# ON A PROBLEM OF PARTITIONS OF $\mathbb{Z}_{m}$ WITH THE SAME REPRESENTATION FUNCTIONS 

CUI-FANG SUN AND MENG-CHI XIONG


#### Abstract

For any positive integer $m$, let $\mathbb{Z}_{m}$ be the set of residue classes modulo $m$. For $A \subseteq \mathbb{Z}_{m}$ and $\bar{n} \in \mathbb{Z}_{m}$, let representation function $R_{A}(\bar{n})$ denote the number of solutions of the equation $\bar{n}=\bar{a}+\overline{a^{\prime}}$ with ordered pairs $\left(\bar{a}, \overline{a^{\prime}}\right) \in A \times A$. In this paper, we determine all sets $A, B \subseteq \mathbb{Z}_{m}$ with $A \cup B=\mathbb{Z}_{m}$ and $|A \cap B|=2$ or $m-2$ such that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$. We also prove that if $m$ is a positive integer with $4 \mid m$, then there exist two distinct sets $A, B \subseteq \mathbb{Z}_{m}$ with $A \cup B=\mathbb{Z}_{m}$ and $|A \cap B|=4$ or $m-4, B \neq A+\frac{\bar{m}}{2}$ such that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$. If $m$ is a positive integer with $2 \| m, A \cup B=\mathbb{Z}_{m}$ and $|A \cap B|=4$ or $m-4$, then $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$ if and only if $B=A+\overline{\frac{m}{2}}$.


Keywords: Representation function, partition, residue class.

## 1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For $S \subseteq \mathbb{N}$ and $n \in S$, let the representation function $R_{S}^{\prime}(n)$ denote the number of solutions of the equation $n=s+s^{\prime}$ with $s, s^{\prime} \in S$. Sárkőzy asked whether there exist two sets $A, B \subseteq \mathbb{N}$ with $|(A \cup B) \backslash(A \cap B)|=\infty$ such that $R_{A}^{\prime}(n)=R_{B}^{\prime}(n)$ for all sufficiently large integers $n$. In 2002, Dombi [2] showed that the answer is negative. There are many other related results (see [1, 3, 4, 5, 6, 7] and the references therein).

For a positive integer $m$, let $\mathbb{Z}_{m}$ be the set of residue classes modulo $m$. For any residue classes $\bar{a}, \bar{b} \in \mathbb{Z}_{m}$, there exist two integers $a^{\prime}, b^{\prime}$ with $0 \leq a^{\prime}, b^{\prime} \leq m-1$ such that $\overline{a^{\prime}}=\bar{a}$ and $\overline{b^{\prime}}=\bar{b}$. We define the ordering $\bar{a} \leq \bar{b}$ if $a^{\prime} \leq b^{\prime}$. For any $\bar{n} \in \mathbb{Z}_{m}$, without loss of generality, we may suppose that $0 \leq n \leq m-1$. For $A \subseteq \mathbb{Z}_{m}$ and $\bar{n} \in \mathbb{Z}_{m}$, let $R_{A}(\bar{n})$ denote the number of solutions of $\bar{n}=\bar{a}+\overline{a^{\prime}}$ with $\bar{a}, \overline{a^{\prime}} \in A$. For $\bar{n} \in \mathbb{Z}_{m}$ and $A \subseteq \mathbb{Z}_{m}$, let $\bar{n}+A=\{\bar{n}+\bar{a}: \bar{a} \in A\}$. For $A, B \subseteq \mathbb{Z}_{m}$ and $\bar{n} \in \mathbb{Z}_{m}$, let $R_{A, B}(\bar{n})$ be the number of solutions of $\bar{n}=\bar{a}+\bar{b}$ with $\bar{a} \in A$ and $\bar{b} \in B$. The characteristic function of $A \subseteq \mathbb{Z}_{m}$ is denoted by

$$
\chi_{A}(n)= \begin{cases}1 & \bar{n} \in A \\ 0 & \bar{n} \notin A .\end{cases}
$$

In 2012, Yang and Chen [8] determined all sets $A, B \subseteq \mathbb{Z}_{m}$ with $|(A \cup B) \backslash(A \cap B)|=m$ such that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$. In 2017, Yang and Tang [9] determined all sets $A, B \subseteq \mathbb{Z}_{m}$ with $|(A \cup B) \backslash(A \cap B)|=2$ or $m-1$ such that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$. Yang and Tang [9] also posed the following problem for further research.

Problem 1.1. Given a positive even integer $m$ and an integer $k$ with $2 \leq k \leq m-1$. Do there exist two distinct sets $A, B \subseteq \mathbb{Z}_{m}$ with $|A|=|B|=k$ and $B \neq A+\left\{\frac{m}{2}\right\}$ such that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$ ?

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E-mail: cuifangsun@163.com, mengchixiong@126.com.

