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Unit groups of some multiquadratic number fields and 2-class groups

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Abstract

Let $p \equiv -q \equiv 5 \pmod 8$ be two prime integers. In this paper, we investigate the unit groups of the fields $L_1 = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-1})$ and $L_1^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. Furthermore , we give the second 2-class groups of the subextensions of L_1 as well as the 2-class groups of the fields $L_n = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \zeta_{2^{n+2}})$ and their maximal real subfields.

Keywords Multiquadratic number fields · Unit group · 2-class group · Hilbert 2-class field tower · Cyclotomic $\mathbb{Z}_2$-extension.

Mathematics Subject Classification 11R04 · 11R27 · 11R29 · 11R37

1 Introduction

Let k be a number field and E_k its unit group. The determination of E_k is a very difficult computational problem that serves to give answers to many problems such as the computation of the class number of k, the capitulation problem and many other problems in algebraic number theory. The most spectacular result that describes the structure of E_k is the well-known Dirichlet unit theorem that says that

$$E_k = \mu(k) \times \mathbb{Z}^{r_1+r_2-1},$$

where $\mu(k)$ is the group of roots of unity contained in k, r_1 is the number of real embeddings and r_2 is the number of conjugate pairs of complex embeddings of k. This is the only known and general result that covers any given number field k. If k is an imaginary J-field, there is a known result of Hasse that gives the difference between the unit group of k and that of its real maximal subfield k^+ i.e., the index $[E_k : \mu(k)E_{k^+}]$ equals 1 or 2.

Unfortunately, these results do not give much information on the generators of the group E_k. For the particular family of multiquadratic number fields there are some useful algorithms

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by Wada ([15]) and Azizi ([3]) that helped to compute the unit groups of many families of real biquadratic number fields and imaginary triquadratic number fields ([4,7]). However, these algorithms became very difficult to apply to real multiquadratic fields of degree at least 8 and imaginary multiquadratic fields of degree at least 16. To the best of our knowledge there is only one example in the literature that explicitly determines the unit groups of some infinite families of such fields (see the recently published paper [9]). In Sect. 2 of this paper we shall modify the algorithms of Wada and Azizi by a process of elimination based on norm maps and class number formulas to explicitly determine the unit groups of the fields $L_1 = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-1})$ and $L_1^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$, where p and q are two primes that satisfy one of the following conditions:

$$p \equiv 5 \pmod 8,\ q \equiv 3 \pmod 8 \text{ and } \left(\frac{p}{q}\right) = 1, \tag{1.1}$$

$$p \equiv 5 \pmod 8,\ q \equiv 3 \pmod 8 \text{ and } \left(\frac{p}{q}\right) = -1. \tag{1.2}$$

In Sect. 3, we determine the 2-class groups and the second 2-class groups of the unramified quadratic extensions of $\mathbb{Q}(\sqrt{2pq}, i)$, as well as we give the 2-class groups of the layers of their cyclotomic $\mathbb{Z}_2$-extension.

Notations

Let k be a number field. We shall use the following notations for the rest of this paper:

- $h_2(k)$: the 2-class number of k,
- $h_2(d)$: the 2-class number of the quadratic field $\mathbb{Q}(\sqrt{d})$,
- ε_d: the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{d})$,
- E_k: the unit group of k,
- FSU: abbreviation of "fundamental system of units",
- $k^{(1)}$: the Hilbert 2-class field of k,
- $k^{(2)}$: the Hilbert 2-class field of $k^{(1)}$,
- k^+: the maximal real subfield of k, whenever k is imaginary,
- $q(k) = (E_k : \prod_i E_{k_i})$ is the unit index of k, if k is multiquadratic, where k_i are the quadratic subfields of k,
- $N_{k'/k}$: the norm map of an extension k'/k.

2 Units of some multiquadratic number fields of degree 8 and 16

Let us start by collecting some results that will be useful in the sequel.

Lemma 2.1 ([2, Lemma 5]) *Let $d > 1$ be a square-free integer and $\varepsilon_d = x + y\sqrt{d}$, where x, y are integers or semi-integers. If $N(\varepsilon_d) = 1$, then $2(x+1)$, $2(x-1)$, $2d(x+1)$ and $2d(x-1)$ are not squares in $\mathbb{Q}$.*

Lemma 2.2 ([3], Proposition 2) *Let K_0 be a real number field, $K = K_0(i)$ a quadratic extension of K_0, $n \geq 2$ an integer and ξ_n a 2^n-th primitive root of unity. Then $\xi_n = \frac{1}{2}(\mu_n + \lambda_n i)$, where $\mu_n = \sqrt{2 + \mu_{n-1}}$, $\lambda_n = \sqrt{2 - \mu_{n-1}}$, $\mu_2 = 0$, $\lambda_2 = 2$ and $\mu_3 = \lambda_3 = \sqrt{2}$. Let n_0 be the greatest integer such that ξ_{n_0} is contained in K, $\{\varepsilon_1, \dots, \varepsilon_r\}$ a fundamental system* *of units of K_0 and ε a unit of K_0 such that $(2+\mu_{n_0})\varepsilon$ is a square in K_0 (if it exists). Then a fundamental system of units of K is one of the following systems:*

1. $\{\varepsilon_1, \dots, \varepsilon_{r-1}, \sqrt{\xi_{n_0}\varepsilon}\}$ *if ε exists, in this case $\varepsilon = \varepsilon_1^{j_1} \cdots \varepsilon_{r-1}^{j_1}\varepsilon_r$, where $j_i \in \{0, 1\}$;*
2. $\{\varepsilon_1, \dots, \varepsilon_r\}$ *else.*

Let us recall the method given in [15] that describes a fundamental system of units of a real multiquadratic field K_0. Let σ_1 and σ_2 be two distinct elements of order 2 of the Galois group of $K_0/\mathbb{Q}$. Let K_1, K_2 and K_3 be the three subextensions of K_0 invariant by σ_1, σ_2 and $\sigma_3 = \sigma_1\sigma_3$, respectively. Let ε denote a unit of K_0. Then

$$\varepsilon^2 = \varepsilon\varepsilon^{\sigma_1}\varepsilon\varepsilon^{\sigma_2}(\varepsilon^{\sigma_1}\varepsilon^{\sigma_2})^{-1},$$

and we have, $\varepsilon\varepsilon^{\sigma_1} \in E_{K_1}$, $\varepsilon\varepsilon^{\sigma_2} \in E_{K_2}$ and $\varepsilon^{\sigma_1}\varepsilon^{\sigma_2} \in E_{K_3}$. It follows that the unit group of K_0 is generated by the elements of E_{K_1}, E_{K_2} and E_{K_3}, and the square roots of elements of $E_{K_1}E_{K_2}E_{K_3}$ which are perfect squares in K_0.
Let us continue by stating the following results.

