Path-pairability of infinite planar grids

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Abstract

For a given integer $k \ge 1$, a graph *G* with at least 2*k* vertices is called *k*-path-pairable, if for any set of *k* disjoint pairs of vertices, $s_i, t_i, 1 \le i \le k$, there exist pairwise edge-disjoint s_i, t_i -paths in *G*. The path-pairability number is the largest *k* such that *G* is *k*-path-pairable. Bounds on the path-pairability number are given here if *G* is the graph of infinite integer grids in the Euclidean plane. We prove that the path-pairability number of the integer quadrant is 4, and we show that the integer half-plane is 6-path-pairable and at most 7-path-pairable.

Keywords Linkage · Infinite grid · Path-pairability · Interconnection networks

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1 Introduction

Problems concerning edge-disjoint paths have several applications in the areas of routing, wavelength assignment, call admission control, point-to-point delivery, real-time communication, etc. (see [5]). In VLSI layout and reconfiguration problems the linkage properties are studied on a planar grid network, that is the underlying graph of the model is the Cartesian product of paths (see [1]). The general concept of path-pairability we are dealing with here is actually originated in a practical problem concerning telecommunications network as first described in [2]. Such a data- or telephone network is a collection of terminals (hosts), transmission links, and intermediate nodes which are assembled so as to enable simultaneous communication between pairs of terminals.

For a given integer $k \ge 1$, a graph G with at least 2k vertices is *k*-path-pairable, if for any set of k disjoint pairs of vertices s_i , t_i , $1 \le i \le k$, called *terminals*, there exist pairwise edge-

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disjoint s_i , t_i -paths in G. The *path-pairability number*, denoted pp(G), is the largest k such that G is k-path-pairable. A summary of early results concerning path-pairability was given in [3], recent papers witness a renewed interest in linkage properties of Cartesian products of graphs (see [10–12]). The parameter pp(G) was investigated recently in [6,7], for finite and infinite grid graphs G, that is, for the Cartesian product of finite or infinite paths.

We use here \mathbb{N} and \mathbb{Z} to denote the one-way and the two-way infinite path on the natural numbers and on the integers, respectively, with an edge between *i* and *j* if and only if |i - j| = 1. The Cartesian product of two of these infinite paths lead to three kinds of infinite grid, $\mathbb{Z} \square \mathbb{Z}$ (integer plane), $\mathbb{N} \square \mathbb{Z}$ (integer half-plane), and $\mathbb{N} \square \mathbb{N}$ (integer quadrant). (Between vertices (p, q) and (i, j) of a product graph there is an edge if and only if |p-i|+|q-j|=1.)

In [6,8] the path-pairability number of the finite rectangular grids, the Cartesian product $P_a \square P_b$, was determined for every pair of positive integers $a, b \ge 2$. From the meticulous proof of $pp(P_6 \square P_6) = 4$ in [8] we immediately obtain the path-pairability number of $\mathbb{N} \square \mathbb{N}$. In Sect. 4 a much shorter transparent proof is presented for $pp(\mathbb{N} \square \mathbb{N}) = 4$. We use two technical lemmas (Lemmas 4.1 and 4.2) whose detailed proofs are given in [9].

Determining the path-pairability number of the integer half-plane and the integer plane remain challenging open problems. The bounds $10 \le pp(\mathbb{Z} \square \mathbb{Z}) \le 14$ were obtained in [7]; here in Sect. 3 we prove $6 \le pp(\mathbb{N} \square \mathbb{Z}) \le 7$ by using similar methods. The lower bound proofs are based on Hall's matching theorem, the upper bound constructions violate cut conditions as explained in Sect. 2. Conjectures on the value of these path-pairability numbers, supported/tested by exhaustive computer search, are proposed in Sect. 5

2 Proof strategies

To prove that a graph *G* is not *k*-path-pairable, we need to exhibit a counterexample consisting of *k* terminal pairs that does not admit edge disjoint linkage. For the proof that no linkage exists we use a necessary condition, called *cut condition*¹ and its variations. As an example, the bound $pp(\mathbb{Z} \square \mathbb{Z}) < 19$ follows from a pairing where one terminal from each pair is located at the vertices of a 4 × 5 grid *R* and the second members of the pairs are located anywhere in $(\mathbb{Z} \square \mathbb{Z}) - R$. Indeed, if there was a linkage for the 19 pairs, 19 edge disjoint paths must exit *R*, but there are only 18 edges between *R* and $(\mathbb{Z} \square \mathbb{Z}) - R$. We might say, there is no edge disjoint linkage because the cut condition is violated by this particular pairing. Sharper upper bounds can be obtained by using stronger necessary conditions derived from the cut condition. The best bound $pp(\mathbb{Z} \square \mathbb{Z}) < 15$ that we know so far was obtained in this way in [7].