In this paper, we consider for which positive even integers $m$ there exist two distinct sets $A, B \subseteq$ $\mathbb{Z}_{m}$ with $A \cup B=\mathbb{Z}_{m}$ and $|A \cap B|=2, m-2,4$ or $m-4$ such that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$ and obtain the following results:
theorem 1.2. Let $m$ be a positive even integer. Let $A, B \subseteq \mathbb{Z}_{m}$ with $A \cup B=\mathbb{Z}_{m}$ and $|A \cap B|=2$ or $m-2$. Then $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$ if and only if $B=A+\frac{\bar{m}}{2}$.
theorem 1.3. Let $m$ be a positive integer with $4 \mid m$. Then there exist two distinct sets $A, B \subseteq \mathbb{Z}_{m}$ with $A \cup B=\mathbb{Z}_{m}, B \neq A+\frac{\bar{m}}{2}$ and $|A \cap B|=4$ or $m-4$ such that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$.
theorem 1.4. Let $m$ be a positive integer with $2 \| m$. Let $A, B \subseteq \mathbb{Z}_{m}$ with $A \cup B=\mathbb{Z}_{m}$ and $|A \cap B|=4$ or $m-4$. Then $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$ if and only if $B=A+\frac{\bar{m}}{2}$.

## 2. Lemmas

Lemma 2.1. Let $m$ be a positive even integer and t be a positive integer with $t<\frac{m}{2}$. Let $A, B \subseteq \mathbb{Z}_{m}$ with $A \cup B=\mathbb{Z}_{m}$ and $|A \cap B|=2 t$. If $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$, then $|\bar{A}|=|B|=\frac{\bar{m}}{2}+t$.

Proof. If $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$, then

$$
|A|^{2}=\sum_{\bar{n} \in \mathbb{Z}_{m}} R_{A}(\bar{n})=\sum_{\bar{n} \in \mathbb{Z}_{m}} R_{B}(\bar{n})=|B|^{2}
$$

Thus $|A|=|B|$. Noting that

$$
|A|+|B|=|A \cup B|+|A \cap B|=m+2 t,
$$

we have $|A|=|B|=\frac{m}{2}+t$.
This completes the proof of Lemma 2.1.
Lemma 2.2. Let $m$ be a positive even integer and t be a positive integer with $t<\frac{m}{2}$. Let $A, B \subseteq \mathbb{Z}_{m}$ with $A \cup B=\mathbb{Z}_{m}$ and $A \cap B=\left\{\overline{r_{1}}, \overline{r_{2}}, \ldots, \overline{r_{2 t}}\right\}$. If $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$, then

$$
\begin{equation*}
\sum_{i=1}^{2 t} \chi_{A}\left(n-r_{i}\right)=t+\frac{1}{2} R_{\left\{\overline{r_{1}}, \overline{r_{2}}, \ldots, \overline{r_{22}}\right\}}(\bar{n}) \tag{2.1}
\end{equation*}
$$

Proof. Noting that $B=\left(\mathbb{Z}_{m} \backslash A\right) \cup\left\{\overline{r_{1}}, \overline{r_{2}}, \ldots, \overline{r_{2 t}}\right\}$, we have

