ON A PROBLEM OF PARTITIONS OF \mathbb{Z}_m WITH THE SAME REPRESENTATION FUNCTIONS

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ABSTRACT. For any positive integer m, let \mathbb{Z}_m be the set of residue classes modulo m. For $A \subseteq \mathbb{Z}_m$ and $\overline{n} \in \mathbb{Z}_m$, let representation function $R_A(\overline{n})$ denote the number of solutions of the equation $\overline{n} = \overline{a} + \overline{a'}$ with ordered pairs $(\overline{a}, \overline{a'}) \in A \times A$. In this paper, we determine all sets $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 2$ or m - 2 such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. We also prove that if m is a positive integer with 4|m, then there exist two distinct sets $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 4$ or $m - 4, B \neq A + \overline{\frac{m}{2}}$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. If m is a positive integer with $2||m, A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 4$ or m - 4, then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if and only if $B = A + \overline{\frac{m}{2}}$.

Keywords: Representation function, partition, residue class.

1. INTRODUCTION

Let \mathbb{N} be the set of all nonnegative integers. For $S \subseteq \mathbb{N}$ and $n \in S$, let the representation function $R'_S(n)$ denote the number of solutions of the equation n = s + s' with $s, s' \in S$. Sárkőzy asked whether there exist two sets $A, B \subseteq \mathbb{N}$ with $|(A \cup B) \setminus (A \cap B)| = \infty$ such that $R'_A(n) = R'_B(n)$ for all sufficiently large integers n. In 2002, Dombi [2] showed that the answer is negative. There are many other related results (see [1, 3, 4, 5, 6, 7] and the references therein).

For a positive integer m, let \mathbb{Z}_m be the set of residue classes modulo m. For any residue classes $\overline{a}, \overline{b} \in \mathbb{Z}_m$, there exist two integers a', b' with $0 \le a', b' \le m - 1$ such that $\overline{a'} = \overline{a}$ and $\overline{b'} = \overline{b}$. We define the ordering $\overline{a} \le \overline{b}$ if $a' \le b'$. For any $\overline{n} \in \mathbb{Z}_m$, without loss of generality, we may suppose that $0 \le n \le m - 1$. For $A \subseteq \mathbb{Z}_m$ and $\overline{n} \in \mathbb{Z}_m$, let $R_A(\overline{n})$ denote the number of solutions of $\overline{n} = \overline{a} + \overline{a'}$ with $\overline{a}, \overline{a'} \in A$. For $\overline{n} \in \mathbb{Z}_m$ and $A \subseteq \mathbb{Z}_m$, let $\overline{n} + A = \{\overline{n} + \overline{a} : \overline{a} \in A\}$. For $A, B \subseteq \mathbb{Z}_m$ and $\overline{n} \in \mathbb{Z}_m$, let $R_{A,B}(\overline{n})$ be the number of solutions of $\overline{n} = \overline{a} + \overline{b}$ with $\overline{a} \in A$ and $\overline{b} \in B$. The characteristic function of $A \subseteq \mathbb{Z}_m$ is denoted by

$$\chi_A(n) = \begin{cases} 1 & \overline{n} \in A, \\ 0 & \overline{n} \notin A. \end{cases}$$

In 2012, Yang and Chen [8] determined all sets $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. In 2017, Yang and Tang [9] determined all sets $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = 2$ or m - 1 such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. Yang and Tang [9] also posed the following problem for further research.

Problem 1.1. Given a positive even integer m and an integer k with $2 \le k \le m-1$. Do there exist two distinct sets $A, B \subseteq \mathbb{Z}_m$ with |A| = |B| = k and $B \ne A + \{\frac{m}{2}\}$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$?