Lemma 2.3 *Let p and q be two primes satisfying (1.1).*

(1) *Let x and y be two integers such that $\varepsilon_{2pq} = x + y\sqrt{2pq}$. Then we have*

  (i) *$p(x-1)$ is a square in $\mathbb{N}$,*
  (ii) *$\sqrt{2\varepsilon_{2pq}} = y_1\sqrt{p} + y_2\sqrt{2q}$ and $2 = 2qy_2^2 - py_1^2$, for some integers y_1 and y_2.*

(2) *Let a and b be two integers such that $\varepsilon_{pq} = a + b\sqrt{pq}$. Then we have*

  (i) *$2p(a+1)$ is a square in $\mathbb{N}$,*
  (ii) *$\sqrt{\varepsilon_{pq}} = b_1\sqrt{p} + b_2\sqrt{q}$ and $1 = pb_1^2 - qb_2^2$, for some integers b_1 and b_2.*

(3) *Let c and d be two integers such that $\varepsilon_{2q} = c + d\sqrt{2q}$. Then we have*

  (i) *$c-1$ is a square in $\mathbb{N}$,*
  (ii) *$\sqrt{2\varepsilon_{2q}} = d_1 + d_2\sqrt{2q}$ and $2 = -d_1^2 + 2qd_2^2$, for some integers d_1 and d_2.*

(4) *Let α and β be two integers such that $\varepsilon_q = \alpha + \beta\sqrt{q}$. Then we have*

  (i) *$\alpha - 1$ is a square in $\mathbb{N}$,*
  (ii) *$\sqrt{2\varepsilon_q} = \beta_1 + \beta_2\sqrt{q}$ and $2 = -\beta_1^2 + q\beta_2^2$, for some integers β_1 and β_2.*

Proof (1) It is known that $N(\varepsilon_{2pq}) = 1$. Then, by the unique factorization in $\mathbb{Z}$ and Lemma 2.1, there exist some integers y_1 and y_2 ($y = y_1y_2$) such that

$$(1):\begin{cases} x \pm 1 = y_1^2, \\ x \mp 1 = 2pqy_2^2, \end{cases} \quad (2):\begin{cases} x \pm 1 = py_1^2, \\ x \mp 1 = 2qy_2^2, \end{cases} \quad \text{or} \quad (3):\begin{cases} x \pm 1 = 2py_1^2, \\ x \mp 1 = qy_2^2. \end{cases}$$

∗ System (1) can not occur since it implies $1 = \left(\frac{y_1^2}{p}\right) = \left(\frac{x\pm1}{p}\right) = \left(\frac{x\mp1\pm2}{p}\right) = \left(\frac{\pm2}{p}\right) = \left(\frac{2}{p}\right) = -1$, which is absurd.

∗ Similarly, system (3) can not occur either since it implies $1 = \left(\frac{q}{p}\right) = \left(\frac{x\mp1}{p}\right) = \left(\frac{\pm2}{p}\right) = \left(\frac{2}{p}\right) = -1$, which is absurd.

∗ Suppose that $\begin{cases} x + 1 = py_1^2, \\ x - 1 = 2qy_2^2. \end{cases}$ Then $1 = \left(\frac{py_1^2}{q}\right) = \left(\frac{x+1}{q}\right) = \left(\frac{x-1+2}{q}\right) = \left(\frac{2}{q}\right) = -1$, which is also impossible.

Thus, the only possible case is $\begin{cases} x-1 = py_1^2, \\ x+1 = 2qy_2^2, \end{cases}$ which implies that $\sqrt{2\varepsilon_{2pq}} = y_1\sqrt{p} + y_2\sqrt{2q}$ and $2 = 2qy_2^2 - py_1^2$.

(2) $N(\varepsilon_{pq}) = 1$. Then, by Lemma 2.1, we have

$$(1): \begin{cases} a \pm 1 = pb_1^2, \\ a \mp 1 = qb_2^2, \end{cases} \quad (2): \begin{cases} a \pm 1 = b_1^2, \\ a \mp 1 = pqb_2^2, \end{cases} \quad \text{or} \quad (3): \begin{cases} a \pm 1 = 2pb_1^2, \\ a \mp 1 = 2qb_2^2, \end{cases}$$

for some integers b_1 and b_2 such that $b = b_1b_2$ or $b = 2b_1b_2$ ($b = 2b_1b_2$ in the cases of system (3)). As above we show that the only possible case is $\begin{cases} a+1 = 2pb_1^2, \\ a-1 = 2qb_2^2. \end{cases}$ From this we infer that $\sqrt{\varepsilon_{pq}} = b_1\sqrt{p} + b_2\sqrt{q}$ and $1 = pb_1^2 - qb_2^2$.

(3) $N(\varepsilon_{2q}) = 1$. Then, using Lemma 2.1 and the same technique as above, we show that there are two integers d_1 and d_2 such that $\begin{cases} c-1 = d_1^2, \\ c+1 = 2qd_2^2. \end{cases}$ Thus, $\sqrt{2\varepsilon_{2q}} = d_1 + d_2\sqrt{2q}$ and $2 = -d_1^2 + 2qd_2^2$.

(4) $N(\varepsilon_q) = 1$. Then, using Lemma 2.1 and the same technique as above, we show that there are two integers β_1 and β_2 such that $\begin{cases} \alpha-1 = \beta_1^2, \\ \alpha+1 = q\beta_2^2. \end{cases}$ Thus, $\sqrt{2\varepsilon_q} = \beta_1 + \beta_2\sqrt{q}$ and $2 = -\beta_1^2 + q\beta_2^2$.

Corollary 2.4 *Let p and q be two primes satisfying* (1.1).

(1) *A FSU of $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ is given by $\{\varepsilon_p, \varepsilon_q, \sqrt{\varepsilon_{pq}}\}$.*
(2) *A FSU of $\mathbb{Q}(\sqrt{2}, \sqrt{q})$ is given by $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$.*
(3) *A FSU of $\mathbb{Q}(\sqrt{p}, \sqrt{2q})$ is given by $\{\varepsilon_p, \varepsilon_{2q}, \sqrt{\varepsilon_{2q}\varepsilon_{2pq}}\}$.*
(4) *A FSU of $\mathbb{Q}(\sqrt{q}, \sqrt{2p})$ is given by $\{\varepsilon_q, \varepsilon_{2p}, \sqrt{\varepsilon_{2pq}}\}$.*
(5) *A FSU of $\mathbb{Q}(\sqrt{2}, \sqrt{pq})$ is given by $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$.*

Proof Note that $\sqrt{2} \notin \mathbb{Q}(\sqrt{p}, \sqrt{q})$ and ε_p has negative norm. So, using Lemma 2.3, one easily verifies that the only element of the form $\varepsilon_{pq}^i\varepsilon_p^j\varepsilon_q^k$, for i, j and $k \in \{0, 1\}$, which is a square in $\mathbb{Q}(\sqrt{p}, \sqrt{q})$, is ε_{pq}. So (1) follows by the method given on Page 3. One can similarly deduce the rest from Lemma 2.3 and [7, Propositions 3.1 and 3.2]. □

Now we are able to state the first important result of this section.

