CENTRALLY HARMONIC SPACES

P. B. GILKEY AND J. H. PARK

Abstract. We construct examples of centrally harmonic spaces by generalizing work of Copson and Ruse. We show that these examples are generically not centrally harmonic at other points. We use this construction to exhibit manifolds which are not conformally flat but such that their density function agrees with Euclidean space.

1. Introduction

1.1. **Notational conventions.** Any 2-dimensional manifold is Einstein; thus this condition imposes no additional restrictions and the case $m = 2$ is often exceptional. We shall therefore sometimes assume that $m \geq 3$ to simplify the analysis. If $\vec{\xi} = (\xi^1, \ldots, \xi^m) \in \mathbb{R}^m$, set:

$$\begin{array}{ll} \|\xi\|^2 := (\xi^1)^2 + \cdots + (\xi^m)^2, & d\xi := d\xi^1 \ldots d\xi^m, \\ g_e := (d\xi^1)^2 + \cdots + (d\xi^m)^2, & \Delta_e^0 := -\partial_{\xi^1}^2 - \cdots - \partial_{\xi^m}^2, \\ S_0(r) := \{\xi : \|\xi\| = r\}. & \end{array}$$

There is a radial solution to the equation $\Delta_e^0 f = 0$ for $\|\xi\| > 0$ given by

$$f(\xi) := \left\{ \begin{array}{lll} \log \|\xi\|^2 & \text{if} & m = 2 \\ \|\xi\|^{2-m} & \text{if} & m > 2 \end{array} \right\}.$$

Ruse [12] was the first to examine radial solutions to the Laplace equation in the more general context of a Riemannian manifold $\mathcal{M} = (M, g)$ of dimension $m \geq 2$. Let $\Delta_{\mathcal{M}}^0$ (resp. $\Delta_{\mathcal{M}}^1$) be the Laplace-Beltrami operator on functions (resp. 1-forms). Let $r_P(Q)$ be the geodesic distance from a point P to another point Q of M. A function f is said to be *radial* if $f(Q) = \check{f}(r_P(Q))$ for some function $\check{f}$ of a single variable; in the interests of notational simplification, we shall identify f with $\check{f}$ when no confusion is likely to result. Let ι_P be the injectivity radius. If there exists a non-constant radial function so that $\Delta_{\mathcal{M}}^0 f = 0$ for $0 < r < \iota_P$, then $\mathcal{M}$ is said to be *centrally harmonic about* P. If $\mathcal{M}$ is centrally harmonic about every point, then $\mathcal{M}$ is said to be a *harmonic space* (see Willmore [15]).

Much of the subsequent work in the field has focussed on harmonic spaces. But in this note, we will go back to the original question and study spaces which are centrally harmonic about a point P. There are a number of useful characterizations of this property. Let $(\xi^1, \ldots, \xi^m)$ be geodesic coordinates centered at a point P of M. Such coordinate systems are characterized by the fact that the curves $t \to t\xi$ are unit speed geodesics from P if $\|\xi\| = 1$ and hence $r_P(\xi) = \|\xi\|$ if $\|\xi\| < \iota_P$. The Riemannian measure defined by g is $\tilde{\Theta}_P d\xi$ where $\tilde{\Theta}_P := \sqrt{\det g_{ij}}$ is the associated *volume density function*. Let $S_P(r) := \{\xi \in T_P M : \|\xi\| = r\}$ be the geodesic sphere of radius r centered at P and let $(r, \theta) \to r \cdot \theta$ define geodesic polar coordinates where $\theta \in S_P(1)$ and $0 < r < \iota_P$. If $d\theta$ is the Euclidean volume element of $S_P(1)$,

2010 *Mathematics Subject Classification.* 53C21.

Key words and phrases. harmonic spaces, density function, centrally harmonic space, Damek-Ricci space.

then $d\xi = r^{m-1}drd\theta$ so the volume density in geodesic polar coordinates is given by $\Theta_P := r^{m-1}\tilde{\Theta}_P$. Let

$$\Xi_P := \partial_r \log(\Theta_P(\xi))\,;$$

Ξ_P is the mean curvature of the geodesic sphere $S_P(\|\xi\|)$ at $\xi \in T_PM$. For $\lambda \in \mathbb{C}$, let $\mathfrak{E}^0_P(\lambda)$ (resp. $\mathfrak{E}^1_P(\lambda)$) be the eigen-space of radial functions (resp. 1-forms) defined by λ, i.e.,

$$\begin{aligned}\mathfrak{E}^0_P(\lambda) &:= \left\{\phi^0 \in C^\infty(0,\iota_P) : \Delta^0_{\mathcal{M}}(\phi^0(r)) = \lambda\phi^0(r)\right\},\\ \mathfrak{E}^1_P(\lambda) &:= \left\{\phi^1 dr \in C^\infty(0,\iota_P)dr : \Delta^1_{\mathcal{M}}(\phi^1(r)dr) = \lambda\phi^1(r)dr\right\}.\end{aligned}$$

Note that we exclude the origin $r = 0$; these functions are permitted to be singular at the center point P. If $\lambda \neq 0$, then d is an isomorphism from $\mathfrak{E}^0_P(\lambda)$ to $\mathfrak{E}^1_P(\lambda)$ so it suffices to study $\mathfrak{E}^0_P(\lambda)$ in this instance. Let $\mathfrak{H}^0_P$ (resp. $\mathfrak{H}^1_P$) be the space of radial harmonic functions (resp. 1-forms), i.e., $\mathfrak{H}^0_P := \mathfrak{E}^0_P(0)$ and $\mathfrak{H}^1_P := \mathfrak{E}^1_P(0)$.

1.2. **Characterizations of centrally harmonic spaces.** The following result was established by the authors previously [8].

Theorem 1.1. The following assertions are equivalent and if any is satisfied, then $\mathcal{M}$ is centrally harmonic about the point P. If they hold at every point, then $\mathcal{M}$ is said to be a harmonic space. Let $\lambda \neq 0$.

(a) Θ_P is radial. (b) $\tilde{\Theta}_P$ is radial. (c) Ξ_P is radial.
(d) $\dim\{\mathfrak{H}^0_P\} = 2$. (e) $\dim\{\mathfrak{H}^0_P\} \geq 2$. (f) $\dim\{\mathfrak{H}^1_P\} = 2$.
(g) $\dim\{\mathfrak{H}^1_P\} \geq 1$. (h) $\dim\{\mathfrak{E}^0_P(\lambda)\} = 2$. (i) $\dim\{\mathfrak{E}^0_P(\lambda)\} \geq 1$.
(j) $\dim\{\mathfrak{E}^1_P(\lambda)\} = 2$. (k) $\dim\{\mathfrak{E}^1_P(\lambda)\} \geq 1$. (l) $\Delta^0_{\mathcal{M}} r$ is radial.
(m) Geodesic spheres about P have constant mean curvature.