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In this paper, we consider for which positive even integers m there exist two distinct sets $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 2, m - 2, 4$ or m - 4 such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ and obtain the following results:

theorem 1.2. Let m be a positive even integer. Let $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 2$ or m - 2. Then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if and only if $B = A + \frac{\overline{m}}{2}$.

theorem 1.3. Let m be a positive integer with 4|m. Then there exist two distinct sets $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m, B \neq A + \frac{\overline{m}}{2}$ and $|A \cap B| = 4$ or m - 4 such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$.

theorem 1.4. Let m be a positive integer with 2||m. Let $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 4$ or m - 4. Then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if and only if $B = A + \frac{\overline{m}}{2}$.

2. Lemmas

Lemma 2.1. Let m be a positive even integer and t be a positive integer with $t < \frac{m}{2}$. Let $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 2t$. If $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$, then $|A| = |B| = \frac{m}{2} + t$.

Proof. If $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$, then

$$|A|^2 = \sum_{\overline{n} \in \mathbb{Z}_m} R_A(\overline{n}) = \sum_{\overline{n} \in \mathbb{Z}_m} R_B(\overline{n}) = |B|^2.$$

Thus |A| = |B|. Noting that

$$|A| + |B| = |A \cup B| + |A \cap B| = m + 2t,$$

we have $|A| = |B| = \frac{m}{2} + t$.

This completes the proof of Lemma 2.1.

Lemma 2.2. Let m be a positive even integer and t be a positive integer with $t < \frac{m}{2}$. Let $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m$ and $A \cap B = \{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}$. If $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$, then

(2.1)
$$\sum_{i=1}^{2t} \chi_A(n-r_i) = t + \frac{1}{2} R_{\{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}}(\overline{n}).$$

Proof. Noting that $B = (\mathbb{Z}_m \setminus A) \cup \{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}$, we have

$$R_B(\overline{n}) = R_{\mathbb{Z}_m \setminus A}(\overline{n}) + 2R_{\mathbb{Z}_m \setminus A, \{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}}(\overline{n}) + R_{\{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}}(\overline{n})$$

$$= |\{(a, a') : \overline{a}, \overline{a'} \in \mathbb{Z}_m \setminus A, 0 \le a, a' \le m - 1, a + a' = n \text{ or } a + a' = n + m\}|$$

$$+ 2\sum_{i=1}^{2t} (1 - \chi_A(n - r_i)) + R_{\{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}}(\overline{n})$$

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$$\begin{split} &= \sum_{0 \leq i \leq \frac{n}{2}} (1 - \chi_A(i))(1 - \chi_A(n-i)) + \sum_{n+1 \leq i \leq \frac{n+m}{2}} (1 - \chi_A(i))(1 - \chi_A(n-i)) \\ &+ \sum_{0 \leq i < \frac{n}{2}} (1 - \chi_A(i))(1 - \chi_A(n-i)) + \sum_{n+1 \leq i < \frac{n+m}{2}} (1 - \chi_A(i))(1 - \chi_A(n-i)) \\ &+ 2\sum_{i=1}^{2t} (1 - \chi_A(n-r_i)) + R_{\{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}}(\overline{n}) \\ &= \sum_{0 \leq i \leq \frac{n}{2}} 1 - \sum_{0 \leq i \leq n} \chi_A(i) - \chi_A(\frac{n}{2}) + \sum_{0 \leq i \leq \frac{n}{2}} \chi_A(i)\chi_A(n-i) \\ &+ \sum_{n+1 \leq i \leq \frac{n+m}{2}} 1 - \sum_{n+1 \leq i \leq m-1} \chi_A(i) - \chi_A(\frac{n+m}{2}) + \sum_{n+1 \leq i \leq \frac{n+m}{2}} \chi_A(i)\chi_A(n-i) \\ &+ \sum_{0 \leq i < \frac{n}{2}} 1 - \sum_{0 \leq i \leq n} \chi_A(i) + \chi_A(\frac{n}{2}) + \sum_{0 \leq i < \frac{n}{2}} \chi_A(i)\chi_A(n-i) \\ &+ \sum_{n+1 \leq i < \frac{n+m}{2}} 1 - \sum_{0 \leq i \leq n} \chi_A(i) + \chi_A(\frac{n+m}{2}) + \sum_{n+1 \leq i < \frac{n+m}{2}} \chi_A(i)\chi_A(n-i) \\ &+ \sum_{n+1 \leq i < \frac{n+m}{2}} 1 - \sum_{n+1 \leq i \leq m-1} \chi_A(i) + \chi_A(\frac{n+m}{2}) + \sum_{n+1 \leq i < \frac{n+m}{2}} \chi_A(i)\chi_A(n-i) \\ &+ 2\sum_{i=1}^{2t} (1 - \chi_A(n-r_i)) + R_{\{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}}(\overline{n}) \\ &= \left[\frac{n}{2}\right] + 1 + \left[\frac{n+m}{2}\right] - n + \left[\frac{n-1}{2}\right] + 1 + \left[\frac{n+m-1}{2}\right] - n - 2|A| \\ &+ R_A(\overline{n}) + 2\sum_{i=1}^{2t} (1 - \chi_A(n-r_i)) + R_{\{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}}(\overline{n}) \\ &= -2t + R_A(\overline{n}) - 2\sum_{i=1}^{2t} \chi_A(n-r_i) + R_{\{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}}(\overline{n}). \end{split}$$