$$
\begin{aligned}
R_{B}(\bar{n})= & R_{\mathbb{Z}_{m} \backslash A}(\bar{n})+2 R_{\mathbb{Z}_{m} \backslash A,\left\{\overline{\bar{r}_{1}}, \overline{r_{2}}, \ldots, \overline{r_{2 t}}\right\}}(\bar{n})+R_{\left\{\overline{r_{1}}, \overline{r_{2}}, \ldots, \overline{\left.r_{2 t}\right\}}\right.}(\bar{n}) \\
= & \mid\left\{\left(a, a^{\prime}\right): \bar{a}, \overline{a^{\prime}} \in \mathbb{Z}_{m} \backslash A, 0 \leq a, a^{\prime} \leq m-1, a+a^{\prime}=n \text { or } a+a^{\prime}=n+m\right\} \mid \\
& +2 \sum_{i=1}^{2 t}\left(1-\chi_{A}\left(n-r_{i}\right)\right)+R_{\left\{\overline{r_{1}}, \overline{r_{2}}, \ldots, \overline{r_{2 t}}\right\}}(\bar{n})
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{0 \leq i \leq \frac{n}{2}}\left(1-\chi_{A}(i)\right)\left(1-\chi_{A}(n-i)\right)+\sum_{n+1 \leq i \leq \frac{n+m}{2}}\left(1-\chi_{A}(i)\right)\left(1-\chi_{A}(n-i)\right) \\
& +\sum_{0 \leq i<\frac{n}{2}}\left(1-\chi_{A}(i)\right)\left(1-\chi_{A}(n-i)\right)+\sum_{n+1 \leq i<\frac{n+m}{2}}\left(1-\chi_{A}(i)\right)\left(1-\chi_{A}(n-i)\right) \\
& +2 \sum_{i=1}^{2 t}\left(1-\chi_{A}\left(n-r_{i}\right)\right)+R_{\left\{\overline{\left.r_{1}, \overline{r_{2}}, \ldots, \overline{\left.r_{2 t}\right\}}\right\}}(\bar{n})\right.}^{=} \sum_{0 \leq i \leq \frac{n}{2}} 1-\sum_{0 \leq i \leq n} \chi_{A}(i)-\chi_{A}\left(\frac{n}{2}\right)+\sum_{0 \leq i \leq \frac{n}{2}} \chi_{A}(i) \chi_{A}(n-i) \\
& +\sum_{n+1 \leq i \leq \frac{n+m}{2}} 1-\sum_{n+1 \leq i \leq m-1} \chi_{A}(i)-\chi_{A}\left(\frac{n+m}{2}\right)+\sum_{n+1 \leq i \leq \frac{n+m}{2}} \chi_{A}(i) \chi_{A}(n-i) \\
& +\sum_{0 \leq i<\frac{n}{2}} 1-\sum_{0 \leq i \leq n} \chi_{A}(i)+\chi_{A}\left(\frac{n}{2}\right)+\sum_{0 \leq i<\frac{n}{2}} \chi_{A}(i) \chi_{A}(n-i) \\
& +\sum_{n+1 \leq i<\frac{n+m}{2}} 1-\sum_{n+1 \leq i \leq m-1} \chi_{A}(i)+\chi_{A}\left(\frac{n+m}{2}\right)+\sum_{n+1 \leq i<\frac{n+m}{2}} \chi_{A}(i) \chi_{A}(n-i) \\
& +2 \sum_{i=1}^{2 t}\left(1-\chi_{A}\left(n-r_{i}\right)\right)+R_{\left\{\overline{r_{1}, ~}, \overline{r_{2}}, \ldots, \overline{\left.r_{2} t\right\}}\right.}(\bar{n}) \\
= & {\left[\frac{n}{2}\right]+1+\left[\frac{n+m}{2}\right]-n+\left[\frac{n-1}{2}\right]+1+\left[\frac{n+m-1}{2}\right]-n-2|A| } \\
& +R_{A}(\bar{n})+2 \sum_{i=1}^{2 t}\left(1-\chi_{A}\left(n-r_{i}\right)\right)+R_{\left\{\overline{r_{1}, \overline{r_{2}}, \ldots, \overline{\left.r_{2} t\right\}}}(\bar{n})\right.}^{2 t} \\
= & -2 t+R_{A}(\bar{n})+2 \sum_{i=1}^{2 t}\left(1-\chi_{A}\left(n-r_{i}\right)\right)+R_{\left\{\overline{r_{1}}, \overline{r_{2}}, \ldots, \overline{\left.r_{2 t}\right\}}\right.}(\bar{n}) \\
= & 2 t+R_{A}(\bar{n})-2 \sum_{i=1}^{2 t} \chi_{A}\left(n-r_{i}\right)+R_{\left\{\overline{\left.r_{1}, \overline{r_{2}}, \ldots, \overline{\left.r_{2 t}\right\}}\right\}}(\bar{n}) .\right.}
\end{aligned}
$$

Since $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$, we have

$$
\sum_{i=1}^{2 t} \chi_{A}\left(n-r_{i}\right)=t+\frac{1}{2} R_{\left\{\overline{r_{1}}, \overline{r_{2}}, \ldots, \overline{r_{2 t}}\right\}}(\bar{n}) .
$$

This completes the proof of Lemma 2.2,

## 3. Proofs

Proof of Theorem 1.1. It is clear that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$ if $B=A+\frac{\bar{m}}{2}$. Now we suppose that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$. Clearly, the result is true for $m=2$. Now we may assume that $m \geq 4$.

Case 1. $|A \cap B|=2$. Let $A \cap B=\left\{\overline{r_{1}}, \overline{r_{2}}\right\}$ with $\overline{r_{1}} \neq \overline{r_{2}}$. By choosing $n=2 r_{1}$ in (2.1), we have

$$
1+\chi_{A}\left(2 r_{1}-r_{2}\right)=\chi_{A}\left(r_{1}\right)+\chi_{A}\left(2 r_{1}-r_{2}\right)=1+\frac{1}{2} R_{\left\{\overline{r_{1}}, \overline{r_{2}}\right\}}\left(\overline{2 r_{1}}\right) \geq \frac{3}{2}
$$

Then $\chi_{A}\left(2 r_{1}-r_{2}\right)=1$ and $\overline{r_{2}}=\overline{r_{1}}+\frac{\bar{m}}{2}$. Let $\bar{k} \in A$ with $\bar{k} \neq \overline{r_{1}}$ and $\bar{k} \neq \overline{r_{2}}$. By choosing $n=k+r_{1}$ in (2.1), we have

$$
\chi_{A}(k)+\chi_{A}\left(k+\frac{m}{2}\right)=1+\frac{1}{2} R_{\left\{\overline{r_{1}}, \overline{r_{2}}\right\}}\left(\overline{k+r_{1}}\right)=1 .
$$

