ON THE NUMBER OF WEIGHTED ZERO-SUM SUBSEQUENCES

A. LEMOS, B.K. MORIYA, A.O. MOURA AND A.T. SILVA*

Abstract. Let G be a finite additive abelian group with exponent $d^k n, d, n > 1$, and k a positive integer. For S a sequence over G and $A = \{1, 2, \ldots, d^k n - 1\} \setminus \{d^k n/d^i : i \in [1, k]\}$, we investigate the lower bound of the number $N_{A,0}(S)$, which denotes the number of A-weighted zero-sum subsequences of S. In particular, we prove that $N_{A,0}(S) \geq 2^{|S| - D_A(G)+1}$, where $D_A(G)$ is the A-weighted Davenport Constant. We also characterize the structures of the extremal sequences for which equality holds for some groups.

1. Introduction

Let G be a finite additive abelian group with exponent n and S be a sequence over G. The enumeration of subsequences with certain prescribed properties is a classical topic in Combinatorial Number Theory going back to Erdős, Ginzburg and Ziv (see [8, 14, 15]) who proved that $2n - 1$ is the smallest integer, such that every sequence S over a cyclic group C_n has a subsequence of length n with zero-sum. This raises the problem of determining the smallest positive integer l, such that every sequence $S = g_1 \cdots g_l$ has a nonempty zero-sum subsequence. Such an integer l is called the *Davenport constant of* G (see [7, 22]), denoted by $D(G)$, which is still unknown for wide class of groups. In an analogous manner, for a nonempty subset $A \subset \mathbb{Z}$, Adhikari *et. al.* defined, see [1], an A-weighted Davenport constant, denoted by $D_A(G)$, to be a smallest $t \in \mathbb{N}$ such that every sequence S over G of length t has nonempty A-weighted zero-sum subsequence.

For any g of G, let $N_{A,g}(S)$ (when $A = \{1\}$ we write $N_g(S)$) denote *the number of weighted subsequences* $T = \prod_{i\in I} g_i$ of $S = g_1 \cdots g_l$ such that $\sum_{i\in I} a_i g_i = g$, where $I \subseteq \{1, \ldots, l\}$ is a nonempty subset and $a_i \in A$. In 1969, Olson, see [23], proved that $N_0(S) \geq 2^{|S|-D(G)+1}$ for every sequence S over G of length $|S| \geq D(G)$. Subsequently, several authors, including [3, 4, 5, 9, 10, 11, 12, 13, 16, 17, 18, 19] obtained a huge variety of results on the number of subsequences with prescribed properties. In 2011, Chang *et al.*, see [6], found the lower bound of $N_g(S)$ for any arbitrary g and classify the extremal sequences for $|G|$ odd. Recently, Lemos *et al.*, see [20], found the lower bound of $N_{A,0}(S)$ for $A = \{1, \ldots, n-1\}$ and classify the extremal sequences for $|G|$ odd. Here we prove that $N_{A,0}(S) \geq 2^{|S|-D_A(G)+1}$, when $A = \{1, 2, \ldots, d^k n-1\} \setminus \{d^k n/d^i : i \in [1, k]\}$, where k is a positive integer. Besides, we classify the sequences such that $N_{A,0}(S) = 2^{|S|-D_A(G)+1}$, where $G = H \oplus C^r_{d^k n}$, with n odd, $\exp(H) \mid d^k$, $\gcd(d, n) \leq d - 1$ and $d^k n \geq 6$.

2. Notations and terminologies

In this section, we will introduce some notations and terminologies. Notations and terminologies are in accordance with [20]. Let $\mathbb{N}_0$ be *the set of non-negative integers*. For integers $a, b \in \mathbb{N}_0$, we define $[a, b] = \{x \in \mathbb{N}_0 : a \leq x \leq b\}$.