Since $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$, we have

$$\sum_{i=1}^{2t} \chi_A(n-r_i) = t + \frac{1}{2} R_{\{\overline{r_1}, \overline{r_2}, \dots, \overline{r_{2t}}\}}(\overline{n}).$$

This completes the proof of Lemma 2.2.

3. Proofs

Proof of Theorem 1.1. It is clear that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if $B = A + \frac{\overline{m}}{2}$. Now we suppose that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. Clearly, the result is true for m = 2. Now we may assume that $m \ge 4$.

Case 1. $|A \cap B| = 2$. Let $A \cap B = \{\overline{r_1}, \overline{r_2}\}$ with $\overline{r_1} \neq \overline{r_2}$. By choosing $n = 2r_1$ in (2.1), we have

$$1 + \chi_A(2r_1 - r_2) = \chi_A(r_1) + \chi_A(2r_1 - r_2) = 1 + \frac{1}{2}R_{\{\overline{r_1}, \overline{r_2}\}}(\overline{2r_1}) \ge \frac{3}{2}$$

Then $\chi_A(2r_1 - r_2) = 1$ and $\overline{r_2} = \overline{r_1} + \frac{\overline{m}}{2}$. Let $\overline{k} \in A$ with $\overline{k} \neq \overline{r_1}$ and $\overline{k} \neq \overline{r_2}$. By choosing $n = k + r_1$ in (2.1), we have

$$\chi_A(k) + \chi_A(k + \frac{m}{2}) = 1 + \frac{1}{2}R_{\{\overline{r_1}, \overline{r_2}\}}(\overline{k + r_1}) = 1.$$

It follows that $B = A + \frac{\overline{m}}{2}$.

Case 2. $|A \cap B| = m - 2$. Let $A \cap B = T$, $A = \{\overline{a}\} \cup T$, $B = \{\overline{b}\} \cup T$ with $\overline{a} \neq \overline{b}$ and $\overline{a}, \overline{b} \notin T$. For any $\overline{n} \in \mathbb{Z}_m$, we have

$$\begin{aligned} R_A(\overline{n}) &= R_T(\overline{n}) + 2R_{T,\{\overline{a}\}}(\overline{n}) + R_{\{\overline{a}\}}(\overline{n}) = R_T(\overline{n}) + 2\chi_T(n-a) + R_{\{\overline{a}\}}(\overline{n}); \\ R_B(\overline{n}) &= R_T(\overline{n}) + 2R_{T,\{\overline{b}\}}(\overline{n}) + R_{\{\overline{b}\}}(\overline{n}) = R_T(\overline{n}) + 2\chi_T(n-b)) + R_{\{\overline{b}\}}(\overline{n}). \end{aligned}$$