Theorem 2.5 *Let p and q be two primes satisfying* (1.1). *Put $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-1})$ and $\mathbb{K}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. Then the following hold.*

(1) (a) $E_{\mathbb{K}^+} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}, \sqrt[4]{\varepsilon_p^2\varepsilon_{2q}\varepsilon_{pq}\varepsilon_{2pq}}\rangle$.
    (b) *The class number of $\mathbb{K}^+$ is odd.*

(2) (a) $E_{\mathbb{K}} = \langle \zeta_{24}$ *or* $\zeta_8, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}, \sqrt[4]{\varepsilon_p^2\varepsilon_{2q}\varepsilon_{pq}\varepsilon_{2pq}}, \sqrt[4]{\zeta_8^2\varepsilon_2^2\varepsilon_q\varepsilon_{2q}}\rangle$, *according to whether $q = 3$ or not.*
    (b) $h_2(\mathbb{K}) = h_2(-pq)$.

Proof (1) Consider the following diagram (see Fig. 1):
Note that by [6, Théorème 6], $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, is a FSU of k_1. By Corollary 2.4, $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ is a FSU of k_2 and a FSU of k_3 is given by $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{2pq}\varepsilon_{pq}}\}$.

$$
\begin{tikzcd}
 & \mathbb{K}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}) & \\
k_1 = \mathbb{Q}(\sqrt{2}, \sqrt{p}) \arrow[ur] & k_2 = \mathbb{Q}(\sqrt{2}, \sqrt{q}) \arrow[u] & k_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq}) \arrow[ul] \\
 & \mathbb{Q}(\sqrt{2}) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Fig. 1 Subfields of $\mathbb{K}^+/\mathbb{Q}(\sqrt{2})$

It follows that

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \varepsilon_{pq}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Note that a FSU of $\mathbb{K}$ consists of seven units chosen from those of k_1, k_2 and k_3, and from the square roots of the units of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{K}$ (cf. Page 3). Thus, we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{K}^+$. Suppose X is an element of $\mathbb{K}^+$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that

$$X^2 = \varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$.

We shall use norm maps from $\mathbb{K}^+$ to its subextensions to eliminate the cases of X^2 which do not occur.

Let τ_1, τ_2 and τ_3 be the elements of $\mathrm{Gal}(\mathbb{K}^+/\mathbb{Q})$ defined by

$$\begin{array}{lll}
\tau_1(\sqrt{2}) = -\sqrt{2}, & \tau_1(\sqrt{p}) = \sqrt{p}, & \tau_1(\sqrt{q}) = \sqrt{q}, \\
\tau_2(\sqrt{2}) = \sqrt{2}, & \tau_2(\sqrt{p}) = -\sqrt{p}, & \tau_2(\sqrt{q}) = \sqrt{q}, \\
\tau_3(\sqrt{2}) = \sqrt{2}, & \tau_3(\sqrt{p}) = \sqrt{p}, & \tau_3(\sqrt{q}) = -\sqrt{q}.
\end{array}$$

Note that $\mathrm{Gal}(\mathbb{K}^+/\mathbb{Q}) = \langle \tau_1, \tau_2, \tau_3\rangle$ and the subfields k_1, k_2 and k_3 are fixed by $\langle\tau_3\rangle$, $\langle\tau_2\rangle$ and $\langle\tau_2\tau_3\rangle$ respectively. Lemma 2.3 is used to compute the norm maps from $\mathbb{K}^+$ to its subextensions. We summarize these computations in Table 1. Let us start by applying the norm map $N_{\mathbb{K}^+/k_2} = 1 + \tau_2$:

$$\begin{aligned}
N_{\mathbb{K}^+/k_2}(X^2) = N_{\mathbb{K}^+/k_2}(X)^2 &= \varepsilon_2^{2a}(-1)^b \cdot 1 \cdot \varepsilon_q^d\varepsilon_{2q}^e \cdot (-1)^f \cdot (-1)^{gv}\varepsilon_2^g \\
&= \varepsilon_2^{2a}\varepsilon_q^d\varepsilon_{2q}^e \cdot (-1)^{b+f+gv}\varepsilon_2^g.
\end{aligned}$$

Note that, by Corollary 2.4, $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ is a FSU of k_2. Thus, ε_q and ε_{2q} are squares in k_2 whereas ε_2 is not. Since $N_{\mathbb{K}^+/k_2}(X^2) > 0$, then $b + f + vg \equiv 0 \pmod 2$ and ε_2^g is a square in k_2. Therefore $g = 0$ and $b = f$. So we have

$$X^2 = \varepsilon_2^a\varepsilon_p^f\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

Similarly, by applying $N_{\mathbb{K}^+/k_3} = 1 + \tau_2\tau_3$, one gets

$$\begin{aligned}
N_{\mathbb{K}^+/k_3}(X^2) &= \varepsilon_2^{2a} \cdot (-1)^f \cdot \varepsilon_{pq}^{2c} \cdot (-1)^d \cdot (-1)^e \cdot \varepsilon_{pq}^f\varepsilon_{2pq}^f \\
&= \varepsilon_2^{2a}\varepsilon_{pq}^{2c}\varepsilon_{pq}^f\varepsilon_{2pq}^f(-1)^{f+d+e} > 0.
\end{aligned}$$

Note that, by Corollary 2.4, $\varepsilon_{pq}\varepsilon_{2pq}$ is a square in k_3. Thus, all that we can deduce is $f+d+e \equiv 0 \pmod 2$. Let us now apply $N_{\mathbb{K}^+/k_4} = 1+\tau_1$, where $k_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. We have

$$\begin{aligned} N_{\mathbb{K}^+/k_4}(X^2) &= (-1)^a \cdot \varepsilon_p^{2f} \cdot \varepsilon_{pq}^{2c} \cdot (-\varepsilon_q)^d \cdot 1 \cdot (\varepsilon_{pq})^f \\ &= \varepsilon_p^{2f}\varepsilon_{pq}^{2c}\varepsilon_{pq}^f \cdot (-1)^{a+d} \cdot \varepsilon_q^d > 0. \end{aligned}$$

Thus, $a+d \equiv 0 \pmod 2$. By Corollary 2.4, ε_{pq} is a square in k_4 and, by Lemma 2.3, $2\varepsilon_q$ is a square in k_4 whereas ε_q is not (in fact $\sqrt{2} \notin k_4$). So $d=0$ and then $a=0$. Since $f+d+e \equiv 0 \pmod 2$, we have $f=e$. Therefore,

$$X^2 = \varepsilon_p^f \varepsilon_{pq}^c \sqrt{\varepsilon_{2q}}^f \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

Note that, by Lemma 2.3, ε_{pq} is a square in $\mathbb{K}^+$, so we may put

$$X^2 = \varepsilon_p^f \sqrt{\varepsilon_{2q}}^f \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

Suppose that $f=0$. Then, by the above discussions and Lemma 2.2, a FSU of $\mathbb{K}^+$ is

$$\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}.$$

Thus, $q(\mathbb{K}^+) = 2^5$. We have $h_2(p) = h_2(q) = h_2(2q) = h_2(2) = 1$ and $h_2(2p) = h_2(pq) = h_2(2pq) = 2$ (cf. [10, Corollaries 18.4, 19.7 and 19.8]),

$$\begin{aligned} h_2(\mathbb{K}^+) &= \frac{1}{2^9} q(\mathbb{K}^+) h_2(2)h_2(p)h_2(q)h_2(2p)h_2(2q)h(pq)h_2(2pq) \\ &= \frac{1}{2^9} \cdot 2^5 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \\ &= \frac{1}{2}, \end{aligned}$$

which is absurd. Thus, $f=1$ and then $q(\mathbb{K}^+) = 2^6$. So we have (1).