For a sequence

$$S = \prod_{i=1}^{m} g_i \in \mathcal{F}(G),$$

where $\mathcal{F}(G)$ is the *free abelian monoid* with basis G, a *subsequence* $T = g_{i_1} \ldots g_{i_k}$ of S, with $I_T = \{i_1, \ldots, i_k\} \subseteq [1, m]$ is denoted by $T|S$; we identify two subsequences S_1 and S_2 if $I_{s_1} = I_{s_2}$. Given subsequences $S_1, \ldots, S_r$ of S, we define *gcd*$(S_1, \ldots, S_r)$ to be the sequence indexed by $I_{S_1} \cap \cdots \cap I_{S_r}$. We say that two subsequences S_1 and S_2 are *disjoint* if $(S_1, S_2) = \lambda$, where λ refers to the *empty sequence*. If S_1 and S_2 are disjoint, then we denote by $S_1 S_2$ the subsequence with set index $I_{s_1} \cup I_{s_2}$; if $S_1|S_2$; we denote by $S_2 S_1^{-1}$ the subsequence with set index $I_{s_2} \setminus I_{s_1}$. Moreover, we define

2010 *Mathematics Subject Classification.* 20K01,11B75.

Key words and phrases. Finite abelian group, Sequences and sets, Extremal 0-complete sequence.

∗ The authors were partially supported by FAPEMIG APQ-02546-21 and RED-00133-21.

(i) $|S| = m$ *the length of* S.
(ii) an *A-weighted sum* is a sum of the form $\sigma^{\mathbf{a}}(S) = \sum_{i=1}^{m} a_i g_i$, with fixed $\mathbf{a} = a_1 \cdots a_m \in \mathcal{F}(A)$, where $\mathcal{F}(A)$ is the free abelian monoid with basis A. When $A = [1, n-1]$, we call S a *fully weighted sequence*.
(iii) $\sum_A (S) = \left\{\sum_{i \in I} a_i g_i : \emptyset \neq I \subseteq [1, m] \text{ and } a_i \in A\right\}$, a *set of nonempty A-weighted subsums of S*.

According to the above definitions, we adopt the convention that $\sigma^{\mathbf{a}}(\lambda) = 0$, for any $\mathbf{a} \in \mathcal{F}(A)$. For convenience, we define $\sum_A^{\bullet}(S) = \sum_A (S) \cup \{0\}$.

The sequence S is called

(i) an *A-weighted zero-sum free sequence* if $0 \notin \sum_A (S)$ and
(ii) an *A-weighted zero-sum sequence* if $\sigma^{\mathbf{a}}(S) = 0$ for some $\mathbf{a} \in \mathcal{F}(A)$.

When $A = \{1\}$, we call S *zero-sum free sequence* and *zero-sum sequence*, respectively. For an element $g \in G$, let

$$N_{A,g}(S) = \left|\left\{I \subseteq [1, m] : \sum_{i \in I} a_i g_i = g,\ a_i \in A\right\}\right|$$

denote *the number of subsequences* T *of* S with $\sigma^{\mathbf{a}}(T) = g$ for some $\mathbf{a} \in \mathcal{F}(A)$.

Definition 2.1. Let n be the exponent of G, $g \in G$, $A \subseteq \mathbb{Z} \setminus \{kn : k \in \mathbb{Z}\}$ and $S \in \mathcal{F}(G)$. We say S is *g-complete sequence with weight in A* if $N_{A,g}(S) \geq 2^{|S| - D_A(G) + 1}$. We call S an *extremal g-complete sequence with respect to A* if $N_{A,g}(S) = 2^{|S| - D_A(G) + 1}$. Let us denote $C_{A,g}(\mathcal{F}(G))$ as the *set of all g-complete sequences with respect to A* and $EC_{A,g}(\mathcal{F}(G))$ as the *set of all extremal g-complete sequences with respect to A*.

Definition 2.2. Let n be the exponent of G and $A \subseteq \mathbb{Z} \setminus \{kn : k \in \mathbb{Z}\}$. We say G is a 0*-complete group with respect to A* if $\mathcal{F}(G) = C_{A,0}(\mathcal{F}(G))$.

When $A = \{1\}$, Olson [23] proved that all finite abelian groups are 0-complete with respect to A. Chang *et al.* [6] proved, that, when $A = \{1\}$, if $g \in \sum_A^{\bullet}(S)$, then $S \in C_{A,g}(\mathcal{F}(G))$ and, if S is extremal h-complete sequence with respect to A for some $h \in G$, then S is g-complete sequence with respect to A for all $g \in G$. Moreover, they classified the sequences in $EC_{A,0}(\mathcal{F}(G))$ when G is a group of odd order.