1.3. **Asymptotic expansion of the volume density function in geodesic coordinates.** If $\mathcal{M}$ a Riemannian manifold, then we can expand

$$\Theta_P(\xi) \sim \|\xi\|^{m-1}\left(1 + \sum_{k=2}^{\infty}\mathcal{H}_k(\xi)\right) \tag{1.a}$$

in a power series about the origin where $\mathcal{H}_k(c\xi) = c^k\mathcal{H}_k(\xi)$ for $c \in \mathbb{R}$; we omit the dependence on the point P in the interests of notational simplification. Let R be the curvature tensor of M and let $\mathcal{J}_k(\xi)$ be the endomorphism of T_PM defined by the identity

$$g(\mathcal{J}_k(\xi)\eta_1,\eta_2) = (\nabla^k R)(\eta_1,\xi,\xi,\eta_2;\xi,\ldots,\xi)\,;$$

$\mathcal{J}_0(\xi)$ is the Jacobi operator and $\mathcal{J}_k(\xi) = (\nabla^k_\xi \mathcal{J}_0)(\xi)$. We have, see for example the discussion on page 229 of [15], that

$$\begin{aligned}\mathcal{H}_2(\xi) &= -\tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)\}}{6},\\ \mathcal{H}_3(\xi) &= -\tfrac{\operatorname{Tr}\{\mathcal{J}_1(\xi)\}}{12},\\ \mathcal{H}_4(\xi) &= \tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)\}^2}{72} - \tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)^2\}}{180} - \tfrac{\operatorname{Tr}\{\mathcal{J}_2(\xi)\}}{40},\\ \mathcal{H}_5(\xi) &= \tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)\}\operatorname{Tr}\{\mathcal{J}_1(\xi)\}}{72} - \tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)\mathcal{J}_1(\xi)\}}{180} - \tfrac{\operatorname{Tr}\{\mathcal{J}_3(\xi)\}}{180},\\ \mathcal{H}_6(\xi) &= -\tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)\}^3}{1296} + \tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)\}\operatorname{Tr}\{\mathcal{J}(\xi)^2\}}{1080} + \tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)\}\operatorname{Tr}\{\mathcal{J}_2(\xi)\}}{240} - \tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)^3\}}{2835}\\ &\quad - \tfrac{\operatorname{Tr}\{\mathcal{J}(\xi)\mathcal{J}_2(\xi)\}}{630} + \tfrac{\operatorname{Tr}\{\mathcal{J}_1(\xi)\}^2}{288} - \tfrac{\operatorname{Tr}\{\mathcal{J}_1(\xi)^2\}}{672} - \tfrac{\operatorname{Tr}\{\mathcal{J}_4(\xi)\}}{1008}.\end{aligned} \tag{1.b}$$

Formulas for $\mathcal{H}_7$ and $\mathcal{H}_8$ were derived in [7]. More generally, one can show that

$$\mathcal{H}_n(\xi) = c_n \operatorname{Tr}\{\mathcal{J}_{n-2}(\xi)\} + \text{lower order terms}$$

In particular, $c_2 = -\frac{1}{6}$, $c_3 = -\frac{1}{12}$, $c_4 = -\frac{1}{40}$, $c_5 = -\frac{1}{180}$, and $c_6 = -\frac{1}{1008}$. We will establish the following result in Section 2; it provides a leading term analysis which will be crucial in what follows.

Lemma 1.2. We have $c_n = -\dfrac{n-1}{(n+1)!}$.

1.4. **Examples of harmonic spaces.** If $\mathcal{M}$ is a simply connected 2-point homogeneous space, i.e., if $\mathcal{M}$ is either $\mathbb{R}^m$ or $\mathcal{M}$ is a rank one symmetric space, then the isometry group of $\mathcal{M}$ acts transitively on the set of unit tangent vectors and hence Θ_P is radial for any P; consequently any 2-point homogeneous space is centrally harmonic about any point and hence is a harmonic space. In negative curvature, the Damek-Ricci spaces are also harmonic spaces; these are solvmanifolds, but need not be 2-point homogeneous spaces. All known harmonic spaces are locally homogeneous and modeled on one of these geometries. We refer to Berndt et. al. [1] for further details.

1.5. **Constructing centrally harmonic spaces.** Copson and Ruse [6] gave examples of centrally harmonic spaces by noting that a radial conformal deformation of the Euclidean metric g_e is centrally harmonic about the origin. More generally, if $\mathcal{M}=(M,g)$ is the germ of a Riemannian manifold, and if ψ is a smooth non-zero function of 1-variable, we define a radial conformal deformation of $\mathcal{M}$ by setting

$$\mathcal{M}_\psi := (M, \psi(r_P^2)^{-2} g)\,.$$

In Section 3, we will establish the following result which shows if $\mathcal{M}$ is centrally harmonic about P, then $\mathcal{M}_\psi$ is centrally harmonic about P as well. Since we can always take the base manifold $\mathcal{M}$ to be a harmonic space, this permits us to construct many centrally harmonic spaces generalizing the examples of Copson and Ruse [6].

Theorem 1.3. If $\mathcal{M}=(M,g)$ is the germ of a Riemannian manifold which centrally harmonic about P, then $\mathcal{M}_\psi$ is centrally harmonic about P as well.

1.6. **Space forms.** $\mathcal{M}$ is said to be a *space form* if $\mathcal{M}$ has constant sectional curvature. Let $\mathcal{U}$ be an open subset of $\mathbb{R}$, let ψ be a non-zero analytic function on $\mathcal{U}$, let $\mathcal{O} := \{\xi \in \mathbb{R}^m : \|\xi\|^2 \in \mathcal{U}\}$, and let $\Psi(\xi) := \psi(\|\xi\|^2) \in C^\infty(\mathcal{O})$. Define a real-analytic radial conformal deformation of the standard Euclidean metric g_e by setting

$$\mathcal{N}_\psi := (\mathcal{O}, \Psi^{-2} g_e)\,.$$

Let $\psi_{a,b}(t) := a + bt$ define $\mathcal{N}_{a,b}$ for $(a,b) \neq (0,0)$. Although the following result is well-known, we present a proof in Section 4 for the sake of completeness since we will need to develop the requisite preliminaries in any event; we suppose $m \geq 3$ as that is the case of interest.