Let *G* be an infinite grid whose vertices are arranged into 'vertical' and 'horizontal' infinite lines (or half-lines). For a proof that $pp(G) \ge k$, let $\pi_i = \{s_i, t_i\}$ be an arbitrary *pairing* of 2k terminals in *G*. Denote by *T* the set of those 2k terminals and include *T* into a (smallest) rectangle $R \subset G$. An edge disjoint linkage of the *k* pairs can be obtained by a general two-stage procedure.

The first stage, the essential part of the proofs, is slightly different for each of the three infinite grids $\mathbb{Z} \square \mathbb{Z}$, $\mathbb{N} \square \mathbb{Z}$, and $\mathbb{N} \square \mathbb{N}$. First we take the edge disjoint linkage of some pairs inside *R*; then every unlinked terminal $u \in T$ is 'mapped' to a vertical or horizontal half-line $\ell(u)$ through *u* going out from *R* in such a way that all these half-lines are pairwise edge-

¹ If the vertex set of G is partitioned into two sets, the number of edges of G between the sets is not smaller than the cardinality of each of these sets.

disjoint and disjoint from the paths used by the linkage inside R. To obtain such a mapping Hall's matching theorem (see [4]) will be used.

The second stage is common for each of the three infinite grids. We take k pairwise disjoint finite paths F_i in the grid, $1 \le i \le k$, each intersecting all the half-lines assigned to unlinked terminals by the mapping in the first stage. For every unlinked terminal $u \in \pi_i$, let $u^* \in \ell(u) \cap F_i$; then the linkage for any unlinked pair π_i can be completed along F_i between s_i^* and t_i^* .

A humble application of Hall's matching theorem in the first stage would lead to small k, a plain lower bound on the path-pairability number of the grid. In order to achieve larger k, the application of Hall's matching theorem will be preceded with an investigation of exceptional terminal pairings, a few annoying cases called here 'obstructions', when Hall's condition is violated.

In order to handle the linkage of terminals of an obstruction, the general procedure is refined as follows. Besides linking a pair inside R we are also allowed to 'move' a terminal into a 'mate' at a more favorable locations along pairwise edge-disjoint 'mating paths' in R. The first stage of the refined procedure starts with the edge disjoint linkage of some pairs inside R, then half-lines 'sticking out' from R are assigned to the unlinked terminals. The head vertices of these half-lines located on the boundary of R will serve as mates; to conclude the first stage edge-disjoint mating paths are built from each unlinked terminal to the head vertex of an appropriate half-line. In this way the unlinked terminals 'exit' through the half-lines headed at their mates, and those pairs leaving R become linked in the standard second stage as described above. The lower bound $pp(\mathbb{N} \square \mathbb{Z}) \ge 6$ in Proposition 3.2 is obtained in this way.

The process of selecting the pool of mates and the strategy of building non-conflicting mating paths from terminals to mates is tailored to the underlying grids under study and depends on specific conditions different from case to case (see [6–10]).

3 The integer half-plane

In this section we prove the bounds $6 \le pp(\mathbb{N} \square \mathbb{Z}) \le 7$.

Proposition 3.1 *The grid* $\mathbb{N} \square \mathbb{Z}$ *is not* 8-*path-pairable.*

Proof Let A(i), i = 1, 2, ..., denote the horizontal lines (also called rows) of the grid $G = \mathbb{N} \square \mathbb{Z}$ from top down. Let $R \subset A(1) \cup A(2)$ be a 2 × 5 rectangle; locate 10 terminals $s_1, t_1, s_2, t_2, s_i, 3 \le i \le 8$, in R as seen in Fig. 1, and let $t_j, 3 \le j \le 8$, be placed arbitrarily in G - R.