It follows that $B=A+\frac{\bar{m}}{2}$.
Case 2. $|A \cap B|=m-2$. Let $A \cap B=T, A=\{\bar{a}\} \cup T, B=\{\bar{b}\} \cup T$ with $\bar{a} \neq \bar{b}$ and $\bar{a}, \bar{b} \notin T$. For any $\bar{n} \in \mathbb{Z}_{m}$, we have

$$
\begin{aligned}
& R_{A}(\bar{n})=R_{T}(\bar{n})+2 R_{T,\{\bar{a}\}}(\bar{n})+R_{\{\bar{a}\}}(\bar{n})=R_{T}(\bar{n})+2 \chi_{T}(n-a)+R_{\{\bar{a}\}}(\bar{n}) ; \\
& \left.R_{B}(\bar{n})=R_{T}(\bar{n})+2 R_{T,\{\bar{b}\}}(\bar{n})+R_{\{\bar{b}\}}(\bar{n})=R_{T}(\bar{n})+2 \chi_{T}(n-b)\right)+R_{\{\bar{b}\}}(\bar{n}) .
\end{aligned}
$$

Then

$$
\begin{equation*}
2 \chi_{T}(n-a)+R_{\{\bar{a}\}}(\bar{n})=2 \chi_{T}(n-b)+R_{\{\bar{b}\}}(\bar{n}) . \tag{3.1}
\end{equation*}
$$

By choosing $n=2 b$ in (3.1), we have

$$
2 \chi_{T}(2 b-a)+R_{\{\bar{a}\}}(\overline{2 b})=2 \chi_{T}(b)+R_{\{\bar{b}\}}(\overline{2 b})=1
$$

Then $2 \chi_{T}(2 b-a)=0$ and $R_{\{\bar{a}\}}(\overline{2 b})=1$. Thus $\bar{b}=\bar{a}+\frac{\bar{m}}{2}$. It follows that $B=A+\frac{\bar{m}}{2}$.
This completes the proof of Theorem 1.2
Proof of Theorem 1.2. Let $m=4 k$ with $k$ a positive integer.
Case 1. $|A \cap B|=4$. Noting that $A \neq B$, we have $k \geq 2$. Let

$$
\begin{aligned}
& A=\{\overline{0}, \overline{1}, \ldots, \overline{k-1}, \bar{k}\} \cup\{\overline{2 k}, \overline{2 k+1}, \ldots, \overline{3 k-1}, \overline{3 k}\} \\
& B=\{\bar{k}, \overline{k+1}, \ldots, \overline{2 k-1}, \overline{2 k}\} \cup\{\overline{3 k}, \overline{3 k+1}, \ldots, \overline{4 k-1}, \overline{0}\} .
\end{aligned}
$$

It is clear that $A, B \subseteq \mathbb{Z}_{m}$ with $A \cup B=\mathbb{Z}_{m}, B \neq A+\frac{\bar{m}}{2}$ and $A \cap B=\{\overline{0}, \bar{k}, \overline{2 k}, \overline{3 k}\}$. Let $S=\{\overline{0}, \overline{1}, \ldots, \overline{k-1}, \bar{k}\}$. Then $A=S \cup(S+\overline{2 k})$ and $B=(S+\bar{k}) \cup(S+\overline{3 k})$. Noting that

$$
\begin{aligned}
R_{S}(\bar{n}) & =R_{S+2 \overline{2 k}}(\bar{n})=R_{S+\bar{k}, S+\overline{3 k}}(\bar{n}) \\
R_{S+\bar{k}}(\bar{n}) & =R_{S, S+2 k}(\bar{n})=R_{S+3 k}(\bar{n})
\end{aligned}
$$

for all $\bar{n} \in \mathbb{Z}_{m}$, we have

$$
\begin{aligned}
R_{A}(\bar{n}) & =R_{S}(\bar{n})+2 R_{S, S+\overline{2 k}}(\bar{n})+R_{S+2 \overline{2 k}}(\bar{n}) \\
& =R_{S+\bar{k}}(\bar{n})+2 R_{S+\bar{k}, S+\overline{3 k}}(\bar{n})+R_{S+\overline{3 k}}(\bar{n}) \\
& =R_{B}(\bar{n}) .
\end{aligned}
$$

Case 2. $|A \cap B|=m-4$. If $k=1$, then by choosing $A=\{\overline{0}, \overline{2}\}, B=\{\overline{1}, \overline{3}\}$, we have $A \cup B=\mathbb{Z}_{4}, B \neq A+\overline{2}$ and $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{4}$. Now let $k \geq 2$ and

$$
\begin{aligned}
A & =\{\overline{0}, \overline{1}, \ldots, \overline{k-1}\} \cup\{\overline{k+1}, \overline{k+2}, \ldots, \overline{3 k-1}\} \cup\{\overline{3 k+1}, \overline{3 k+2}, \ldots, \overline{4 k-1}\} \\
B & =\{\overline{1}, \overline{2}, \ldots, \overline{2 k-1}\} \cup\{\overline{2 k+1}, \overline{2 k+2}, \ldots, \overline{4 k-1}\} .
\end{aligned}
$$