Remark 2.3. *Take an A-weighted zero-sum free sequence* U *over* G *with* $|U| = D_A(G) - 1$. *Thus, for* $S = U0^{|S| - D_A(G) + 1}$ *and for any* $g \in \sum_A^{\bullet}(U)$, *we have* $S \in C_{A,g}(\mathcal{F}(G))$ *and* $S \in EC_{A,0}(\mathcal{F}(G))$.

We write a finite abelian group G as direct sum $G = H \oplus C_n^r$, where C_n^r denotes r copies of the cyclic group of order n denoted by C_n and $H = C_{n_1} \oplus \cdots \oplus C_{n_t}$ with $1 < n_1 | n_2 | \cdots | n_t | n = exp(G)$ and $n_t < n$.

We have some auxiliary results, which are as follows.

Lemma 2.4. *[Theorem 5.2 [21]] Let* $G = H \oplus C_n^r$, *where* $H = C_{n_1} \oplus \cdots \oplus C_{n_t}$ *with* $1 < n_1 | n_2 | \cdots | n_t | n = exp(G)$ *and* $n_t < n$. *Then,* $D_A(G) = r + 1$.

A subsequence T of S is called an *extremal A-weighted zero-sum free subsequence* if $|T| = D_A(G) - 1$ and T is A-weighted zero-sum free.

It is worth mentioning the following important result for the fully weighted 0-complete sequences, which was proved in [20].

Theorem 2.5. *All finite abelian group* G *with exponent* n *is* 0*-complete with respect to* $A = [1, n-1]$.

In [20] the authors conjectured that Theorem 2.5 holds for any A. In the Section 3, we proved that such a theorem is true for $G = H \oplus C_{d^k n}^r$, with n odd, $\exp(H) | d^k, \gcd(d, n) < d - 1$, $d^k n < 6$ and $A = \{1, 2, \ldots, d^k n - 1\} \setminus \{d^k n / d^i : i \in [1, k]\}$, where k is a positive integer.

3. Lower bound

We start this section by presenting an important theorem.

Theorem 3.1 (Adhikari *et al.*,Theorem 4.1, item (i) [2])**.** *Let* G *be a finite and nontrivial abelian group and let* $S \in \mathcal{F}(G)$ *be a sequence. If* $|S| \geq \log_2 |G| + 1$ *and* G *is not an elementary* 2*-group, then* S *contains a proper, nontrivial* $\{\pm 1\}$*-weighted zero-sum subsequence.*

To find the lower bound for $N_{A,0}(S)$, with $S \in \mathcal{F}(G)$, we used the value of $D_A(G)$.

Theorem 3.2. *Let $G = H \oplus C^r_{d^k n}$, where $\exp(H) \mid d^k$, $\gcd(d,n) \le d-1$ and $d^k n \ge 6$, where k is a positive integer. Then $D_A(G) = r+1$, for $A = \{1, 2, \dots, d^k n - 1\} \setminus \{d^k n/d^i : i \in [1,k]\}$.*

Proof. Since the canonical sequence $\prod_{i=1}^r e_i \in \mathcal{F}(C^r_{d^k n})$ does not have zero-sum subsequence with respect to weights in A, $D_A(G) \ge r+1$. Let $S = (h_i, g_i)_{i=1}^{r+1} \in \mathcal{F}(G)$. Consider a canonical homomorphism $\phi : G \to C_n^r$. Let $A' = \{1, 2, \dots, n-1\}$. Since $D_{A'}(C_n^r) = r+1$, by Lemma 2.4, we get a non-empty subsequence $T = (g_{i_k})_{k=1}^t$ of S such that $\sum_{j=1}^t a_j \phi(g_{i_j}) = 0 \in C_n^r$, where $a_j \in A', \forall j$. Hence, using the fact that $\exp(H) \mid d^k$ we have, $\sum_{j=1}^t d^k a_j (h_{i_j}, g_{i_j}) = 0 \in G$ (Note that $\phi(g) \equiv g \pmod n$), which as a result gives, $d^k \phi(g) \equiv d^k \cdot g \pmod{d^k \cdot n}$). Since $\gcd(d,n) \le d-1$, it follows that $d^k a_j \in A, \forall j$, which proves the theorem. □

The hypothesis $d^k n \ge 6$ in Theorem 3.2 is necessary, as on the contrary we have the following proposition.

