

SOME  $q$ -CONGRUENCES ARISING FROM CERTAIN IDENTITIES

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ABSTRACT. In this paper, by constructing some identities, we prove some  $q$ -analogues of some congruences. For example, for any odd integer  $n > 1$ , we show that

$$\sum_{k=0}^{n-1} \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^k \equiv (-1)^{(n+1)/2} q^{(n^2-1)/4} - (1+q)[n] \pmod{\Phi_n(q)^2},$$

$$\sum_{k=0}^{n-1} \frac{(q^3; q^2)_k}{(q; q)_k} q^k \equiv (-1)^{(n+1)/2} q^{(n^2-9)/4} + \frac{1+q}{q^2} [n] \pmod{\Phi_n(q)^2},$$

where the  $q$ -Pochhammer symbol is defined by  $(x; q)_0 = 1$  and  $(x; q)_k = (1-x)(1-xq) \cdots (1-xq^{k-1})$  for  $k \geq 1$ , the  $q$ -integer is defined by  $[n] = 1 + q + \cdots + q^{n-1}$  and  $\Phi_n(q)$  is the  $n$ -th cyclotomic polynomial. The  $q$ -congruences above confirm some recent conjectures of Gu and Guo.

## 1. INTRODUCTION

In 2010, Sun and Tauraso [11] studied some congruence properties of sums concerning central binomial coefficients  $\binom{2k}{k}$  where  $k \in \mathbb{N} = \{0, 1, \dots\}$ . For example, let  $p$  be an odd prime and  $r \in \mathbb{Z}^+$ , they proved that for any  $m \in \mathbb{Z}$  with  $p \nmid m$ ,

$$\sum_{k=0}^{p^r-1} \frac{\binom{2k}{k+d}}{m^k} \equiv u_{p^r-|d|}(m-2) \pmod{p},$$

where  $|d| \in \{0, \dots, p^r\}$  and the sequence of polynomials  $u_n(x)$  ( $n \in \mathbb{N}$ ) is defined as follows:

$$u_0(x) = 0, \quad u_1(x) = 1, \quad \text{and} \quad u_{n+1}(x) = xu_n(x) - u_{n-1}(x) \quad (n = 2, 3, \dots).$$

In particular, they obtained that

$$\sum_{k=0}^{p^r-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p^r-1)/2} \pmod{p}. \quad (1.1)$$

Later, Sun [10] further proved that (1.1) also holds modulo  $p^2$ .

Throughout the paper, the  $q$ -integer  $[n]_q$  is defined as  $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$ , while the  $q$ -pochhammer symbol ( $q$ -shifted factorial) is defined by  $(x; q)_0 = 1$  and  $(x; q)_k =$

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$(1-x)(1-xq)\cdots(1-xq^{k-1})$  for  $k \geq 1$ . And recall that the  $n$ -th *cyclotomic polynomial*  $\Phi_n(q)$  is defined as

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ \gcd(n,k)=1}} (q - \zeta^k),$$

where  $\zeta$  is an  $n$ -th primitive root of unity.

As we all know, identities or congruences usually have nice  $q$ -analogues. In recent years,  $q$ -analogues of identities and congruences have been investigated by various authors (cf. for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13]). In 2010, Guo and Zeng [8] gave the following  $q$ -analogue of (1.1):

$$\sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q; q)_k} q^k \equiv (-1)^{(n-1)/2} q^{(n^2-1)/4} \pmod{\Phi_n(q)}, \quad (1.2)$$

where  $n$  is a positive odd integer. Moreover, Guo [7] established the following generalization of (1.2):

$$\sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q; q)_k} q^k \equiv (-1)^{(n-1)/2} q^{(n^2-1)/4} \pmod{\Phi_n(q)^2}. \quad (1.3)$$

It should be pointed out that (1.3) for odd primes  $n$  was first conjectured by Tauraso [13] in 2013.