It is clear that $A \cup B=\mathbb{Z}_{m}, B \neq A+\frac{\bar{m}}{2}$ and

$$
A \cap B=\{\overline{1}, \ldots, \overline{k-1}\} \cup\{\overline{k+1}, \ldots, \overline{2 k-1}\} \cup\{\overline{2 k+1}, \ldots, \overline{3 k-1}\} \cup\{\overline{3 k+1}, \ldots, \overline{4 k-1}\} .
$$

Noting that $R_{A \cap B,\{\overline{0}, \overline{2 k}\}}(\bar{n})=R_{A \cap B,\{\bar{k}, \overline{3 k}\}}(\bar{n})$ and $R_{\{\overline{0}, \overline{2 k}\}}(\bar{n})=R_{\{\bar{k}, \overline{3 k}\}}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$, we have

$$
\begin{aligned}
R_{A}(\bar{n}) & =R_{\{\overline{0}, \overline{2 k}\}}(\bar{n})+2 R_{A \cap B,\{\overline{0}, \overline{2 k}\}}(\bar{n})+R_{A \cap B}(\bar{n}) \\
& =R_{\{\bar{k}, \overline{3 k}\}}(\bar{n})+2 R_{A \cap B,\{\bar{k}, \overline{3 k}\}}(\bar{n})+R_{A \cap B}(\bar{n}) \\
& =R_{B}(\bar{n}) .
\end{aligned}
$$

This completes the proof of Theorem 1.3
Proof of Theorem 1.3. It is clear that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$ if $B=A+\frac{\bar{m}}{2}$. Now we suppose that $R_{A}(\bar{n})=R_{B}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{m}$. By $A \neq B$ and $2 \| m$, we have $m \geq 6$.

Case 1. $|A \cap B|=4$. Let $A \cap B=\left\{\overline{r_{1}}, \overline{r_{2}}, \overline{r_{3}}, \overline{r_{4}}\right\}$. By Lemma[2.2] we have

$$
\begin{equation*}
\chi_{A}\left(n-r_{1}\right)+\chi_{A}\left(n-r_{2}\right)+\chi_{A}\left(n-r_{3}\right)+\chi_{A}\left(n-r_{4}\right)=2+\frac{1}{2} R_{\left\{\overline{r_{1}}, \overline{r_{2}}, \overline{r_{3}}, \overline{r_{4}}\right\}}(\bar{n}) . \tag{3.2}
\end{equation*}
$$

Then $2 \mid R_{\left\{\overline{r_{1}}, \overline{r_{2}}, \overline{r_{3}}, \overline{r_{4}}\right\}}(\bar{n})$ for any $\bar{n} \in \mathbb{Z}_{m}$. Thus $2 \mid R_{\left\{\overline{\left.r_{1}, \overline{r_{2}}, \overline{r_{3}}, \overline{r_{4}}\right\}},\right.}\left(\overline{2 r_{i}}\right)$ for $i \in\{1,2,3,4\}$. Without loss of generality, we may assume that $\overline{r_{3}}=\overline{r_{1}}+\frac{\bar{m}}{2}$ and $\overline{r_{4}}=\overline{r_{2}}+\frac{\bar{m}}{2}$. Clearly, the result is true for $m=6$. Now let $m=2 k$ with $2 \nmid k$ and $k \geq 5$. Let

$$
l=\min \left\{i \geq 2:\left\{\overline{r_{2}}, \overline{r_{2}}+\frac{\bar{m}}{2}\right\}=\left\{\overline{i r_{1}-(i-1) r_{2}}, \overline{i r_{1}-(i-1) r_{2}}+\frac{\bar{m}}{2}\right\}\right\}
$$

By $2 \| m$, we have $3 \leq l \leq k$ and $l \mid k$. Now we discuss the following two subcases according to $l$.
Subcase 1.1 $l=k$. Then

$$
\mathbb{Z}_{m}=\bigcup_{i=1}^{k}\left\{\overline{i r_{1}-(i-1) r_{2}}, \overline{i r_{1}-(i-1) r_{2}}+\frac{\bar{m}}{2}\right\}
$$

By choosing $n-r_{2}=i r_{1}-(i-1) r_{2}$ for $i=1,2, \ldots, k$ in (3.2), we have

$$
\begin{aligned}
& \chi_{A}\left(r_{2}\right)+\chi_{A}\left(r_{2}+\frac{m}{2}\right)+\chi_{A}\left(r_{1}\right)+\chi_{A}\left(r_{1}+\frac{m}{2}\right)=4, \\
& \chi_{A}\left(r_{1}\right)+\chi_{A}\left(r_{1}+\frac{m}{2}\right)+\chi_{A}\left(2 r_{1}-r_{2}\right)+\chi_{A}\left(2 r_{1}-r_{2}+\frac{m}{2}\right)=3, \\
& \chi_{A}\left(2 r_{1}-r_{2}\right)+\chi_{A}\left(2 r_{1}-r_{2}+\frac{m}{2}\right)+\chi_{A}\left(3 r_{1}-2 r_{2}\right)+\chi_{A}\left(3 r_{1}-2 r_{2}+\frac{m}{2}\right)=2, \\
& \cdots \\
& \chi_{A}\left((k-2) r_{1}-(k-3) r_{2}\right)+\chi_{A}\left((k-2) r_{1}-(k-3) r_{2}+\frac{m}{2}\right) \\
& \quad+\quad \chi_{A}\left((k-1) r_{1}-(k-2) r_{2}\right)+\chi_{A}\left((k-1) r_{1}-(k-2) r_{2}+\frac{m}{2}\right)=2, \\
& \chi_{A}\left((k-1) r_{1}-(k-2) r_{2}\right)+\chi_{A}\left((k-1) r_{1}-(k-2) r_{2}+\frac{m}{2}\right)+\chi_{A}\left(r_{2}\right)+\chi_{A}\left(r_{2}+\frac{m}{2}\right)=3 .
\end{aligned}
$$

