# Complete monotonicity of some entropies

Ioan Raşa

Department of Mathematics, Technical University of Cluj-Napoca, Memorandumului Street 28, 400114 Cluj-Napoca, Romania, ioan.rasa@math.utcluj.ro

#### Abstract

It is well-known that the Shannon entropies of some parameterized probability distributions are concave functions with respect to the parameter. In this paper we consider a family of such distributions (including the binomial, Poisson, and negative binomial distributions) and investigate the Shannon, Rényi, and Tsallis entropies of them with respect to the complete monotonicity.

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## 1 Introduction

Let  $c \in \mathbb{R}$ ,  $I_c := \left[0, -\frac{1}{c}\right]$  if c < 0, and  $I_c := \left[0, +\infty\right)$  if  $c \ge 0$ . For  $\alpha \in \mathbb{R}$  and  $k \in \mathbb{N}_0$  the binomial coefficients are defined as usual by

$$\binom{\alpha}{k} := \frac{\alpha(\alpha - 1)\dots(\alpha - k + 1)}{k!} \quad \text{if } k \in \mathbb{N}, \text{ and } \binom{\alpha}{0} := 1.$$

Let n > 0 be a real number such that n > c if  $c \ge 0$ , or n = -cl with some  $l \in \mathbb{N}$  if c < 0.

For  $k \in \mathbb{N}_0$  and  $x \in I_c$  define

$$p_{n,k}^{[c]}(x) := (-1)^k \binom{-\frac{n}{c}}{k} (cx)^k (1+cx)^{-\frac{n}{c}-k}, \quad \text{if } c \neq 0,$$

$$p_{n,k}^{[0]}(x) := \lim_{c \to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!} e^{-nx}$$

Details and historical notes concerning these functions can be found in [3], [7], [21] and the references therein. In particular,

$$\frac{d}{dx}p_{n,k}^{[c]}(x) = n\left(p_{n+c,k-1}^{[c]}(x) - p_{n+c,k}^{[c]}(x)\right).$$
(1)

Moreover,

$$\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1;$$
(2)

$$\sum_{k=0}^{\infty} k p_{n,k}^{[c]}(x) = nx,$$
(3)

so that  $\left(p_{n,k}^{[c]}(x)\right)_{k\geq 0}$  is a parameterized probability distribution. Its associated Shannon entropy is

$$H_{n,c}(x) := -\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \log p_{n,k}^{[c]}(x),$$

while the Rényi entropy of order 2 and the Tsallis entropy of order 2 are given, respectively, by (see [18], [20])

$$R_{n,c}(x) := -\log S_{n,c}(x); \quad T_{n,c}(x) := 1 - S_{n,c}(x),$$

where

$$S_{n,c}(x) := \sum_{k=0}^{\infty} \left( p_{n,k}^{[c]}(x) \right)^2, \quad x \in I_c.$$

The cases c = -1, c = 0, c = 1 correspond, respectively, to the binomial, Poisson, and negative binomial distributions. For other details see also [15], [16].

In this paper we investigate the above entropies with respect to the complete monotonicity.

# 2 Shannon entropy

### A. Let's start with the case c < 0.

 $H_{n,-1}$  is a concave function; this is a special case of the results of [19]; see also [6], [8], [9] and the references therein.

Here we shall determine the signs of all the derivatives of  $H_{n,c}$ .

**Theorem 1** Let c < 0. Then, for all  $k \ge 0$ ,

$$H_{n,c}^{(2k+2)}(x) \le 0, \quad x \in \left(0, -\frac{1}{c}\right),$$
(4)

$$H_{n,c}^{(2k+1)}(x) = \begin{cases} \geq 0 & x \in (0, -\frac{1}{2c}], \\ \leq 0 & x \in [-\frac{1}{2c}, -\frac{1}{c}). \end{cases}$$
(5)

**Proof** We have n = -cl with  $l \in \mathbb{N}$ . As in [10], let us represent  $\log(l!)$  by integrals:

$$\log\left(l!\right) = \int_0^\infty \left(l - \frac{1 - e^{-ls}}{1 - e^{-s}}\right) \frac{e^{-s}}{s} ds = \int_0^1 \left(\frac{1 - (1 - t)^l}{t} - l\right) \frac{dt}{\log\left(1 - t\right)}.$$
(6)

Now using (2), (3) and (6) we get

$$H_{n,c}(x) = H_{l,-1}(-cx) = -l\left[(-cx)\log\left(-cx\right) + (1+cx)\log\left(1+cx\right)\right] + \int_{0}^{1} \frac{-t}{\log\left(1-t\right)} \frac{(1+cxt)^{l} + (1-t-cxt)^{l} - 1 - (1-t)^{l}}{t^{2}} dt.$$