(2) Keep the notations of Lemma 2.2. Note that the greatest integer n_0 such that $\zeta_{2^{n_0}}$ is contained in $\mathbb{K}$ equals 3, therefore $\mu_{n_0} = \sqrt{2}$. So, according to Lemma 2.2, we should find an element Y, if it exists, which is in $\mathbb{K}^+$ such that

$$Y^2 = (2+\sqrt{2})\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^f\sqrt[4]{\varepsilon_p^2\varepsilon_{2q}\varepsilon_{pq}\varepsilon_{2pq}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$. So firstly we shall use norm maps to eliminate some cases (see Table 1).

- We have $N_{\mathbb{K}^+/k_2} = 1+\tau_2$. So, by applying $N_{\mathbb{K}^+/k_2}$, we get

$$\begin{aligned} N_{\mathbb{K}^+/k_2}(Y^2) &= (2+\sqrt{2})^2\varepsilon_2^{2a}(-1)^b\varepsilon_q^c\varepsilon_{2q}^d \cdot (-1)^e \cdot (-1)^{fv}\varepsilon_2^f(-1)^{gs}\sqrt{\varepsilon_{2q}}^g, \\ &= (2+\sqrt{2})^2\varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d(-1)^{b+e+fv+gs}\varepsilon_2^f\sqrt{\varepsilon_{2q}}^g > 0. \end{aligned}$$

Thus, $b+e+fv+gs = 0 \pmod 2$. By Corollary 2.4, $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ is a FSU of k_2. Since ε_2, $\sqrt{\varepsilon_{2q}}$ and $\varepsilon_2\sqrt{\varepsilon_{2q}}$ are not squares in k_2, we have $f=g=0$ and so $b=e$. Therefore,

$$Y^2 = (2+\sqrt{2})\varepsilon_2^a\varepsilon_p^e\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e.$$

We have $N_{\mathbb{K}^+/k_4} = 1 + \tau_1$ with $k_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. So

$$N_{\mathbb{K}^+/k_4}(Y^2) = (4-2)(-1)^a \varepsilon_p^{2e}(-1)^c \varepsilon_q^c \cdot 1 \cdot \varepsilon_{pq}^e,$$
$$= \varepsilon_p^{2e}\varepsilon_{pq}^e(-1)^{a+c} \cdot 2 \cdot \varepsilon_q^c > 0.$$

So $a+c = 0 \pmod 2$. Since $\sqrt{2} \notin k_4$ and, by Lemma 2.3, $\sqrt{2\varepsilon_q} \in k_4$, then $c = 1$. Therefore $a = c = 1$ and we have

$$Y^2 = (2+\sqrt{2})\varepsilon_2\varepsilon_p^e\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e.$$

By applying the norm map, $N_{\mathbb{K}^+/k_3} = 1 + \tau_2\tau_3$, we get

$$N_{\mathbb{K}^+/k_3}(Y^2) = (2+\sqrt{2})^2\varepsilon_2^2 \cdot (-1)^e \cdot (-1) \cdot (-1)^d.(-1)^e.\varepsilon_{pq}^e,$$
$$= (2+\sqrt{2})^2\varepsilon_2^2 \cdot (-1)^{1+d} \cdot \varepsilon_{pq}^e > 0.$$

Thus, $1 + d = 0 \pmod 2$. So $d = 1$. As, by Corollary 2.4, ε_{pq} is a not a square in k_3, then $e = 0$. It follows that

$$Y^2 = (2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}.$$

Let us now verify that $(2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}$ is a square in $\mathbb{K}^+$.

- Note that by [5, Theorem 5.5], the 2-class group of $L_{pq} := \mathbb{Q}(\sqrt{pq}, \sqrt{2}, i)$ is cyclic. Since $\mathbb{K}$ is an unramified quadratic extension of L_{pq}, this implies that the Hilbert 2-class field of L_{pq}(i.e., $L_{pq}^{(1)}$) and $\mathbb{K}$ have the same Hilbert 2-class field. So $h_2(L_{pq}) = 2h_2(\mathbb{K})$. Therefore, again by [8, Lemma 3], we have $2h_2(\mathbb{K}) = 2h_2(-pq)$. Thus,

$$h_2(\mathbb{K}) = h_2(-pq). \tag{2.1}$$

  Assume that $(2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}$ is not a square in $\mathbb{K}^+$. Then, by Lemma 2.2 and the above discussions, $\mathbb{K}^+$ and $\mathbb{K}$ have the same fundamental system of units. Thus, $q(\mathbb{K}) = 2^7$. We have $h_2(-1) = h_2(-2) = h_2(-q) = 1$, $h_2(-p) = h_2(-2p) = h_2(-2q) = 2$ and $h_2(-2pq) = 4$ by [10, Corollary 18.4], [10, Corollary 19.6] and [12, p. 353] respectively. So, by the class number formula (cf. [15, p. 201]) and the above setting on the 2-class numbers of real quadratic fields (Page 6), we get

$$\begin{aligned}h_2(\mathbb{K}) &= \frac{1}{2^{16}}q(\mathbb{K})h_2(-1)h_2(2)h_2(-2)h_2(p)h_2(-p)h_2(q)h_2(-q)h_2(2p)\\ &\quad h_2(-2p)h_2(2q)h_2(-2q)h_2(pq)h_2(-pq)h_2(2pq)h_2(-2pq)\\ &= \frac{1}{2^{16}} \cdot 2^7 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot h_2(-pq) \cdot 2 \cdot 4\\ &= \frac{1}{2}h_2(-pq).\end{aligned}$$

  This contradicts (2.1). It follows that $(2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}$ is a square in $\mathbb{K}^+$. Hence Lemma 2.2 completes the proof. □

To prove our second main result of this section, we need the following lemma and corollary.

Lemma 2.6 *Let p and q be two primes satisfying* (1.2).

