = A(\Psi_F X,Y) - \Psi_F \circ A(X,Y).$$
(3.13)

We prove firstly that A is a D_F -valued (0, 2)-tensor field. From (3.7) and:

$$\left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right] = \frac{\partial N^i_j}{\partial y^k} \frac{\partial}{\partial y^i} = \frac{\partial^2 \gamma^i_{00}}{\partial y^j \partial y^k} \frac{\partial}{\partial y^i}$$
(3.14)

we obtain:

$$A\left(\frac{\delta}{\delta x^{j}},\frac{\delta}{\delta x^{k}}\right) = A\left(\frac{\partial}{\partial y^{j}},\frac{\partial}{\partial y^{k}}\right) = 0, \quad A\left(\frac{\delta}{\delta x^{j}},\frac{\partial}{\partial y^{k}}\right) = R_{jk}^{i}\frac{\partial}{\partial y^{i}}$$
(3.15)

which means that $\eta^1 \circ A = 0$ and:

$$A = R^i_{jk} dx^j \wedge \delta y^k \otimes \frac{\partial}{\partial y^i}.$$
(3.16)

A main identity in Finsler geometry is:

$$y_i R^i_{ab} = 0 \tag{3.17}$$

and then $\eta^2 \circ A = 0$ which conclude the first part of the proof.

Secondly, we search for the framework of Proposition 1.4. The torsion tensor S on D_F is:

$$S(X,Y) = N_{\varphi}(X,Y) - \eta^{1}([X,Y])\xi_{1} - \eta^{2}([X,Y])\xi_{2}$$

with:

$$N_{\varphi}(X,Y) = [\Psi_F X, \Psi_F Y] + \varphi^2([X,Y]) - \varphi \circ A(X,Y).$$

Since φ is an element of a framed *f*-structure we get:

$$N_{\varphi}(X,Y) = [\Psi_F X, \Psi_F Y] - [X,Y] + \eta^1([X,Y])\xi_1 + \eta^2([X,Y])\xi_2 - \varphi \circ A(X,Y)$$

and from the definition (2.7_3) of φ it follows:

$$S(X,Y) = [\Psi_F X, \Psi_F Y] - [X,Y] - (\Psi_F + \eta^1 \otimes \xi_2 - \eta^2 \otimes \xi_1) \circ A(X,Y) = N_{\Psi_F}(X,Y).$$
(3.18)
In local coordinates we have:

local coordinates we have:

$$N_{\Psi_F} = R^i_{jk} \delta y^j \wedge \delta y^k \otimes \frac{\partial}{\partial y^i}$$
(3.19)

and then N_{Ψ_F} has components only when applied on the pair (v_a, v_b) . A long but straightforward computation yields:

$$N_{\Psi_F}(v_a, v_b) = 2 \left[R^i_{ab} + \frac{1}{F^2} (R^i_a y_b - R^i_b y_a) \right] \frac{\partial}{\partial y^i}$$
(3.20)

and therefore the normality condition is:

$$F^2 R^i_{ab} = R^i_b y_a - R^i_a y_b (3.21)$$

which can be expressed as:

$$N_{\Psi_F} = \eta^2 \wedge \left(R_k^i \delta y^k \otimes \frac{\partial}{\partial y^i} \right). \tag{3.22}$$

The relation (3.10) yields:

$$R_k^i = \lambda (F^2 X_k^i - y^a X_a^i y_k) \tag{3.23}$$

and then, both sides of (3.21) are equal with $\lambda F^2(X_k^i y_j - X_j^i y_k)$ which gives the final conclusion. The condition (3.11) corresponds to the relation (3.17).

Let us also point out that the condition (3.10) gives the following expression for the Nijenhuis tensor:

$$N_{\Psi_F} = 2\lambda F^2 \eta^2 \wedge \left(X_j^i \delta y^j \otimes \frac{\partial}{\partial y^i} \right)$$
(3.24)

which yields again the vanishing of N_{Ψ_F} on D_F due to the presence of η^2 . Concerning the tensor field A we have:

$$A = \lambda F^2 \left[\eta^1 \wedge (X^i_j \delta y^j \otimes \frac{\partial}{\partial y^i}) - (X^i_j dx^j \otimes \frac{\partial}{\partial y^i}) \wedge \eta^2 \right]$$
(3.25)

which proves the relations: $\eta^1 \circ A = \eta^2 \circ A = 0$.

