# ISOMETRIES BETWEEN COMPLETELY REGULAR VECTOR-VALUED FUNCTION SPACES 

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#### Abstract

In this paper, first we study surjective isometries (not necessarily linear) between completely regular subspaces $A$ and $B$ of $C_{0}(X, E)$ and $C_{0}(Y, F)$ where $X$ and $Y$ are locally compact Hausdorff spaces and $E$ and $F$ are normed spaces, not assumed to be neither strictly convex nor complete. We show that for a class of normed spaces $F$ satisfying a new defined property related to their $T$-sets, such an isometry is a (generalized) weighted composition operator up to a translation. Then we apply the result to study surjective isometries between $A$ and $B$ whenever $A$ and $B$ are equipped with certain norms rather than the supremum norm. Our results unify and generalize some recent results in this context.


## 1. Introduction

Considerable works have been done to characterize linear isometries between various Banach spaces of functions. The result for surjective isometries between Ba nach spaces of all continuous functions was initiated by Banach and Stone as the weighted composition operators. There are various generalizations of this theorem based on different techniques. For instance, for the characterization of (surjective) isometries between subspaces of continuous scalar-valued functions endowed with the supremum norm or special complete norms, see $[6,9,14,16,21,20]$. The first vector-valued version of the Banach-Stone theorem was given by Jerison in [13]. By Jerison's result, if $E$ is a strictly convex Banach space, then any surjective linear isometry on the Banach space $C(X, E)$ of all $E$-valued continuous functions on a compact Hausdorff space $X$ is a generalized weighted composition operator. There

[^0]are similar results for the case where the dual space of $E$ is strictly convex [17] and in general for the case that $E$ has trivial centralizer, see [8, Cor. 7.4.11].

Surjective linear isometries between subspaces of vector-valued Lipschitz functions with respect to particular complete norms have been studied, for instance, in $[1,2$, $15,19]$. In $[1,15]$ the target spaces are assumed to be strictly convex. In [2] the values of the Lipschitz functions are taken in a quasi-sub-reflexive Banach space with trivial centralizer and this result has been improved in [19] without the quasi-sub-reflexivity assumption. We should note that in a strictly convex normed space $E$, any maximal convex subset of the unit sphere of $E$ is a singleton. However, in [12] there are some results for surjective supremum norm isometries between vector-valued spaces of continuous functions with values in a Banach space $E$ whose unit sphere contains a maximal convex subset which is a singleton. Characterizatin of surjective isometries on spaces of vector-valued continuously differentiable functions with values in a finite-dimensional real Hilbert space can be found in [3]. Surjective isometries (not necessarily linear) between spaces of vector-valued absolutely continuous functions with values in a strictly convex normed space have been studied in [11]. In the recent paper [10] of Hatori, he studies linear isometries between certain Banach algebras with values in $C(Y)$, where $Y$ is a compact Hausdorff space. These Banach algebras include the $C(Y)$-valued Banach algebra of Lipschitz functions and $C(Y)$ valued Banach algebra of continuously differentiable functions. Recently, in [18], isometries on certain subspaces of vector-valued continuous functions with respect to the supremum norm and the other (not necessarily complete) norms have been characterized. We should note that, in [18], neither the target space itself nor its dual space is assumed to be strictly convex, but they satisfy a mild condition related to the maximal convex subsets of the unit spheres.

In this paper we deal with surjective (not necessarily linear) isometries between completely regular subspaces of vector-valued continuous functions endowed with either the supremum norm or other certain norms. Introducing a property, called $\left(D_{w}\right)$, on the target spaces which is considerably weaker than the strict convexity, we obtain some characterizations for such isometries as generalized weighted composition operators.

## 2. Preliminaries

Throughout this paper $\mathbb{K}$ stands for the scalar fields $\mathbb{R}$ and $\mathbb{C}$. For a normed space $E$ over $\mathbb{K}$, we denote its closed unit ball by $E_{1}$ and its unit sphere by $S(E)$. We use the notations $E^{*}$ and $\operatorname{ext}\left(E_{1}^{*}\right)$ for the dual space of $E$ and the set of extreme points of $E_{1}^{*}$, respectively.

For a $\mathbb{K}$-normed space $E$, a subset $S$ of $E$ is said to be norm additive if for any finite collection $e_{1}, e_{2}, \ldots, e_{n}$ of elements of $S$,

$$
\left\|e_{1}+\cdots+e_{n}\right\|=\left\|e_{1}\right\|+\cdots\left\|e_{n}\right\| .
$$

A maximal norm additive subset of $E$ is called a $T$-set in $E$. If $e_{1}, e_{2} \in E$ such that $\left\|e_{1}+e_{2}\right\|=\left\|e_{1}\right\|+\left\|e_{2}\right\|$, then for all $s, t \geq 0$ we have $\left\|s e_{1}+t e_{2}\right\|=s\left\|e_{1}\right\|+t\left\|e_{2}\right\|$, see [13, Lemma 4.1]. Hence for any $T$-set $S$ in $E$ and any $t \geq 0$ we have $t S \subseteq S$.

