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New characterizations of ruled real hypersurfaces in complex projective space

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Abstract

We consider real hypersurfaces M in complex projective space equipped with both the Levi–Civita and generalized Tanaka–Webster connections. For any nonnull constant k and any symmetric tensor field of type $(1, 1)$ L on M, we can define two tensor fields of type $(1, 2)$ on M, $L_F^{(k)}$ and $L_T^{(k)}$, related to both connections. We study the behaviour of the structure operator ϕ with respect to such tensor fields in the particular case of $L = A$, the shape operator of M, and obtain some new characterizations of ruled real hypersurfaces in complex projective space.

Keywords g-Tanaka–Webster connection · Complex projective space · Real hypersurface · kth Cho operator · Torsion operator · Ruled real hypersurfaces

Mathematics Subject Classification 53C15 · 53B25

1 Introduction

Let $\mathbb{C}P^m$, $m \geq 2$, be the complex projective space endowed with the Kaehlerian structure (J, g), where g is the Fubini–Study metric of constant holomorphic sectional curvature 4. Let M be a connected real hypersurface of $\mathbb{C}P^m$ without boundary, g the restriction of the metric on $\mathbb{C}P^m$ to M and ∇ the Levi–Civita connection on M. Take a locally defined unit normal vector field N on M and let $\xi = -JN$. This is a tangent vector field to M called the structure (or Reeb) vector field on M. If X is a vector field on M, we write $JX = \phi X + \eta(X)N$, where ϕX denotes the tangent component of JX. Then $\eta(X) = g(X, \xi)$, ϕ is called the structure tensor on M and (ϕ, ξ, η, g) is an almost contact metric structure on M induced by the Kaehlerian structure of $\mathbb{C}P^m$. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [5, 19–21]. His classification contains 6 types of real hypersurfaces. Among them we find type (A_1) real hypersurfaces that are

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geodesic hyperspheres of radius $r, 0 < r < \frac{\pi}{2}$, and type (A_2) real hypersurfaces that are tubes of radius $r, 0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n, 0 < n < m-1$. We will call both types of real hypersurfaces type (A) real hypersurfaces. They are Hopf, that is, the structure vector field is principal, and are the unique real hypersurfaces in $\mathbb{C}P^m$ such that $A\phi = \phi A$, see [11].

Ruled real hypersurfaces in $\mathbb{C}P^m$ satisfy that the maximal holomorphic distribution on M, $\mathbb{D}$, given at any point by the vectors orthogonal to ξ, is integrable and its integral manifolds are totally geodesic $\mathbb{C}P^{m-1}$. Equivalently, $g(A\mathbb{D}, \mathbb{D}) = 0$. For examples of ruled real hypersurfaces see [6] or [8].

The Tanaka–Webster connection, [22, 24], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold. As a generalization of this connection, Tanno [23], defined the generalized Tanaka–Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y \tag{1.1}$$

for any vector fields X, Y on the manifold.

Using the almost contact metric structure on M and the naturally extended affine connection of Tanno's generalized Tanaka–Webster connection, Cho defined the kth generalized Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface M in $\mathbb{C}P^m$, see [3, 4], by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.2}$$

for any X,Y tangent to M where k is a nonzero real number. Then $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)} g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the kth generalized Tanaka–Webster connection coincides with the Tanaka–Webster connection.

Here we can consider the tensor field of type $(1, 2)$ given by the difference of the connections $F^{(k)}(X, Y) = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any X, Y tangent to M, see [7] Proposition 7.10, pp. 234–235. We will call this tensor the kth Cho tensor on M. Associated to it, for any X tangent to M and any nonnull real number k, we can consider the tensor field of type $(1, 1)$ $F^{(k)}_X$, given by $F^{(k)}_X Y = F^{(k)}(X, Y)$ for any $Y \in TM$. This operator will be called the kth Cho operator corresponding to X. Notice that if $X \in \mathbb{D}$, the corresponding Cho operator does not depend on k and we simply write F_X. The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $T^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$ for any X, Y tangent to M. We define the kth torsion operator associated to X to the operator given by $T^{(k)}_X Y = T^{(k)}(X, Y)$, for any X, Y tangent to M.

Let $\mathcal{L}$ denote the Lie derivative on M. Therefore, $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$ for any X, Y tangent to M. Now we can define on M a differential operator of first order, associated to the kth generalized Tanaka–Webster connection, given by

$$\mathcal{L}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X = \mathcal{L}_X Y + T^{(k)}_X Y$$

for any X, Y tangent to M. We will call it the derivative of Lie type associated to the kth generalized Tanaka–Webster connection.

Let now L be a symmetric tensor of type $(1, 1)$ defined on M. We can consider then the type $(1, 2)$ tensor $L^{(k)}_F$ associated to L in the following way:

$$L^{(k)}_F(X, Y) = [F^{(k)}_X, L]Y = F^{(k)}_X LY - LF^{(k)}_X Y$$

for any X, Y tangent to M. We also can consider another tensor of type $(1, 2)$, $L_T^{(k)}$, associated to L, by

$$L_T^{(k)}(X, Y) = [T_X^{(k)}, L]Y = T_X^{(k)}LY - LT_X^{(k)}Y$$

for any X, Y tangent to M. Notice that if $X \in \mathbb{D}$, $L_F^{(k)}$ does not depend on k. We will write it simply L_F.