Recently, Gu and Guo [3] provided some  $q$ -congruences formally analogous to (1.2) by making use of Carlitz's transformation formula (cf. [2]). For any odd integer  $n > 1$ , they proved that

$$\sum_{k=0}^{n-1} \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^k \equiv (-1)^{(n+1)/2} q^{(n^2-1)/4} \pmod{\Phi_n(q)}, \quad (1.4)$$

$$\sum_{k=0}^{n-1} \frac{(q^3; q^2)_k}{(q; q)_k} q^k \equiv (-1)^{(n+1)/2} q^{-(n-3)^2/4} \pmod{\Phi_n(q)}. \quad (1.5)$$

Our first theorem concerns the generalization of (1.4).

**Theorem 1.1.** *For any odd integer  $n > 1$  we have*

$$\sum_{k=0}^{n-1} \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^k \equiv (-1)^{(n+1)/2} q^{(n^2-1)/4} - (1+q)[n] \pmod{\Phi_n(q)^2}, \quad (1.6)$$

$$\sum_{k=0}^{n-1} \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^{2k} \equiv (-1)^{(n+1)/2} q^{(n^2+3)/4} - 2q[n] \pmod{\Phi_n(q)^2}. \quad (1.7)$$

Letting  $q \rightarrow 1$  in (1.6) or (1.7) we obtain the following congruence.

**Corollary 1.1.** *Let  $p$  be an odd prime and  $r \in \mathbb{Z}^+$ . Then*

$$\sum_{k=0}^{p^r-1} \frac{1}{2^k(2k-1)} \binom{2k}{k} \equiv (-1)^{(p-1)/2} + 2p^r \pmod{p^2}. \quad (1.8)$$

*Remark 1.1.* (1.6) and (1.8) were conjectured by Gu and Guo in [3]. (1.7) is actually a different  $q$ -analogue of (1.8).

Gu and Guo [3] also attempted to find a mod  $\Phi_n(q)^2$  extension of (1.5) but failed. The next theorem gives a different  $q$ -analogue of (1.1) and generalizes (1.5).

**Theorem 1.2.** *Let  $n > 1$  be an odd integer. Then*

$$\sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q; q)_k} q^{2k} \equiv (-1)^{(n-1)/2} q^{(n^2-5)/4} + \frac{q-1}{q} [n] \pmod{\Phi_n(q)^2}, \quad (1.9)$$

$$\sum_{k=0}^{n-1} \frac{(q^3; q^2)_k}{(q; q)_k} q^k \equiv (-1)^{(n+1)/2} q^{(n^2-9)/4} + \frac{1+q}{q^2} [n] \pmod{\Phi_n(q)^2}. \quad (1.10)$$

Letting  $q \rightarrow 1$  we have the following corollary which confirms [3, (1.8)].

**Corollary 1.2.** *Let  $p$  be an odd prime and  $r \in \mathbb{Z}^+$ . Then*

$$\sum_{k=0}^{p^r-1} \frac{(2k+1)}{2^k} \binom{2k}{k} \equiv (-1)^{(p+1)/2} + 2p^r \pmod{p^2}. \quad (1.11)$$

*Remark 1.2.* (1.10) is essentially an extension of (1.5). In fact, noting that  $q^n \equiv 1 \pmod{\Phi_n(q)}$ , we immediately get

$$q^{(n^2-9)/4} \equiv q^{-(n-3)^2/4} \pmod{\Phi_n(q)}.$$

Differently from Gu and Guo's method, we will not use Carlitz's transformation. Our strategy is to find some new identities linking the  $q$ -congruences to be solved with (1.3). Assume that  $F(k, q)$  is a rational function in  $q$  such that  $F(k, q)/F(k-1, q)$  can be written as a ratio of two polynomials in  $q$ . Now we want to find a polynomial  $R(k, q)$  such that  $\sum_{k=0}^n F(k, q)R(k, q)$  has a closed form. Consider the summation  $\sum_{k=0}^n F(k, q)$ . By the definition of  $F$ , we may write