For each $e \in S(E)$ the star-like set $\operatorname{St}(e)$ is defined as

$$
\operatorname{St}(e)=\left\{e^{\prime} \in S(E):\left\|e+e^{\prime}\right\|=2\right\} .
$$

It is well-known that $\operatorname{St}(e)$ is the union of all maximal convex subsets of $S(E)$ containing $e$. Clearly, in the case that $E$ is strictly convex, we have $\operatorname{St}(e)=\{e\}$ for all $e \in S(E)$. We also note that if $e \in S(E)$ such that $\operatorname{St}(e)=\{e\}$, then $e$ is an extreme point of $E_{1}$, that is $E$ is strictly convex if and only if $\operatorname{St}(e)=\{e\}$ for all $e \in S(E)$. For each $e \in S(E)$ and $e^{\prime} \in \operatorname{St}(e)$ we have $\left\|r e+e^{\prime}\right\|>r=\|r e\|$ for $r>0$. Motivated by this, for each $u \in E$ we put

$$
\operatorname{St}_{w}(u)=\left\{e^{\prime} \in S(E):\left\|u+e^{\prime}\right\|>\|u\|\right\}
$$

It should be noted that if $e^{\prime} \in \operatorname{St}_{w}(u)$, then $\left\|u+r e^{\prime}\right\|>\|u\|$ for all $r \geq 1$, that is $e^{\prime} \in \operatorname{St}_{w}\left(\frac{u}{r}\right)$ for all $r \geq 1$.

For a topological space $X$ and a normed space $E$ over $\mathbb{K}$, let $C(X, E)$ be the space of all continuous $E$-valued functions on $X$. For an element $v \in E$, the constant function $x \mapsto v$ in $C(X, E)$ will be denoted by $\hat{v}$. In the case that $X$ is locally compact, $C_{0}(X, E)$ denotes the normed space of all continuous $E$-valued functions on $X$ vanishing at infinity, with the supremum norm $\|\cdot\|_{\infty}$. By [7, Theorem 2.3.5], for $\mathcal{Z}=C_{0}(X, E)$ we have

$$
\operatorname{ext}\left(\mathcal{Z}_{1}^{*}\right)=\left\{v^{*} \circ \delta_{x}: v^{*} \in \operatorname{ext}\left(E_{1}^{*}\right), x \in X\right\},
$$

where for each $x \in X, \delta_{x}: C_{0}(X, E) \longrightarrow E$ is defined by $\delta_{x}(f)=f(x)$. Moreover, if $A$ is a subspace of $C_{0}(X, E)$ then, by [7, Corollary 2.3.6], we have

$$
\operatorname{ext}\left(A_{1}^{*}\right) \subseteq\left\{v^{*} \circ \delta_{x}: v^{*} \in \operatorname{ext}\left(E_{1}^{*}\right), x \in X\right\}
$$

The Choquet boundary of $A$ which is denoted by $\operatorname{ch}(A)$, consists of all points $x \in X$ such that $v^{*} \circ \delta_{x}$ is an extreme point of $A_{1}^{*}$ for some $v^{*} \in \operatorname{ext}\left(E_{1}^{*}\right)$. Then $\operatorname{ch}(A)$ is a boundary for $A$, that is for each $f \in A$ there exists a point $x \in \operatorname{ch}(A)$ such that $\|f\|_{\infty}=\|f(x)\|$.

By [8, Lemma 7.2.2] for a locally compact Hausdorff space $X$ and a normed space $E$, if $S$ is a $T$-set in $E$ and $x \in X$, then the set

$$
(S, x)=\left\{f \in C_{0}(X, E): f(x) \in S,\|f\|_{\infty}=\|f(x)\|\right\}
$$

is a $T$-set in $C_{0}(X, E)$. Conversely, any $T$-set in $C_{0}(X, E)$ is of this form.

For any $T$-set $S$ in a normed space $E$, we put

$$
\Gamma_{S}=\left\{v^{*} \in S\left(E^{*}\right): v^{*}(u)=\|u\| \text { for all } u \in S\right\} .
$$

By [8, Lemma 7.2.4] we have $\Gamma_{S} \cap \operatorname{ext}\left(E_{1}^{*}\right) \neq \emptyset$. We should note that the Lemmas 7.2.2 and 7.2.4 in [8] have been stated for the case that $E$ is a Banach space, however, the completeness of $E$ has no role in the proofs. It is obvious that for any $T$-set $R$ in $E$, the corresponding set $\Gamma_{R}$ is a convex subset of $S\left(E^{*}\right)$ which is norm additive.

The following proposition, which states some elementary properties of $T$-sets, is easily verified. For the sake of completeness we state it here.