Assume there is a linkage in *G* for each pair π_i , $1 \le i \le 8$. Observe that inside *R* there is a linkage either for π_1 or for π_2 , but not for both. Therefore, at least eight terminals must leave *R* using the nine exits (two exits on *A*(1) and *A*(2), and one exit on each vertical half-line). By symmetry, we may assume that the s_1 , t_1 -path P_1 of the solution is contained by *R*. Observe that between $\{s_1, s_2\}$ and $R \setminus \{s_1, s_2\}$ there are two edges, and one is used by P_1 . Thus at most two among the three exits at s_1 , s_2 can be accessed by the remaining terminals. The same is true considering the three exits at t_1 , t_2 . Hence only seven terminals are able to leave *R*, a contradiction.

The vertices of a $p \times q$ rectangular grid R are considered as entries of a $p \times q$ matrix, and (i, j) $1 \le i \le p$, $1 \le j \le q$, denotes the vertex in row i and column j. The rows and the

Fig.1 $pp(\mathbb{N} \square \mathbb{Z}) < 8$



Proposition 3.2 *The grid* $\mathbb{N} \square \mathbb{Z}$ *is* 6-*path-pairable.*

Proof Let *T* be a set of at most 12 terminals in the grid $G = \mathbb{N} \square \mathbb{Z}$; let L(T) be the set of all horizontal lines and vertical half-lines in *G* containing a terminal. We wish to exhibit a mapping $\psi : T \longrightarrow L(T)$ such that $u \in \psi(u)$, for every $u \in T$, furthermore, at most one terminal is mapped to each vertical half-line of L(T), and at most two terminals are mapped to each horizontal line of L(T). Let $S \subseteq T$ and assume that L(S) contains *p* horizontal lines and *q* vertical half-lines. The existence of the mapping ψ would follow from Hall's condition provided

$$|S| \le \min\{12, pq\} \le 2p + q \tag{1}$$

holds for every $S \subseteq T$.

A grid formed by any p rows and any q columns of G, and holding a set $S \subseteq T$ of terminals is considered to be an obstruction, provided |S| > 2p + q. We may successively reduce an obstruction to a minimal one as follows. If one or two terminals are contained in a row, then they are mapped into one half-line or two disjoint half-lines emanating from these terminals; if a single terminal is contained by a column, then it is mapped to the vertical half-line emanating from this terminal. After discarding the terminals and the assigned rows and columns the procedure is repeated until an obstruction is reduced to minimal.

Thus a (minimal) *obstruction* is defined as a $p \times q$ rectangle *R* containing a set of terminals, $S \subseteq T$, such that each row has at least 3 terminals and each column has at least 2 terminals, furthermore, |S| > 2p + q. In particular, an obstruction satisfies $2 \le p \le 4$ and $3 \le q \le 6$.

Case a: p = 2. Let R_q be a minimal $2 \times q$ obstruction containing a set S_q of terminals. Each column of R_q has 2 terminals, thus we have $|S_q| = 2q \le |T| = 12$. Since $|S_q| = 2q \ge 1 + (2p + q) = q + 5$, it follows that q = 5 or 6. Let $R \subset R_q$ be a 2 × 4 grid. Since T contains 6 pairs, and ||R|| = 8, there is a pair $\pi_1 \subset R$, by the pigeon hole principle.

We label the vertices of R along the 8-path $Q^* = (v_1, v_2, ..., v_7, v_8)$ as in the top left in Fig. 2. Let $P_1 \subset Q^*$ be the (unique) linkage for π_1 along Q^* . For j = 1, 8 we define $Q_j \subset Q^*$ to be the mating path (possibly of length 0) from the terminal at v_j to the closest end vertex of P_1 . Then we assign to the unlinked terminals and mates the vertical half-lines at v_i , for $3 \le i \le 6$, and the two disjoint horizontal half-lines at v_2 and at v_7 (see the top right in Fig. 2).

For q = 5, a mapping described above for a 2 × 4 subgrid $R \subset R_5$ can be extended with a vertical half-line disjoint from R and a horizontal half-line in the second row of R assigned to the two terminals not in R. For q = 6, let $R \subset R_6$ be the 2 × 4 grid obtained by the





Fig. 2 Exit a 2×4 rectangle and exit a 2×6 obstruction



Fig. 3 Solving the 4×3 obstruction

removal of the first and last column from R_6 . Then the mapping described for $R \subset R_6$ can be extended with two vertical half-lines disjoint from R and two horizontal half-lines in the second row of R_6 assigned to the four terminals not in R (see the third picture in Fig. 2).