It is a matter of calculus to prove that

$$\begin{aligned} H_{n,c}''(x) &= cl\left(\frac{1}{x} - \frac{c}{1+cx}\right) \\ &+ c^2l(l-1)\int_0^1 \frac{-t}{\log\left(1-t\right)}\left[(1+cxt)^{l-2} + (1-t-cxt)^{l-2}\right]dt, \end{aligned}$$

and for  $k\geq 0$ 

$$H_{n,c}^{(2k+2)}(x) = cl(2k)! \left( \frac{1}{x^{2k+1}} - \left( \frac{c}{1+cx} \right)^{2k+1} \right) + l(l-1) \dots (l-2k-1)c^{2k+2} \int_0^1 \frac{-t}{\log(1-t)} \left[ (1+cxt)^{l-2k-2} + (1-t-cxt)^{l-2k-2} \right] t^{2k} dt.$$

For 0 < t < 1 we have

$$0 < \frac{-t}{\log(1-t)} < 1,$$
(7)

so that

$$H_{n,c}^{(2k+2)}(x) \le cl(2k)! \left(\frac{1}{x^{2k+1}} - \left(\frac{c}{1+cx}\right)^{2k+1}\right) +$$
(8)

$$+l(l-1)\dots(l-2k-1)c^{2k+2}\int_0^1 \left[(1+cxt)^{l-2k-2}+(1-t-cxt)^{l-2k-2}\right]t^{2k}dt.$$

Repeated integration by parts yields

$$\int_0^1 (1+cxt)^{l-2k-2} t^{2k} dt \le \frac{(2k)!}{(l-2)(l-3)\dots(l-2k-1)(cx)^{2k}} \int_0^1 (1+cxt)^{l-2} dt,$$

and so

$$\int_0^1 (1+cxt)^{l-2k-2} t^{2k} dt \le \frac{(2k)! \left[ (1+cx)^{l-1} - 1 \right]}{(l-1)(l-2)\dots(l-2k-1)(cx)^{2k+1}}.$$
 (9)

Replacing x by  $-\frac{1}{c} - x$  we obtain

$$\int_0^1 (1-t-cxt)^{l-2k-2} t^{2k} dt \le \frac{(2k)! \left[1-(-cx)^{l-1}\right]}{(l-1)(l-2)\dots(l-2k-1)(1+cx)^{2k+1}}.$$
 (10)

From (8), (9) and (10) it follows that

$$H_{n,c}^{(2k+2)}(x) \le cl(2k)! \left[ \frac{(1+cx)^{l-1}}{x^{2k+1}} - \frac{c^{2k+1}(-cx)^{l-1}}{(1+cx)^{2k+1}} \right] \le 0,$$

and this proves (4). It is easy to verify that  $H_{n,c}^{(2k+1)}\left(-\frac{1}{2c}\right) = 0$ . Since  $H_{n,c}^{(2k+2)} \leq 0$ , it follows that  $H_{n,c}^{(2k+1)}$  is decreasing, and this implies (5).

### B. Consider the case c = 0.

 $H_{n,0}$  is the Shannon entropy of the Poisson distribution. The derivative of this function is completely monotonic: see, e.g., [2, p. 2305]. For the sake of completeness we insert here a short proof.

**Theorem 2**  $H'_{n,0}$  is completely monotonic, i.e.,

$$(-1)^k H_{n,0}^{(k+1)}(x) \ge 0, \quad k \ge 0, \quad x > 0.$$
 (11)

**Proof** Let us remark that  $H_{n,0}(y) = H_{1,0}(ny)$ ; so it suffices to investigate the derivatives of  $H_{1,0}(x)$ .

According to [10, (2.5)],

$$H_{1,0}(x) = x - x \log x + \int_0^\infty \frac{e^{-t}}{t} \left( x - \frac{1 - \exp(x(e^{-t} - 1))}{1 - e^{-t}} \right) dt$$
  
=  $x - x \log x - \int_0^1 \left( x - \frac{1 - e^{-sx}}{s} \right) \frac{ds}{\log(1 - s)}.$ 

It follows that

$$H'_{1,0}(x) = -\log x - \int_0^1 \left(1 - e^{-sx}\right) \frac{ds}{\log(1-s)}$$

and for  $k \geq 1$ ,

$$H_{1,0}^{(k+1)}(x) = (-1)^k \left( \frac{(k-1)!}{x^k} + \int_0^1 s^k e^{-sx} \frac{ds}{\log(1-s)} \right).$$
(12)