(1) *Let x and y be two integers such that $\varepsilon_{2pq} = x + y\sqrt{2pq}$. Then we have*

Table 1 Norms when p and q satisfy conditions (1.1)

ε	ε^{τ_1}	ε^{τ_2}	ε^{τ_3}	$\varepsilon^{1+\tau_1}$	$\varepsilon^{1+\tau_2}$	$\varepsilon^{1+\tau_3}$	$\varepsilon^{1+\tau_1\tau_2}$	$\varepsilon^{1+\tau_1\tau_3}$	$\varepsilon^{1+\tau_2\tau_3}$
ε_2	$\frac{-1}{\varepsilon_2}$	ε_2	ε_2	-1	ε_2^2	ε_2^2	-1	-1	ε_2^2
ε_p	ε_p	$\frac{-1}{\varepsilon_p}$	ε_p	ε_p^2	-1	ε_p^2	-1	ε_p^2	-1
$\sqrt{\varepsilon_q}$	$-\sqrt{\varepsilon_q}$	$\sqrt{\varepsilon_q}$	$\frac{-1}{\sqrt{\varepsilon_q}}$	$-\varepsilon_q$	ε_q	-1	$-\varepsilon_q$	1	-1
$\sqrt{\varepsilon_{2q}}$	$\frac{1}{\sqrt{\varepsilon_{2q}}}$	$\sqrt{\varepsilon_{2q}}$	$\frac{-1}{\sqrt{\varepsilon_{2q}}}$	1	ε_{2q}	-1	1	$-\varepsilon_{2q}$	-1
$\sqrt{\varepsilon_{pq}}$	$\sqrt{\varepsilon_{pq}}$	$\frac{-1}{\sqrt{\varepsilon_{pq}}}$	$\frac{1}{\sqrt{\varepsilon_{pq}}}$	ε_{pq}	-1	1	-1	1	$-\varepsilon_{pq}$
$\sqrt{\varepsilon_{2pq}}$	$\frac{1}{\sqrt{\varepsilon_{2pq}}}$	$\frac{1}{\sqrt{\varepsilon_{2pq}}}$	$\frac{-1}{\sqrt{\varepsilon_{2pq}}}$	1	1	-1	ε_{2pq}	$-\varepsilon_{2pq}$	$-\varepsilon_{2pq}$
$\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}$	$(-1)^u\sqrt{\frac{\varepsilon_p}{\varepsilon_2\varepsilon_{2p}}}$	$(-1)^v\sqrt{\frac{\varepsilon_2}{\varepsilon_p\varepsilon_{2p}}}$	$\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}$	$(-1)^u\varepsilon_p$	$(-1)^v\varepsilon_2$	$\varepsilon_2\varepsilon_p\varepsilon_{2p}$			
$\sqrt[4]{\varepsilon_p^2\varepsilon_{2q}\varepsilon_{pq}\varepsilon_{2pq}}$	$(-1)^r\sqrt[4]{\frac{\varepsilon_p^2\varepsilon_{pq}}{\varepsilon_{2q}\varepsilon_{pq}}}$	$(-1)^s\sqrt[4]{\frac{\varepsilon_{2q}}{\varepsilon_p^2\varepsilon_{pq}\varepsilon_{2pq}}}$	$(-1)^t\sqrt[4]{\frac{\varepsilon_p^2}{\varepsilon_{2q}\varepsilon_{pq}\varepsilon_{2pq}}}$	$(-1)^r\varepsilon_p\sqrt{\varepsilon_{pq}}$	$(-1)^s\sqrt{\varepsilon_{2q}}$	$(-1)^t\varepsilon_p$			

(i) $2p(x-1)$ *is a square in* $\mathbb{N}$,
(ii) $\sqrt{2\varepsilon_{2pq}} = y_1\sqrt{2p} + y_2\sqrt{q}$ *and* $2 = -2py_1^2 + qy_2^2$, *for some integers* y_1 *and* y_2.

(2) *Let* a *and* b *be two integers such that* $\varepsilon_{pq} = a + b\sqrt{pq}$. *Then we have*

(i) $p(a+1)$ *is a square in* $\mathbb{N}$,
(ii) $\sqrt{2\varepsilon_{pq}} = b_1\sqrt{p} + b_2\sqrt{q}$ *and* $2 = pb_1^2 - qb_2^2$, *for some integers* b_1 *and* b_2.

(3) *Let* c *and* d *be two integers such that* $\varepsilon_{2q} = c + d\sqrt{2q}$. *Then we have*

(i) $c-1$ *is a square in* $\mathbb{N}$,
(ii) $\sqrt{2\varepsilon_{2q}} = d_1 + d_2\sqrt{2q}$ *and* $2 = -d_1^2 + 2qd_2^2$, *for some integers* d_1 *and* d_2.

(4) *Let* α *and* β *be two integers such that* $\varepsilon_q = \alpha + \beta\sqrt{q}$. *Then we have*

(i) $\alpha - 1$ *is a square in* $\mathbb{N}$,
(ii) $\sqrt{2\varepsilon_q} = \beta_1 + \beta_2\sqrt{q}$ *and* $2 = -\beta_1^2 + q\beta_2^2$, *for some integers* β_1 *and* β_2.

Proof We proceed similarly as in the proof of Lemma 2.3. □

Corollary 2.7 *Let* p *and* q *be two primes satisfying* (1.2).

(1) *A FSU of* $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ *is given by* $\{\varepsilon_p, \varepsilon_q, \sqrt{\varepsilon_q\varepsilon_{pq}}\}$.
(2) *A FSU of* $\mathbb{Q}(\sqrt{2}, \sqrt{q})$ *is given by* $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$.
(3) *A FSU of* $\mathbb{Q}(\sqrt{p}, \sqrt{2q})$ *is given by* $\{\varepsilon_p, \varepsilon_{2q}, \sqrt{\varepsilon_{2pq}}\}$.
(4) *A FSU of* $\mathbb{Q}(\sqrt{q}, \sqrt{2p})$ *is given by* $\{\varepsilon_q, \varepsilon_{2p}, \sqrt{\varepsilon_q\varepsilon_{2pq}}\}$.
(5) *A FSU of* $\mathbb{Q}(\sqrt{2}, \sqrt{pq})$ *is given by* $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$.

Proof We proceed similarly as in the proof of Corollary 2.4. □

We can now state and prove the second main theorem of this section.

Theorem 2.8 *Let* p *and* q *be two primes satisfying* (1.2). *Put* $\mathbb{L} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-1})$ *and* $\mathbb{L}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. *Then the following hold.*

(1) (a) $E_{\mathbb{L}^+} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}, \sqrt[4]{\varepsilon_2^2\varepsilon_p^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}\rangle$.
(b) $h_2(\mathbb{L}^+) = 1$.

(2) (a) $E_{\mathbb{L}} = \langle \zeta_{24}$ *or* $\zeta_8, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}, \sqrt[4]{\varepsilon_2^2\varepsilon_p^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}, \sqrt[4]{\zeta_8^2\varepsilon_2^2\varepsilon_q\varepsilon_{2q}}\rangle$, *according to whether* $q = 3$ *or not.*
(b) $h_2(\mathbb{L}) = h_2(-pq) = 2$.

Proof (1) We consider an analogous diagram as in Fig. 1. Note that, by [6, Théorème 6] and Corollary 2.7, a FSU of $k_1 = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ is given by $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, a FSU of $k_2 = \mathbb{Q}(\sqrt{2}, \sqrt{q})$ is given by $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and a FSU of $k_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$ is given by $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{2pq}\varepsilon_{pq}}\}$. It follows that

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \varepsilon_{pq}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Note that, by Lemma 2.7, ε_{pq} is a square in $\mathbb{L}^+$. So we shall find elements X of $\mathbb{L}^+$, if they exist, such that

$$X^2 = \varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^e\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^f,$$

where a, b, c, d, e and f are in $\{0, 1\}$. Let us define τ_1, τ_2 and τ_3 similarly as in the proof of Theorem 2.5. We shall use Table 2.