Case b: q = 3 and $p \ge 3$. Let R_p be the $p \times 3$ minimal obstruction containing a set S_p of terminals. Since $12 \ge |S_p| = 3p \ge 1 + 2p + q = 2p + 4$, we obtain p = 4. Thus we assume that the only minimal obstruction is formed by 12 terminals located at the vertices of a 4 × 3 rectangle *R*. The strategy of handling the obstruction is similar to the one in Case a.

We label the vertices of $R \setminus A(4)$ along the 8-path $Q^* = (v_1, v_2, \ldots, v_7, v_8)$ as on the left in Fig. 3. There is a pair $\pi_1 \subset Q^*$, by the pigeon hole principle. Let $P_1 \subset Q^*$ be the (unique) linkage for π_1 along Q^* . For j = 1, 8, we define $Q_j \subset Q^*$ to be the mating path (possibly of length 0) from the terminal at v_j to the closest end vertex of P_1 . Then we assign to the unlinked terminals and mates in Q^* the vertical half-line at v_5 , and the disjoint horizontal half-lines at v_i for $2 \le i \le 7$, $i \ne 5$. Terminals (4, 1), (4, 3) are assigned to vertical half-lines, terminals (1, 3), (4, 2) exit along horizontal half-lines going right, each emanating from them (see examples in Fig. 3).

Case c: p = 3, q = 4, and |S| = 11 or 12. Let *R* be the 3×4 rectangle with a terminal at each vertex except possible one position. The strategy is similar to the ones in Case a and b.



L(w)

A(3)

 \overline{u}

Fig. 4 Solving 3×4 obstructions

Fig. 5 Notations in Case d

We label the vertices of $R \setminus A(3)$ along the 8-path $Q^* = (v_1, v_2, \ldots, v_7, v_8)$, see the left of Fig. 4. There is a pair $\pi_1 \subset Q^*$ by the pigeon hole principle. Let $P_1 \subset Q^*$ be the (unique) linkage for π_1 along Q^* . For j = 1, 8, we define $Q_j \subset Q^*$ to be the mating path from the terminal at v_j to the closest end vertex of P_1 . Then we assign to the unlinked terminals and mates in Q^* the vertical half-lines at v_4, v_5 , and the disjoint horizontal half-lines emanating from v_i for $i \in \{2, 3, 6, 7\}$. Vertical half-lines are assigned to terminals (3, 1), (3, 4), and terminals (3, 2), (3, 3) exit along disjoint horizontal half-lines emanating from them (see examples in Fig. 4).

There remain several minimal obstructions with parameters p = 3, q = 5, and |S| = 12. Each column of a 3×5 rectangle *R* contains at most one, and each row of *R* contains at most two terminal-free vertices we will call *holes*. These obstructions are handled next in Case d and Case e, according to the number of holes in row A(3) of *R*.

Let $M = \{(i, j) \mid 1 \le i \le 2 \le j \le 4\}$ be the middle rectangle containing the six interior vertices of *R*. A vertex $w \in R \setminus M$ is called a boundary vertex; each boundary vertex is the head of a half-line L(w) sticking out from *R*. A terminal located at a boundary vertex of *R* is called a boundary terminal.

Case d: ||A(3)|| = 3. Since A(3) contains two holes, there is a third hole in $R \setminus A(3)$. Depending on the location of the third hole, there are 6 or 7 boundary terminals to be associated with some of the 11 horizontal or vertical half-lines emanating from the boundary vertices. Then there remain 5 or 4 unassigned half-lines available to escape terminals from M. Our strategy consists of finding a linkage for an appropriate pair $\pi_1 \subset R \setminus A(3)$ inside $R \setminus A(3)$ and 'move down' (at most) 4 further terminals to their mates $m_i \in A(3)$, $1 \le i \le 4$, (see Fig. 5).

First we discuss how to distribute the 7 horizontal and vertical half-lines at the vertices of A(3) among the three terminals $t_1, t_2, t_3 \in A(3)$ and four fixed mate positions $m_1, m_2, m_3, m_4 \in A(3)$ not necesserly distinct from terminals (indexing is from left to right along A(3)). Terminals t_1 and t_3 are mated along A(3) to be associated with disjoint horizontal half-lines emanating from (3, 1) and (3, 5), respectively. Furthermore, the vertical half-line at m_i is associated to the terminal mated with m_i , for every $1 \le i \le 4$. We are able to finish the assignment of the half-lines in two particular cases as follows.