In [15], respectively [12], we proved the nonexistence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that, for the tensors of type $(1, 2)$ associated to the shape operator, $A_F^{(k)} = 0$, respectively $A_T^{(k)} = 0$, for any nonnull real number k. Further results on such tensors were obtained in [13, 14].

The purpose of the present paper is to study the behaviour of both tensors with respect to the structure operator ϕ. We will say that $A_F^{(k)}$ is pure with respect to ϕ if $A_F^{(k)}(\phi X, Y) = A_F^{(k)}(X, \phi Y)$, for any X, Y tangent to M, Tachibana [18], see also [16, 17]. We will say that $A_F^{(k)}$ is η-pure with respect to ϕ if $A_F^{(k)}(\phi X, Y) = A_F^{(k)}(X, \phi Y)$, for any $X, Y \in \mathbb{D}$. Analogously, we will say that $A_F^{(k)}$ is hybrid with respect to ϕ if $A_F^{(k)}(\phi X, Y) + A_F^{(k)}(X, \phi Y) = 0$ for any X, Y tangent to M, Tachibana [18], and it is η-hybrid if $A_F^{(k)}(\phi X, Y) + A_F^{(k)}(X, \phi Y) = 0$ for any $X, Y \in \mathbb{D}$. We will prove

Theorem 1.1 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then A_F is η-pure with respect to ϕ if and only if M is locally congruent to a ruled real hypersurface.*

Also we will prove

Theorem 1.2 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then A_F is η-hybrid with respect to ϕ if and only if M is locally congruent to one of the following real hypersurfaces:*

1. *a tube of radius $\frac{\pi}{4}$ around a complex submanifold of $\mathbb{C}P^m$;*
2. *a real hypersurface of type (A);*
3. *a ruled real hypersurface.*

On the other hand, we also have

Theorem 1.3 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $A_F(\phi X, Y) = \phi A_F(X, Y)$ for any $X, Y \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

Concerning the tensor $A_T^{(k)}$, we will prove

Theorem 1.4 *There does not exist any real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, such that $A_T^{(k)}$ is η-pure with respect to ϕ, for any nonnull real number k.*

Also we will obtain

Theorem 1.5 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then $A_T^{(k)}$ is η-hybrid with respect to ϕ if and only if M is locally congruent to a real hypersurface of type (A).*

As in the case of A_F, we can prove

Theorem 1.6 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then $A_T^{(k)}(\phi X, Y) = \phi A_T^{(k)}(X, Y)$, for any $X, Y \in \mathbb{D}$, if and only if M is locally congruent to a ruled real hypersurface.*

We also prove

Theorem 1.7 *There does not exist any real hypersurface in* $\mathbb{C}P^m$, $m \geq 3$, *such that* $A_T^{(k)}(X, \phi Y) = \phi A_T^{(k)}(X, Y)$ *for any* $X, Y \in \mathbb{D}$ *and any nonnull real number* k.

2 Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi–Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M, we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M, see [1]. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any vectors X, Y tangent to M. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the Codazzi equation is given by

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{2.5}$$

for any vectors X, Y tangent to M. We will call the maximal holomorphic distribution $\mathbb{D}$ on M the following one: at any $p \in M$, $\mathbb{D}(p) = \{X \in T_pM | g(X, \xi) = 0\}$. We will say that M is Hopf if ξ is principal, that is, $A\xi = \alpha\xi$ for a certain function α on M.

In the sequel we need the following result:

Theorem 2.1 ([9]) *If* ξ *is a principal curvature vector with corresponding principal curvature* α *and* $X \in \mathbb{D}$ *is principal with principal curvature* λ, *then* $2\lambda - \alpha \neq 0$ *and* ϕX *is principal with principal curvature* $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.

3 Proofs of results concerning A_F

In order to prove Theorem 1.1, we should have $F_{\phi X}AY - AF_{\phi X}Y = F_X A\phi Y - AF_X\phi Y$, for any $X, Y \in \mathbb{D}$. This yields

$$\begin{aligned} &g(\phi A\phi X, AY)\xi - \eta(AY)\phi A\phi X - g(\phi A\phi X, Y)A\xi \\ &\quad = g(\phi AX, A\phi Y)\xi - \eta(A\phi Y)\phi AX - g(AX, Y)A\xi \end{aligned} \tag{3.1}$$