Example 3.2 Recall that in dimension 2 the Nijenhuis tensor field of any almost complex structure vanishes. Then every 2-dimensional Finsler manifold (M_2, F) satisfies the condition of Theorem 3.1. Let V(TM) be spanned by the vector fields Γ and V respectively H(TM) be spanned by the vector fields S_F and H. Then D_F is spanned by V and H and:

$$J_F(H) = -V, \quad J_F(V) = H.$$
 (3.26)

We have that H is a linear combination of h_1 and h_2 while V is a linear combination of v_1 and v_2 . \Box

In order to consider examples in any dimension we remark that a solution of condition (3.11) is:

$$X_{j}^{i} = \mu \delta_{j}^{i} + (1 - \mu) \frac{y^{i} y_{j}}{F^{2}}$$
(3.27)

again with μ a smooth function on T_0M . It follows:

Example 3.3 If $\mu = 1$ then $X_j^i = \delta_j^i$ and the Finsler manifold (M, F) is of scalar flag curvature λ since:

$$R^i_{jk} = \lambda(\delta^i_k y_j - \delta^i_j y_k) \tag{3.28}$$

and then:

$$R_k^i = \lambda(\delta_k^i F^2 - y^i y_k). \tag{3.29}$$

Corollary 3.4 If (M, F) is of scalar flag curvature then $(D_F = (span\{S_F, \Gamma\})^{\perp G_F}, J_F)$ is a CR-structure on T_0M .

Remark also that the hypothesis of scalar flag curvature yields:

$$N_{\Psi_F} = 2\lambda F^2 \eta^2 \wedge \pi_{V(TM)} \tag{3.30}$$

where $\pi_{V(TM)}$ is the projector on the vertical part in the G_F -orthogonal decomposition $T(T_0M) = H(TM) \oplus V(TM)$ i.e $\pi_{V(TM)} = \delta y^i \otimes \frac{\partial}{\partial y^i}$. However, Ψ_F is integrable only in the flat case (i.e. $\lambda = 0$) since $N_{\Psi_F}(\Gamma, v_a) = 2\lambda F^2 v_a$. The integrability of Ψ_F as a tensor field of (1, 1)-type which is equivalent with the integrability of the Cartan nonlinear connection of (M, F) and then (T_0M, Ψ_F, G_F) is a Kähler manifold.

Particular case 3.5 (Riemannian geometry) Let $g = (g_{ij}(x))$ be a Riemannian metric on M. Then $\gamma_{ik}^i(x, y) = \Gamma_{ik}^i(x)$ the Riemannian Christoffel symbols and:

$$R^{i}_{jk}(x,y) = R^{i}_{jka}(x)y^{a}$$
(3.31)

with $R_g = (R_{jka}^i)$ the Riemannian curvature tensor of g. It results that a Riemannian geometry $(M, F = (g_{ij}(x)y^iy^j)^{\frac{1}{2}})$ is of scalar flag curvature if and only if g is of constant curvature. Therefore on the slit tangent bundle of a space form (M, g) there exists a CR-structure on the distribution complementary (with respect to the Sasaki lift of g) to the distribution generated by the Liouville vector field and the geodesic spray S_q . \Box

Example 3.6 Returning to the general non-Riemannian case (3.27) with $\mu = 0$ we get:

$$X_{j}^{i} = \frac{y^{i}y_{j}}{F^{2}} \tag{3.32}$$

and then $R_{jk}^i = 0$ which means that (M, F) is flat, a situation belonging also to the Example 3.3 for vanishing scalar curvature. \Box

For the general μ we have:

$$N_{\Psi_F} = 2\lambda F^2 \eta^2 \wedge \left[\mu \pi_{V(TM)} + (1-\mu)\eta^2 \otimes \Gamma\right] = 2\lambda \mu F^2 \eta^2 \wedge \mu \pi_{V(TM)}.$$
(3.33)

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4. A 1-parametric generalization

Let $\alpha > 0$ and $\beta > 0$ two positive numbers as well as the smooth function $v : [0, +\infty) \to \mathbb{R}$ which, following the approach of [14], will be considered as $v = v(\tau)$ with $\tau = F^2$. Supposing that:

$$\alpha + 2\tau v(\tau) > 0 \tag{4.1}$$

for any τ in the cited paper is constructed the smooth function:

$$w = -\frac{\beta v}{\alpha + \tau v} \tag{4.2}$$

and the Riemannian metric on T_0M :

$$\bar{G} = G_{ij}dx^i \otimes dx^j + H_{ij}\delta y^i \otimes \delta y^j \tag{4.3}$$

where:

$$\begin{cases}
G_{ij} = \frac{1}{\beta}g_{ij} + \frac{v}{\alpha\beta}y_iy_j \\
H_{ij} = \beta g_{ij} + wy_iy_j.
\end{cases}$$
(4.4)