R.

М

 m_i



Fig. 6 Assigning half-lines to terminals and mates in A(3)



Fig. 7 $\pi_1 \subset B(2) \cup B(3)$

- (i) If $(3, 3) \neq m_i$ (i = 2, 3), then t_2 is mated to (3, 3) and is associated with the vertical half-line at (3, 3) (see Fig. 6 (i)).
- (ii) If $t_1 = (3, 1)$ and $(3, 1) \neq m_1$, then t_2 is mated to (3, 1) and the vertical half-line at (3, 1) is assigned to t_2 (see Fig. 6 (ii)).

(d.1) Assume there is a pair in M, say $\pi_1 \subset M$. By symmetry, we have either $\pi_1 \subset B(2)$, or $\pi_1 \subset B(2) \cup B(3)$, or $\pi_1 \subset B(2) \cup B(4)$. If $\pi_1 \subset B(2)$ or $\pi_1 \subset B(2) \cup B(3)$, there are five patterns for the 4 unlinked terminals. Mating for the terminals in B(4) is common for them: $(1, 4) \rightsquigarrow (3, 5)$ and $(2, 4) \rightsquigarrow (3, 4)$, where the symbol $t \rightsquigarrow w$ means the mating of the terminal at location $t \in M$ into a mate at location $w \in A(3)$.

The mating paths for each is shown in Fig. 7 (the top left picture shows the solution for $\pi_1 \subset B(2)$, the other ones are shown without repeating the identical columns B(4), B(5)).

For $\pi_1 \subset B(2) \cup B(4)$ the terminals are mated according to $(1, 3) \rightsquigarrow (3, 1), (2, 3) \rightsquigarrow (3, 5)$, furthermore, $(i, j) \rightsquigarrow (3, j)$, for i = 1, 2, j = 2, 4, and $(i, j) \notin \pi_1$. By symmetry, π_1 has three different locations, the mating paths are shown in Fig. 8. In each case (3, 3) is not assigned as a mate, thus the assignment satisfies (i).

(d.2) There is no terminal pair in M, and M contains no hole. By symmetry, we may assume that the only hole in $R \setminus A(3)$ is $h \in \{(1, 1), (2, 1)\}$. By the pigeon hole principle, there is a terminal pair (s_1, t_1) among the seven terminals of $(R \setminus (A(3) \cup B(3)))$, furthermore, $s_1 \in \{(2, 1), (1, 1)\}$ and $t_1 \in M$. Notice that (3, 1) is a terminal.

The mating plan for the case when $s_1, h \in B(1) \setminus A(3)$ is as follows



Fig. 10 $s_1 \in B(1)$ and $t_1, h \in M$

 $(1, 2) \rightsquigarrow (1, 1), (1, 3) \rightsquigarrow (3, 5), (1, 3) \rightsquigarrow (1, 1), (1, 4) \rightsquigarrow (3, 5),$ $(i, j) \rightsquigarrow (3, j)$ for i = 1, 2, j = 2, 3, 4 and $(i, j) \neq t_1$.

Notice that the left horizontal half-line emanating from (1, 1) is unused, because $(1, 1) \in \{s_1, h\}$, hence this location is an available mate in addition to those in A(3). In each case the mating satisfies the half-line assignment (ii). The mating paths for the 12 possible patterns of π_1 are very similar, Fig. 9 shows two examples.

(d.3) Assume there is no terminal pair in M, and $h \in M$ is a hole. The strategy is similar to (d.2), it starts with linking a pair $\pi_1 = (s_1, t_1)$ such that $t_1 \in M$ and $s_1 \in (B(1) \cup B(5)) \setminus A(3)$. We mate the other 4 terminals from M to $A(3) \setminus \{(3, 3)\}$, in order to satisfy the assignment in (i).

If $h \in B(j)$, $2 \le j \le 4$, then w.l.o.g. we may assume that there exists a pair $\pi_1 = (s_1, t_1)$ such that $s_1 \in \{(1, 1), (2, 1)\}$ and $t_1 \in (R \setminus B(j)) \cap M$, since $||(R \setminus B(j)) \cap M|| = 4$ and ||A(3)|| = 3. Let $t_1 \in B(k)$, $j \ne k$, and set $\ell \in \{2, 3, 4\} \setminus \{j, k\}$. Notice that M has two terminals in $B(\ell)$ to mate, and one terminal from each of B(j) and B(k). According to the values $\ell = 2, 3$, or 4, we propose the following mating plans, which have realizations with mating paths in each case.