Proposition 2.1. Let $E$ be a normed space over $\mathbb{K}$. Then
(i) For any $T$-set $R$ in $E$ and $w^{*} \in \Gamma_{R}$ we have $R=\left\{u \in E: w^{*}(u)=\|u\|\right\}$.
(ii) For distinct $T$-sets $R_{1}$ and $R_{2}$ in $E$, if $R_{1} \cap R_{2} \neq\{0\}$, then for any $w_{1}^{*} \in \Gamma_{R_{1}}$ and any $w_{2}^{*} \in \Gamma_{R_{2}}$ we have $w_{1}^{*} \in \operatorname{St}\left(w_{2}^{*}\right)$.
(iii) For distinct T-sets $R_{1}$ and $R_{2}$ in $E$ we have $\Gamma_{R_{1}} \cap \Gamma_{R_{2}}=\emptyset$.
(iv) If $E^{*}$ is strictly convex, then for any $T$-set $R$ in $E, \Gamma_{R}$ is a singleton.

Proof. (i) It is obvious that $R \subseteq\left\{u \in E: w^{*}(u)=\|u\|\right\}$. Now since for each $u_{1}, \ldots, u_{n} \in E$ with $w^{*}\left(u_{i}\right)=\left\|u_{i}\right\|, i=1, \ldots, n$, we have

$$
\left\|u_{1}+\cdots+u_{n}\right\|=\left\|u_{1}\right\|+\cdots+\left\|u_{n}\right\|
$$

the maximality of $R$ implies that $R=\left\{u \in E: w^{*}(u)=\|u\|\right\}$.
(ii) Let $u \in R_{1} \cap R_{2}$ be nonzero. Since $\left(w_{1}^{*}+w_{2}^{*}\right)(u)=2\|u\|$ it follows that $\left\|w_{1}^{*}+w_{2}^{*}\right\|=2$ that is $w_{1}^{*} \in \operatorname{St}\left(w_{2}^{*}\right)$.
(iii) Assume on the contrary that $\Gamma_{R_{1}} \cap \Gamma_{R_{2}} \neq \emptyset$ and let $w^{*}$ be a point in this intersection. Then by (i) we have

$$
R_{1}=\left\{u \in E: w^{*}(u)=\|u\|\right\}=R_{2},
$$

which is a contradiction.
(iv) Let $R$ be a $T$-set in $E$ and $w_{1}^{*}, w_{2}^{*} \in \Gamma_{R}$ be distinct. Being $E^{*}$ strictly convex we have $\left\|w_{1}^{*}+w_{2}^{*}\right\|<2$ while by (i) for any nonzero $u \in R$ we have $w_{1}^{*}(u)+w_{2}^{*}(u)=2\|u\|$, a contradiction.

Two $T$-sets $S$ and $R$ in a normed space $E$ are said to be discrepant if either $S \cap R=\{0\}$, or there exists a T-set $L$ in $E$ such that $R \cap L=S \cap L=\{0\}$. A normed space $E$ is said to satisfy property $(D)$ if any two T-sets in $E$ are discrepant (see Definition 7.2.10 in [8]). Since in a strictly convex space $E$ each $T$-set is of the form $\{t u: t \geq 0\}$ for some nonzero $u \in E$, it follows that for any two distinct $T$-sets $S$ and $R$ in $E$ we have $S \cap R=\{0\}$, that is all strictly convex spaces have property $(D)$. For some examples of non-strictly convex normed spaces with this property, see Examples 7.2.11 in [8].

## 3. Main Results

In this section, introducing a property, called $\left(D_{w}\right)$, which is weaker than the property $(D)$, we characterize surjective (not necessarily linear) isometries $T: A \longrightarrow$ $B$ between certain subspaces $A$ and $B$ of of $C_{0}(X, E)$ and $C_{0}(Y, F)$ where $F$ satisfies the property $\left(D_{w}\right)$.

Definition 3.1. We say that a normed space $E$ satisfies property $\left(D_{w}\right)$ if there exists a $T$-set $R_{0}$ in $E$ which is discrepant to any other $T$-set in $E$, that is for any $T$-set $R$ in $E$ distinct from $R_{0}$ we have either $R_{0} \cap R=\{0\}$ or there exists a $T$-set $L$ in $E$ such that $R_{0} \cap L=R \cap L=\{0\}$.

Remark. (i) We should note that if $E$ is a normed space with a $T$-set $R_{0}$ such that $R_{0} \cap R=\{0\}$ for all $T$-sets $R$ in $E$ distinct from $R_{0}$, then $E$ clearly satisfies the property $(D)$. In particular, if $E$ is a normed space whose unit sphere $S(E)$ contains an element $e$ with $\operatorname{St}(e)=\{e\}$, then $E$ has property $(D)$. For an example of a non-strictly convex space $E$ such that $\operatorname{St}(e)=\{e\}$ for some $e \in S(E)$, see [12].
(ii) If $E$ is a normed space and $w_{0}^{*} \in \cup_{R} \Gamma_{R}$, where the union is taken over all $T$-sets of $E$, such that $\operatorname{St}\left(w_{0}^{*}\right)=\left\{w_{0}^{*}\right\}$ then, using Proposition 2.1, for the $T$-set $R_{0}$ in $E$ containing $w_{0}^{*}$ we have $R_{0} \cap R=\{0\}$ for all $T$-sets $R$ in $E$ distinct from $R_{0}$. Hence $E$ has property $(D)$. In particular, if $E^{*}$ is strictly convex, then any pair of distinct $T$-sets of $E$ has trivial intersection and consequently $E$ has property $(D)$.