Lemma 1.4. *Let $m \geq 3$.*

1. *If ψ is linear, then $\mathcal{N}_\psi$ is a space form.*
2. *If $\mathcal{N}_\psi$ is a space form, then ψ is linear.*
3. *$\mathcal{N}_{a,b}$ has constant sectional curvature $4ab$.*
4. *If $\mathcal{M}$ has constant sectional curvature κ, $\mathcal{M}$ is locally isometric to $\mathcal{N}_{\psi_{1,4\kappa}}$.*

1.7. **Radial conformal deformations which are centrally harmonic about an intermediate point.** Let L_ξ be the second fundamental form of the geodesic sphere $S_P(\|\xi\|)$ about P which passes through ξ. We say $S_P(\|\xi\|)$ is *totally umbillic* at ξ if L_ξ is a multiple of the identity. As noted by Copson and Ruse [6], a radial conformal deformation of Euclidean space is in general not centrally harmonic about any other point. Recall that every harmonic space is Einstein and that every Einstein manifold is real analytic. We will prove the following result in Section 5.

Theorem 1.5. Let P be a point of an Einstein manifold $\mathcal{M} = (M, g)$ of dimension $m \geq 3$. Assume that ψ is real analytic and that $\mathcal{M}_\psi$ is centrally harmonic about some vector ξ with $0 < \|\xi\| < \iota_P$ in geodesic coordinates.

(1) If $S_P(\|\xi\|)$ is not totally umbillic at ξ, then ψ is constant.
(2) If $\mathcal{M}$ is a space form, then $\mathcal{M}_\psi$ is a space form.

1.8. **Totally umbillic geodesic spheres.** The Jacobi operator $\mathcal{J}_0(\xi)$ is a self-adjoint endomorphism of T_PM. Let $\tilde{\mathcal{J}}_0(\xi)$ be the restriction of $\mathcal{J}_0$ to $\xi^\perp$, let $m_P(\xi)$ (resp. $M_P(\xi)$) be the smallest (resp. largest) eigenvalue of $\tilde{\mathcal{J}}_0(\xi)$, and let

$$s_P := \inf_{|\xi|=1} \{M_P(\xi) - m_P(\xi)\}$$

be the minimal difference the largest and the smallest eigenvalue of $\tilde{\mathcal{J}}_0(\xi)$ for ξ a unit tangent vector at P. We will establish the following result in Section 6.

Lemma 1.6.

(1) $\mathcal{M}$ is a space form, then every geodesic sphere is totally umbillic.
(2) If every sufficiently small geodesic sphere is totally umbillic and if $m \geq 3$, then $\mathcal{M}$ is a space form.
(3) If an irreducible symmetric space $\mathcal{M}$ admits a totally umbilical hypersurface $\mathcal{N}$, then both $\mathcal{M}$ and $\mathcal{N}$ are space forms.
(4) If $s_P > 0$, then there exists $\varepsilon > 0$ so that geodesic spheres of radius less than ε at P are not totally umbillic at any point.
(5) If $\mathcal{M}$ is a rank one symmetric space or $\mathcal{M}$ is a Damek-Ricci space, and if $\mathcal{M}$ is not a space form, then $s_P > 0$.

1.9. **Radial conformal deformations of the sphere.** Theorem 1.3 and Theorem 1.5 deal with points within the injectivity radius. Let $\mathcal{S} := (S^m, g_{S^m})$ where g_{S^m} is the standard round metric on the unit sphere S^m of $\mathbb{R}^{m+1}$. Denote the north and south poles of S^m by $P_\pm := (\pm 1, 0, \ldots, 0)$, respectively; $d_{P_\pm}(\xi) = \arccos(\pm\xi^1)$ and $\iota_\pm = \pi$. Let ψ be a positive real analytic function of 1-variable and let $\mathcal{S}_\psi := (S^m, \psi((\xi^1)^2)^{-2} g_{S^m})$. We will establish the following result in Section 7.

Lemma 1.7. $\mathcal{S}_\psi$ is centrally harmonic about the points $P_\pm$. If $\mathcal{S}_\psi$ is not a space form and if $m \geq 3$, then $\mathcal{S}_\psi$ is centrally harmonic about no points of the sphere other than $P_\pm$.

1.10. **A non-flat example with trivial volume density function.** We will use Theorem 1.3 to establish the following result in Section 8.

Theorem 1.8. If $m \geq 4$ is even, then there exists a Riemannian metric g on $\mathbb{R}^m$ which is centrally harmonic about the origin, which is not conformally flat, and which has $\Theta_0 = r^{m-1}$.

Remark 1.9. Since the metric g of Theorem 1.8 is not conformally flat, g is not flat. Since any harmonic space with trivial volume density function is flat, g is not a harmonic metric. We will show in Section 8 that g is essentially geodesically incomplete in dimensions 4, 6, and 8.

2. The proof of Lemma 1.2: A leading term analysis

We use Equation (1.b) to assume $n \geq 7$ in the proof of Lemma 1.2. We express $\mathcal{H}_n(\xi) = c_n \operatorname{Tr}\{\mathcal{J}_{n-2}(\xi)\} +$ lower order terms. By considering product formulas, we see that the coefficients c_n are dimension free so we may take $m = 2$. We set $ds^2 = dr^2 + f(r,\theta)d\theta^2$ where $f(r,\theta) := \{r(1 + b_n(\theta)r^n)\}^2$. We then have $\Theta(r,\theta) = r(1 + b_n(\theta)r^n)$ so $\mathcal{H}_n(\partial_r^\theta) = b_n(\theta)$ where ∂_r^θ is the radial vector field pointing from the origin to $\theta \in S^1$. We adapt an argument from Gilkey and

Park [7]. Let $f_r := \partial_r^\theta f$, $f_{rr} = \partial_r^\theta \partial_r^\theta f$, and so forth. We use the Koszul formula to compute:

$$\begin{array}{llll} \Gamma_{rrr} = 0, & \Gamma_{rr\theta} = 0, & \Gamma_{rr}{}^r = 0, & \Gamma_{rr}{}^\theta = 0, \\ \Gamma_{r\theta r} = 0, & \Gamma_{r\theta\theta} = \frac12 f_r, & \Gamma_{r\theta}{}^r = 0, & \Gamma_{r\theta}{}^\theta = \frac12 f_r f^{-1}, \\ \Gamma_{\theta\theta r} = -\frac12 f_r, & \Gamma_{\theta\theta\theta} = \frac12 f_\theta, & \Gamma_{\theta\theta}{}^r = -\frac12 f_r, & \Gamma_{\theta\theta}{}^\theta = \frac12 f_\theta f^{-1}. \end{array}$$