Inspired by [14] we define also:

$$\begin{cases} \bar{\xi}_1 = (\beta + w\tau)S_F, & \bar{\xi}_2 = \Gamma = \xi_2\\ \bar{\eta}^1 = \frac{1}{\tau}y_i dx^i = \eta^1, & \bar{\eta}^2 = (\frac{\beta}{\tau} + w)y_i \delta y^i\\ \bar{\Psi}_F(\frac{\delta}{\delta x^i}) = -G_i^a \frac{\partial}{\partial y^a}, & \bar{\Psi}_F(\frac{\partial}{\partial y^i}) = H_i^a \frac{\delta}{\delta x^a} \end{cases}$$
(4.5)

where the lift of indices in the third line is constructed with $g^{-1} = (g^{ab})$. In fact, the only difference between us and [14] is with respect to 1-form $\bar{\eta}^i$; in order to reobtain that of Section 2 we divide with τ the 1-forms of Peyghan-Zhong. With a computation similar to that of Theorem 4.8 of Peyghan-Zhong we derive that $(\bar{G}, \bar{\varphi}, \bar{\xi}_a, \bar{\eta}^a)$ with:

$$\bar{\varphi} = \bar{\Psi}_F + \bar{\eta}^1 \otimes \bar{\xi}_2 - \bar{\eta}^2 \otimes \bar{\xi}_1 \tag{4.6}$$

is a metric framed f-structure on T_0M if and only if:

$$\beta + \tau w = 1. \tag{4.7}$$

From this condition we get that $\bar{\xi}_a = \xi_a$ and $\bar{\eta}^a = \eta^a$. From (4.2) and (4.7) we obtain:

$$v(\tau) = \frac{\alpha(\beta - 1)}{\tau}, \quad w(\tau) = \frac{1 - \beta}{\tau}.$$
(4.8)

In the particular case $\alpha = \beta = 1$ we recover the metric framed *f*-structure of Anastasiei since $v = w \equiv 0$.

Now, under condition (4.7) we have the same structural distribution D_F but the expression of the tensor field:

$$\bar{A}(X,Y) := [X,\bar{\Psi}_F Y] + [\bar{\Psi}_F X,Y]$$

$$(4.9)$$

is more complicated. More detailed:

$$\begin{cases} \bar{A}(\frac{\delta}{\delta x^{j}},\frac{\delta}{\delta x^{k}}) = \left(\frac{\delta G_{y}^{v}}{\delta x^{k}} - \frac{\delta G_{k}^{v}}{\delta x^{j}} + G_{j}^{u}\frac{\partial N_{k}^{v}}{\partial y^{u}} - G_{k}^{u}\frac{\partial N_{j}^{v}}{\partial y^{u}}\right)\frac{\partial}{\partial y^{v}} \\ \bar{A}(\frac{\partial}{\partial y^{j}},\frac{\partial}{\partial y^{k}}) = \left(\frac{\partial H_{k}^{v}}{\partial y^{j}} - \frac{\partial H_{j}^{v}}{\partial y^{k}}\right)\frac{\delta}{\delta x^{v}} + \left(H_{j}^{u}\frac{\partial N_{u}^{v}}{\partial y^{k}} - H_{k}^{u}\frac{\partial N_{u}^{v}}{\partial y^{j}}\right)\frac{\partial}{\partial y^{v}} \\ \bar{A}(\frac{\delta}{\delta x^{j}},\frac{\partial}{\partial y^{k}}) = \frac{\delta H_{k}^{v}}{\delta x^{j}}\frac{\delta}{\delta x^{v}} + \left(H_{k}^{u}R_{ju}^{v} + \frac{\partial G_{j}^{v}}{\partial y^{k}}\right)\frac{\partial}{\partial y^{v}}$$
(4.10)

where, with (4.7):

$$\begin{cases} G_{ij} = \frac{1}{\beta} g_{ij} + \frac{\beta - 1}{\beta \tau} y_i y_j, & H_{ij} = \beta g_{ij} + \frac{1 - \beta}{\tau} y_i y_j \\ G_j^a = \frac{1}{\beta} \delta_j^a + \frac{\beta - 1}{\beta \tau} y^a y_j, & H_j^a = \beta \delta_j^a + \frac{1 - \beta}{\tau} y^a y_j \\ \bar{\Psi}_F(\frac{\delta}{\delta x^i}) = -\frac{1}{\beta} \frac{\partial}{\partial y^i} + \frac{1 - \beta}{\beta \tau} y_i \Gamma, & \bar{\Psi}_F(\frac{\partial}{\partial y^i}) = \beta \frac{\delta}{\delta x^i} + \frac{1 - \beta}{\tau} y_i S_F. \end{cases}$$
(4.11)

It results that α disappears and this motives the title of this Section, namely 1-parametric generalization and not 2-parametric. Note that $\bar{\Psi}_F(h_i) = -\frac{1}{\beta}v_i$ and $\bar{\Psi}_F(v_i) = \beta h_i$.