It is clear that property $\left(D_{w}\right)$ is weaker than the property $(D)$. We give an example which shows that the converse statement does not necessarily hold.

Example 3.2. Let $E$ be a normed space whose closed unit ball is the subset $K$ of $\mathbb{R}^{3}$ as in Figure 1 with the origin in the center of $K$. Indeed, since $K$ is a symmetric compact convex subset of $\mathbb{R}^{3}$ and origin is an interior point of $K$, it suffices to consider $E=\mathbb{R}^{3}$ with the norm $\|\cdot\|$ defined by $\|0\|=0$ and for each nonzero point $x \in \mathbb{R}^{n},\|x\|=\frac{1}{\max \{t \in \mathbb{R}: t x \in K\}}$. Then $K$ is the closed unit ball of $E$ with respect to this norm.

The set $K$ consists of a cube and two pyramid on up and down. Hence the unit sphere of $E$ has twelve maximal convex subsets (four faces of cube, four faces of upper pyramid and four faces of bottom pyramid), that is $E$ has twelve $T$-sets. We note that $E$ does not satisfy the property $(D)$. Indeed, letting $R$ and $S$ be the $T$-sets corresponding to two adjacent faces in the cube, we have $R \cap S=\{0\}$. Meanwhile, the other $T$-sets clearly have non-empty intersection with at least one of $R$ or $S$. Thus $E$ does not satisfy the property $(D)$. However, considering $R_{0}$ as the $T$-set corresponding to one of the upper pyramid faces we see that $E$ satisfies the property $\left(D_{w}\right)$.


Figure 1.

For a subspace $A$ of $C_{0}(X, E)$ we say that $A$ is $E$-separating if for distinct points $x_{1} \neq x_{2}$ in $X$ and any $u \in E$ there exists $f \in A$ such that $f\left(x_{1}\right)=u,\|f\|_{\infty}=\|u\|$ and $f\left(x_{2}\right)=0$. We say that $A$ is completely regular if for any $x$ in $X$, any $u \in E$ and any neighbourhood $U$ of $x$ there exists $f \in A$ such that $f(x)=u,\|f\|_{\infty}=\|u\|$ and $f=0$ on $X \backslash U$. It is clear that any completely regular subspace of $C_{0}(X, E)$ is $E$-separating.

For examples of completely regular subspaces of $C(K, E)$, where $K$ is a compact Hausdorff space, we can refer to the spaces $\operatorname{Lip}^{\alpha}(K, E)$ of $E$-valued Lipschitz functions of order $\alpha \in[0,1]$ on a compact Hausdorff space $K$, the space $C^{n}([0,1], E)$ of $E$-valued $n$-times continuously differentiable functions on $[0,1]$, and its subspace $\operatorname{Lip}^{n}([0,1], E)$ consisting of those functions whose derivatives are also Lipschitz functions. The space $\mathrm{AC}(K, E)$ of all absolutely continuous $E$-valued functions on the compact subset $K$ of the real-line is also completely regular. On the other hand, by [5], for any locally compact Hausdorff space $X$, the kernel of each continuous complex-valued regular Borel measure on $X$ is a completely regular subspace of $C_{0}(X)$.

Next Lemma gives the general form of $T$-sets in $E$-separating subspaces of $C_{0}(X, E)$.
Lemma 3.3. Let $X$ be a locally compact Hausdorff space, $E$ be a real or complex normed space and $A$ be an E-separating subspace of $C_{0}(X, E)$. Then for any $T$-set $S$ in $E$ and any point $x \in X$, the set $(S, x) \cap A$ is a $T$-set in $A$ and conversely, any $T$-set in $A$ is of this form.

Proof. First assume that $R$ is a $T$-set in $A$. Since $R$ is a norm additive subset of $C_{0}(X, E)$, it is contained in a $T$-set in $C_{0}(X, E)$. Hence there exists a $T$-set $S$ in $E$ and a point $x \in X$ such that $R \subseteq(S, x)$. We note that $R \subseteq(S, x) \cap A$ and clearly $(S, x) \cap A$ is a norm additive subset of $A$. Hence it follows from the maximality of $R$ that $R=(S, x) \cap A$.