Thus we have that

$$\begin{array}{l} \nabla_\theta \nabla_r \partial_r^\theta = 0, \\ \nabla_r \nabla_\theta \partial_r^\theta = \nabla_r \{\Gamma_{\theta r}{}^\theta \partial_\theta\} = \{\frac12 f_{rr} f^{-1} - \frac12 f_r f_r f^{-2} + \frac14 f_r f_r f^{-2}\}\partial_\theta, \\ R(\partial_\theta, \partial_r^\theta, \partial_r^\theta, \partial_\theta) = -\frac12 f_{rr} + \frac14 f_r f_r f^{-1}, \\ \operatorname{Tr}\{\mathcal{J}_0(\partial_r^\theta)\} = f^{-1}\{-\frac12 f_{rr} + \frac14 f_r f_r f^{-1}\}\,. \end{array}$$

We compute:

$$\begin{array}{l} f(r,\theta) = r^2 + 2b_n(\theta) r^{n+2} + O(r^{n+3}), \\ f^{-1}(r,\theta) {=} r^{-2}(1 - 2b_n(\theta)r^n + O(r^{n+1})), \\ -\frac12 f_{rr} = -1 - (n+2)(n+1)b_n(\theta)r^n + O(r^{n+1}), \\ \frac14 f_r^2 f^{-1} = (r + (n+2)b_n(\theta)r^{n+1} + O(r^{n+2}))^2 r^{-2}(1 - 2b_n(\theta)r^n + O(r^{n+1})) \\ \quad = 1 + (2(n+2) - 2)b_n(\theta)r^n + O(r^{n+1}), \\ -\frac12 f_{rr} + \frac14 f_r^2 f^{-1} = b_n(\theta)(-(n+2)(n+1) + 2(n+1))r^n + O(r^{n+1}) \\ \quad = -n(n+1)b_n(\theta)r^n + O(r^{n+1}), \\ \operatorname{Tr}\{\mathcal{J}(\partial_r^\theta)\} = f^{-1}\{-\frac12 f_{rr} + \frac14 f_r^2 f^{-1}\} \\ \quad = r^{-2}(1 - 2b_n(\theta)r^n + O(r^{n+1}))(-n(n+1)b_n(\theta)r^n + O(r^{n+1})) \\ \quad = -n(n+1)b_n(\theta)r^{n-2} + O(r^{n-1}), \\ \nabla_{\partial_r^\theta}^{n-2} \operatorname{Tr}\{\mathcal{J}(\partial_r^\theta)|_{r=0}\} = -\frac{(n+1)!}{n-1} b_n(\theta). \end{array}$$

Consequently, $c_n = -\frac{n-1}{(n+1)!}$. □

3. Proof of Theorem 1.3: Constructing centrally harmonic spaces

Let (r,θ) be geodesic polar coordinates centered at a point P. Choose local coordinates $\theta = (\theta^1, \dots, \theta^{m-1})$ on the unit sphere to express $g = dr^2 + g_{ab}(r,\theta)d\theta^a d\theta^b$ and $\Theta_P(r,\theta) = \det(g_{ab}(r,\theta))^{\frac12}\nu(\theta)$ where $d\theta = \nu(\theta)d\theta^1 \cdots d\theta^{m-1}$. Let $\mathfrak{r}(r)$ satisfy $\mathfrak{r}(0) = 0$ and $d\mathfrak{r} = \psi(r^2)^{-1}dr$. Let $r(\mathfrak{r})$ be the inverse function. We have

$$\begin{array}{rcl} g_\psi & = & \psi(r^2)^{-2} g = \psi^{-2} dr^2 + \psi^{-2} g_{ab}(r,\theta) d\theta^a d\theta^b \\ & = & d\mathfrak{r}^2 + \psi^{-2}(r(\mathfrak{r})^2) g_{ab}(r(\mathfrak{r}),\theta) d\theta^a d\theta^b\,. \end{array}$$

Consequently, $(\mathfrak{r},\theta) \to r(\mathfrak{r})\cdot\theta$ gives geodesic polar coordinates for the metric g_ψ and $\mathfrak{r}$ is the geodesic distance function for g_ψ. We then have

$$\Theta_{P,g_\psi}(\mathfrak{r},\theta) = \psi(r(\mathfrak{r})^2)^{1-m}\Theta_{P,g}(r(\mathfrak{r})) \tag{3.a}$$

and g_ψ is harmonic at the point P as well. □

4. Proof of Lemma 1.4: Space forms

We adopt the following notational conventions in Section 4. Let $\mathcal{M} = (M,g)$ be a Riemannian manifold and let $\mathcal{M}_\psi := (M, \Psi^{-2}g)$ be a conformal radial deformation of $\mathcal{M}$. Let ρ and ρ_ψ be the Ricci tensors of g and g_ψ. If ϕ is a smooth function on M, let $\operatorname{Hess}_g(\phi) := \nabla^2\phi$ be the Hessian of ϕ with respect to g;

$$\operatorname{Hess}_g(\phi) = \nabla^2\phi = \{\partial_{\xi^i}\partial_{\xi^j}\phi - \Gamma_{ij}{}^k \partial_{\xi^k}\phi\} d\xi^i \otimes d\xi^j\,. \tag{4.a}$$

Fix $\xi \in T_PM$ with $0 < \|\xi\| < \iota_P$. Choose the coordinate system on T_PM so $\xi = (\|\xi\|, 0, \dots, 0)$. The following is a crucial technical result that will play a central role in the proof of Lemma 1.4 and of Theorem 1.5.

Lemma 4.1.