Then

(3.1)
$$2\chi_T(n-a) + R_{\{\overline{a}\}}(\overline{n}) = 2\chi_T(n-b) + R_{\{\overline{b}\}}(\overline{n}).$$

By choosing n = 2b in (3.1), we have

$$2\chi_T(2b-a) + R_{\{\overline{a}\}}(\overline{2b}) = 2\chi_T(b) + R_{\{\overline{b}\}}(\overline{2b}) = 1$$

Then $2\chi_T(2b-a) = 0$ and $R_{\{\overline{a}\}}(\overline{2b}) = 1$. Thus $\overline{b} = \overline{a} + \overline{\frac{m}{2}}$. It follows that $B = A + \overline{\frac{m}{2}}$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.2. Let m = 4k with k a positive integer.

Case 1. $|A \cap B| = 4$. Noting that $A \neq B$, we have $k \geq 2$. Let

$$A = \{\overline{0}, \overline{1}, \dots, \overline{k-1}, \overline{k}\} \cup \{\overline{2k}, \overline{2k+1}, \dots, \overline{3k-1}, \overline{3k}\},$$

$$B = \{\overline{k}, \overline{k+1}, \dots, \overline{2k-1}, \overline{2k}\} \cup \{\overline{3k}, \overline{3k+1}, \dots, \overline{4k-1}, \overline{0}\}.$$

It is clear that $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m, B \neq A + \frac{\overline{m}}{2}$ and $A \cap B = \{\overline{0}, \overline{k}, \overline{2k}, \overline{3k}\}$. Let $S = \{\overline{0}, \overline{1}, \dots, \overline{k-1}, \overline{k}\}$. Then $A = S \cup (S + \overline{2k})$ and $B = (S + \overline{k}) \cup (S + \overline{3k})$. Noting that

$$\begin{array}{lll} R_{S}(\overline{n}) & = & R_{S+\overline{2k}}(\overline{n}) = R_{S+\overline{k},S+\overline{3k}}(\overline{n}), \\ R_{S+\overline{k}}(\overline{n}) & = & R_{S,S+\overline{2k}}(\overline{n}) = R_{S+\overline{3k}}(\overline{n}) \end{array}$$

for all $\overline{n} \in \mathbb{Z}_m$, we have

$$\begin{aligned} R_A(\overline{n}) &= R_S(\overline{n}) + 2R_{S,S+\overline{2k}}(\overline{n}) + R_{S+\overline{2k}}(\overline{n}) \\ &= R_{S+\overline{k}}(\overline{n}) + 2R_{S+\overline{k},S+\overline{3k}}(\overline{n}) + R_{S+\overline{3k}}(\overline{n}) \\ &= R_B(\overline{n}). \end{aligned}$$

Case 2. $|A \cap B| = m - 4$. If k = 1, then by choosing $A = \{\overline{0}, \overline{2}\}, B = \{\overline{1}, \overline{3}\}$, we have $A \cup B = \mathbb{Z}_4, B \neq A + \overline{2}$ and $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_4$. Now let $k \ge 2$ and

$$A = \{\overline{0}, \overline{1}, \dots, \overline{k-1}\} \cup \{\overline{k+1}, \overline{k+2}, \dots, \overline{3k-1}\} \cup \{3k+1, \overline{3k+2}, \dots, \overline{4k-1}\}, \\ B = \{\overline{1}, \overline{2}, \dots, \overline{2k-1}\} \cup \{\overline{2k+1}, \overline{2k+2}, \dots, \overline{4k-1}\}.$$

It is clear that $A \cup B = \mathbb{Z}_m, B \neq A + \overline{\frac{m}{2}}$ and

$$A \cap B = \{\overline{1}, \dots, \overline{k-1}\} \cup \{\overline{k+1}, \dots, \overline{2k-1}\} \cup \{\overline{2k+1}, \dots, \overline{3k-1}\} \cup \{\overline{3k+1}, \dots, \overline{4k-1}\}.$$