for any $X, Y \in \mathbb{D}$. If M is Hopf with $A\xi = \alpha\xi$, the scalar product of (3.1) and ξ gives $g(\phi A\phi X, AY) - \alpha g(\phi A\phi X, Y) = g(\phi AX, A\phi Y) - \alpha g(\phi AX, Y)$ for any $X, Y \in \mathbb{D}$. Let us suppose that $X \in \mathbb{D}$ satisfies $AX = \lambda X$. Then $A\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, and we obtain $-\lambda\mu + \alpha\mu = \lambda\mu - \alpha\lambda$. That is, $2\lambda\mu = \alpha(\mu+\lambda)$. This implies $\frac{2\alpha\lambda^2+4\lambda}{2\lambda-\alpha} = \alpha\left(\frac{\alpha\lambda+2}{2\lambda-\alpha}+\lambda\right) = \alpha\left(\frac{2(1+\lambda)^2}{2\lambda-\alpha}\right)$. Thus, $\alpha\lambda^2 + 2\lambda = \alpha\lambda^2 + \alpha$, and so, $\lambda = \frac{\alpha}{2}$. As $2\lambda\mu = \alpha(\mu+\lambda)$, we get $\alpha\mu = \alpha(\mu+\lambda)$. Then $\alpha\lambda = \frac{\alpha^2}{2} = 0$, that is, $\alpha = 0$ and also $\lambda = 0$, a contradiction with the fact $2\lambda - \alpha \neq 0$.

This means that M must be non-Hopf. Therefore, locally we can write $A\xi = \alpha\xi + \beta U$, U being a unit vector field in $\mathbb{D}$, α and β functions on M and $\beta \neq 0$. We also define $\mathbb{D}_U$ as the orthogonal complementary distribution in $\mathbb{D}$ to the one spanned by U and ϕU. With this in mind (3.1) becomes

$$\begin{aligned} &g(\phi A\phi X, AY)\xi - \beta g(Y,U)\phi A\phi X - g(\phi A\phi X, Y)A\xi \\ &\quad = g(\phi AX, A\phi Y)\xi - \beta g(\phi Y, U)\phi AX - g(AX, Y)A\xi \end{aligned} \tag{3.2}$$

for any $X, Y \in \mathbb{D}$. The scalar product of (3.2) and ϕU gives $-\beta g(Y,U)g(A\phi X, U) = -\beta g(\phi Y, U)g(AX, U)$ for any $X, Y \in \mathbb{D}$. Taking $Y = U$, we obtain $-\beta g(AU, \phi X) = 0$ for any $X \in \mathbb{D}$. As we suppose $\beta \neq 0$ and changing X by ϕX, we have $g(AU, X) = 0$ for any $X \in \mathbb{D}$. This means that

$$AU = \beta\xi. \tag{3.3}$$

The scalar product of (3.2) and U yields $-\beta g(Y,U)g(\phi A\phi X, U) - \beta g(\phi A\phi X, Y) = -\beta g(\phi Y, U)g(\phi AX, U) - \beta g(AX, Y)$, for any $X, Y \in \mathbb{D}$. As $\beta \neq 0$, we have

$$g(Y,U)g(A\phi U, \phi X) + g(A\phi Y, \phi X) = g(\phi Y, U)g(A\phi U, X) - g(AX, Y) \tag{3.4}$$

for any $X, Y \in \mathbb{D}$. If we take $X = U$ in (3.4), it follows $2g(A\phi U, \phi X) = -g(AU, X)$ for any $X \in \mathbb{D}$. From (3.3), changing X by ϕX, we obtain $g(A\phi U, X) = 0$ for any $X \in \mathbb{D}$. Therefore

$$A\phi U = 0. \tag{3.5}$$

Now the scalar product of (3.2) and $Z \in \mathbb{D}_U$ implies $-\beta g(Y,U)g(\phi A\phi X, Z) = -\beta g(\phi Y, U)g(\phi AX, Z)$, for any $X, Y \in \mathbb{D}$, $Z \in \mathbb{D}_U$. If $Y = \phi U$, we obtain $\beta g(\phi AX, Z) = 0$ for any $X \in \mathbb{D}$, $Z \in \mathbb{D}_U$. If we change Z by ϕZ and bear in mind that $\beta \neq 0$, it follows $g(AZ, X) = 0$ for any $Z \in \mathbb{D}_U$, $X \in \mathbb{D}$. Therefore,

$$AZ = 0 \tag{3.6}$$

for any $Z \in \mathbb{D}_U$. From (3.3), (3.5) and (3.6), M is locally congruent to a ruled real hypersurface. The converse is trivial and we have finished the proof of Theorem 1.1.

Now if A_F is η-hybrid, we have

$$\begin{aligned} &g(\phi A\phi X, AY)\xi - \eta(AY)\phi A\phi X - g(\phi A\phi X)A\xi + g(\phi AX, A\phi Y)\xi \\ &\quad -\eta(A\phi Y)\phi AX - g(AX, Y)A\xi = 0 \end{aligned} \tag{3.7}$$

for any $X, Y \in \mathbb{D}$. Let us suppose that M is Hopf and write $A\xi = \alpha\xi$. If we take the scalar product of (3.7) and ξ, it follows $g(\phi A\phi X, AY) - \alpha g(\phi A\phi X, Y) + g(\phi AX, A\phi Y) - \alpha g(AX, Y) = 0$, for any $X, Y \in \mathbb{D}$. This means that $A\phi A\phi X - \alpha\phi A\phi X - \phi A\phi AX - \alpha AX = 0$ for any $X \in \mathbb{D}$. If we take $X \in \mathbb{D}$ such that $AX = \lambda X$, as $A\phi X = \mu\phi X$, we get $-\lambda\mu + \alpha\mu + \lambda\mu - \alpha\lambda = 0$. That is, $\alpha(\mu - \lambda) = 0$. Thus, either $\alpha = 0$, and by Cecil and Ryan [2] we have (1) in Theorem 1.2, or $\mu = \lambda$. This means that $A\phi = \phi A$ and in this case we have (2) in Theorem 1.2.