(1) *If $\mathcal{M}$ is centrally harmonic about P, then $\mathcal{M}$ is Einstein at P.*
(2) $\rho_{g_\psi} - \rho_g = \Psi^{-1}(m-2)\operatorname{Hess}_g(\Psi) + \{-\Psi^{-1}\Delta^0_{\mathcal{M}}\Psi - (m-1)\Psi^{-2}\|d\Psi\|^2_g\}g.$
(3) $L_\xi(\partial_{\xi^i}, \partial_{\xi^j}) = -\|\xi\|^{-1}\delta_{ij} + \Gamma_{ij}{}^1(\xi).$
(4) *Assume $\mathcal{M}$ and $\mathcal{M}_\psi$ are Einstein at ξ and that $m \geq 3$.*
  (a) *If L_ξ is not a multiple of the identity, then $\psi'(\|\xi\|^2) = 0$.*
  (b) *If $\psi'(\|\xi\|^2) = 0$, then $\psi''(\|\xi\|^2) = 0$.*

Proof. If $\mathcal{M}$ is centrally harmonic about P, then $\mathcal{H}_2$ only depends on $\|\xi\|$ so we shall write $\mathcal{H}_2(\xi) = \mathcal{H}_2(\|\xi\|)$. In particular, by Equation (1.b), $\rho_g(\xi,\xi) = \operatorname{Tr}\{\mathcal{J}_g(\xi)\}$ only depends on $\|\xi\|$ so $\rho_g(\xi,\xi) = c\|\xi\|^2$ and Assertion (1) follows. We refer to Kühnel and Rademacher [9] for the proof of Assertion (2). If $i > 1$, let $\sigma_i(\theta) := \|\xi\|\cos(\theta)\partial_{\xi^1} + \|\xi\|\sin(\theta)\partial_{\xi^i}$ define a curve in $S_P(\|\xi\|)$ with $\dot\sigma_i(0) = \|\xi\|\partial_{\xi^i}$. Assertion (3) follows by polarizing the identity

$$\begin{aligned} L_\xi(\partial_{\xi^i}, \partial_{\xi^i}) &= \|\xi\|^{-2}\left\{g(\nabla_{g,\dot\sigma_i}\dot\sigma_i, \partial_{\xi^1})\right\}\Big|_{\theta=0} \\ &= \|\xi\|^{-2}\left\{(\partial^2_\theta\sigma_i, \partial_{\xi^1}) + \|\xi\|^2(\nabla_{g,\partial_{\xi^i}}\partial_{\xi^i}, \partial_{\xi^1})\right\}\Big|_{\theta=0} \\ &= -\|\xi\|^{-1} + \Gamma_{ii}{}^1\,. \end{aligned}$$

Suppose that $\mathcal{M}$ and $\mathcal{M}_\psi$ are Einstein at ξ. Since $m \geq 3$, Assertion (2) implies that $\operatorname{Hess}_g(\Psi)(\xi)$ is a multiple of g; if $m = 2$, then we obtain no information from the Einstein condition and it is for this reason we assume $m \geq 3$ henceforth whenever using Lemma 4.1. Since $\Psi = \psi((\xi^1)^2 + \dots + (\xi^m)^2)$ and we are evaluating at $\xi = (\|\xi\|, 0, \dots, 0)$, we use Equation (4.a) to compute:

$$\begin{aligned} &\operatorname{Hess}_g(\Psi)(\xi) = \{\partial_{\xi^i}\partial_{\xi^j}\Psi - \Gamma_{ij}{}^k\partial_{\xi^k}\Psi\}(\xi)d\xi^i \otimes d\xi^j \\ &\quad = \{2\delta_{ij}\psi'(\|\xi\|^2) + 4\|\xi\|^2\delta_{1i}\delta_{1j}\psi''(\|\xi\|^2) - 2\|\xi\|\Gamma_{ij}{}^1\psi'(\|\xi\|^2)\}d\xi^i \otimes d\xi^j \\ &\quad = (2\psi'(\|\xi\|^2) + 4\|\xi\|^2\psi''(\|\xi\|^2))dr \otimes dr \\ &\qquad -2\psi'(\|\xi\|^2)\ \|\xi\|\textstyle\sum_{i,j>1} L(\partial_i, \partial_j)d\xi^i \otimes d\xi^j\,. \end{aligned} \tag{4.b}$$

Suppose first that L is not a multiple of g and that $\psi'(\|\xi\|^2) \neq 0$. We may then use Equation (4.b) to see that $\operatorname{Hess}_g(\Psi)(\xi)$ is not a multiple of g. Since $m \geq 3$, Assertion 2 then shows $\rho_{g_\psi} - \rho_g$ is not a multiple of g. This contradicts the assumption that $\mathcal{M}$ and $\mathcal{M}_\psi$ are Einstein at ξ and establishes Assertion (4a). Suppose finally that $\psi'(\|\xi\|^2) = 0$ and that $\psi''(\|\xi\|^2) \neq 0$. Again, examining Equation (4.b) shows that $\operatorname{Hess}_g(\Psi)(\|\xi\|^2)$ is not a multiple of g which is a contradiction; this establishes Assertion (4b). □

4.1. **Analytic radial conformal deformations of $\mathbb{R}^m$.** We adopt the notation of Section 1.6 for the remainder of this section. Let $g_{a,b} := (a + b\|\xi\|^2)^{-2}g_e$ on the appropriate domain for $(a,b) \neq (0,0)$.

Lemma 4.2. Let $c \neq 0$.

(1) $g_{a,b}$ and $g_{b,a}$ are isometric.
(2) $g_{a,b}$ and $g_{ac^{-1},bc}$ are isometric.
(3) $g_{ca,cb}$ are homothetic.

Proof. Let $\eta = \|\xi\|^{-2}\xi$ for $\xi \neq 0$ define *inversion* about the origin. Express $\xi = r \cdot \theta$ and $\eta = t \cdot \theta$ in polar coordinates where $r = \|\xi\|$, $t = \|\eta\|$, $\theta = \xi/\|\xi\| = \eta/\|\eta\|$, and

$rt = 1$. We prove Assertion (1) by computing:

$$g_{a,b} \quad = \quad \frac{dr^2 + r^2 d\theta^2}{(a + br^2)^2} = \frac{t^{-4}dt^2 + t^{-2}d\theta^2}{(a + bt^{-2})^2} = \frac{dt^2 + t^2 d\theta^2}{(at^2 + b)^2} = g_{b,a}\,.$$

Next set $\xi = c\eta$. Since $r = ct$, we prove Assertion (2) by computing:

$$g_{a,b} \quad = \quad \frac{dr^2 + r^2 d\theta^2}{(a + br^2)^2} = \frac{c^2 dt^2 + c^2 t^2 d\theta^2}{(a + bc^2 t^2)^2} = \frac{dt^2 + t^2 d\theta^2}{(ac^{-1} + bct^2)^2} = g_{ac^{-1},bc}\,.$$