By applying the norm map $N_{\mathbb{L}^+/k_2} = 1+\tau_2$, where $k_2 = \mathbb{Q}(\sqrt{2}, \sqrt{q})$, we get

$$\begin{aligned} N_{\mathbb{L}^+/k_2}(X^2) &= \varepsilon_2^{2a}(-1)^b \cdot \varepsilon_q^c \varepsilon_{2q}^d \cdot (-1)^e \cdot (-1)^{fv}\varepsilon_2^f \\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d \cdot (-1)^{b+e+fv}\varepsilon_2^f > 0. \end{aligned}$$

We have $b+e+fv \equiv 0 \pmod 2$. By Corollary 2.7, the units ε_q and ε_{2q} are squares in k_2 whereas ε_2 is not. Then $f=0$ and $b=e$. Therefore

$$X^2 = \varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^b,$$

$N_{\mathbb{L}^+/k_4} = 1+\tau_1$, where $k_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. We have

$$\begin{aligned} N_{\mathbb{L}^+/k_4}(X^2) &= (-1)^a \cdot \varepsilon_p^{2b} \cdot (-1)^c \cdot \varepsilon_q^c \cdot 1 \cdot \varepsilon_{pq}^b \\ &= \varepsilon_p^{2b} \cdot (-1)^{a+c} \cdot \varepsilon_q^c\varepsilon_{pq}^b > 0. \end{aligned}$$

Then $a+c = 0 \pmod 2$, so $a=c$. Since, by Corollary 2.7, the units ε_q and ε_{pq} are not squares in k_4, we have $c=b$. Thus, $a=b=c$ and

$$X^2 = \varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}^a\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^a.$$

Let us now apply $N_{\mathbb{L}^+/k_3} = 1+\tau_2\tau_3$, where $k_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$. Then

$$N_{\mathbb{L}^+/k_4}(X^2) = \varepsilon_2^{2a} \cdot (-1)^a \cdot (-1)^a \cdot (-1)^d \cdot \varepsilon_{pq}^a\varepsilon_{2pq}^a = \varepsilon_2^{2a} \cdot (-1)^d \cdot \varepsilon_{pq}^a\varepsilon_{2pq}^a > 0.$$

Thus, $d=0$. Hence, $X^2 = \varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}^a\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^a$.

Let us suppose that $a=0$. Then a FSU of $\mathbb{L}^+$ is

$$\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}.$$

Thus, $q(\mathbb{L}^+) = 2^5$. We have $h_2(p) = h_2(q) = h_2(2q) = h_2(2) = 1$ and $h_2(2p) = h_2(pq) = h_2(2pq) = 2$ (cf. [10, Corollaries 18.4, 19.7 and 19.8]). So the class number formula (cf. [15, p. 201]) gives

$$\begin{aligned} h_2(\mathbb{L}^+) &= \frac{1}{2^9}q(\mathbb{L}^+)h_2(2)h_2(p)h_2(q)h_2(2p)h_2(2q)h(pq)h_2(2pq) \\ &= \frac{1}{2^9}\cdot 2^5 \cdot 1\cdot 1\cdot 1\cdot 2\cdot 1\cdot 2\cdot 2 \\ &= \frac{1}{2}, \end{aligned}$$

which is absurd. So necessarily $a=1$ and then $q(\mathbb{L}^+) = 2^6$. Therefore we have (1).

(2) We shall proceed as in the second part of the proof of Theorem 2.5. So let

$$Y^2 = (2+\sqrt{2})\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^f\sqrt[4]{\varepsilon_2^2\varepsilon_p^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$. According to Lemma 2.2, we should find an element Y, if it exists, which is in $\mathbb{L}^+$. So firstly we shall use norm maps to eliminate some cases (see Table 2).

- We have $N_{\mathbb{L}^+/k_2} = 1+\tau_2$. Note that, by Corollary 2.7, $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ is a FSU of k_2. So we have

$$\begin{aligned} N_{\mathbb{L}^+/k_2}(Y^2) &= (2+\sqrt{2})^2\varepsilon_2^{2a}(-1)^b\varepsilon_q^c \cdot \varepsilon_{2q}^d \cdot (-1)^e \cdot (-1)^{fv}\varepsilon_2^f(-1)^{gs}\varepsilon_2^g\sqrt{\varepsilon_{2q}}^g, \\ &= (2+\sqrt{2})^2\varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d(-1)^{b+e+fv+gs}\varepsilon_2^{f+g}\sqrt{\varepsilon_{2q}}^g > 0. \end{aligned}$$

Thus, $b+e+fv+gs=0 \pmod 2$. We have

* $f=g=1$ is impossible. In fact, $\sqrt{\varepsilon_q}$ is not square in k_2.
* $f\neq g$ is impossible. In fact, $\varepsilon_2\sqrt{\varepsilon_{2q}}$ and ε_2 are not squares in k_2.

Thus, $f=g=0$ and $b=e$. It follows that

$$Y^2=(2+\sqrt{2})\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^b.$$

We have $N_{\mathbb{L}^+/k_4}=1+\tau_1$ with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. Note that, by Corollary 2.7, $\{\varepsilon_p,\varepsilon_q,\sqrt{\varepsilon_q\varepsilon_{pq}}\}$ is a FSU of k_4. We have

$$\begin{aligned}N_{\mathbb{L}^+/k_4}(Y^2)&=(4-2)(-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot\varepsilon_q^c\cdot 1\cdot(-1)^b\cdot\varepsilon_{pq}^b\\&=\varepsilon_p^{2b}\cdot(-1)^{a+b+c}\cdot 2\cdot\varepsilon_q^c\cdot\varepsilon_{pq}^b>0.\end{aligned}$$

So $a+b+c=0 \pmod 2$. Since $\varepsilon_q\varepsilon_{pq}$ is a square in k_4 whereas 2 is not, this implies $c\neq b$. So $a=1$ and we have

$$Y^2=(2+\sqrt{2})\varepsilon_2\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^b,$$

with $c\neq b$. Let us now apply $N_{\mathbb{L}^+/k_3}=1+\tau_2\tau_3$ to obtain

$$\begin{aligned}N_{\mathbb{L}^+/k_3}(Y^2)&=(2+\sqrt{2})^2\cdot\varepsilon_2^2\cdot(-1)^b\cdot(-1)^c\cdot(-1)^d.(-1)^b.\varepsilon_{pq}^b\\&=(2+\sqrt{2})^2\cdot\varepsilon_2^2\cdot(-1)^{c+d}\cdot\varepsilon_{pq}^b>0.\end{aligned}$$

Then $c+d=0 \pmod 2$ and so $c=d$. By Corollary 2.7, ε_{pq} is a not a square in k_3, therefore $b=0$. Since $c\neq b$, we have $c=d=1$. Therefore

$$Y^2=(2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}.$$

Let us now verify that $(2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}$ is a square in $\mathbb{L}^+$.