Noting that

$$
\chi_{A}\left(r_{2}\right)+\chi_{A}\left(r_{2}+\frac{m}{2}\right)=\chi_{A}\left(r_{1}\right)+\chi_{A}\left(r_{1}+\frac{m}{2}\right)=2,
$$

we have

$$
\chi_{A}\left(i r_{1}-(i-1) r_{2}\right)+\chi_{A}\left(i r_{1}-(i-1) r_{2}+\frac{m}{2}\right)=1
$$

for $i=2,3, \ldots, k-1$. It follows that $B=A+\frac{\bar{m}}{2}$.
Subcase 1.2 $3 \leq l<k$. Then $k=l s$ with $s \geq 3$ and $2 \nmid s$. Thus

$$
\mathbb{Z}_{m}=\bigcup_{i=1}^{l} \bigcup_{j=0}^{s-1}\left\{\overline{i r_{1}-(i-1) r_{2}+j}, \overline{i r_{1}-(i-1) r_{2}+j}+\frac{\bar{m}}{2}\right\} .
$$

By choosing $n-r_{2}=i r_{1}-(i-1) r_{2}$ for $i=1,2, \ldots, l$ in (3.2), we have

$$
\begin{aligned}
& \chi_{A}\left(r_{2}\right)+\chi_{A}\left(r_{2}+\frac{m}{2}\right)+\chi_{A}\left(r_{1}\right)+\chi_{A}\left(r_{1}+\frac{m}{2}\right)=4, \\
& \chi_{A}\left(r_{1}\right)+\chi_{A}\left(r_{1}+\frac{m}{2}\right)+\chi_{A}\left(2 r_{1}-r_{2}\right)+\chi_{A}\left(2 r_{1}-r_{2}+\frac{m}{2}\right)=3, \\
& \chi_{A}\left(2 r_{1}-r_{2}\right)+\chi_{A}\left(2 r_{1}-r_{2}+\frac{m}{2}\right)+\chi_{A}\left(3 r_{1}-2 r_{2}\right)+\chi_{A}\left(3 r_{1}-2 r_{2}+\frac{m}{2}\right) \\
& =2+\frac{1}{2} R_{\left\{r_{1}, \overline{r_{1}+\frac{m}{2}}, \overline{r_{2}}, \overline{\left.r_{2}+\frac{m}{2}\right\}}\right\}}\left(\overline{3 r_{1}-r_{2}}\right), \\
& \cdots \\
& \chi_{A}\left((l-1) r_{1}-(l-2) r_{2}\right)+\chi_{A}\left((l-1) r_{1}-(l-2) r_{2}+\frac{m}{2}\right)+\chi_{A}\left(r_{2}\right)+\chi_{A}\left(r_{2}+\frac{m}{2}\right)=3 .
\end{aligned}
$$

Noting that

$$
\chi_{A}\left(r_{2}\right)+\chi_{A}\left(r_{2}+\frac{m}{2}\right)=\chi_{A}\left(r_{1}\right)+\chi_{A}\left(r_{1}+\frac{m}{2}\right)=2,
$$

we have

$$
\chi_{A}\left(i r_{1}-(i-1) r_{2}\right)+\chi_{A}\left(i r_{1}-(i-1) r_{2}+\frac{m}{2}\right)=1
$$

for $i=2, \ldots, l-1$.
For any $j \in\{1,2, \ldots, s-1\}$, by choosing $n-r_{2}=i r_{1}-(i-1) r_{2}+j$ for $i=1,2, \ldots, l$ in (3.2), we have

$$
\begin{aligned}
& \chi_{A}\left(r_{2}+j\right)+\chi_{A}\left(r_{2}+j+\frac{m}{2}\right)+\chi_{A}\left(r_{1}+j\right)+\chi_{A}\left(r_{1}+j+\frac{m}{2}\right)=2, \\
& \chi_{A}\left(r_{1}+j\right)+\chi_{A}\left(r_{1}+j+\frac{m}{2}\right)+\chi_{A}\left(2 r_{1}-r_{2}+j\right)+\chi_{A}\left(2 r_{1}-r_{2}+j+\frac{m}{2}\right)=2, \\
& \chi_{A}\left(2 r_{1}-r_{2}+j\right)+\chi_{A}\left(2 r_{1}-r_{2}+j+\frac{m}{2}\right)+\chi_{A}\left(3 r_{1}-2 r_{2}+j\right) \\
& \quad \quad+\chi_{A}\left(3 r_{1}-2 r_{2}+j+\frac{m}{2}\right)=2, \\
& \ldots \\
& \chi_{A}\left((l-1) r_{1}-(l-2) r_{2}+j\right)+\chi_{A}\left((l-1) r_{1}-(l-2) r_{2}+j+\frac{m}{2}\right) \\
& \quad+\chi_{A}\left(r_{2}+j\right)+\chi_{A}\left(r_{2}+j+\frac{m}{2}\right)=2
\end{aligned}
$$