Asseretion (3) is immediate. □

4.2. **The proof of Lemma 1.4 (1).** We must show that a radial conformal deformation of the Euclidean metric defined by a linear function is a space form. Since $g_{1,0} = g_e$ is the Euclidean flat metric, $g_{1,0}$ is a space form metric. Stereographic projection shows that $g_{\frac{1}{2},\frac{1}{2}} = 4(1 + \|\xi\|^2)^{-2} g_e$ is the standard round metric on the sphere of radius 1 and hence is a space form metric. The hyperbolic metric on the unit disk is $g_{\frac{1}{2},-\frac{1}{2}} = 4(1 - \|\xi\|^2)^{-2} g_e$ and hence is a space form metric. Inversion about the origin, which was discussed the proof of Lemma 4.2, interchanges the region $0 < \|\xi\|^2 < 1$ and $\|\xi\|^2 > 1$ and shows $g_{\frac{1}{2},-\frac{1}{2}}$ is a space form metric on the region $\|\xi\|^2 > 1$ as well. Thus $g_{\frac{1}{2},\pm\frac{1}{2}}$ are space form metrics on the appropriate domains. Any metric homothetic or isometric to a space form metric is again a space form metric. Thus Lemma 4.2 applies to show $g_{c^{-1}d,\pm cd}$ is a space form metric if $c \neq 0$ and $d \neq 0$. Set

$$c = \left|\frac{b}{a}\right|^{1/2}, \quad d = a\left|\frac{b}{a}\right|^{1/2}, \quad \varepsilon := \operatorname{sign}\left(\frac{b}{a}\right) = \pm\,.$$

We then have $a = c^{-1}d$ and $b = \varepsilon cd$. This shows that $g_{a,b} = g_{c^{-1}d,\varepsilon cd}$ is a space form metric. If $a \neq 0$, then $g_{a,0}$ is homothetic to the Euclidean metric and is a space form metric. Finally by Lemma 4.2 (1), $g_{a,0}$ and $g_{0,a}$ are isometric and hence $g_{0,a}$ is a space form metric. □

4.3. **The proof of Lemma 1.4 (2).** Suppose that a radial analytic conformal deformation $\mathcal{N}_\psi$ of Euclidean space is a space form and $m \geq 3$. Since g_ψ and g_e are Einstein at any point ξ in the domain of definition and since $m \geq 3$, we may apply Lemma 4.1 to see $\operatorname{Hess}_{g_e}(\Psi)$ is a multiple of g_e. Since $\Gamma_{ij}{}^1(g_e) = 0$, Equation (4.b) shows that $\psi''(\|\xi\|^2) = 0$ and hence ψ is linear. □

4.4. **Proof of Lemma 1.4 (3,4).** We must show that the metric $g_{a,b}$ has constant sectional curvature $4ab$. The metrics $g_{a,0}$ and $g_{0,b}$ are flat and have sectional curvature 0. We may therefore assume $a \neq 0$ and $b \neq 0$. The metrics $4(1 \pm \|\xi\|^2)^{-2} ds_e^2$ have constant sectional curvature ± 1, i.e. $g_{\frac{1}{2},\pm\frac{1}{2}}$ has constant sectional curvature ± 1. Thus Assertion (3) holds if $(a,b) = (\frac{1}{2}, \pm\frac{1}{2})$. Isometric metrics have the same sectional curvature and thus by Lemma 4.2 (2), $g_{\frac{1}{2}c^{-1},\pm\frac{1}{2}c}$ has constant sectional curvature ± 1. Rescaling the metric by a homothetic constant d^{-2} rescales the sectional curvature by d^2. Thus $g_{\frac{1}{2}c^{-1}d,\pm\frac{1}{2}cd}$ has constant sectional curvature $\pm d^2$. The argument of Section 4.2 now establishes the result in general. Since any two manifolds of the same constant sectional curvature are locally isometric, Assertion (4) follows from Assertion (3). □

5. Proof of Theorem 1.5: Radial conformal deformations

The notation of Equation (1.b) for the covariant deformation of the Jacobi operator does not distinguish between the two metrics g and g_Ψ. We evaluate at ξ.

Let η, η_1, and η_2 belong to $T_\xi M$. We define the following endomorphisms of $T_\xi M$:

$$g(\mathcal{J}_{k,g}(\eta)\eta_1,\eta_2) := \{\nabla_g^k R_g(\xi)\}(\eta_1,\eta,\eta,\eta_2;\eta,\dots,\eta),$$
$$g_\Psi(\mathcal{J}_{k,g_\Psi}(\eta)\eta_1,\eta_2) = \{\nabla_{g_\Psi}^k R_{g_\Psi}(\xi)\}(\eta_1,\eta,\eta,\eta_2;\eta,\dots,\eta)\,.$$

We emphasize that everything is evaluated at ξ. We continue our discussion.

Lemma 5.1. *If $\mathcal{M}$ is Einstein, if $m \ge 3$, if $\psi'(\|\xi\|^2) = 0$, and if $\mathcal{M}_\psi$ is centrally harmonic about ξ, then $\psi^{(k)}(\|\xi\|^2) = 0$ for all k.*

Proof. By Assertion (4a) of Lemma 4.1, $\psi''(\|\,\xi\|^2) = 0$ as well. Suppose the Lemma is false. Choose $n \ge 2$ minimal so that $\psi^{(j)}(\|\xi\|^2) = 0$ for $1 \le j \le n$ but so that $\psi^{(n+1)}(\|\xi\|^2) \ne 0$. We argue for a contradiction. Since $\mathcal{M}$ is Einstein, $\nabla_g^j \rho_g$ vanishes identically for $j \ge 1$. Since $\psi^{(j)}(\|\xi\|^2) = 0$ for $1 \le j \le n$, we have $\nabla_g^j = \nabla_{g_\psi}^j$ at ξ for $1 \le j \le n$ and we need not distinguish the two. We may covariantly differentiate Assertion (2) of Lemma 4.1 to see

$$\{\nabla^j \rho_{g_\psi}\}(\xi) = \nabla^j\{\rho_{g_\psi} - \rho_g\}(\xi) = 0 \text{ for } j \le n-2\,. \tag{5.a}$$

Since $\mathcal{M}_\psi$ is centrally harmonic about ξ, $\mathcal{H}_{g_\psi,\xi,n+1}(\eta)$ is a multiple of $\|\eta\|^{n+1}$. Since $\mathcal{M}_\psi$ is Einstein, $\mathcal{J}_0(\eta) = c\|\xi\|^2\,\mathrm{id}$. We use Equation (5.a) and Assertion (2) of Lemma 4.1 to see that $\mathcal{J}_j(\eta)$ is zero and hence a multiple of id for $1 \le j \le n-2$ (if $n = 2$, this assertion is vacuous). We may therefore use Lemma 1.2 to see that $\mathrm{Tr}\{\mathcal{J}_{g_\psi,\xi,n-1}(\eta)\}$ is a multiple of $\|\eta\|^{n-1}$. In particular, $\mathrm{Tr}\{\mathcal{J}_{g_\psi,\xi,n-1}(\eta)\} = 0$ if n is even.