Proposition 3.3. *If $G = C_2^s \oplus C_4^r$, then $D_A(G) = 2r + s + 1$ for $A = \{1,3\} = \{1,-1\}$.*

Proof. This upper bound is a immediate consequence of Theorem 3.1. For the lower bound we observe that the sequence $S = \prod_{i=1}^{s+r} e_i \prod_{i=s+1}^{s+r} 2e_i$ does not have $\{1,-1\}$-weighted zero-sum subsequece, where $\{e_1, \dots, e_{s+r}\}$ is the canonical base of G. □

Remark 3.4. *Note that, if $B \subset A$ then $D_A(G) \le D_B(G)$.*

As a consequence of Theorem 3.2, we get the following corollary.

Corollary 3.5. *Let $G = H \oplus C^r_{d^k n}$, where $\exp(H) \mid d^k$, $\gcd(d,n) \le d-1$ and $d^k n \ge 6$, where k is a positive integer. Then $D_A(G) = r+1$, for all A containing $B = \{1, 2, \dots, d^k n - 1\} \setminus \{d^k n/d^i : i \in [1,k]\}$.*

One can easily see that Theorem 3.2 does not hold true if $\exp(H) \nmid d^k$, in fact, the next proposition provides infinitely many examples such that $D_{A\setminus\{n\}}(G) \ne D_A(G)$.

Proposition 3.6. *Let $G = L \oplus C_n \oplus C^r_{2n}$, where $n > 1$ an odd number and $L = C_{n_1} \oplus \cdots \oplus C_{n_t}$ with $n_1 | n_2 | \cdots | n_t | n$. Then, $D_{A\setminus\{n\}}(G) \ge r+2$.*

Proof. Since n is an odd number one can easily prove that $(e_{t+1} + e_{t+2})(e_{t+1} + ne_{t+2})$ is zero-sum free with respect to weights in $A \setminus \{n\}$ and which in turn gives rise to a $A \setminus \{n\}$-weighted zero-sum free sequence $(e_{t+1} + e_{t+2})(e_{t+1} + ne_{t+2}) \prod_{i=3}^{r+1} e_{t+i}$. □

Theorem 3.7 below provides one more case for which Conjecture 4.3 of [20] holds.

Theorem 3.7. *Let $G = H \oplus C^r_{d^k n}$, where $\exp(H) \mid d^k$, $\gcd(d,n) \le d-1$ and $d^k n \ge 6$, where k is a positive integer. Then G is 0-complete with respect to $A = \{1, 2, \dots, d^k n - 1\} \setminus \{d^k n/d^j : j \in [1,k]\}$.*

Proof. Let $S \in \mathcal{F}(G)$ be a sequence. According to Corollary 3.5, we can write $D_A(G) = r+1$. If $|S| \le r$, then $N_{A,0}(S) \ge 1 \ge 2^{|S|-r}$. If $|S| = r+1$, then there is an A-weighted zero-sum nonempty subsequence T of S. Thus, $N_{A,0}(S) \ge 2 = 2^{|S|-r}$.

Suppose now $r+1 < |S|$. Let $S = TW \in \mathcal{F}(G)$ be a sequence such that T is a maximal A-weighted zero-sum free with $|T| \le r$ or $T = \lambda$.

Then, for each element $g|W$, we have two possibilities:

a) If $o(g) \in A$, then g is an A-weighted zero-sum subsequence.

b) If $o(g) \notin A$ for $j = 1, \dots k$, then Tg has an A-weighted zero-sum subsequence with g being one of its elements.

In both possibilities, there is $V|Tg$, such that V is an A-weighted zero-sum subsequence whose coefficient of g is $a_g \in A$. Then, $a_g g$ is an A-weighted sum of some subsequence of T:

$$a_g g = \sum\nolimits_{i \in I_g} a_i g_i; I_g \subset I_T \text{ and } a_i \in A.$$