For $\ell = 2$: $(1, 2) \rightsquigarrow s_1, (2, 2) \rightsquigarrow (3, 2),$

- $(i, 3) \rightsquigarrow (3, 5)$, where i = 1, 2 and $(i, 3) \notin \{h, t_1\}$,
- $(i, 4) \rightsquigarrow (3, 4)$, where i = 1, 2 and $(i, 4) \notin \{h, t_1\}$.

For $\ell = 3$: (1, 3) $\rightsquigarrow s_1$, (2, 3) \rightsquigarrow (3, 5), (*i*, 2) \rightsquigarrow (3, 2), where i = 1, 2 and (*i*, 2) $\notin \{h, t_1\}$, (*i*, 4) \rightsquigarrow (3, 4), where i = 1, 2 and (*i*, 4) $\notin \{h, t_1\}$. For $\ell = 4$: (1, 4) \rightsquigarrow (3, 5), (2, 4) \rightsquigarrow (3, 4), (*i*, 2) \rightsquigarrow (3, 2), where i = 1, 2 and (*i*, 2) $\notin \{h, t_1\}$, (*i*, 3) $\rightsquigarrow s_1$, where i = 1, 2 and (*i*, 3) $\notin \{h, t_1\}$.

Figure 10 shows examples of realizing the mating plan, two for each value of ℓ . This concludes the proof of Case d when ||A(3)|| = 3. For the proof of the last case $||A(3)|| \ge 4$ we need a technical lemma.

Lemma 3.3 Let $D \subset \mathbb{N} \square \mathbb{Z}$ be a 2 × 4 rectangle of eight terminals labeled as below.



Assume that there is a pair π_1 among the terminals of *D*. If $y_0 \notin \pi_1$, then there is a linkage in *D* for π_1 , and there exist edge disjoint mating paths mating the other six terminals to the head of distinct horizontal and vertical half-lines exiting *D* subject to either one of the following restrictions:

- (A) The horizontal left half-line at x_0 is not assigned to any terminal.
- (B) The horizontal half-lines at y₀ and y₃ are not assigned to any terminal, furthermore, edges x₁y₁ and x₂y₂ are not used by the linkage and by any mating paths.
- **Proof** (A) If $x_0 \notin \pi_1$, then we assign to the terminal x_0 the vertical half-line at x_0 and we assign to y_0 the left horizontal half-line at y_0 . To complete a solution as required by (A) we consider the 6-path $Q^* = (x_1, y_1, y_2, y_3, x_3, x_2)$ and let $P_1 \subset Q^*$ be the (unique) linkage in Q^* for π_1 .

For $x_j \notin \pi_1$, j = 1, 2, we define $Q_j \subset Q^*$ to be the mating path from the terminal at x_j to the closest end vertex of P_1 . Then we assign to the four terminals and mates in $Q^* - \pi_1$ the three vertical half-lines at y_i , for $1 \le i \le 3$, and the horizontal right half-line at x_3 (see examples in Fig. 11).

Among the six possible cases when $s_1 = x_0$ and $t_1 = x_i$ or y_i , $1 \le i \le 3$, we show a solution for $t_1 \in \{y_1, x_2, y_2\}$ in Fig. 12. Obvious variations of these examples by extending or shortening P_1 lead to a solution for $t_1 \in \{x_1, x_3, y_3\}$.

(B) Let $Q^* = (x_1, x_0, y_0, y_1, y_2, y_3, x_3, x_2)$ and let $P_1 \subset Q^*$ be the (unique) linkage for π_1 along Q^* . For $x_j \notin \pi_1$, j = 1, 2, we define $Q_j \subset Q^*$ to be the mating path from the terminal at x_j to the closest end vertex of P_1 . Then we assign to the six terminals and mates the four vertical half-lines at y_i , $0 \le i \le 3$, and the two disjoint horizontal half-lines at x_0 and at x_3 (see examples in Fig. 13).