To prove the converse statement, let $S$ be a $T$-set in $E$ and let $x \in X$. Since $(S, x) \cap A$ is a norm additive subset of $A$ there exists a T-set $R$ in $A$ such that $(S, x) \cap A \subseteq R$. By the first part, there are a $T$-set $S_{0}$ in $E$ and a point $x_{0} \in X$ such that $R=\left(S_{0}, x_{0}\right) \cap A$. We claim that $S=S_{0}$ and $x=x_{0}$. Assume that $x \neq x_{0}$. Choosing a nonzero element $e \in S$ it follows from the hypothesis that there exists $f \in A$ such that $f(x)=e,\|f\|_{\infty}=\|e\|$ and $f\left(x_{0}\right)=0$. Thus $f \in$ $(S, x) \cap A \subseteq\left(S_{0}, x_{0}\right) \cap A$. Hence $\|f\|_{\infty}=\left\|f\left(x_{0}\right)\right\|=0$, a contradiction. This shows that $x_{0}=x$. Since $S$ and $S_{0}$ are both $T$-sets in $E$, to prove that $S=S_{0}$ it suffices to show that $S \subseteq S_{0}$. For this suppose that $e \in S$, and choose $f \in A$ such that $f(x)=e,\|f\|_{\infty}=\|e\|$. Then $f \in(S, x) \cap A \subseteq\left(S_{0}, x_{0}\right) \cap A$. In particular, $e=f(x)=f\left(x_{0}\right) \in S_{0}$, that is $S \subseteq S_{0}$, as desired.

Proposition 3.4. Let $X$ be a locally compact Hausdorff space, $E$ be a real or complex normed space and $A$ be an $E$-separating subspace of $C_{0}(X, E)$. Then for any $x \in X$ and any $T$-set $S$ in $E$ there exists $v^{*} \in \Gamma_{S} \cap \operatorname{ext}\left(E_{1}^{*}\right)$ such that $v^{*} \circ \delta_{x} \in \operatorname{ext}\left(A_{1}^{*}\right)$. In particular, $\operatorname{ch}(A)=X$.

Proof. Let $x \in X$ and $S$ be a T-set in $E$. By Lemma 3.3, $(S, x) \cap A$ is a T-set in $A$. Hence that there exists $l \in \operatorname{ext}\left(A_{1}^{*}\right)$ such that $l(f)=\|f\|_{\infty}$ for all $f \in(S, x) \cap A$. Since $l \in \operatorname{ext}\left(A_{1}^{*}\right)$, there are $z \in X$ and $v^{*} \in \operatorname{ext}\left(E_{1}^{*}\right)$ such that $l=v^{*} \circ \delta_{z}$. Thus for all $f \in(S, x) \cap A$ we have

$$
\begin{equation*}
v^{*}(f(z))=v^{*} \circ \delta_{z}(f)=l(f)=\|f\|_{\infty} . \tag{3.1}
\end{equation*}
$$

We show that $z=x$ and $v^{*} \in \Gamma_{S}$. Suppose that $z \neq x$ and let $u$ be a nonzero element of $S$. By hypothesis, there exists $f \in A$ such that $f(x)=u, f(z)=0$ and $\|f\|_{\infty}=\|u\|$. Then $f \in(S, x) \cap A$ and it follows from (3.1) that $0=v^{*}(f(z))=$ $\|f\|_{\infty}=\|u\|$, a contradiction. Hence $z=x$. To show that $v^{*} \in \Gamma_{S}$, let $u \in S$ and choose $f \in A$ such that $f(x)=u$ and $\|f\|_{\infty}=\|u\|$. Then $f \in(S, x) \cap A$ and it follows from (3.1) that $v^{*}(u)=\|u\|$. This shows that $v^{*} \in \Gamma_{S}$, as desired.

Proposition 3.5. Let $X$ and $Y$ be locally compact Hausdorff spaces, $E$ and $F$ be real or complex normed spaces (not necessarily complete) and $A$ and $B$ be $E$ separating and $F$-separating subspaces of $C_{0}(X, E)$ and $C_{0}(Y, F)$, respectively. Let $T: A \longrightarrow B$ be a surjective real-linear isometry. If $x \in X$ and $y \in Y$ such that $T((S, x) \cap A)=(R, y) \cap B$ where $S$ and $R$ are $T$-sets in $E$ and $F$, respectively, then there are $v^{*} \in \Gamma_{S} \cap \operatorname{ext}\left(E_{1}^{*}\right)$ and $w^{*} \in \Gamma_{R} \cap \operatorname{ext}\left(F_{1}^{*}\right)$ such that $T^{*}\left(w^{*} \circ \delta_{y}\right)=v^{*} \circ \delta_{x}$.