Noting that $2 \nmid l$, we have

$$
\chi_{A}\left(i r_{1}-(i-1) r_{2}+j\right)+\chi_{A}\left(i r_{1}-(i-1) r_{2}+j+\frac{m}{2}\right)=1
$$

for $i=1,2, \ldots, l-1$. It follows that $B=A+\frac{\bar{m}}{2}$.
Case 2. $|A \cap B|=m-4$. Let $A \cap B=T, A=\left\{\overline{a_{1}}, \overline{a_{2}}\right\} \cup T, B=\left\{\overline{b_{1}}, \overline{b_{2}}\right\} \cup T$ with $\left\{\overline{a_{1}}, \overline{a_{2}}\right\} \cap\left\{\overline{b_{1}}, \overline{b_{1}}\right\}=\varnothing$ and $\left\{\overline{a_{1}}, \overline{a_{2}}, \overline{\bar{b}_{1}}, \overline{b_{2}}\right\} \cap T=\varnothing$. For any $\bar{n} \in \mathbb{Z}_{m}$, we have

$$
\begin{aligned}
R_{A}(\bar{n}) & =R_{T}(\bar{n})+2 R_{T,\left\{\overline{a_{1}}, \overline{a_{2}}\right\}}(\bar{n})+R_{\left\{\overline{1_{1}}, \overline{a_{2}}\right\}}(\bar{n}) \\
& =R_{T}(\bar{n})+2 \chi_{T}\left(n-a_{1}\right)+2 \chi_{T}\left(n-a_{2}\right)+R_{\left\{\overline{\overline{1}_{1}}, \overline{a_{2}}\right\}}(\bar{n}) ; \\
R_{B}(\bar{n}) & =R_{T}(\bar{n})+2 R_{T,\left\{\overline{b_{1}}, \overline{\left.b_{2}\right\}}\right.}(\bar{n})+R_{\left\{\overline{b_{1}}, \overline{\left.b_{2}\right\}}\right.}(\bar{n}) \\
& =R_{T}(\bar{n})+2 \chi_{T}\left(n-b_{1}\right)+2 \chi_{T}\left(n-b_{2}\right)+R_{\left\{\overline{b_{1}}, \overline{\left.b_{2}\right\}}\right.}(\bar{n}) .
\end{aligned}
$$

Then

$$
\begin{equation*}
2 \chi_{T}\left(n-a_{1}\right)+2 \chi_{T}\left(n-a_{2}\right)+R_{\left\{\overline{a_{1}}, \overline{a_{2}}\right\}}(\bar{n})=2 \chi_{T}\left(n-b_{1}\right)+2 \chi_{T}\left(n-b_{2}\right)+R_{\left\{\overline{b_{1}}, \overline{b_{2}}\right\}}(\bar{n}) \tag{3.3}
\end{equation*}
$$

By choosing $n=a_{1}+a_{2}$ in (3.3), we have

$$
2 \chi_{T}\left(a_{1}+a_{2}-b_{1}\right)+2 \chi_{T}\left(a_{1}+a_{2}-b_{2}\right)+R_{\left\{\overline{b_{1}}, \overline{b_{2}}\right\}}\left(\overline{a_{1}}+\overline{a_{2}}\right)=2
$$

If $\chi_{T}\left(a_{1}+a_{2}-b_{1}\right)=1$, then

$$
\chi_{T}\left(a_{1}+a_{2}-b_{2}\right)=R_{\left\{\overline{b_{1}}, \overline{b_{2}}\right\}}\left(\overline{a_{1}}+\overline{a_{2}}\right)=0
$$

Thus $\overline{a_{1}}+\overline{a_{2}}-\overline{b_{2}} \in\left\{\overline{b_{1}}, \overline{b_{2}}\right\}$ and $\overline{a_{1}}+\overline{a_{2}} \notin\left\{\overline{2 b_{1}}, \overline{b_{1}}+\overline{b_{2}}, \overline{2 b_{2}}\right\}$, which is a contradiction. It follows that $\chi_{T}\left(a_{1}+a_{2}-b_{1}\right)=0$. Similarly, we can get $\chi_{T}\left(a_{1}+a_{2}-b_{2}\right)=0$. Therefore $R_{\left\{\overline{b_{1}}, \overline{b_{2}}\right\}}\left(\overline{a_{1}}+\overline{a_{2}}\right)=2$. It means that $\overline{a_{1}}+\overline{a_{2}}=\overline{2 b_{1}}=\overline{2 b_{2}}$ or $\overline{a_{1}}+\overline{a_{2}}=\overline{b_{1}}+\overline{b_{2}}$.