If M is non-Hopf, following the same steps as in Theorem 1.1 we obtain (3) in Theorem 1.2, finishing its proof.

If we suppose that M satisfies the condition in Theorem 1.3, we must have $F_{\phi X}AY - AF_{\phi X}Y = \phi F_X AY - \phi AF_X Y$ for any $X, Y \in \mathbb{D}$. This yields

$$g(\phi A\phi X, AY)\xi - \eta(AY)\phi A\phi X - g(\phi A\phi X, Y)A\xi = -\eta(AY)\phi^2 AX$$
$$-g(\phi AX, Y)\phi A\xi \tag{3.8}$$

for any $X, Y \in \mathbb{D}$. If we suppose that M is Hopf, the scalar product of (3.8) and ξ gives $g(\phi A\phi X, AY) - \alpha g(\phi A\phi X, Y) = 0$. Therefore, $A\phi A\phi X - \alpha\phi A\phi X = 0$, for any $X \in \mathbb{D}$. If we suppose that $X \in \mathbb{D}$ satisfies $AX = \lambda X$ we obtain $\mu(\alpha - \lambda) = 0$. Therefore, either $\mu = 0$ and then $\alpha \neq 0$ and $\lambda = -\frac{2}{\alpha}$, or if $\mu \neq 0$, $\alpha = \lambda$ and then $\mu = \frac{\alpha^2+2}{\alpha}$. Moreover, all principal curvatures are constant and, by Kimura [5], M must be locally congruent to a real hypersurface appearing among the six types in Takagi's list. Looking at such types, none has our principal curvatures, Takagi [20], proving that M must be non-Hopf.

We write as above $A\xi = \alpha\xi + \beta U$, with the same conditions. Then (3.8) becomes

$$g(A\phi A\phi X, Y)\xi - \beta g(Y, U)\phi A\phi X - g(\phi A\phi X, Y)A\xi$$
$$= -\beta g(Y, U)\phi^2 AX - \beta g(\phi AX, Y)\phi U \tag{3.9}$$

for any $X, Y \in \mathbb{D}$. The scalar product of (3.9) and ϕU gives, bearing in mind that $\beta \neq 0$,

$$g(Y, U)g(AU, \phi X) = g(Y, U)g(\phi AX, U) - g(\phi AX, Y), \tag{3.10}$$

for any $X, Y \in \mathbb{D}$. If $X = Y = U$, we get $g(AU, \phi U) = -2g(AU, \phi U)$. Thus,

$$g(AU, \phi U) = 0. \tag{3.11}$$

If we take $Y = U$, $X \in \mathbb{D}$ and orthogonal to U in (3.10), we have $g(\phi AU, X) = 2g(A\phi U, X)$ for such an X. From (3.11) the same is true for $X = U$. Therefore, $2A\phi U - \phi AU$ has no component in $\mathbb{D}$. As its scalar product with ξ also vanishes, we get

$$2A\phi U = \phi AU. \tag{3.12}$$

If we take $Y = \phi U$, $X \in \mathbb{D}$ in (3.10), it follows $g(AX, U) = 0$ for any $X \in \mathbb{D}$. Thus,

$$AU = \beta\xi \tag{3.13}$$

and, from (3.11),

$$A\phi U = 0. \tag{3.14}$$

The scalar product of (3.9) and U, bearing in mind (3.13) and (3.14), gives $g(\phi A\phi X, Y) = g(Y, U)g(\phi^2 AX, U) = -g(Y, U)g(AX, U) = 0$, for any $X \in \mathbb{D}$. Taking $\phi X \in \mathbb{D}_U$ instead of X, we obtain $\phi AX = 0$. Applying ϕ, we get

$$AX = 0 \tag{3.15}$$

for any $X \in \mathbb{D}_U$. From (3.13), (3.14) and (3.15), M must be locally congruent to a ruled real hypersurface and we have finished the proof of Theorem 1.3.

Remark 3.1 With proofs similar to the proof of Theorem 1.3, we can obtain other characterizations of ruled real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, if we consider any of the following conditions:

1. $A_F(\phi X, Y) + \phi A_F(X, Y) = 0$, for any $X, Y \in \mathbb{D}$;

2. $A_F(X, \phi Y) = \phi A_F(X, Y)$, for any $X, Y \in \mathbb{D}$;
3. $A_F(X, \phi Y) + \phi A_F(X, Y) = 0$, for any $X, Y \in \mathbb{D}$.