Proof. Since the real dual of a complex normed space is isometrically isomorphic to its complex dual, without loss of generality we assume that $E$ and $F$ are real normed spaces. Let $A^{*}$ and $B^{*}$ denote the (real) duals of $A$ and $B$, respectively, and $T^{*}: B^{*} \longrightarrow A *$ be the adjoint of $T$ as a bounded real-linear operator. Then $T^{*}$ is a surjective real-linear isometry. By Proposition 3.4 for the T-set $R$ in $F$ and the
point $y \in Y$ there exists $w^{*} \in \Gamma_{R}$ such that $w^{*} \circ \delta_{y} \in \operatorname{ext}\left(B_{1}^{*}\right)$. Since $T^{*}\left(w^{*} \circ \delta_{y}\right)$ is an extreme point of the unit ball of $A^{*}$ there are $v^{*} \in \operatorname{ext}\left(E_{1}^{*}\right)$ and $x_{0} \in X$ such that $T^{*}\left(w^{*} \circ \delta_{y}\right)=v^{*} \circ \delta_{x_{0}}$, that is

$$
\begin{equation*}
w^{*}(T f(y))=v^{*}\left(f\left(x_{0}\right)\right) \quad(f \in A) . \tag{3.2}
\end{equation*}
$$

It suffices to show that $v^{*} \in \Gamma_{S}$ and $x_{0}=x$. Suppose that $x_{0} \neq x$. Considering a nonzero element $u \in S$ we can find a function $f \in A$ such that $f(x)=u, f\left(x_{0}\right)=0$ and $\|f\|_{\infty}=\|u\|$. Hence $f \in(S, x) \cap A$ and consequently $T f \in(R, y) \cap B$, that is $\|T f(y)\|=\|T f\|_{\infty}$ and $T f(y) \in R$. Thus, using (3.2) we have

$$
\|u\|=\|f\|_{\infty}=\|T f\|_{\infty}=\|T f(y)\|=w^{*}(T f(y))=v^{*}\left(f\left(x_{0}\right)\right)=0
$$

which is a contradiction. This implies that $x_{0}=x$.
Now to show that $v^{*} \in \Gamma_{S}$, let $u \in S$ be given and choose $f \in A$ such that $f(x)=u$ and $\|f\|_{\infty}=\|u\|$. Then $f \in(S, x) \cap A$ and consequently $T f \in(R, y) \cap B$. Using (3.2), once again, it follows that $v^{*}(u)=\|u\|$, as desired.

Lemma 3.6. Let $X$ be a locally compact Hausdorff space, $E$ be a real or complex normed space and $A$ be a completely regular subspace of $C_{0}(X, E)$. If $S$ and $R$ are $T$-sets in $E$ and $x, z \in X$ such that $(S, x) \cap(R, z) \cap A=\{0\}$, then $x=z$ and $S \cap R=\{0\}$.

Proof. First assume that $x \neq z$ and choose disjoint neighbourhoods $U$ and $V$ of $x$ and $z$, respectively in $X$. Since any T-set is closed under positive multiples, there are nonzero elements $u \in S$ and $v \in R$ such that $\|u\|=\|v\|$. By hypothesis, there exist $f, g \in A$ such that $f(x)=u,\|f\|_{\infty}=\|f(x)\|$ and $f=0$ on $X \backslash U$, and similarly $g(z)=v,\|g\|_{\infty}=\|g(z)\|$ and $g=0$ on $X \backslash V$. We put $F=f+g$. Then clearly $\|F\|_{\infty}=\|u\|=\|v\|, F(x)=u$ and $F(z)=v$. Hence $F \in(S, x) \cap(R, z) \cap A=\{0\}$, a contradiction.

To show that $S \cap R=\{0\}$, let $u$ be a nonzero element in $S \cap R$. By hypothesis, there exists $f \in A$ such that $f(x)=u,\|f\|_{\infty}=\|u\|$, that is $f \in(S, x) \cap(R, z) \cap A=\{0\}$, which is again a contradiction.

Let $X$ and $Y$ be locally compact Hausdorff spaces, $E$ and $F$ be real or complex normed spaces (not necessarily complete), $A$ and $B$ be $E$-separating and $F$ separating subspaces of $C_{0}(X, E)$ and $C_{0}(Y, F)$, respectively. Let $T: A \longrightarrow B$ be a surjective real-linear isometry. Being $T^{-1}$ an isometry, it maps any $T$-set in $B$ to a $T$-set in $A$. Hence for any $T$-set $R$ in $F$ and a point $y \in Y$, there exist a $T$-set $S$ in $E$ and a point $x \in X$ such that $T^{-1}((R, y) \cap B)=(S, x) \cap A$. Using the separating property, it is easy to see that the $T$-set $S$ and the point $x \in X$ satisfying this equality are uniquely determined. Hence for each $T$-set $R$ in $F$ we can define
a function $\varphi_{R}: Y \longrightarrow X$ such that for each $y \in Y$, there exists a $T$-set $S$ in $E$ satisfying $T^{-1}((R, y) \cap B)=\left(S, \varphi_{R}(y)\right) \cap A$.

Lemma 3.7. Let $X$ and $Y$ be locally compact Hausdorff spaces, $E$ and $F$ be real or complex normed spaces not assumed to be complete. Let $A$ be a completely regular subspace of $C_{0}(X, E), B$ be an $F$-separating subspace of $C_{0}(Y, F)$ and $T: A \longrightarrow B$ be a surjective real-linear isometry. If $F$ satisfies the property $\left(D_{w}\right)$, then $\varphi_{R_{1}}=\varphi_{R_{2}}$ for all $T$-sets $R_{1}$ and $R_{2}$ in $F$.