- As in the second part of the proof of Theorem 2.5, we show that

$$h_2(\mathbb{L})=h_2(-pq)=2. \tag{2.2}$$

  Assume that $(2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}$ is not a square in $\mathbb{L}^+$. Then, by Lemma 2.2 and the above discussions, $\mathbb{L}^+$ and $\mathbb{L}$ have the same fundamental system of units. Therefore $q(\mathbb{L})=2^7$. We have $h_2(-1)=h_2(-2)=h_2(-q)=1$, $h_2(-p)=h_2(-2p)=h_2(-2q)=h_2(-pq)=2$ and $h_2(-2pq)=4$ by [10, Corollary 18.4], [10, Corollary 19.6] and [12, p. 353] respectively. So, by the class number formula (cf. [15, p. 201]) and the above setting on the 2-class numbers of real quadratic fields (Page 11), we get

$$\begin{aligned}h_2(\mathbb{L})&=\frac{1}{2^{16}}q(\mathbb{L})h_2(-1)h_2(2)h_2(-2)h_2(p)h_2(-p)h_2(q)h_2(-q)h_2(2p)\\&\quad h_2(-2p)h_2(2q)h_2(-2q)h_2(pq)h_2(-pq)h_2(2pq)h_2(-2pq)\\&=\frac{1}{2^{16}}\cdot 2^7\cdot 1\cdot 1\cdot 1\cdot 1\cdot 2\cdot 1\cdot 1\cdot 2\cdot 2\cdot 1\cdot 2\cdot 2\cdot 2\cdot 2\cdot 4=1.\end{aligned}$$

  This contradicts (2.2). It follows that $(2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}$ is a square in $\mathbb{L}^+$ and so Lemma 2.2 completes the proof.

□

Table 2 Norms when p and q satisfy conditions (1.2)

ε	ε^{τ_1}	ε^{τ_2}	ε^{τ_3}	$\varepsilon^{1+\tau_1}$	$\varepsilon^{1+\tau_2}$	$\varepsilon^{1+\tau_3}$	$\varepsilon^{1+\tau_1\tau_2}$	$\varepsilon^{1+\tau_1\tau_3}$	$\varepsilon^{1+\tau_2\tau_3}$
ε_2	$\frac{-1}{\varepsilon_2}$	ε_2	ε_2	-1	ε_2^2	ε_2^2	-1	-1	ε_2^2
ε_p	ε_p	$\frac{-1}{\varepsilon_p}$	ε_p	ε_p^2	-1	ε_p^2	-1	ε_p^2	-1
$\sqrt{\varepsilon_q}$	$-\sqrt{\varepsilon_q}$	$\sqrt{\varepsilon_q}$	$\frac{-1}{\sqrt{\varepsilon_q}}$	$-\varepsilon_q$	ε_q	-1	$-\varepsilon_q$	1	-1
$\sqrt{\varepsilon_{2q}}$	$\frac{1}{\sqrt{\varepsilon_{2q}}}$	$\sqrt{\varepsilon_{2q}}$	$\frac{-1}{\sqrt{\varepsilon_{2q}}}$	1	ε_{2q}	-1	1	$-\varepsilon_{2q}$	-1
$\sqrt{\varepsilon_{pq}}$	$-\sqrt{\varepsilon_{pq}}$	$\frac{-1}{\sqrt{\varepsilon_{pq}}}$	$\frac{1}{\sqrt{\varepsilon_{pq}}}$	$-\varepsilon_{pq}$	-1	1	1	-1	$-\varepsilon_{pq}$
$\sqrt{\varepsilon_{2pq}}$	$\frac{-1}{\sqrt{\varepsilon_{2pq}}}$	$\frac{1}{\sqrt{\varepsilon_{2pq}}}$	$\frac{-1}{\sqrt{\varepsilon_{2pq}}}$	-1	1	-1	$-\varepsilon_{2pq}$	ε_{2pq}	$-\varepsilon_{2pq}$
$\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}$	$(-1)^u\sqrt{\frac{\varepsilon_p}{\varepsilon_2\varepsilon_{2p}}}$	$(-1)^v\sqrt{\frac{\varepsilon_2}{\varepsilon_p\varepsilon_{2p}}}$	$\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}$	$(-1)^u\varepsilon_p$	$(-1)^v\varepsilon_2$	$\varepsilon_2\varepsilon_p\varepsilon_{2p}$			
$\sqrt[4]{\varepsilon_2^2\varepsilon_p^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}$	$(-1)^r\sqrt[4]{\frac{\varepsilon_p^2\varepsilon_q\varepsilon_{pq}}{\varepsilon_2^2\varepsilon_{2pq}}}$	$(-1)^s\sqrt[4]{\frac{\varepsilon_2^2\varepsilon_q}{\varepsilon_p^2\varepsilon_{pq}\varepsilon_{2pq}}}$	$(-1)^t\sqrt[4]{\frac{\varepsilon_2^2\varepsilon_p^2}{\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}}$	$(-1)^r\varepsilon_p\sqrt{\varepsilon_q\varepsilon_{pq}}$	$(-1)^s\varepsilon_2\sqrt{\varepsilon_q}$	$(-1)^t\varepsilon_2\varepsilon_p$			

3 Remarks on Hilbert 2-class field towers and cyclotomic $\mathbb{Z}_2$-extensions

Let k be an algebraic number field and $\mathrm{Cl}_2(k)$ the 2-Sylow subgroup of its ideal class group $\mathrm{Cl}(k)$. Let $k^{(1)}$ (resp. $k^{(2)}$) be the first (resp. second) Hilbert 2-class field of k and put $G = \mathrm{Gal}(k^{(2)}/k)$. Then if G' denotes the commutator subgroup of G, we have by class field theory that $G' \simeq \mathrm{Gal}(k^{(2)}/k^{(1)})$ and $G/G' \simeq \mathrm{Gal}(k^{(1)}/k) \simeq \mathrm{Cl}_2(k)$. Assume in all what follows that $\mathrm{Cl}_2(k)$ is of type $(2, 2)$. Then in [13] Kisilevsky showed that G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, Q_m, D_m or S_m, where Q_m, D_m, and S_m denote the quaternion, dihedral and semidihedral groups, respectively, of order 2^m, where $m \geq 3$ and $m \geq 4$ for S_m. Let F_1, F_2 and F_3 be the three unramified quadratic extensions of k and assume that the 2-class group of F_1 is cyclic. Then, using some known results of group theory, one can easily deduce from [13, Theorem 2] that we have the following remark (cf. [9, Remark 2.2]):

Remark 3.1 The 2-class groups of the two fields F_2 and F_3 are cyclic if and only if $k^{(1)} = k^{(2)}$ or $k^{(1)} \neq k^{(2)}$ and $G \simeq Q_3$. In the other cases the 2-class groups F_2 and F_3 are of type $(2, 2)$ (whereas that of F_1 is cyclic).

Set the following notations:

(1) L_{pq}: $\mathbb{Q}(\sqrt{2}, \sqrt{qp}, i)$,
(2) F_{pq}: $\mathbb{Q}(\sqrt{p}, \sqrt{2q}, i)$,
(3) K_{pq}: $\mathbb{Q}(\sqrt{2p}, \sqrt{q}, i)$.