By using (7) we get

$$\int_0^1 \frac{s^k e^{-sx}}{\log(1-s)} ds \ge -\int_0^1 s^{k-1} e^{-sx} ds =$$
$$= -\int_0^x \frac{t^{k-1}}{x^k} e^{-t} dt \ge -\int_0^\infty \frac{1}{x^k} t^{k-1} e^{-t} dt = -\frac{(k-1)!}{x^k}.$$

Combined with (12), this proves (11) for  $k \ge 1$ . In particular, we see that  $H_{n,0}$  is concave and non-negative on  $[0, +\infty)$ ; it follows that  $H'_{n,0} \ge 0$  and so (11) is completely proved.

## C. Let now c > 0.

**Theorem 3** For c > 0,  $H'_{n,c}$  is completely monotonic.

**Proof** Since  $H_{m,c}(y) = H_{\frac{m}{c},1}(cy)$ , it suffices to study the derivatives of  $H_{n,1}(x)$ . By using (2), (3) and

$$\log A = \int_0^\infty \frac{e^{-x} - e^{-Ax}}{x} dx, \quad A > 0,$$

we get

$$H_{n,1}(x) = n\left((1+x)\log(1+x) - x\log x\right) + \int_0^\infty \frac{e^{-ns} - e^{-s}}{s(1-e^{-s})} \left(1 - (1+x-xe^{-s})^{-n}\right) ds$$

$$= n\left((1+x)\log(1+x) - x\log x\right) + \int_0^1 \frac{1 - (1-t)^{n-1}}{t\log(1-t)} \left(1 - (1+tx)^{-n}\right) dt.$$

It follows that, for  $j \ge 1$ ,

$$\frac{1}{n}H_{n,1}^{(j+1)}(x) = (-1)^{j-1}(j-1)!\left((x+1)^{-j} - x^{-j}\right) +$$

$$+(-1)^{j-1}(n+1)(n+2)\dots(n+j)\int_0^1 \frac{-t}{\log(1-t)} \left[1-(1-t)^{n-1}\right](1+xt)^{-n-j-1}t^{j-1}dt.$$

Using again (7), we get

$$\begin{split} &(-1)^{j-1} \frac{1}{n} H_{n,1}^{(j+1)}(x) \leq (j-1)! \left( (x+1)^{-j} - x^{-j} \right) + \\ &+ (n+1)(n+2) \dots (n+j) \int_0^1 \left[ 1 - (1-t)^{n-1} \right] (1+xt)^{-n-j-1} t^{j-1} dt \\ &= u(x) + v(x), \end{split}$$

where

$$u(x) := \frac{(j-1)!}{(x+1)^j} - (n+1)(n+2)\dots(n+j) \int_0^1 t^{j-1} (1-t)^{n-1} (1+xt)^{-n-j-1} dt,$$
$$v(x) := (n+1)(n+2)\dots(n+j) \int_0^1 t^{j-1} (1+xt)^{-n-j-1} dt - \frac{(j-1)!}{x^j}.$$

We shall prove that  $u(x) \leq 0$  and  $v(x) \leq 0$ , x > 0. Let us remark that

$$\int_0^1 t^{j-1} (1-t)^{n-1} (1+xt)^{-n-j-1} dt \ge \int_0^1 t^{j-1} (1-t)^n (1+xt)^{-n-j-1} dt, \quad (13)$$

and integration by parts yields

$$\int_0^1 \frac{t^{j-1}(1-t)^n}{(1+xt)^{n+j+1}} dt = \frac{j-1}{(n+1)(x+1)} \int_0^1 \frac{t^{j-2}(1-t)^{n+1}}{(1+xt)^{n+j+1}} dt.$$

Applying repeatedly this formula we obtain

$$\int_0^1 \frac{t^{j-1}(1-t)^n}{(1+xt)^{n+j+1}} dt = \frac{(j-1)!}{(n+1)(n+2)\dots(n+j)} \frac{1}{(x+1)^j}.$$
 (14)

Now (13) and (14) imply  $u(x) \leq 0$ . Using again integration by parts we get

$$\int_0^1 t^{j-1} (1+xt)^{-n-j-1} dt \le \frac{j-1}{(n+j)x} \int_0^1 t^{j-2} (1+xt)^{-n-j} dt$$
$$\le \dots \le \frac{(j-1)!}{(n+1)(n+2)\dots(n+j)} \frac{1}{x^j},$$

which shows that  $v(x) \leq 0$ .