Proof. Since $F$ satisfies the property $\left(D_{w}\right)$, there exists a $T$-set $R_{0}$ which is discrepant to all $T$-sets in $F$. It suffices to show that $\varphi_{R}=\varphi_{R_{0}}$ for all $T$-sets $R$ in $F$. Let $R$ be a given $T$-set in $F$ and $y \in Y$. Put $x=\varphi_{R}(y)$ and $x_{0}=\varphi_{R_{0}}(y)$. Hence there are $T$-sets $S$ and $S_{0}$ in $E$ such that $T^{-1}((R, y) \cap B)=(S, x) \cap A$ and $T^{-1}\left(\left(R_{0}, y\right) \cap B\right)=$ $\left(S_{0}, x_{0}\right) \cap A$. Since the T-sets $R$ and $R_{0}$ are discrepant, the same arguments as in [8, Theorem 7.2.13] together with Lemma 3.6 imply that $x=x_{0}$.

Using the above lemma, in the case that $A$ is completely regular, $B$ is $F$-separating and $F$ satisfies the property $\left(D_{w}\right)$, we can define a function $\varphi: Y \longrightarrow X$ such that for each $T$-set $R$ in $F$ there exists a $T$-set $S$ in $E$ satisfying $T^{-1}((R, y) \cap B)=$ $(S, \varphi(y)) \cap A$. It is easy to see that $\varphi$ is surjective. Indeed, given $x \in X$, let $S$ be an arbitrary $T$-set in $E$. Then there exist a $T$-set $R$ in $F$ and a point $y \in Y$ such that $T((S, x) \cap A)=(R, y) \cap B$, that is $\left.T^{-1}(R, y) \cap B\right)=(S, x) \cap A$ which concludes that $\varphi(y)=\varphi_{R}(y)=x$.

Lemma 3.8. Under the assumptions of Lemma 3.7 if $f \in A$ and $y \in Y$ such that $f(\varphi(y))=0$, then $(T f)(y)=0$.

Proof. We put $x=\varphi(y)$ and $u=T f(y)$. Assume on the contrary that $u \neq 0$. Choose a $T$-set $R$ in $F$ containing $u$. Then, by the definition of $\varphi$, there exists a T-set $S$ in $E$ such that $T^{-1}((R, y) \cap B)=(S, x) \cap A$. Now by Proposition 3.5 there are $v^{*} \in \Gamma_{S}$ and $w^{*} \in \Gamma_{R}$ such that $T^{*}\left(w^{*} \circ \delta_{y}\right)=v^{*} \circ \delta_{x}$. Hence we have

$$
\|u\|=w^{*}(u)=w^{*}(T f(y))=v^{*}(f(x))=0
$$

a contradiction.
Theorem 3.9. Let $X$ and $Y$ be locally compact Hausdorff spaces, $E$ and $F$ be real or complex normed spaces (not necessarily complete), A be a completely regular subspace of $C_{0}(X, E)$ and $B$ be an $F$-separating subspace of $C_{0}(Y, F)$. Let $T: A \longrightarrow B$ be a surjective real-linear isometry. If $F$ satisfies the property $\left(D_{w}\right)$, then there exist a continuous map $\varphi: Y \longrightarrow X$, a family $\left\{V_{y}\right\}_{y \in Y}$ of bounded real-linear operators from $E$ to $F$ with $\left\|V_{y}\right\| \leq 1$ such that for each $y \in Y$

$$
T f(y)=V_{y}(\underset{9}{f(\varphi(y)))} \quad(f \in A)
$$

Moreover, if $E$ also satisfies the property $\left(D_{w}\right)$ and $B$ is completely regular, then $\varphi$ is a homeomorphism and each $V_{y}$ is a surjective isometry.

Proof. Let $\varphi: Y \longrightarrow X$ be the function defined before. For each $y \in Y$ and each $e \in E$ by the hypotheses there exists $f \in A$ such that $f(\varphi(y))=e$. Put $V_{y}(e)=T f(y)$. We note that, by Lemma 3.8, the definition of $V_{y}(e)$ is independent of the function $f \in A$ satisfying $f(\varphi(y))=e$. Then $V_{y}: E \longrightarrow F$ is clearly a real-linear operator satisfying

$$
\begin{equation*}
T f(y)=V_{y}(f(\varphi(y)) \quad(f \in A) \tag{3.3}
\end{equation*}
$$

Since for each $y \in Y$ and $e \in S(E)$ we can choose a function $f \in A$ with $\|f\|_{\infty}=1$ and $f(\varphi(y))=e$, it follows easily that $\left\|V_{y}\right\| \leq 1$.

To prove that $\varphi$ is continuous, assume that $y_{0} \in Y$ and $U$ is an open neighbourhood of $\varphi\left(y_{0}\right)$ in $X$. Since $V_{y_{0}} \neq 0$ there exists $e \in E$ such that $V_{y_{0}}(e) \neq 0$. Choose $f \in A$ such that $f\left(\varphi\left(y_{0}\right)\right)=e$ and $f=0$ on $X \backslash U$. Then $W=\{y \in Y: T f(y) \neq 0\}$ is a neighbourhood of $y_{0}$ and the equality (3.3) implies that $\varphi(W) \subseteq U$. Hence $\varphi$ is continuous.