Then:

$$\begin{cases}
\bar{A}(\frac{\delta}{\delta x^{j}},\frac{\delta}{\delta x^{k}}) = \frac{\beta-1}{\beta\tau} \left[\frac{\delta}{\delta x^{k}} \left(y_{j} y^{v} \right) - \frac{\delta}{\delta x^{j}} \left(y_{k} y^{v} \right) \right] \frac{\partial}{\partial y^{v}} \\
\bar{A}(\frac{\partial}{\partial y^{j}},\frac{\partial}{\partial y^{k}}) = (1-\beta) \left[\frac{\partial}{\partial y^{j}} \left(\frac{y_{k} y^{v}}{\tau} \right) - \frac{\partial}{\partial y^{k}} \left(\frac{y_{j} y^{v}}{\tau} \right) \right] \frac{\delta}{\delta x^{v}} \\
\bar{A}(\frac{\delta}{\delta x^{j}},\frac{\partial}{\partial y^{k}}) = \frac{1-\beta}{\tau} \frac{\delta}{\delta x^{j}} \left(y_{k} y^{v} \right) \frac{\delta}{\delta x^{v}} + \left[\beta R_{jk}^{v} + \frac{1-\beta}{\tau} y_{k} y^{u} R_{ju}^{v} + \frac{\beta-1}{\beta} \frac{\partial}{\partial y^{k}} \left(\frac{y_{j} y^{v}}{\tau} \right) \right] \frac{\partial}{\partial y^{v}}.$$
(4.12)

Choosing $\alpha = 1$ the second main result is:

Theorem 4.1 Let $\beta > \frac{1}{2}$ and the smooth functions $v(\tau) = -w(\tau) = \frac{\beta-1}{\tau}$. If for any $X, Y \in D_F$ we have: 1) $\bar{A}(X,Y) \in D_F$, 2) $N_{\bar{\Psi}_F}(X,Y) = 0$,

then $(D_F, \bar{J}_F = \bar{\Psi}_F|_{D_F})$ is a CR-structure on T_0M .

Proof The condition in β is the expression of (4.1). Exactly as in the proof of Theorem 3.1 we have:

$$S(X,Y) = N_{\bar{\Psi}_F}(X,Y) - \eta^1(\bar{A}(X,Y))\xi_2 + \eta^2(\bar{A}(X,Y))\xi_1.$$
(4.13)

and the conclusion follows directly. Let us note that 1) corresponds to the condition (1.1_1) while 2) corresponds to the condition (1.1_2) . \Box

Let us remark that:

$$\beta \eta^2 \circ \bar{A}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = \eta^1 \circ \bar{A}\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right) - \eta^1 \circ \bar{A}\left(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}\right).$$
(4.14)

and then, the vanishing of $\eta^1 \circ \bar{A}\left(\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^b}\right)$ implies the vanishing of $\eta^2 \circ \bar{A}\left(\frac{\delta}{\delta x^u}, \frac{\delta}{\delta x^v}\right)$. The vanishing of the former expression means that y_k are eigenvalue for $\frac{\delta}{\delta x^j}$:

$$\frac{\delta y_k}{\delta x^j} = \left(-\frac{N_j^a y_a}{F^2}\right) y_k \tag{4.15}$$

and then y_k are eigenvalues for the geodesic spray:

$$S_F(y_k) = \left(-\frac{N_j^a y^j y_a}{F^2}\right) y_k.$$
(4.16)

Such condition holds in the Euclidian space $(\mathbb{R}^m, g_{ij} = \delta_{ij})$ but here the expression $\eta^2 \circ \overline{A}(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k})$ is non-vanishing since:

$$y_v \frac{\partial}{\partial y^k} \left(\frac{y_j y^v}{F^2} \right) = \delta_{jk} - \frac{y_j y^k}{F^2} \neq 0$$
(4.17)

and then it remains an open problem to find Riemannian and/or Finsler manifolds satisfying the Theorem 4.1 with $\beta \neq 1$.

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