$$\frac{F(k, q)}{F(k-1, q)} = \frac{S(k, q)}{T(k, q)},$$

or equivalently,

$$F(k, q)T(k, q) = F(k-1, q)S(k, q), \quad (1.12)$$

where  $S(k, q)$  and  $T(k, q)$  are polynomials of  $q$ . Then summing both sides of (1.12) from  $k=1$  to  $n$  and via some simple computation we find that

$$\sum_{k=0}^n (T(k, q) - S(k+1, q))F(k, q) = F(0, q)T(0, q) - F(n, q)S(n+1, q). \quad (1.13)$$

Here  $T(k, q) - S(k+1, q)$  is the polynomial that we hope to find.

The proofs of Theorem 1.1 and 1.2 will be given in Sections 2 and 3 respectively.

## 2. PROOF OF THEOREM 1.1

**Lemma 2.1.** *For any positive integer  $n$ , we have the following identities*

$$\sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q; q)_k} q^k + \frac{1}{1-q} \sum_{k=0}^{n-1} \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^k (1-q^k) = \frac{(q; q^2)_{n-1}}{(q; q)_{n-1}} q^{n-1} \quad (2.1)$$

and

$$\frac{1}{1-q} \sum_{k=0}^{n-1} \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^k (q - q^k) = -\frac{(q; q^2)_{n-1}}{(q; q)_{n-1}}. \quad (2.2)$$

*Proof.* Set

$$F_1(k, q) = \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^k.$$

It is easy to verify that

$$(1 - q^k) F_1(k, q) = q(1 - q^{2k-3}) F_1(k-1, q). \quad (2.3)$$

Summing both sides of (2.3) from  $k = 1$  to  $n-1$  and noting that the left-hand side of (2.3) vanishes when  $k = 0$ , we arrive at

$$\sum_{k=0}^{n-1} (1 - q^k) F_1(k, q) = \sum_{k=1}^{n-1} q(1 - q^{2k-3}) F_1(k-1, q) = \sum_{k=0}^{n-2} q(1 - q^{2k-1}) F_1(k, q),$$

or equivalently,

$$\sum_{k=0}^{n-1} q(1 - q^{2k-1}) F_1(k, q) - \sum_{k=0}^{n-1} (1 - q^k) F_1(k, q) = q(1 - q^{2n-3}) F_1(n-1, q).$$

Then (2.1) follows by noting that

$$q(1 - q^{2k-1}) F_1(k, q) = (q-1) \frac{(q; q^2)_k}{(q; q)_k} q^k$$

for all  $k$  among  $0, 1, \dots, n-1$ .

To show (2.2) we set

$$F_2(k, q) = \frac{(q^{-1}; q^2)_k}{(q; q)_k}.$$

Now we find that

$$(1 - q^k) F_2(k, q) = (1 - q^{2k-3}) F_2(k-1, q).$$

Then we may obtain (2.2) by some similar arguments as above.  $\square$

**Lemma 2.2.** *For any odd integer  $n > 1$ , we have*

$$\frac{(q; q^2)_{n-1}}{(q; q)_{n-1}} \equiv -q[n] \pmod{\Phi_n(q)^2}. \quad (2.4)$$

*Proof.* Clearly,

$$\frac{(q; q^2)_{n-1}}{(q; q)_{n-1}} = \frac{(q; q)_{2n-2}}{(-q; q)_{n-1}(q; q)_{n-1}^2}. \quad (2.5)$$

Note that

$$q^n \equiv 1 \pmod{\Phi_n(q)}$$

and

$$q^j \not\equiv 1 \pmod{\Phi_n(q)} \quad \text{for all } j = 1, 2, \dots, n-1.$$