Thus, for every nonempty subsequence U of W, we have

$$d^k \sum\nolimits_{g|U} a_g g = \sum\nolimits_{g|U} d^k \sum\nolimits_{i \in I_g} a_i g_i = \sum\nolimits_{i \in I_{V'}} d^k b_i g_i; I_{V'} \subset I_T,$$

with $d^k b_i \pmod{d^k n} \in A$ or $d^k b_i \equiv 0 \pmod{d^k n}$, i. e., the A-weighted sum $d^k \sum_{g|U} a_g g$ is an A-weighted sum of some subsequence $V' = V_U$ of T. Therefore, UV_U is an A-weighted zero-sum subsequence of S. Notice that if $V_U = \lambda$, then U is an A-weighted zero-sum subsequence. Therefore, if we include the empty subsequence, we obtain a minimum of $2^{|W|} = 2^{|S|-|T|}$ distinct A-weighted zero-sum subsequences of S. This proves that $N_{A,0}(S) \geq 2^{|S|-r}$. □

4. Characterization of extremal 0-complete sequences

We shall start by mentioning one of the main results obtained in the case which $\exp(G)$ is an odd positive integer (see Theorem 4.2 of [20]).

Theorem 4.1. *Let G be a finite abelian group with $exp(G) = n$ an odd number. If $S \in EC_{A,0}(\mathcal{F}(G))$, with $A = [1, n-1]$, $0 \nmid S$ and $o(g) = n$ for all $g|S$, then $r \leq |S| \leq 2r$ and there is $T = \prod_{i=1}^{r} g_i$ an extremal A-weighted zero-sum free, such that*

$$S = T \prod_{j=1}^{k} h_j, \tag{4.1}$$

where $k \in [1, r]$, $b_j h_j = \sum_{i \in I_j} a_i g_i$ with $a_i, b_j \in A$, $I_j \subset [1, r]$ and I_j's are pairwise disjoint ($I_j = \emptyset$ for all j implies that $S = T$).

In this section, our aim is to prove a variant of the result above in case $G = H \oplus C_{d^k n}^r$ be a finite abelian group, where k is a positive integer, $\exp(H) \mid d^k$, n is an odd number, $\gcd(d, n) \leq d-1, d^k n \geq 6$ and $A = \{1, 2, \ldots, d^k n - 1\} \setminus \{d^k n/d^j : j \in [1, k]\}$, which will be established in Theorem 4.4.

First, we consider a modification of the Proposition 4.1 (see [20]), which will be the main tool to prove the Theorem 4.4.

As $N_{A,0}(S) = 2N_{A,0}(S0^{-1})$ and $N_{A,0}(S) = 2N_{A,0}(Sg^{-1})$, if $o(g) \in A$, it suffices to consider sequences S, such that $0 \nmid S$ and $o(g) \notin A$ for all $g|S$.

Proposition 4.2. *Let $G = H \oplus C_{d^k n}^r$ be a finite abelian group where $\exp(H) \mid d^k$, $\gcd(d, n) \leq d-1$ and $d^k n \geq 6$, where k is a positive integer. If $S \in EC_{A,0}(\mathcal{F}(G \setminus \{0\}))$, with $A = \{1, 2, \ldots, d^k n - 1\} \setminus \{d^k n/d^j : j \in [1, k]\}$ and $o(g) \notin A$ for all $g|S$, then $r \leq |S|$ and there is $T = \prod_{i=1}^{r} g_i$ an extremal A-weighted zero-sum free such that*

$$S = T \prod_{j=1}^{\nu} h_j, \tag{4.2}$$

where $\nu \in [1, r]$, $b_j h_j = \sum_{i \in I_j} a_i g_i$ with $a_i, b_j \in A$, $I_j \subset [1, r]$.

The proposition above is a mere consequence of $D_A(G) = r + 1$.

Let us see below an example where we show an extreme sequence with respect to $N_{A,0}(S)$ for a group of order 72.

Example 4.3. *Let $S = e_2 e_3 (2e_2)(2e_3)$ be a sequence over $G = C_2 \oplus C_6^2$, where $\{e_1, e_2, e_3\}$ is the canonical basis of G. Note that, $D_A(G) = 3$ where $A = \{1, 2, 4, 5\}$, $|S| = 4 = 2(D_A(G) - 1)$, and $N_{A,0}(S) = 2^{|S| - D_A(G) + 1} = 2^2 = 4$. In this case, $T = e_2 e_3$ is an extremal A-weighted zero-sum free.*

The example above motivates us to establish the theorem below.