4 Results concerning $A_T^{(k)}$

If we suppose that $A_T^{(k)}$ is η-pure with respect to ϕ, we will have $F_{\phi X} AY - F_{AY}^{(k)} \phi X - AF_{\phi X} Y + AF_Y \phi X = F_X A\phi Y - F_{A\phi Y}^{(k)} X - AF_X \phi Y + AF_{\phi Y} X$ for any $X, Y \in \mathbb{D}$. This yields

$$\begin{aligned} &g(\phi A\phi X, AY)\xi - \eta(AY)\phi A\phi X - g(A^2Y, X)\xi - k\eta(AY)X - g(\phi A\phi X, Y)A\xi \\ &\quad + g(AY, X)A\xi = g(\phi AX, A\phi Y)\xi - \eta(A\phi Y)\phi AX - g(\phi A^2\phi Y, X)\xi \\ &\quad + k\eta(A\phi Y)\phi X - g(AX, Y)A\xi + g(\phi A\phi X)A\xi \end{aligned} \tag{4.1}$$

for any $X, Y \in \mathbb{D}$. Let us suppose that M is Hopf with $A\xi = \alpha\xi$. Then (4.1) becomes $g(\phi A\phi X, Y)\xi - g(A^2Y, X)\xi - \alpha g(\phi A\phi X, Y)\xi + \alpha g(AY, X)\xi = g(\phi AX, A\phi Y)\xi - g(\phi A^2\phi Y, X)\xi - \alpha g(AX, Y)\xi + \alpha g(\phi A\phi Y, X)\xi$, for any $X, Y \in \mathbb{D}$. Let us suppose that $X \in \mathbb{D}$ satisfies $AX = \lambda X$. Then $A\phi X = \mu\phi X$ and from the last equation we obtain $-\lambda\mu - \lambda^2 + 2\alpha\mu + 2\alpha\lambda = \lambda\mu + \mu^2$. That is $(\mu + \lambda)^2 - 2\alpha(\mu + \lambda) = 0$. Thus, $(\mu + \lambda)(\mu + \lambda - 2\alpha) = 0$. If $\mu + \lambda = 0$, as $\mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}$, we get $2\lambda^2 + 2 = 0$, which is impossible. Therefore $\mu + \lambda = 2\alpha$ and the value of μ yields $\lambda^2 - 2\alpha\lambda + 1 + \alpha^2 = 0$. This equation has no real solutions and this implies that our real hypersurface must be non-Hopf.

As in the previous section, we write locally $A\xi = \alpha\xi + \beta U$, with the same conditions, and also make the following computations locally. The scalar product of (4.1) and ϕU gives $-\eta(AY)g(A\phi X, U) - k\eta(AY)g(X, \phi U) = -\eta(A\phi Y)g(AX, U) + k\eta(A\phi Y)g(X, U)$ for any $X, Y \in \mathbb{D}$. That is, bearing in mind that $\beta \neq 0$,

$$g(Y, U)g(A\phi X, U) + kg(Y, U)g(X, \phi U) = g(\phi Y, U)g(AX, U) - kg(\phi Y, U)g(X, U) \tag{4.2}$$

for any $X, Y \in \mathbb{D}$. Take $Y = \phi U$ in (4.2) to obtain $g(AX, U) - kg(X, U) = 0$, for any $X \in \mathbb{D}$. Therefore,

$$AU = \beta\xi + kU. \tag{4.3}$$

Now the scalar product of (4.1) and U yields

$$\begin{aligned} &-g(Y, U)g(\phi A\phi X, U) - kg(Y, U)g(X, U) - g(\phi A\phi X, Y) + g(AY, X) \\ &\quad = -g(\phi Y, U)g(\phi AX, U) + kg(\phi Y, U)g(\phi X, U) - g(AX, Y) + g(\phi A\phi Y, X) \end{aligned} \tag{4.4}$$

for any $X, Y \in \mathbb{D}$. Taking $Y = U$ in (4.4) we obtain

$$-kg(X, U) - 3g(\phi A\phi X, U) + 2g(AU, X) = 0 \tag{4.5}$$

for any $X \in \mathbb{D}$. Taking $X \in \mathbb{D}_U$ and changing X by ϕX in (4.5) we get $g(A\phi U, X) = 0$ for any $X \in \mathbb{D}_U$. If $X = U$ in (4.5), we have $-k + 3g(A\phi U, \phi U) + 2k = 0$. Bearing in mind (4.3) we have obtained

$$A\phi U = -\frac{k}{3}\phi U. \tag{4.6}$$

Moreover, the scalar product of (4.1) and $\phi Z \in \mathbb{D}_U$ implies

$$-g(Y, U)g(A\phi X, Z) - kg(Y, U)g(X, \phi Z) = -g(\phi Y, U)g(AX, Z) + kg(\phi Y, U)g(X, Z) \tag{4.7}$$

for any $X, Y \in \mathbb{D}$, $Z \in \mathbb{D}_U$. Taking $Y = \phi U$ in (4.7) we obtain $g(AZ, X) - kg(Z, X) = 0$ for any $Z \in \mathbb{D}_U$, $X \in \mathbb{D}$, and this yields

