On two supercongruences of double binomial sums

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Abstract. In this note, we confirm two conjectural supercongruences on double sums of binomial coefficients due to El Bachraoui.

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1 Introduction

Recall that the q-shifted factorials are given by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \ge 1$ and $(a;q)_0 = 1$, and the q-integers are defined by $[n] = (1-q^n)/(1-q)$. For polynomials $A_1(q), A_2(q), P(q) \in \mathbb{Z}[q]$, the q-congruence

$$A_1(q)/A_2(q) \equiv 0 \pmod{P(q)}$$

is understood as $A_1(q)$ is divisible by $P(q)$ and $A_2(q)$ is coprime with $P(q)$. In general, for rational functions $A(q), B(q) \in \mathbb{Z}(q)$,

$$A(q) \equiv B(q) \pmod{P(q)} \Longleftrightarrow A(q) - B(q) \equiv 0 \pmod{P(q)}.$$

In the past few years, q-congruence for sums of binomial coefficients as well as hypergeometric series attracted many experts' attention (see, for instance, [1–7]). In particular, Guo and Zudilin [5] developed a creative microscoping method to prove many interesting q-congruences, such as

$$\sum_{k=0}^{n-1}(-1)^k\frac{(q;q^2)_k(-q;q^2)_k^2}{(q^4;q^4)_k(-q^4;q^4)_k^2}[6k+1]q^{3k^2} \equiv 0 \pmod{[n]}, \tag{1.1}$$

and

$$\sum_{k=0}^{n-1}\frac{(q^2;q^4)_k(-q;q^2)_k^2}{(q^4;q^4)_k(-q^4;q^4)_k^2}[6k+1]q^{k^2} \equiv 0 \pmod{[n]}, \tag{1.2}$$

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for any odd positive integer n.

Motivated by (1.1) and (1.2), El Bachraoui [1, Theorems 1 and 2] established the following two q-congruences:

$$\sum_{k=0}^{n-1}\sum_{j=0}^{k} c_q(j)c_q(k-j) \equiv 0 \pmod{[n]}, \tag{1.3}$$

and

$$\sum_{k=0}^{n-1}\sum_{j=0}^{k} c'_q(j)c'_q(k-j) \equiv 0 \pmod{[n]}, \tag{1.4}$$

where $c_q(k)$ and $c'_q(k)$ denote the k-th term of the summations on the left-hand sides of (1.1) and (1.2), respectively.

Suppose p is an odd prime. Letting $q \to 1$ and $n = p$ in (1.3) and (1.4) gives

$$\sum_{k=0}^{p-1}\left(-\frac{1}{8}\right)^k \sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1) \equiv 0 \pmod{p}, \tag{1.5}$$

and

$$\sum_{k=0}^{p-1}\left(\frac{1}{4}\right)^k \sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1) \equiv 0 \pmod{p}. \tag{1.6}$$

El Bachraoui [1, Conjectures 1 and 2] also conjectured two extensions of (1.5) and (1.6) as follows:

Conjecture 1.1 *For any odd prime p, we have*

$$\sum_{k=0}^{p-1}\left(-\frac{1}{8}\right)^k \sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1) \equiv -\frac{p}{2} \pmod{p^2}, \tag{1.7}$$

and

$$\sum_{k=0}^{p-1}\left(\frac{1}{4}\right)^k \sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1) \equiv p \pmod{p^2}. \tag{1.8}$$

The above two conjectural supercongruences motivate us to establish the following more general result, which includes (1.7) and (1.8) as special cases.

Theorem 1.2 *For any positive integer n, we have*

$$\sum_{k=0}^{n-1}\left(\frac{q}{4}\right)^k \sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1)$$
$$= \frac{(9n^2-15n+8)q^{n+2}-(18n^2-12n-8)q^{n+1}+(9n^2+3n+2)q^n-2(2q+1)^2}{2(1-q)^3}. \tag{1.9}$$

Letting $q \to -\frac{1}{2}$ and $q \to 1$ in (1.9), we obtain the following two combinatorial identities:

Corollary 1.3 *For any positive integer n, we have*

$$\sum_{k=0}^{n-1} \left(-\frac{1}{8}\right)^k \sum_{j=0}^{k} \binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1) = \left(-\frac{1}{2}\right)^n n(1-3n), \tag{1.10}$$

and

$$\sum_{k=0}^{n-1} \left(\frac{1}{4}\right)^k \sum_{j=0}^{k} \binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1) = \frac{n(3n^2-3n+2)}{2}. \tag{1.11}$$

It is clear that (1.7) and (1.8) can be deduced from (1.10) and (1.11) directly. We shall prove Theorem 1.2 in the next section.

2 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following preliminary result.

Lemma 2.1 *For any non-negative integer n, we have*

$$\sum_{j=0}^{k} \binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1) = 4^k\left(\frac{9}{2}k^2+\frac{3}{2}k+1\right). \tag{2.1}$$

Proof. Note that central binomial coefficients possess the following generating function:

$$\frac{1}{\sqrt{1-4x}} = \sum_{j=0}^{\infty} \binom{2j}{j} x^j. \tag{2.2}$$

On the other hand,

$$\frac{1}{1-4x} = \sum_{k=0}^{\infty} (4x)^k.$$

Thus,

$$\left(\sum_{j=0}^{\infty} \binom{2j}{j} x^j\right)^2 = \sum_{k=0}^{\infty} (4x)^k. \tag{2.3}$$