Consequently we must differentiate the coefficients appearing in Assertion (2) of Lemma 4.1 to study $\nabla^{n-1}\{\rho_{g_\psi} - \rho_g\}(\xi)$. We have

$$\begin{aligned}&\nabla^{(n-1)}\rho_{g_\psi}(\partial_{\xi^i},\partial_{\xi^i};\partial_{\xi^i},\dots,\partial_{\xi^i})(\xi)\\ &\quad= \nabla^{(n-1)}\{\rho_{g_\psi} - \rho_g\}(\partial_{\xi^i},\partial_{\xi^i};\partial_{\xi^i},\dots,\partial_{\xi^i})(\xi)\\ &\quad= \Psi^{-1}(\xi)\left\{\begin{array}{lll}(m-1)\psi^{n+1}(\|\xi\|^2) & \text{if} & i=1\\ 0 & \text{if} & i>1\end{array}\right\}.\end{aligned}$$

Since this must depend only on $\|\partial_{\xi^i}\|$, we conclude as desired $\psi^{(n+1)}(\|\xi\|^2) = 0$. □

5.1. **The proof of Theorem 1.5 (1).** Let $\mathcal{M} = (M, g)$ be an Einstein manifold of dimension $m \ge 3$. Suppose that ψ is real analytic and that $\mathcal{M}_\psi$ is centrally harmonic about some ξ with $0 < \|\xi\| < \iota_P$. Assume the geodesic sphere about P is not totally umbillic at ξ. By Lemma 4.1 4, we have $\psi'(\|\xi\|^2) = \psi''(\|\xi\|^2) = 0$. By Lemma 5.1, we have $\psi^{(k)}(\|\xi\|^2) = 0$ for all $k \ge 1$. Since ψ is real analytic, this implies ψ is constant. □

5.2. **The proof of Theorem 1.5 (2).** We may work locally and assume without loss of generality that $\mathcal{M}$ is flat space and $g = g_e$. Suppose ψ is real analytic and that $\mathcal{M}_\psi$ is centrally harmonic about some ξ with $0 < \|\xi\| < \iota_P$. We assume $m \ge 3$ and use Lemma 5.1.

5.2.1. *Suppose that $\psi(\|\xi\|^2) \ne \|\xi\|^2\psi'(\|\xi\|^2)$.* Express $\mathcal{M}_\psi = \{\mathcal{N}_{1,a}\}_{\phi_a}$ where we set $\phi_a(t) := (1 + at)^{-1}\psi(t)$. We then have

$$\phi_a'(t) = \frac{-a\psi(t) + (1 + at)\psi'(t)}{(1 + at)^2}\,.$$

We solve the equation $\phi_a'(\|\xi\|^2) = 0$ to obtain

$$a = -\frac{\psi'(\|\xi\|^2)}{\|\xi\|^2\psi'(\|\xi\|^2) - \psi(\|\xi\|^2)}\,.$$

By Lemma 1.4, $\mathcal{N}_{1,a}$ is a space form. Since $\mathcal{N}_{\phi_a}$ is centrally harmonic about ξ and $\phi_a'(\|\xi\|^2) = 0$, we may use Lemma 5.1 to see that $\phi_a'(\|\xi\|^2) = 0$ and hence ϕ_a is constant so $\mathcal{M}$ is homothetic to $\mathcal{N}_{1,a}$ and hence is a space form.

5.2.2. *Suppose that* $\psi(\|\xi\|^2) = \|\xi\|^2\psi'(\|\xi\|^2)$. Express $\mathcal{M}_\psi = \{\mathcal{N}_{0,1}\}_\phi$ where we set $\phi(t) = t^{-1}\psi(t)$. We then have $\phi'(\|\xi\|^2) = \{t^{-2}\{-\psi(t) + t\psi'(t)\}\}|_{t=\|\xi\|^2} = 0$ and again we can use Lemma 5.1 to complete the proof. □

6. The proof of Lemma 1.6: Totally umbillic geodesic spheres

The local isometry group of a space form acts transitively on the unit tangent bundle of the geodesic spheres; consequently, the geodesic spheres in a space form are totally umbillic. This proves Assertion (1). We refer to Chen and Vanhecke [5], Kulkarni [10], and Vanhecke and Willmore [14] for the proof the converse assertion to establish Assertion (2). We use Chen [4] to establish Assertion (3). Let $\sigma_{ab}(r\xi)$ be the second fundamental form of the geodesic sphere about P passing thru the point $r\xi$. Chen and Vanhecke [5] show $\sigma_{ab} = r^{-1}\delta_{ab} - \frac{r}{3}R_{\xi a\xi b}(P) + O(r^2)$. Since $s_P > 0$, $R_{\xi a\xi b}$ is not a multiple of δ_{ab} and show Assertion (4) follows. We use Berndt, Tricerri and Vanhecke [1] to derive Assertion (5) from Assertion (4). The eigenvalues of the Jacobi operator are given in the first Theorem on page 96 of Section 4.2. By hypothesis, $\mathcal{M}$ does not have constant sectional curvature but the remaining rank one symmetric spaces are included. There are 6 cases in the classification (i)–(vi). In cases (i)–(v), the eigenvalues of the Jacobi operator are $\{0, -\frac{1}{4}, -1\}$ and the eigenvalue 0 appears with multiplicity 1 which yields the eigenvalues of the reduced Jacobi operator are $\{-\frac{1}{4}, -1\}$ so $M(\xi) - m(\xi) = \frac{3}{4}$. The situation in case (vi) is more complicated. Still, there is a 4-dimensional subspace where the eigenvalues are $\{0, -\frac{1}{4}, -1\}$ where $-\frac{1}{4}$ has multiplicity 2. The computation of the remaining eigenvalues is more difficult. Nevertheless, we obtain $M(\xi) - m(\xi) \geq \frac{3}{4}$ so $s > 0$ as desired. □