$$AZ = kZ \tag{4.8}$$

for any $Z \in \mathbb{D}_U$. Take $Z \in \mathbb{D}_U$. Then $AZ = kZ$ and $A\phi Z = k\phi Z$. From the Codazzi equation, $\nabla_{\phi Z}(kZ) - A\nabla_{\phi Z}Z - \nabla_Z(k\phi Z) + A\nabla_Z\phi Z = 2\xi$. Its scalar product with ξ gives $-kg(Z, \phi A\phi Z) - g(\nabla_{\phi Z}Z, \alpha\xi + \beta U) + kg(\phi Z, \phi AZ) + g(\nabla_Z\phi Z, \alpha\xi + \beta U) = 2$. Then, $\beta g([Z, \phi X], U) + 2k^2 + \alpha g(Z, \phi A\phi Z) - \alpha g(\phi Z, \phi AZ) = 2$. Therefore

$$g([Z, \phi Z], U) = \frac{2 - 2k^2 + 2\alpha k}{\beta}. \tag{4.9}$$

Moreover, its scalar product with U implies $-kg([Z, \phi Z], U) - g(\nabla_{\phi Z}Z, \beta\xi + kU) + g(\nabla_Z\phi Z, \beta\xi + kU) = 0$. This gives $\beta g(Z, \phi A\phi Z) - \beta g(\phi Z, \phi AZ) = 0$ or $2\beta k = 0$, which is impossible and proves Theorem 1.4.

Suppose now that $A_T^{(k)}$ is η-hybrid with respect to ϕ. Then we have

$$\begin{aligned} &g(\phi A\phi X, AY)\xi - \eta(AY)\phi A\phi X - g(A^2Y, X)\xi - k\eta(AY)X + g(\phi AX, A\phi Y)\xi \\ &\quad -\eta(A\phi Y)\phi AX - g(\phi A^2\phi Y, X)\xi + k\eta(A\phi Y)\phi X = 0 \end{aligned} \tag{4.10}$$

for any $X, Y \in \mathbb{D}$. Let us suppose that M is Hopf. Then (4.10) gives $g(A\phi A\phi X, Y) - g(A^2X, Y) - g(\phi A\phi AX, Y) - g(\phi A^2\phi X, Y) = 0$, for any $X, Y \in \mathbb{D}$. Then $A\phi A\phi X - A^2X - \phi A\phi AX - \phi A^2\phi X = 0$ for any $X \in \mathbb{D}$. If $X \in \mathbb{D}$ satisfies $AX = \lambda X$, we obtain $-\lambda\mu - \lambda^2 + \lambda\mu + \mu^2 = 0$. Therefore, $\lambda^2 = \mu^2$. As in the previous theorem, $\lambda + \mu = 0$ gives a contradiction. This means that $\lambda = \mu$ and $\phi A = A\phi$. This yields that M must be locally congruent to a real hypersurface of type (A). The converse is immediate.

Suppose then that M is non-Hopf and $A\xi = \alpha\xi + \beta U$. Taking the scalar product of (4.10) and ϕU we have

$$\begin{aligned} &-g(Y, U)g(A\phi X, U) - kg(Y, U)g(X, \phi U) - g(\phi Y, U)g(AX, U) \\ &\quad +kg(\phi Y, U)g(X, U) = 0 \end{aligned} \tag{4.11}$$

for any $X, Y \in \mathbb{D}$. If $Y = \phi U$ in (4.11), we obtain $g(AU, X) - kg(U, X) = 0$ for any $X \in \mathbb{D}$ and this yields

$$AU = \beta\xi + kU. \tag{4.12}$$

Following the above proof step by step, we can also see that $A\phi U = k\phi U$ and $AZ = kZ$, for any $Z \in \mathbb{D}_U$. If we apply again the Codazzi equation to Z and ϕZ, $Z \in \mathbb{D}_U$, we obtain $k\beta = 0$, which is impossible and finishes the proof of Theorem 1.5.

The condition in Theorem 1.6 implies

$$\begin{aligned} &g(\phi A\phi X, AY)\xi - \eta(AY)\phi A\phi X - g(A^2Y, X)\xi - g(A^2Y, X)\xi \\ &\quad -g(\phi A\phi X, Y)A\xi + g(AX, Y)A\xi = -\eta(AY)\phi^2AX - g(\phi AX, Y)\phi A\xi \\ &\quad +g(\phi AY, X)\phi A\xi \end{aligned} \tag{4.13}$$

for any $X, Y \in \mathbb{D}$. If M is Hopf, (4.13) yields $g(A\phi A\phi X, Y)\xi - g(A^2X, Y)\xi - \alpha g(\phi A\phi X, Y)\xi - \alpha g(AX, Y)\xi = 0$ for any $X, Y \in \mathbb{D}$. Its scalar product with ξ shows that $A\phi A\phi X - A^2X - \alpha\phi A\phi X + \alpha AX = 0$, for any $X \in \mathbb{D}$: If $X \in \mathbb{D}$ satisfies $AX = \lambda X$, $A\phi X = \mu\phi X$ and we obtain $(\alpha - \lambda)(\mu + \lambda) = 0$. As we saw before, $\lambda + \mu \neq 0$. Therefore, $\lambda = \alpha$ and as $\mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}$, we get $\mu = \frac{\alpha^2 + 2}{\alpha}$. Thus, M has two distinct constant principal curvatures. From [2, 10] M must be locally congruent to a geodesic hypersphere. In this case