Noting that $R_{A\cap B,\{\overline{0},\overline{2k}\}}(\overline{n}) = R_{A\cap B,\{\overline{k},\overline{3k}\}}(\overline{n})$ and $R_{\{\overline{0},\overline{2k}\}}(\overline{n}) = R_{\{\overline{k},\overline{3k}\}}(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$, we have

$$R_{A}(\overline{n}) = R_{\{\overline{0},\overline{2k}\}}(\overline{n}) + 2R_{A\cap B,\{\overline{0},\overline{2k}\}}(\overline{n}) + R_{A\cap B}(\overline{n})$$

$$= R_{\{\overline{k},\overline{3k}\}}(\overline{n}) + 2R_{A\cap B,\{\overline{k},\overline{3k}\}}(\overline{n}) + R_{A\cap B}(\overline{n})$$

$$= R_{B}(\overline{n}).$$

This completes the proof of Theorem 1.3.

Proof of Theorem 1.3. It is clear that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if $B = A + \frac{\overline{m}}{2}$. Now we suppose that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. By $A \neq B$ and 2 || m, we have $m \geq 6$. **Case 1.** $|A \cap B| = 4$. Let $A \cap B = \{\overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{r_4}\}$. By Lemma 2.2, we have

(3.2)
$$\chi_A(n-r_1) + \chi_A(n-r_2) + \chi_A(n-r_3) + \chi_A(n-r_4) = 2 + \frac{1}{2} R_{\{\overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{r_4}\}}(\overline{n}).$$

Then $2|R_{\{\overline{r_1},\overline{r_2},\overline{r_3},\overline{r_4}\}}(\overline{n})$ for any $\overline{n} \in \mathbb{Z}_m$. Thus $2|R_{\{\overline{r_1},\overline{r_2},\overline{r_3},\overline{r_4}\}}(\overline{2r_i})$ for $i \in \{1,2,3,4\}$. Without loss of generality, we may assume that $\overline{r_3} = \overline{r_1} + \frac{\overline{m}}{2}$ and $\overline{r_4} = \overline{r_2} + \frac{\overline{m}}{2}$. Clearly, the result is true for m = 6. Now let m = 2k with $2 \nmid k$ and $k \ge 5$. Let

$$l = \min\{i \ge 2 : \{\overline{r_2}, \overline{r_2} + \overline{\frac{m}{2}}\} = \{\overline{ir_1 - (i-1)r_2}, \overline{ir_1 - (i-1)r_2} + \overline{\frac{m}{2}}\}\}.$$

By 2||m|, we have $3 \le l \le k$ and l|k. Now we discuss the following two subcases according to l. Subcase 1.1 l = k. Then

$$\mathbb{Z}_m = \bigcup_{i=1}^k \{ \overline{ir_1 - (i-1)r_2}, \overline{ir_1 - (i-1)r_2} + \frac{\overline{m}}{2} \}.$$

By choosing $n - r_2 = ir_1 - (i - 1)r_2$ for i = 1, 2, ..., k in (3.2), we have

$$\begin{split} \chi_A(r_2) &+ \chi_A(r_2 + \frac{m}{2}) + \chi_A(r_1) + \chi_A(r_1 + \frac{m}{2}) = 4, \\ \chi_A(r_1) &+ \chi_A(r_1 + \frac{m}{2}) + \chi_A(2r_1 - r_2) + \chi_A(2r_1 - r_2 + \frac{m}{2}) = 3, \\ \chi_A(2r_1 - r_2) &+ \chi_A(2r_1 - r_2 + \frac{m}{2}) + \chi_A(3r_1 - 2r_2) + \chi_A(3r_1 - 2r_2 + \frac{m}{2}) = 2, \\ \cdots \\ \chi_A((k-2)r_1 - (k-3)r_2) + \chi_A((k-2)r_1 - (k-3)r_2 + \frac{m}{2}) \\ &+ \chi_A((k-1)r_1 - (k-2)r_2) + \chi_A((k-1)r_1 - (k-2)r_2 + \frac{m}{2}) = 2, \\ \chi_A((k-1)r_1 - (k-2)r_2) + \chi_A((k-1)r_1 - (k-2)r_2 + \frac{m}{2}) = 2, \end{split}$$