By choosing $n=b_{1}+b_{2}$ in (3.3), we have

$$
2 \chi_{T}\left(b_{1}+b_{2}-a_{1}\right)+2 \chi_{T}\left(b_{1}+b_{2}-a_{2}\right)+R_{\left\{\overline{a_{1}}, \overline{a_{2}}\right\}}\left(\overline{b_{1}}+\overline{b_{2}}\right)=2
$$

If $\chi_{T}\left(b_{1}+b_{2}-a_{1}\right)=1$, then

$$
\chi_{T}\left(b_{1}+b_{2}-a_{2}\right)=R_{\left\{\overline{a_{1}}, \overline{a_{2}}\right\}}\left(\overline{b_{1}}+\overline{b_{2}}\right)=0
$$

Thus $\overline{b_{1}}+\overline{b_{2}}-\overline{a_{2}} \in\left\{\overline{a_{1}}, \overline{a_{2}}\right\}$ and $\overline{b_{1}}+\overline{b_{2}} \notin\left\{\overline{2 a_{1}}, \overline{a_{1}}+\overline{a_{2}}, \overline{2 a_{2}}\right\}$, which is a contradiction. It follows that $\chi_{T}\left(b_{1}+b_{2}-a_{1}\right)=0$. Similarly, we can get $\chi_{T}\left(b_{1}+b_{2}-a_{2}\right)=0$. Therefore $R_{\left\{\overline{a_{1}}, \overline{a_{2}}\right\}}\left(\overline{b_{1}}+\overline{b_{2}}\right)=2$. It means that $\overline{b_{1}}+\overline{b_{2}}=\overline{2 a_{1}}=\overline{2 a_{2}}$ or $\overline{b_{1}}+\overline{b_{2}}=\overline{a_{1}}+\overline{a_{2}}$.

If $\overline{a_{1}}+\overline{a_{2}}=\overline{2 b_{1}}=\overline{2 b_{2}}$, then $\overline{b_{1}}+\overline{b_{2}}=\overline{2 a_{1}}=\overline{2 a_{2}}$. Thus $\overline{a_{2}}=\overline{a_{1}}+\overline{\frac{m}{2}}, \overline{b_{2}}=\overline{b_{1}}+\overline{\frac{m}{2}}$ and $\overline{2 a_{1}}+\overline{\frac{m}{2}}=\overline{2 b_{1}}$, which contradicts $2 \| m$. It follows that $\overline{a_{1}}+\overline{a_{2}}=\overline{b_{1}}+\overline{b_{2}}$. By choosing $n=2 a_{1}$ in (3.3), we have

$$
\begin{equation*}
2 \chi_{T}\left(2 a_{1}-a_{2}\right)+R_{\left\{\overline{a_{1}}, \overline{a_{2}}\right\}}\left(\overline{2 a_{1}}\right)=2 \chi_{T}\left(2 a_{1}-b_{1}\right)+2 \chi_{T}\left(2 a_{1}-b_{2}\right)+R_{\left\{\overline{b_{1}}, \overline{b_{2}}\right\}}\left(\overline{2 a_{1}}\right) \tag{3.4}
\end{equation*}
$$

If $R_{\left\{\overline{a_{1}}, \overline{a_{2}}\right\}}\left(\overline{2 a_{1}}\right)=2$, then $\overline{2 a_{1}}=\overline{2 a_{2}}$ and $\chi_{T}\left(2 a_{1}-a_{2}\right)=\chi_{T}\left(a_{2}\right)=0$. Thus $R_{\left\{\overline{b_{1}}, \overline{\left.b_{2}\right\}}\right.}\left(\overline{2 a_{1}}\right)=0$ and $\chi_{T}\left(2 a_{1}-b_{1}\right)=\chi_{T}\left(2 a_{1}-b_{2}\right)=1$, which contradicts (3.4).

If $R_{\left\{\overline{a_{1}}, \overline{a_{2}}\right\}}\left(\overline{2 a_{1}}\right)=1$, then $\overline{2 a_{1}} \neq \overline{2 a_{2}}$. By (3.4), we have $R_{\left\{\overline{b_{1}}, \overline{b_{2}}\right\}}\left(\overline{2 a_{1}}\right)=1$. Then $\overline{2 b_{1}} \neq \overline{2 b_{2}}$ and $\overline{2 a_{1}} \in\left\{\overline{2 b_{1}}, \overline{2 b_{2}}\right\}$. Without loss of generality, we may assume that $\overline{2 a_{1}}=\overline{2 b_{1}}$. Then $\overline{b_{1}}=\overline{a_{1}}+\frac{\bar{m}}{2}$ and $\overline{b_{2}}=\overline{a_{2}}+\frac{\bar{m}}{2}$. It follows that $B=A+\overline{\frac{m}{2}}$.

This completes the proof of Theorem 1.4

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