Remark 3.2 Let p and q be two primes satisfying conditions (1.1) or (1.2). Note that, by Lemmas 2.3 and 2.6, $x \pm 1$ is not a square in $\mathbb{N}$, where x and y are the two integers such that $\varepsilon_{2pq} = x + y\sqrt{2pq}$. Thus, the Hasse unit index of $\Bbbk = \mathbb{Q}(\sqrt{2pq}, i)$ equals 1 (cf. [3, 3.(1) p. 19]). So by [1] the 2-class group of $\Bbbk$ is of type $(2, 2)$. We similarly deduce, by using Lemmas 2.3 and 2.6, [9, Lemma 4.1] and [3, 3.(1) p. 19], that the condition on the Hasse unit index of $\Bbbk$, in [2, Théorème 21], is always verified (so, in particular, the conditions of this theorem hold).

Theorem 3.3 *Let p and q be two primes satisfying (1.1). Then the following hold.*

(1) *The 2-class group of L_{pq} is $\mathbb{Z}/2^{m+1}\mathbb{Z}$, with $h_2(-pq) = 2^m$.*
(2) *The 2-class group of F_{pq} is of type $(2, 2)$.*
(3) *The 2-class group of K_{pq} is of type $(2, 2)$.*

Proof Let $\Bbbk = \mathbb{Q}(\sqrt{2pq}, i)$. By Remark 3.2, the 2-class group of $\Bbbk$ is of type $(2, 2)$. Note that L_{pq}, F_{pq} and K_{pq} are the three unramified quadratic extensions of $\Bbbk$. Note that, by [8, Theorem 10], the 2-class group L_{pq} is cyclic. So the above preliminaries complete the proof. $\square$

Since the 2-class group of L_{pq} is cyclic, the Hilbert 2-class field tower of L_{pq} terminates at the first layer. Now we shall determine the structure of the groups $\mathrm{Gal}(F_{pq}^{(2)}/F_{pq})$ and $\mathrm{Gal}(K_{pq}^{(2)}/K_{pq})$.

Theorem 3.4 *Let p and q be two primes and m such that and $h_2(-pq) = 2^m$.*

(1) *If p and q satisfy (1.1), then we have*

$$\mathrm{Gal}(F_{pq}^{(2)}/F_{pq}) \simeq \mathrm{Gal}(K_{pq}^{(2)}/K_{pq}) \simeq Q_{m+1}.$$

(2) *If p and q satisfy (1.2), then we have*

$$\mathrm{Gal}(F_{pq}^{(2)}/F_{pq}) \simeq \mathrm{Gal}(K_{pq}^{(2)}/K_{pq}) \simeq \mathbb{Z}/4\mathbb{Z}.$$

Proof Let $L_{1,pq}=\mathbb{Q}(\sqrt{2},\sqrt{q},\sqrt{p},i)$. Since the 2-class group of L_{pq} is cyclic (so its Hilbert 2-class field tower terminates at the first layer) and $L_{1,pq}$ is an unramified extension of L_{pq}, we have that the 2-class group of $L_{1,pq}$ is also cyclic. As $L_{1,pq}$ is also a quadratic unramified extension of both F_{pq} and K_{pq}, then by [9, Proposition 2.2] and Theorems 2.5 and 2.8 we have $|\mathrm{Gal}(F_{pq}^{(2)}/F_{pq})|=|\mathrm{Gal}(K_{pq}^{(2)}/K_{pq})|=2\cdot h_2(L_1)=2\cdot h_2(-pq)$. Since $h_2(-pq)$ is even, and $h_2(-pq)=2$ if and only if $\left(\frac{p}{q}\right)=-1$ (cf. [10, Corollaries 18.4 and 19.6]), then we have the orders of the groups in question in both cases. Suppose that $\left(\frac{p}{q}\right)=1$. Then, by [2, Théorème 21], Remark 3.2 and the above discussions, the two groups in question are subgroups of index 2 of Q_{m+2}. So they are also quaternions of order 2^{m+1}.

Suppose now that $\left(\frac{p}{q}\right)=-1$. Then, by [2, Théorème 21] and Remark 3.2, the groups in question are subgroups of Q_3 of index 2. So they are cyclic. This completes the proof. □

Remark 3.5 Put $\Bbbk=\mathbb{Q}(\sqrt{2pq},i)$ and assume that p and q are two primes satisfying (1.1). The author of [2] did not determine the order of $\mathrm{Gal}(\Bbbk^{(2)}/\Bbbk)$. Now it is easy to deduce that it is of order $|\mathrm{Gal}(\Bbbk^{(2)}/\Bbbk)|=2^{m+2}$ (i.e., $\mathrm{Gal}(\Bbbk^{(2)}/\Bbbk)\simeq Q_{m+2}$), where m is such that $h_2(-pq)=2^m$.

Theorem 3.6 *Let p and q be two primes satisfying* (1.1) *or* (1.2). *Put $\pi_1=2$, $\pi_2=2+\sqrt{2}$,…, $\pi_n=2+\sqrt{\pi_{n-1}}$, $L_n=\mathbb{Q}(\sqrt{q},\sqrt{p},\zeta_{2^{n+2}})$ and $L_n^+=\mathbb{Q}(\sqrt{p},\sqrt{q},\sqrt{\pi_n})$. Then the following hold.*

(1) *For all $n\geq 1$, the 2-class group of L_n^+ is trivial.*
(2) *For all $n\geq 1$, the 2-class group of L_n is $\mathbb{Z}/2^{n+m-1}\mathbb{Z}$, where $h_2(-pq)=2^m$.*

Proof (1) We claim that the 2-class group of $k=\mathbb{Q}(\sqrt{p},\sqrt{q})$ is trivial. In fact, by Corollaries 2.4 and 2.7, [10, Corollaries 18.4 and 19.7] and Kuroda's class number formula ([14, p. 247]), we obtain

$$h_2(k)=\frac{1}{4}q(k)h_2(p)h_2(q)h_2(pq)=\frac{1}{4}\cdot 2\cdot 1\cdot 1\cdot 2=1.$$

By Theorems 2.5 and 2.8, the class number of $\mathbb{Q}(\sqrt{p},\sqrt{q},\sqrt{2})$ the first step of the cyclotomic $\mathbb{Z}_2$-extension of k is odd. So we have proved (1) by [11, Theorem 1].

(2) Note that L_n is the genus field of $L_{n,pq}=\mathbb{Q}(\sqrt{pq},\zeta_{2^{n+2}})$ and $[L_n:L_{n,pq}]=2$. By [8, Theorem 10], the 2-class group of $L_{n,pq}$ is isomorphic to a cyclic group of order 2^{n+m}, therefore that of L_n is also cyclic and $h_2(L_n)=\frac{h_2(L_{n,pq})}{2}=2^{n+m-1}$. So (2) is proved. □

Remark 3.7 Let p and q be two primes satisfying (1.1) or (1.2). By the above theorem, the Iwasawa invariants λ_2 and ν_2 of the fields F_{pq}, K_{pq} and $\mathbb{Q}(\sqrt{p},\sqrt{q},i)$ are equal to 1 and $m-1$, respectively.

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