7. The proof of Lemma 1.7

We adopt the notation of Section 1.9. The round sphere $\mathcal{S}$ is a space form. Since $\mathcal{S}_\psi$ is conformally radially rotationally symmetric about the north and south poles $P_\pm$, $\mathcal{S}_\psi$ is centrally harmonic about these two points by Theorem 1.3. Suppose $\mathcal{S}_\psi$ is centrally harmonic about some other point. Since we are within the injectivity radius, we can apply Theorem 1.5 to see $\mathcal{S}_\psi$ is a space form as we have assumed $m \geq 3$. This is a contradiction. □

8. The proof of Theorem 1.8: A non-flat example with trivial volume density function

Let $m = 2\mathfrak{m} \geq 4$. Let $\mathcal{M} := (\mathbb{CP}^{\mathfrak{m}} - \mathbb{CP}^{\mathfrak{m}-1}, g)$ where g is the Fubini-Study metric. We have removed the cut-locus and consequently, the underlying manifold is an open geodesic ball of radius $\frac{\pi}{2}$. Choose ψ so $\psi(r^2)^{-1}\tilde{\Theta}_{P,g}(r) = 1$. Then the Equation (3.a) ensures $\tilde{\Theta}_{P,g_\psi} = 1$. □

Remark 8.1. We examine $\mathbb{CP}^{\frac{1}{2}m}$ near the cut locus by setting set $u = \frac{\pi}{2} - r$. Set

$$\Theta(u) := \sin\left(\frac{\pi}{2} - u\right)^{(m-1)} \cos\left(\frac{\pi}{2} - u\right),$$
$$\Psi(u) = \frac{\sin\left(\frac{\pi}{2} - u\right)}{\frac{\pi}{2} - u} \cos\left(\frac{\pi}{2} - u\right)^{1/(m-1)}.$$

Then $g_\psi(\partial_u,\partial_u)=\psi(u)^{-2}$ so the curves $\gamma(u)=(u,0,\dots,0)$ have length

$$\int_{u=0}^{\frac{\pi}{2}}\psi^{-1}(u)du\,.$$

Since $\psi(u)=\frac{2}{\pi}u^{\frac{1}{m-1}}+O(1)$, the unparametrized geodesics have finite length and the resulting manifold is not geodesically complete. We use Lemma 4.1 to compute

$$\begin{aligned}
&\rho_g(\psi(u)\partial_u,\psi(u)\partial_u)=O(1),\\
&\rho_{g_\psi}(\psi(u)\partial_u,\psi(u)\partial_u)=(\rho_{g_\psi}-\rho_g)(\psi(u)\partial_u,\psi(u)\partial_u)\\
=\quad&(m-2)\psi(u)\psi''(u)+\psi(u)\Theta(u)^{-1}\partial_u\{\Theta(u)\psi(u)\}-(m-1)\psi'(u)^2+O(1)
\end{aligned}$$

A mathematica computation yields

$$\rho_g(\psi(u)\partial_u,\psi(u)\partial_u)=\left\{\begin{array}{ll}-\frac{28}{9\pi^2}u^{-\frac{4}{3}}+O(u^{-\frac{1}{3}}) & \text{if } m=4\\ -\frac{84}{25\pi^2}u^{-\frac{8}{5}}+O(u^{-\frac{3}{5}}) & \text{if } m=6\\ -\frac{172}{49\pi^2}u^{-12/7}+O(u^{-\frac{5}{7}}) & \text{if } m=8\end{array}\right\}$$

so this is singular at $u=0$ and $\mathcal{M}_\psi$ is essentially geodesically incomplete.

Research support. Research partially supported by PID 2019-105138GB-C21 (Spain) and by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2019R1A2C1083957)

References

[1] J. Berndt, F. Tricerri and L. Vanhecke, "Generalized Heisenberg Groups and Damek Ricci Harmonic Spaces", Lecture Notes in Math. 1598, Springer-Verlag, Berlin 1995.
[2] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergeb. Math. Grenzgeb. **93**, Springer-Verlag, Berlin-New York, 1978.
[3] H. W. Brinkmann, "On Riemann spaces conformal to Euclidean spaces", *Proc. Nat. Acad. Sci. USA* **9** (1923), 1–3.
[4] B.Y. Chen, "Classification of totally umbilical submanifolds in symmetric spaces", *J. Austral. Math. Soc.* **30** (1980), 129–136.
[5] B. Y. Chen and L. Vanhecke, "Differential geometry of geodesic spheres", *J. Reine Angew. Math.* **325** (1981), 28–67.
[6] E. T. Copson and H. S. Ruse, "Harmonic Riemannian spaces", *Proc. Roy. Soc. Edinburgh* **60** (1940), 117–133.
[7] P. Gilkey and J. H. Park, "Harmonic spaces and density function", *Results Math* (2020), 75:121. doi: 10.1007/s00025-020-01248-7.
[8] P. Gilkey and J. H. Park, "Harmonic radial vector fields on harmonic spaces" *J. Math. Anal. Appl.*, in press. arxiv.org/abs/2009.02879.
[9] W. Kühnel and H. Rademacher, "Conformal transformations of pseudo-Riemannian manifolds", **Recent developments in pseudo-Riemannian geometry** (2008), p261–298. ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008. doi: 10.4171/051-1/8.
[10] R. S. Kulkarni, "A finite version of Schur's theorem", *Proc. Amer. Math. Soc.* **53** (1975), 440–442.
[11] H. S. Ruse, *General solutions of Laplace's equation in a simply harmonic manifold*, The Quarterly Journal of Mathematics **14** (1963), 181–192.
[12] H. S. Ruse, *On the elementary solution of Laplace's equation*, Proc. Edinburgh Math. Soc. **2** (1931), 135–139.
[13] L. da Silva and J. da Silva, "Characterization of manifolds of constant curvature by spherical curves", *Annali di Matematica Pura ed Applicata* (2020), 199: 217–229.
[14] L. Vanhecke and T.J. Willmore, "Jacobi fields and geodesic spheres", *P. Roy. Soc. Edinb. A* **82**, (1979), 233–240.
[15] T. J. Willmore, *Riemannian geometry*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1993).

PG: Mathematics Department, University of Oregon, Eugene OR 97403-1222, USA
Email address: gilkey@uoregon.edu

JHP: Department of Mathematics, Sungkyunkwan University, Suwon, 16419 Korea.
Email address: parkj@skku.edu