M has only a principal curvature on $\mathbb{D}$. That means that $\alpha = \frac{\alpha^2+2}{\alpha}$, which is impossible. Therefore M must be non-Hopf and as above, we write $A\xi = \alpha\xi + \beta U$. In this case (4.13) looks as follows:

$$\begin{aligned} &g(\phi A\phi X, AY)\xi - \beta g(Y,U)\phi A\phi X) - g(A^2Y,X)\xi - g(\phi A\phi X)A\xi \\ &\quad + g(AX,Y)A\xi = -\beta g(Y,U)\phi^2AX - \beta g(\phi AX,Y)\phi U + \beta g(\phi AY,X)\phi U \end{aligned} \tag{4.14}$$

for any $X, Y \in \mathbb{D}$. Bearing in mind that $\beta \neq 0$, the scalar product of (4.14) and ϕU gives $-g(Y,U)g(A\phi X,U) = -g(Y,U)g(\phi AX,U) - g(\phi AX,Y) + g(\phi AY,X)$ for any $X, Y \in \mathbb{D}$. Taking $Y = U$, we get $g(\phi AU,X) = -g(\phi AX,U) - g(\phi AX,U) + g(\phi AU,X)$. Therefore, $g(A\phi U,X) = 0$ for any $X \in \mathbb{D}$, and

$$A\phi U = 0. \tag{4.15}$$

Taking $Y = \phi U$ in the last equation, we get $0 = -g(AX,U) + g(\phi A\phi U,X)$, for any $X \in \mathbb{D}$. Bearing in mind (4.15), this implies $g(AU,X) = 0$ for any $X \in \mathbb{D}$ and so

$$AU = \beta\xi. \tag{4.16}$$

From (4.15) and (4.16), $\mathbb{D}_U$ is A-invariant. If in the equality used to find (4.15) and (4.16) we take $Y \in \mathbb{D}_U$, we obtain $0 = g(\phi AY,X) + g(A\phi Y,X)$ for any $Y \in \mathbb{D}_U$, $X \in \mathbb{D}$. Then, $\phi AY + A\phi Y = 0$ for any $Y \in \mathbb{D}_U$. If we suppose that $AY = \lambda Y$, we get $A\phi Y = -\lambda\phi Y$.

The scalar product of (4.14) and $Z \in \mathbb{D}_U$ gives

$$-g(Y,U)g(\phi A\phi X,Z) = g(Y,U)g(AX,Z) \tag{4.17}$$

for any $X, Y \in \mathbb{D}$, $Z \in \mathbb{D}_U$. Taking $Y = U$, $X = Z$, we obtain $g(A\phi Z,\phi Z) = g(AZ,Z)$. This yields $\lambda = -\lambda$. Therefore $\lambda = 0$ and M is locally congruent to a ruled real hypersurface. This finishes the proof of Theorem 1.6.

The condition in Theorem 1.7 yields

$$\begin{aligned} &g(\phi AX, A\phi Y)\xi - \eta(A\phi Y)\phi AX - g(\phi A^2\phi Y,X)\xi + k\eta(A\phi Y)\phi X \\ &\quad - g(AX,Y)A\xi + g(\phi A\phi Y,X)A\xi = -\eta(AY)\phi^2AX - k\eta(AY)X \\ &\quad - g(\phi AX,Y)\phi A\xi + g(\phi AY,X)\phi A\xi \end{aligned} \tag{4.18}$$

for any $X, Y \in \mathbb{D}$. If we suppose that M is Hopf with $A\xi = \alpha\xi$, (4.18) becomes

$$g(\phi AX, A\phi Y)\xi - g(\phi A^2\phi Y,X)\xi - \alpha g(AX,Y)\xi + \alpha g(\phi A\phi Y,X)\xi = 0 \tag{4.19}$$

for any $X, Y \in \mathbb{D}$. This gives $-\phi A\phi AX - \phi A^2\phi X - \alpha AX + \alpha\phi A\phi X = 0$ for any $X \in \mathbb{D}$. If $X \in \mathbb{D}$ satisfies $AX = \lambda X$, we obtain $(\mu - \alpha)(\lambda + \mu) = 0$. As in previous theorems, this case leads to a contradiction.