Noting that

$$\chi_A(r_2) + \chi_A(r_2 + \frac{m}{2}) = \chi_A(r_1) + \chi_A(r_1 + \frac{m}{2}) = 2,$$

we have

$$\chi_A(ir_1 - (i-1)r_2) + \chi_A(ir_1 - (i-1)r_2 + \frac{m}{2}) = 1$$

for i = 2, 3, ..., k - 1. It follows that $B = A + \frac{\overline{m}}{2}$. Subcase 1.2 $3 \le l < k$. Then k = ls with $s \ge 3$ and $2 \nmid s$. Thus

$$\mathbb{Z}_m = \bigcup_{i=1}^l \bigcup_{j=0}^{s-1} \{ \overline{ir_1 - (i-1)r_2 + j}, \overline{ir_1 - (i-1)r_2 + j} + \overline{\frac{m}{2}} \}.$$

By choosing $n - r_2 = ir_1 - (i - 1)r_2$ for i = 1, 2, ..., l in (3.2), we have

$$\begin{split} \chi_A(r_2) &+ \chi_A(r_2 + \frac{m}{2}) + \chi_A(r_1) + \chi_A(r_1 + \frac{m}{2}) = 4, \\ \chi_A(r_1) &+ \chi_A(r_1 + \frac{m}{2}) + \chi_A(2r_1 - r_2) + \chi_A(2r_1 - r_2 + \frac{m}{2}) = 3, \\ \chi_A(2r_1 - r_2) &+ \chi_A(2r_1 - r_2 + \frac{m}{2}) + \chi_A(3r_1 - 2r_2) + \chi_A(3r_1 - 2r_2 + \frac{m}{2}) \\ &= 2 + \frac{1}{2} R_{\{\overline{r_1}, \overline{r_1 + \frac{m}{2}}, \overline{r_2}, \overline{r_2 + \frac{m}{2}}\}} (\overline{3r_1 - r_2}), \\ \dots \\ \chi_A((l-1)r_1 - (l-2)r_2) + \chi_A((l-1)r_1 - (l-2)r_2 + \frac{m}{2}) + \chi_A(r_2) + \chi_A(r_2 + \frac{m}{2}) = 3. \end{split}$$

Noting that

$$\chi_A(r_2) + \chi_A(r_2 + \frac{m}{2}) = \chi_A(r_1) + \chi_A(r_1 + \frac{m}{2}) = 2,$$

we have

$$\chi_A(ir_1 - (i-1)r_2) + \chi_A(ir_1 - (i-1)r_2 + \frac{m}{2}) = 1$$

for i = 2, ..., l - 1.

For any $j \in \{1, 2, ..., s - 1\}$, by choosing $n - r_2 = ir_1 - (i - 1)r_2 + j$ for i = 1, 2, ..., l in (3.2), we have

$$\begin{split} \chi_A(r_2+j) + \chi_A(r_2+j+\frac{m}{2}) + \chi_A(r_1+j) + \chi_A(r_1+j+\frac{m}{2}) &= 2, \\ \chi_A(r_1+j) + \chi_A(r_1+j+\frac{m}{2}) + \chi_A(2r_1-r_2+j) + \chi_A(2r_1-r_2+j+\frac{m}{2}) &= 2, \\ \chi_A(2r_1-r_2+j) + \chi_A(2r_1-r_2+j+\frac{m}{2}) + \chi_A(3r_1-2r_2+j) \\ &+ \chi_A(3r_1-2r_2+j+\frac{m}{2}) &= 2, \\ \cdots \\ \chi_A((l-1)r_1 - (l-2)r_2+j) + \chi_A((l-1)r_1 - (l-2)r_2+j+\frac{m}{2}) \\ &+ \chi_A(r_2+j) + \chi_A(r_2+j+\frac{m}{2}) &= 2 \end{split}$$