Case e: $||A(3)|| \ge 4$. Assume that each row of the 3×5 rectangle *R* contains one hole, and let x = (2, a) and y = (3, b) be holes for some $a \ne b$ and $1 \le a, b \le 5$. Then mate (1, a) to x and (1, b) to y along their columns. There remain two terminals in the top line A(1), we assign to these terminals disjoint horizontal half-lines emanating from them. By symmetry, we may assume that $b \in \{1, 3, 4\}$. Let $D = R \setminus (A(1) \cup B(1))$ be the 2×4 grid with vertices labeled as in Lemma 3.3. Among the six pairs of terminals (and mates) eight





Fig. 14 Each row contains a hole (Case e)

are located at the vertices of D, thus there are at least two pairs in D; one of them, say π_1 , does not contain y_0 .

For b = 1 we obtain a half-line assignment by Lemma 3.3 (A) which can be completed by a horizontal left half-line at (2, 1) and a vertical half-line at (3, 1). For b = 3 or 4 we use Lemma 3.3(B); in each case the assignment (B) can be completed by the horizontal left half-line at (3, 1) and the vertical half-line at (2, 1). Figure 14 shows examples for a = 3, b = 1 and for a = 2, b = 4.

A solution is obtained similarly if row $A(\ell)$ contains two holes, where $\ell = 1$ or 2. Let the third hole in R be w = (i, b), where $i \neq \ell$ and $b \in \{1, 3, 4\}$ (w.l.o.g., by symmetry). We mate (ℓ, b) to w, and we assign disjoint horizontal half-lines emanating from the remaining two terminals in $A(\ell)$. Then we apply Lemma 3.3(A) or (B) with $D = R \setminus (A(\ell) \cup B(1))$ as before. Examples for (A) and (B) are presented in Fig. 15, where in picture (B) $\ell = 2, b = 4$ and $t_1 = (2, 4)$. This concludes the proof of Case e, and the proposition follows.



Fig. 15 Reducing a 3×5 obstruction to a 2×4 grid

Fig. 16 $pp(\mathbb{N} \Box \mathbb{N}) < 5$





4 The integer quadrant

In this section we sketch a proof for $pp(\mathbb{N} \square \mathbb{N}) = 4$. We need two technical lemmas applied in [8] for determining the path-pairability number of the 6×6 grid $P_6 \square P_6$. These lemmas, called there 'exit lemmas', serve to handle the corner of $G = \mathbb{N} \square \mathbb{N}$ crowded with terminals; their detailed proofs are presented in [9].

Let Q be a 3×3 subgrid of G with a horizontal bottom line A and a vertical right boundary line B. The 5 vertices of $A \cup B$ are called *exit vertices* of Q. Recall that mating a terminal t to a vertex $t' \in G$ means fixing a t, t'-path called a *mating path*. In the lemmas below $T = \bigcup \{\pi_i \mid 1 \le i \le 4\}$ is a set of 8 terminals (members of 4 pairs) located in G; and $\|Q\| = |T \cap Q|$ denotes the number of terminals in Q.

Lemma 4.1 [9] If ||Q|| = 7 or 8, then there is a linkage of two or more pairs in Q, and there are edge disjoint mating paths for the unlinked terminals to distinct exit vertices of $A \cup B$. \Box

Lemma 4.2 [9] If ||Q|| = 6, then there is a linkage of one or more pairs in Q, and there are edge disjoint mating paths for the unlinked terminals to distinct exit vertices of $A \cup B$ such that $B \setminus A$ contains at most one exit vertex.

Proposition 4.3 $pp(\mathbb{N} \Box \mathbb{N}) = 4$.

Proof The bound $pp(\mathbb{N} \square \mathbb{N}) < 5$ follows from the pairing of 10 terminals in Fig. 16. If there was a solution P_i , $1 \le i \le 5$, then the path P_5 starting from terminal s_5 will either head right or down from s_5 ; without loss of generality assume it heads down to t_3 . Either P_4 from s_4 or P_2 from s_2 will have to exit the upper-left 2×2 cluster along the edge down toward s_1 . From that location either this path or the path P_1 starting from s_1 will then be forced to head



Fig. 17 Exit from a 2×3 or a 2×4 grid

right to t_3 . There are now three different paths at that location but only two edges go out, a contradiction.