Thus, M must be non-Hopf and, as usual, we write $A\xi = \alpha\xi + \beta U$. In this case (4.18) implies

$$\begin{aligned} &g(\phi AX, A\phi Y)\xi - \beta g(\phi Y,U)\phi AX - g(\phi A^2\phi Y,X)\xi + k\beta g(\phi Y,U)\phi X \\ &\quad - g(AX,Y)A\xi + g(\phi A\phi Y,X)A\xi = \beta g(Y,U)AX - \beta^2 g(Y,U)g(X,U)\xi \\ &\quad - k\beta g(Y,U)X - \beta g(\phi AX,Y)\phi U + \beta g(\phi AY,X)\phi U \end{aligned} \tag{4.20}$$

for any $X, Y \in \mathbb{D}$. The scalar product of (4.20) and ϕU, bearing in mind that $\beta \neq 0$, gives

$$\begin{aligned} &-g(\phi Y,U)g(AX,U) + kg(\phi Y,U)g(X,U) = g(Y,U)g(A\phi U,X) \\ &\quad - kg(Y,U)g(\phi U,X) - g(\phi AX,Y) + g(\phi AY,X) \end{aligned} \tag{4.21}$$

for any $X, Y \in \mathbb{D}$. If we take $Y \in \mathbb{D}_U$ in (4.21), we obtain

$$g(AX, \phi Y) - g(A\phi X, Y) = 0 \tag{4.22}$$

for any $X \in \mathbb{D}$, $Y \in \mathbb{D}_U$.

Taking $X \in \mathbb{D}_U$ in (4.21), we infer

$$-g(\phi Y, U)g(AU, X) = g(Y, U)g(A\phi U, X) + g(A\phi Y, X) + g(\phi AY, X) \tag{4.23}$$

for any $X \in \mathbb{D}_U$, $Y \in \mathbb{D}$.

Take $X = U$ and ϕY instead of Y in (4.22) to have

$$g(AU, Y) + g(A\phi U, \phi Y) = 0 \tag{4.24}$$

for any $Y \in \mathbb{D}_U$. Take $Y = \phi U$ in (4.23). Then

$$g(AU, X) + g(A\phi U, \phi X) = -g(AU, X) \tag{4.25}$$

for any $X \in \mathbb{D}_U$. From (4.24) and (4.25), we derive

$$g(AU, X) = g(A\phi U, X) = 0 \tag{4.26}$$

for any $X \in \mathbb{D}_U$.

The scalar product of (4.20) and U yields

$$\begin{aligned} &g(\phi Y, U)g(A\phi U, X) + kg(\phi Y, U)g(\phi X, U) - g(AX, Y) + g(\phi A\phi Y, X) \\ &\quad = g(Y, U)g(AX, U) - kg(Y, U)g(X, U) \end{aligned} \tag{4.27}$$

for any $X, Y \in \mathbb{D}$. In (4.27) we take $Y = U$ and obtain $2g(AU, X) + g(A\phi U, \phi X) = kg(U, X)$ for any $X \in \mathbb{D}$. If $X = \phi U$ we get

$$g(AU, \phi U) = 0 \tag{4.28}$$

and if $X = U$, we have

$$2g(AU, U) + g(A\phi U, \phi U) = k. \tag{4.29}$$

Taking $Y = \phi U$ in (4.27), we obtain $2\,g(A\phi U, X) + g(\phi AU, X) = kg(\phi U, X)$. If $X = \phi U$, we conclude

$$2g(A\phi U, \phi U) + g(AU, U) = k. \tag{4.30}$$

From (4.29) and (4.30),

$$g(AU, U) = g(A\phi U, \phi U) = \frac{k}{3}. \tag{4.31}$$

From (4.26), (4.28) and (4.31), we obtain

$$\begin{aligned} AU &= \beta\xi + \frac{k}{3}U, \\ A\phi U &= \frac{k}{3}\phi U. \end{aligned} \tag{4.32}$$

The scalar product of (4.20) and ξ yields

$$g(\phi AX, A\phi Y) - g(\phi A^2\phi Y, X) - \alpha g(AX, Y) + \alpha g(\phi A\phi Y, X) = 0 \tag{4.33}$$

for any $X, Y \in \mathbb{D}$. Taking $X = U$ in (4.33), we obtain $g(\phi AU, A\phi Y) + g(A\phi Y, A\phi U) - \alpha g(AU, Y) - \alpha g(A\phi U, \phi Y) = 0$ for any $Y \in \mathbb{D}$. From (4.32) we get $\left(\frac{2k^2}{9} - \frac{2k\alpha}{3}\right) g(U, Y) = 0$ for any $Y \in \mathbb{D}$. Taking $U = Y$,

$$k = 3\alpha. \tag{4.34}$$

If we take $X = \phi U$ in (4.33), we have $g(\phi A\phi U, A\phi Y) - g(AU, A\phi Y) - \frac{k\alpha}{3} g(\phi U, Y) + \alpha g(A\phi Y, U) = 0$ for any $Y \in \mathbb{D}$. From (4.32) and (4.34) it follows $\beta^2 g(U, \phi Y) = 0$ for any $Y \in \mathbb{D}$. If $Y = \phi U$, then $\beta = 0$, which is impossible and this finishes the proof of Theorem 1.7.

Remark 4.1 With proofs similar to the ones appearing in this section, we could also obtain non-existence results for real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, satisfying any of the following conditions:

1. $A_T^{(k)}(\phi X, Y) + \phi A_T^{(k)}(X, Y) = 0$ for any $X, Y \in \mathbb{D}$ and any nonnull real number k;
2. $A_T^{(k)}(X, \phi Y) + \phi A_T^{(k)}(X, Y) = 0$ for any $X, Y \in \mathbb{D}$ and any nonnull real number k.

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