Noting that $2 \nmid l$, we have

$$\chi_A(ir_1 - (i-1)r_2 + j) + \chi_A(ir_1 - (i-1)r_2 + j + \frac{m}{2}) = 1$$

for i = 1, 2, ..., l - 1. It follows that $B = A + \frac{\overline{m}}{2}$. **Case 2.** $|A \cap B| = m - 4$. Let $A \cap B = T, A = \{\overline{a_1}, \overline{a_2}\} \cup T, B = \{\overline{b_1}, \overline{b_2}\} \cup T$ with $\{\overline{a_1}, \overline{a_2}\} \cap \{\overline{b_1}, \overline{b_1}\} = \emptyset$ and $\{\overline{a_1}, \overline{a_2}, \overline{b_1}, \overline{b_2}\} \cap T = \emptyset$. For any $\overline{n} \in \mathbb{Z}_m$, we have

$$R_{A}(\overline{n}) = R_{T}(\overline{n}) + 2R_{T,\{\overline{a_{1}},\overline{a_{2}}\}}(\overline{n}) + R_{\{\overline{a_{1}},\overline{a_{2}}\}}(\overline{n})$$

$$= R_{T}(\overline{n}) + 2\chi_{T}(n - a_{1}) + 2\chi_{T}(n - a_{2}) + R_{\{\overline{a_{1}},\overline{a_{2}}\}}(\overline{n});$$

$$R_{B}(\overline{n}) = R_{T}(\overline{n}) + 2R_{T,\{\overline{b_{1}},\overline{b_{2}}\}}(\overline{n}) + R_{\{\overline{b_{1}},\overline{b_{2}}\}}(\overline{n})$$

$$= R_{T}(\overline{n}) + 2\chi_{T}(n - b_{1}) + 2\chi_{T}(n - b_{2}) + R_{\{\overline{b_{1}},\overline{b_{2}}\}}(\overline{n}).$$

Then

$$(3.3) \ 2\chi_T(n-a_1) + 2\chi_T(n-a_2) + R_{\{\overline{a_1},\overline{a_2}\}}(\overline{n}) = 2\chi_T(n-b_1) + 2\chi_T(n-b_2) + R_{\{\overline{b_1},\overline{b_2}\}}(\overline{n}).$$

By choosing $n = a_1 + a_2$ in (3.3), we have

$$2\chi_T(a_1 + a_2 - b_1) + 2\chi_T(a_1 + a_2 - b_2) + R_{\{\overline{b_1}, \overline{b_2}\}}(\overline{a_1} + \overline{a_2}) = 2.$$

If $\chi_T(a_1 + a_2 - b_1) = 1$, then

$$\chi_T(a_1 + a_2 - b_2) = R_{\{\overline{b_1}, \overline{b_2}\}}(\overline{a_1} + \overline{a_2}) = 0.$$

Thus $\overline{a_1} + \overline{a_2} - \overline{b_2} \in {\overline{b_1}, \overline{b_2}}$ and $\overline{a_1} + \overline{a_2} \notin {\overline{2b_1}, \overline{b_1} + \overline{b_2}, \overline{2b_2}}$, which is a contradiction. It follows that $\chi_T(a_1 + a_2 - b_1) = 0$. Similarly, we can get $\chi_T(a_1 + a_2 - b_2) = 0$. Therefore $R_{\{\overline{b_1},\overline{b_2}\}}(\overline{a_1}+\overline{a_2})=2$. It means that $\overline{a_1}+\overline{a_2}=\overline{2b_1}=\overline{2b_2}$ or $\overline{a_1}+\overline{a_2}=\overline{b_1}+\overline{b_2}$.