We conclude that

$$(-1)^{j-1}H_{n,1}^{(j+1)}(x) \le 0, \quad j \ge 1, x > 0.$$
 (15)

In particular, (15) shows that  $H_{n,1}$  is concave on  $[0, +\infty)$ ; it is also nonnegative, which means that  $H'_{n,1} \ge 0$ . Combined with (15), this shows that  $H'_{n,1}$  is completely monotonic, and the proof is finished.

**Remark 3.1** (14) can be obtained alternatively by using the change of variables y = (1 - t)/(1 + xt) and the properties of the Beta function. An alternative proof of the inequality  $v(x) \leq 0$  follows from

$$\int_0^1 t^{j-1} (1+xt)^{-n-j-1} dt \le \frac{1}{x^{j-1}} \int_0^\infty \frac{(xt)^{j-1}}{(1+xt)^{n+j+1}} dt =$$
$$= \frac{1}{x^j} \int_0^\infty \frac{s^{j-1}}{(1+s)^{j+n+1}} ds = \frac{1}{x^j} B(j,n+1) = \frac{1}{x^j} \frac{(j-1)!n!}{(n+j)!}.$$

**Corollary 3.1** The following inequalities are valid for x > 0 and  $c \ge 0$ :

$$\log \frac{x}{cx+1} \le \sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x) \log \frac{k+1}{ck+n} \le \log \frac{nx+1}{ncx+n}.$$
 (16)

In particular, for c = 0 and n = 1,

$$\log x \le \sum_{k=0}^{\infty} e^{-x} \frac{x^k}{k!} \log (k+1) \le \log (x+1).$$

**Proof** We have seen that  $H'_{n,c}(x) \ge 0$ . An application of (1) yields

$$H'_{n,c}(x) = n\left(\log\frac{1+cx}{x} + \sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x)\log\frac{k+1}{n+ck}\right).$$

This proves the first inequality in (16); the second is a consequence of Jensen's inequality applied to the concave function  $\log t$ .

## **3** Rényi entropy and Tsallis entropy

The following conjecture was formulated in [13]:

Conjecture 3.1  $S_{n,-1}$  is convex on [0,1].

Th. Neuschel [11] proved that  $S_{n,-1}$  is decreasing on  $\left[0,\frac{1}{2}\right]$  and increasing on  $\left[\frac{1}{2},1\right]$ . The conjecture and the result of Neuschel can be found also in [5].

A proof of the conjecture was given by G. Nikolov [12], who related it with some new inequalities involving Legendre polynomials. Another proof can be found in [4].

Using the important results of Elena Berdysheva [3], the following extension was obtained in [17]:

**Theorem 4** ([17, Theorem 9]). For c < 0,  $S_{n,c}$  is convex on  $\left[0, -\frac{1}{c}\right]$ .

A stronger conjecture was formulated in [14] and [17]:

**Conjecture 4.1** For  $c \in \mathbb{R}$ ,  $S_{n,c}$  is logarithmically convex, i.e.,  $\log S_{n,c}$  is convex.

It was validated for  $c \ge 0$  by U. Abel, W. Gawronski and Th. Neuschel [1], who proved a stronger result:

**Theorem 5** ([1]). For  $c \ge 0$ , the function  $S_{n,c}$  is completely monotonic, *i.e.*,

$$(-1)^m \left(\frac{d}{dx}\right)^m S_{n,c}(x) > 0, \quad x \ge 0, m \ge 0.$$

Consequently, for  $c \geq 0$ ,  $S_{n,c}$  is logarithmically convex, and hence convex.

Summing up, for the Rényi entropy  $R_{n,c} = -\log S_{n,c}$  and Tsallis entropy  $T_{n,c} = 1 - S_{n,c}$ , we can state

- **Corollary 5.1** i) Let  $c \ge 0$ . Then  $R_{n,c}$  is increasing and concave, while  $T'_{n,c}$  is completely monotonic on  $[0, +\infty)$ .
- ii)  $T_{n,c}$  is concave for all  $c \in \mathbb{R}$ .

#### Proof

- i) Apply Theorem 5.
- ii) For c < 0, apply Theorem 4. For  $c \ge 0$ , Theorem 5 shows that  $S_{n,c}$  is convex, so that  $T_{n,c}$  is concave.

**Remark 5.1** As far as we know, Conjecture 4.1 is still open for c < 0, so that the concavity of  $R_{n,c}$ , c < 0, remains to be investigated.

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