The second part of the theorem is easily verified.
In the next theorem, we consider the compact case and, using Theorem 3.9, we characterize surjective isometries between certain subspaces $A$ and $B$ of $C(X, E)$ and $C(Y, F)$, respectively endowed with some norms rather than supremum norms. Motivated by the property $\mathbf{P}$ introduced in [1] and the property $\mathbf{Q}$ introduced in [2] for an isometry $T: A \longrightarrow B$, we consider the property ( St ) introduced in the earlier work of the authors [18]. We have compared the new defined property ( St ) with the properties $\mathbf{P}$ and $\mathbf{Q}$ in [18]. Indeed, the property $\mathbf{Q}$ implies the property (St) and in the case where $F$ is strictly convex (this is assumed in [1]) the property $\mathbf{P}$ also implies the property ( St ).

Definition 3.10. [18, Definition 3.4] Let $X$ and $Y$ be compact Hausdorff spaces and let $E$ and $F$ be real or complex normed spaces. Assume that $A$ and $B$ are subspaces of $C(X, E)$ and $C(Y, F)$, respectively, equipped with the norms of the form

$$
\|\cdot\|_{A}=\max \left(\|\cdot\|_{\infty}, p(\cdot)\right) \text { and }\|\cdot\|_{B}=\max \left(\|\cdot\|_{\infty}, q(\cdot)\right)
$$

where $p$ and $q$ are seminorms on $A$ and $B$, respectively, whose kernels contain the constant functions. We say that a surjective real-linear isometry $T: A \longrightarrow B$ has property (St) if
(St) For each $u \in F$ and $y_{0} \in Y$ there exists $v \in S(E)$ such that $\left\|T \hat{v}\left(y_{0}\right)+u\right\|>\|u\|$, i.e. $\frac{T(\hat{v})\left(y_{0}\right)}{\left\|T(\hat{v})\left(y_{0}\right)\right\|} \in \operatorname{St}_{w}(u)$.

Proposition 3.11. [18, Proposition 3.5] Let $X$ and $Y$ be compact Hausdorff spaces and let $E$ and $F$ be real or complex normed spaces, not assumed to be complete.

Assume that $A$ and $B$ are subspaces of $C(X, E)$ and $C(Y, F)$, respectively containing constants and $\|\cdot\|_{A}$ and $\|\cdot\|_{B}$ are norms on $A$ and $B$ such that

$$
\|\cdot\|_{A}=\max \left(\|\cdot\|_{\infty}, p(\cdot)\right) \text { and }\|\cdot\|_{B}=\max \left(\|\cdot\|_{\infty}, q(\cdot)\right)
$$

for some seminorms $p$ and $q$ on $A$ and $B$, respectively, whose kernels contain the constants. If $T: A \longrightarrow B$ is a surjective real-linear isometry and $T$ and $T^{-1}$ satisfy (St), then $T$ is an isometry with respect to the supremum norms on $A$ and $B$.

Using the above proposition we get the next result concerning surjective isometries between completely regular subspaces of functions with respect to some norms.

Theorem 3.12. Let $X$ and $Y$ be compact Hausdorff spaces, $E$ and $F$ be real or complex normed spaces, not necessarily complete such that $F$ satisfies the property $\left(D_{w}\right)$. Let $A$ and $B$ be subspaces of $C(X, E)$ and $C(Y, F)$, respectively containing the constants such that $A$ is completely regular and $B$ is $F$-separating. Let $\|\cdot\|_{A}$ and $\|\cdot\|_{B}$ be norms on $A$ and $B$, respectively, of the form

$$
\|\cdot\|_{A}=\max \left(\|\cdot\|_{\infty}, p(\cdot)\right) \text { and }\|\cdot\|_{B}=\max \left(\|\cdot\|_{\infty}, q(\cdot)\right)
$$

for some seminorms $p$ and $q$ on $A$ and $B$, respectively, whose kernels contain the constants. Then for any surjective real-linear isometry $T: A \longrightarrow B$ such that $T$ and $T^{-1}$ satisfy (St) there exist a surjective continuous map $\varphi: Y \longrightarrow X$, a family $\left\{V_{y}\right\}_{y \in Y}$ of bounded real-linear operators from $E$ to $F$ with $\left\|V_{y}\right\| \leq 1$ such that for each $y \in Y$

$$
T f(y)=V_{y}(f(\varphi(y))) \quad(f \in A)
$$

Moreover, if $E$ also satisfies the property $\left(D_{w}\right)$ and $B$ is completely regular, then $\varphi$ is a homeomorphism and each $V_{y}$ is a surjective isometry.

Proof. It follows immediately from Proposition 3.11 and Theorem 3.9.

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