By choosing $n = b_1 + b_2$ in (3.3), we have

$$2\chi_T(b_1 + b_2 - a_1) + 2\chi_T(b_1 + b_2 - a_2) + R_{\{\overline{a_1}, \overline{a_2}\}}(b_1 + b_2) = 2.$$

If $\chi_T(b_1 + b_2 - a_1) = 1$, then

$$\chi_T(b_1 + b_2 - a_2) = R_{\{\overline{a_1}, \overline{a_2}\}}(\overline{b_1} + \overline{b_2}) = 0$$

Thus $\overline{b_1} + \overline{b_2} - \overline{a_2} \in \{\overline{a_1}, \overline{a_2}\}$ and $\overline{b_1} + \overline{b_2} \notin \{\overline{2a_1}, \overline{a_1} + \overline{a_2}, \overline{2a_2}\}$, which is a contradiction. It follows that $\chi_T(b_1 + b_2 - a_1) = 0$. Similarly, we can get $\chi_T(b_1 + b_2 - a_2) = 0$. Therefore $R_{\{\overline{a_1},\overline{a_2}\}}(\overline{b_1}+\overline{b_2})=2$. It means that $\overline{b_1}+\overline{b_2}=\overline{2a_1}=\overline{2a_2}$ or $\overline{b_1}+\overline{b_2}=\overline{a_1}+\overline{a_2}$.

If $\overline{a_1} + \overline{a_2} = \overline{2b_1} = \overline{2b_2}$, then $\overline{b_1} + \overline{b_2} = \overline{2a_1} = \overline{2a_2}$. Thus $\overline{a_2} = \overline{a_1} + \overline{\frac{m}{2}}, \overline{b_2} = \overline{b_1} + \overline{\frac{m}{2}}$ and $\overline{2a_1} + \overline{\underline{m}} = \overline{2b_1}$, which contradicts 2||m|. It follows that $\overline{a_1} + \overline{a_2} = \overline{b_1} + \overline{b_2}$. By choosing $n = 2a_1$ in (3.3), we have

$$(3.4) \quad 2\chi_T(2a_1 - a_2) + R_{\{\overline{a_1}, \overline{a_2}\}}(\overline{2a_1}) = 2\chi_T(2a_1 - b_1) + 2\chi_T(2a_1 - b_2) + R_{\{\overline{b_1}, \overline{b_2}\}}(\overline{2a_1}).$$

If $R_{\overline{a_1,a_2}}(\overline{2a_1}) = 2$, then $\overline{2a_1} = \overline{2a_2}$ and $\chi_T(2a_1 - a_2) = \chi_T(a_2) = 0$. Thus $R_{\overline{b_1,b_2}}(\overline{2a_1}) = 0$ and $\chi_T(2a_1 - b_1) = \chi_T(2a_1 - b_2) = 1$, which contradicts (3.4).

If $R_{\{\overline{a_1},\overline{a_2}\}}(\overline{2a_1}) = 1$, then $\overline{2a_1} \neq \overline{2a_2}$. By (3.4), we have $R_{\{\overline{b_1},\overline{b_2}\}}(\overline{2a_1}) = 1$. Then $\overline{2b_1} \neq \overline{2b_2}$ and $\overline{2a_1} \in \{\overline{2b_1}, \overline{2b_2}\}$. Without loss of generality, we may assume that $\overline{2a_1} = \overline{2b_1}$. Then $\overline{b_1} = \overline{a_1} + \frac{\overline{m}}{2}$ and $\overline{b_2} = \overline{a_2} + \frac{\overline{m}}{2}$. It follows that $B = A + \frac{\overline{m}}{2}$.

This completes the proof of Theorem 1.4.

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