Comparing the coefficient of x^k on both sides of (2.3), we obtain

$$\sum_{j=0}^{k} \binom{2j}{j}\binom{2k-2j}{k-j} = 4^k. \tag{2.4}$$

Differentiating both sides of (2.2) respect to x, we obtain

$$\frac{2x}{\sqrt{(1-4x)^3}} = \sum_{j=0}^{\infty} j\binom{2j}{j}x^j. \tag{2.5}$$

On the other hand,

$$\frac{4x^2}{(1-4x)^3} = 4x^2\sum_{k=0}^{\infty}\binom{-3}{k}(-4x)^k.$$

Since

$$\binom{-3}{k} = \frac{(-1)^k(k+1)(k+2)}{2},$$

we have

$$\frac{4x^2}{(1-4x)^3} = \sum_{k=0}^{\infty}\frac{4^{k+1}(k+1)(k+2)}{2}x^{k+2}. \tag{2.6}$$

Noting (2.5) and (2.6), we find that

$$\left(\sum_{j=0}^{\infty} j\binom{2j}{j}x^j\right)^2 = \sum_{k=0}^{\infty}\frac{4^{k+1}(k+1)(k+2)}{2}x^{k+2}. \tag{2.7}$$

Comparing the coefficient of x^k on both sides of (2.7), we obtain

$$\sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j}j(k-j) = \frac{4^k k(k-1)}{8}. \tag{2.8}$$

Finally, using (2.4) and (2.8) we arrive at

$$\begin{aligned}
&\sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1)\\
&= (6k+1)\sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j} + 36\sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j}j(k-j)\\
&= 4^k\left(\frac{9}{2}k^2+\frac{3}{2}k+1\right),
\end{aligned}$$

as desired. $\square$

Proof of Theorem 1.2. By (2.1), we have

$$\begin{aligned}
&\sum_{k=0}^{n-1}\left(\frac{q}{4}\right)^k\sum_{j=0}^{k}\binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1)\\
&=\sum_{k=0}^{n-1}q^k\left(\frac{9}{2}k^2+\frac{3}{2}k+1\right). \qquad (2.9)
\end{aligned}$$

Let

$$S_n=\sum_{k=0}^{n-1}q^k k \quad \text{and} \quad T_n=\sum_{k=0}^{n-1}q^k k^2.$$

Note that

$$\begin{aligned}
(1-q)S_n&=\sum_{k=0}^{n-1}q^k k-\sum_{k=0}^{n-1}q^{k+1}k\\
&=\sum_{k=1}^{n-1}q^k k-\sum_{k=1}^{n}q^k(k-1)\\
&=\sum_{k=1}^{n-1}q^k k-\sum_{k=1}^{n-1}q^k(k-1)-q^n(n-1)\\
&=\sum_{k=1}^{n-1}q^k-q^n(n-1).
\end{aligned}$$

Thus,

$$S_n=\frac{q(1-q^{n-1})}{(1-q)^2}-\frac{q^n(n-1)}{1-q}. \qquad (2.10)$$

On the other hand,

$$\begin{aligned}
(1-q)T_n&=\sum_{k=0}^{n-1}q^k k^2-\sum_{k=0}^{n-1}q^{k+1}k^2\\
&=\sum_{k=1}^{n-1}q^k k^2-\sum_{k=1}^{n}q^k(k-1)^2\\
&=\sum_{k=1}^{n-1}q^k k^2-\sum_{k=1}^{n-1}q^k(k-1)^2-q^n(n-1)^2\\
&=\sum_{k=1}^{n-1}q^k(2k-1)-q^n(n-1)^2. \qquad (2.11)
\end{aligned}$$

Combining (2.10) and (2.11), we obtain

$$T_n = \frac{2q(1-q^{n-1})}{(1-q)^3} - \frac{2q^n(n-1)}{(1-q)^2} - \frac{q(1-q^{n-1})}{(1-q)^2} - \frac{q^n(n-1)^2}{1-q}. \tag{2.12}$$

Finally, substituting (2.10) and (2.12) into (2.9), we arrive at

$$\sum_{k=0}^{n-1} \left(\frac{q}{4}\right)^k \sum_{j=0}^{k} \binom{2j}{j}\binom{2k-2j}{k-j}(6j+1)(6k-6j+1)$$
$$= \frac{(9n^2-15n+8)q^{n+2} - (18n^2-12n-8)q^{n+1} + (9n^2+3n+2)q^n - 2(2q+1)^2}{2(1-q)^3},$$

as desired. □

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References

[1] M. El Bachraoui, On supercongruences for truncated sums of squares of basic hypergeometric series, Ramanujan J., online, doi: 10.1007/s11139-019-00226-0.

[2] V.J.W. Guo, Proof of some q-supercongruences modulo the fourth power of a cyclotomic polynomial, Results Math. 75 (2020), Art. 77.

[3] V.J.W. Guo and J.-C. Liu, q-Analogues of two Ramanujan-type formulas for $1/\pi$, J. Difference Equ. Appl. 24 (2018), 1368–1373.

[4] V.J.W. Guo and M. J. Schlosser, Some new q-congruences for truncated basic hypergeometric series: even powers, Results Math. 75 (2020), Art. 1.

[5] V.J.W. Guo and W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.

[6] J.-C. Liu and Z.-Y. Huang, A truncated identity of Euler and related q-congruences, Bull. Aust. Math. Soc., online, doi:10.1017/S0004972720000301.

[7] J.-C. Liu and F. Petrov, Congruences on sums of q-binomial coefficients, Adv. in Appl. Math. 116 (2020), 102003.