Next we verify that $G = \mathbb{N} \Box \mathbb{N}$ is 4-path-pairable. Let *T* be a set of 8 distinct terminals in *G* partitioned into pairs π_i , $1 \le i \le 4$. Let L(T) be the set of all horizontal and vertical half-lines in *G* containing a terminal. If a half-line $\ell \in L(T)$ contains just one terminal, then assign ℓ to this terminal, then remove it from *T*. Repeating this reduction reduces *T* to a subset of terminals $S \subset T$ contained by a $p \times q$ rectangle *R* such that each row and column of *R* contains either zero or at least two terminals. We continue 'exiting' terminals from *R* by mating interior terminals to free half-lines starting at the boundary of *R*, meanwhile, we are allowed (or urged) to link appropriate pair(s) in *R*. The linkage of the pairs is completed using the second stage of the procedure described in Sect. 2. Rows and columns with no terminal can be discarded, thus w.l.o.g. we my assume that each row and column of *R* contains at least two terminals. Consequently we have $2 \le p$, $q \le 4$, and $|S| \le \min\{8, pq\}$. By symmetry, we may assume that $p \le q$.

If q = 2, then R is a 2 × 2 square and |S| = 4. Then there are four edge disjoint half-lines in G emanating from the four terminals. Thus q > 2 can be assumed.

If p = 4, then q = 4, and thus $||\ell|| = 2$, for every $\ell \in L(S)$. Consider the 2-regular 4×4 bipartite graph H where one partite set corresponds to the rows and the other partite set corresponds to the columns formed by the lines in L(S), and the edges of H correspond to the terminals in S. Then H is the union of even cycles. For each even cycle we define disjoint half-lines emanating from consecutive terminals by assigning alternately horizontal and vertical half-lines in G. The remaining cases are: p = 2 or p = 3.

Case a: p = 2. Since each column of R has two terminals, we have q = 3 or 4.

The left picture of Fig. 17 shows the 6-path $Q^* = (v_1, \ldots, v_6)$ covering the terminals in the first three columns of R. The path Q^* contains a pair π_1 , by the pigeon hole principle. Let $P_1 \subset Q^*$ be the (unique) linkage for π_1 in Q^* . For $v_j \notin \pi_1$, j = 1, 6, we define $Q_j \subset Q^*$ to be the mating path from the terminal at v_j to the closest end vertex of P_1 . Then we assign to the unlinked four terminals and mates the three vertical half-lines at $v_i \notin \pi_1$, for $2 \le i \le 4$, and the horizontal right half-line at v_5 , provided $v_5 \notin \pi_1$. For q = 4 there are two more terminals in B(4), they are assigned to the vertical half-line B(4), and the horizontal right half-line in A(2) (see examples in Fig.17).

Case b: p = 3. Let Q be the 3×3 square formed by the rows and the first three columns of R. Let $A = A(3) \cap Q$ and $B = B(3) \cap Q$. Since each column contains two or more terminals, we have $||Q|| \ge 6$.

We apply Lemma 4.1 for ||Q|| = 7 or 8, and we apply Lemma 4.2, for ||Q|| = 6. For q = 3 we assign half-lines to the at most four exits (mates and unlinked terminals) in $A \cup B$ among the the three vertical and three horizontal right half-lines emanating from $A \cup B$. For q = 4, since each column of R has two mates, we have ||Q|| = 6. The vertical half-lines emanating from A are assigned to the exits in A, and a horizontal half-line is assigned to the only exit in $B \setminus A$. The two terminals in B(4) can be mated easily to the head of the two unused horizontal half-lines emanating from B(4) and the vertical half-line in B(4).

5 Conclusion

It was proved in [7] that the grid $\mathbb{Z} \square \mathbb{Z}$ is 10-path-pairable and not 15-path-pairable. These bounds and those proved here in Propositions 3.1, 3.2 and 4.3 are summarized as follows:

 $pp(\mathbb{N} \square \mathbb{N}) = 4, \quad 6 \le pp(\mathbb{N} \square \mathbb{Z}) \le 7, \quad 10 \le pp(\mathbb{Z} \square \mathbb{Z}) \le 14.$

The strategy of applying Hall's theorem (as in the proof of Proposition 3.2) is well-suited for deriving sharper lower bounds by using computer to resolve the rapidly increasing number of obstructions. A computer program confirmed that the half-plane is 7-path-pairable, thus we are making the 'safe conjecture' that $pp(\mathbb{N} \square \mathbb{Z}) = 7$. For the plane the upper and lower bounds are improved by a computer program to $11 \le pp(\mathbb{Z} \square \mathbb{Z}) \le 12$. We think that $pp(\mathbb{Z} \square \mathbb{Z}) = 12$ is the right value, meanwhile, computer checking whether the plane is 12-path-pairable is beyond our computing capacity.

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