Thus we have

$$\frac{(q; q)_{2n-2}}{(q; q)_{n-1}^2} = (1 - q^n) \frac{\prod_{j=1}^{n-2} (1 - q^{n+j})}{\prod_{j=1}^{n-1} (1 - q^j)} \equiv \frac{1 - q^n}{1 - q^{-1}} = -q[n] \pmod{\Phi_n(q)^2}. \quad (2.6)$$

By [1, Corollary 10.2.2(c)] we have

$$(-q; q)_{n-1} = \frac{(-q; q)_n}{1 + q^n} \equiv \frac{1}{2} \sum_{k=0}^n \frac{(q; q)_n q^{n(n+1)/2}}{(q; q)_k (q; q)_{n-k}} \equiv 1 \pmod{\Phi_n(q)}. \quad (2.7)$$

Substituting (2.6) and (2.7) into (2.5) we immediately obtain the desired lemma.  $\square$

*Proof of Theorem 1.1.* We first prove (1.6). Combining (2.1) and (2.2) and with the help of Lemma 2.2 we obtain that

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q; q)_k} q^k + \sum_{k=0}^{n-1} \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^k &= \frac{(q; q^2)_{n-1}}{(q; q)_{n-1}} (1 + q^{n-1}) \\ &\equiv -q[n](1 + q^{-1}) \\ &= -(1 + q)[n] \pmod{\Phi_n(q)^2}. \end{aligned}$$

Now (1.6) follows from (1.3).

With the help of (2.2), we have

$$q \sum_{k=0}^{n-1} \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^k - \sum_{k=0}^{n-1} \frac{(q^{-1}; q^2)_k}{(q; q)_k} q^{2k} = (q - 1) \frac{(q; q^2)_{n-1}}{(q; q)_{n-1}}.$$

Then we obtain (1.7) by noting (1.6) and Lemma 2.2.

The proof of Theorem 1.1 is now complete.  $\square$

### 3. PROOF OF THEOREM 1.2

**Lemma 3.1.** *For any positive integer  $n$  we have the following identities.*

$$(1 - q) \sum_{k=0}^{n-1} \frac{(q^3; q^2)_k}{(q; q)_k} q^k - \frac{1}{q} \sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q; q)_k} q^k (1 - q^k) = \frac{(q^3; q^2)_{n-1}}{(q; q)_{n-1}} (q^{n-1} - q^n) \quad (3.1)$$

and

$$\sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q; q)_k} q^k (1 - q^{k+1}) = (1 - q) \frac{(q^3; q^2)_{n-1}}{(q; q)_{n-1}}. \quad (3.2)$$

*Proof.* Set

$$G_1(k, q) = \frac{(q; q^2)_k}{(q; q)_k} q^k.$$

Then we may easily check that

$$(1 - q^k)G_1(k, q) = (q - q^{2k})G_1(k - 1, q).$$

Summing both sides of the above identity from  $k = 1$  to  $n - 1$  we have

$$q \sum_{k=0}^{n-1} (1 - q^{2k+1})G_1(k, q) - \sum_{k=0}^{n-1} (1 - q^k)G_1(k, q) = (q - q^{2n})G_1(n - 1, q).$$

Thus we obtain (3.1) by noting

$$(1 - q)(q^3; q^2)_k = (1 - q^{2k+1})(q; q^2)_k$$

for all  $k$  among  $0, 1, \dots, n - 1$ .

Also, (3.2) can be deduced by setting

$$G_2(k, q) = \frac{(q; q^2)_k}{(q; q)_k}$$

and noting that

$$(1 - q^k)G_2(k, q) = (1 - q^{2k-1})G_2(k - 1, q).$$

□

*Proof of Theorem 1.2.* By (3.2) we have

$$\sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q; q)_k} q^k - q \sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q; q)_k} q^{2k} = (1 - q^{2n-1}) \frac{(q; q^2)_{n-1}}{(q; q)_{n-1}}.$$

Then (1.9) follows from (1.3) and Lemma 2.2. Substituting (1.3) and (1.9) we immediately get (1.10).

The proof of Theorem 1.2 is now complete. □

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