Theorem 4.4. *Let $G = H \oplus C_{d^k n}^r$ be a finite abelian group where n is an odd number, $\exp(H) \mid d^k$, $\gcd(d, n) \leq d - 1$ and $d^k n \geq 6$, where $k \in \mathbb{N}$. If $S \in EC_{A,0}(\mathcal{F}(G \setminus \{0\}))$, with $A = \{1, 2, \ldots, d^k n - 1\} \setminus \{d^k n/d^j : j \in [1, k]\}$ and $o(g) \notin A$ for all $g|S$, then $r \leq |S| \leq 2r$ and there is $T = \prod_{i=1}^{r} g_i$ an extremal A-weighted zero-sum free such that*

$$S = T \prod_{j=1}^{\nu} h_j, \tag{4.3}$$

where $\nu \in [1, r]$, $b_j h_j = \sum_{i \in I_j} a_i g_i$ with $a_i, b_j \in A$, $I_j \subset [1, r]$ and I_j's are pairwise disjoint ($I_j = \emptyset$ for all j implies that $S = T$).

Proof. Let S be a sequence over $G \setminus \{0\}$, $o(g) \notin A$ for all $g|S$ and $N_{A,0}(S) = 2^{|S|-D_A(G)+1} = 2^{|S|-r}$. We know, by Proposition 4.2, that $S = T\prod_{j=1}^{\nu} h_j$ where $\nu \in \mathbb{N}_0$, $b_j h_j = \sum_{i\in I_j} a_i g_i$ with $a_i, b_j \in A$, $I_j \subset [1, r]$ and $T = \prod_{i=1}^{r} g_i$ is an extremal A-weighted zero-sum free.

Now, we will prove that the I_j's are pairwise disjoint. If $|S| = D_A(G) - 1 = r$, then $N_{A,0}(S) = 1$, $I_j = \emptyset$ for all $j \in [1, \nu]$ and $S = T$. Suppose that $|S| = D_A(G) = r + 1$ then, $I_j \neq \emptyset$ for only one j, $N_{A,0}(S) = 2$ and $S = Th_j$. Finally, suppose $S = T\prod_{j=1}^{\nu} h_j$ with $\nu \geq 2$ and $I_{j_1} \cap I_{j_2} \neq \emptyset$ for some $j_1, j_2 \in [1, \nu]$, with $j_1 \neq j_2$ and where

$$a_{j_1} h_{j_1} = \sum_{i\in I_{j_1}} a_i g_i \text{ and } a_{j_2} h_{j_2} = \sum_{i\in I_{j_2}} b_i g_i$$

with $a_{j_1}, a_{j_2}, a_i, b_i \in A$, since $D_A(G) = r + 1$.

By hypothesis we have $\binom{\nu}{0} + \binom{\nu}{1} + \cdots + \binom{\nu}{\nu} = 2^{\nu} = 2^{|S|-r}$ A-weighted zero-sum subsequences of S, which can be obtained as in the proof of Theorem 3.7. Since $I_{j_1} \cap I_{j_2} \neq \emptyset$, we have $I_x, I_y \subset I_{j_1} \cup I_{j_2}$ such that

$$d^k(a_{j_1} h_{j_1} + a_{j_2} h_{j_2}) = d^k \left(\sum_{i\in I_x} c_i g_i \right) \pmod{d^k n} \tag{4.4}$$

and

$$d^k(a_{j_1} h_{j_1} - a_{j_2} h_{j_2}) = d^k \left(\sum_{i\in I_y} d_i g_i \right) \pmod{d^k n}. \tag{4.5}$$

Since $d^k a_{j_1}, d^k a_{j_2} \pmod{d^k n} \in A$ one can easily verify that $d^k c_i \equiv 0 \pmod{d^k n}$ or $d^k c_i \pmod{d^k n} \in A$ and $d^k d_i \equiv 0 \pmod{d^k n}$ or $d^k d_i \pmod{d^k n} \in A$. If $d^k c_i \equiv 0 \pmod{d^k n}$, for all $i \in I_x$ and $d^k d_i \equiv 0 \pmod{d^k n}$ for all $i \in I_y$, then $d^k(a_{j_1} h_{j_1} + a_{j_2} h_{j_2}) = 0$ and $d^k(a_{j_1} h_{j_1} - a_{j_2} h_{j_2}) = 0$. But, this implies that $2d^k a_{j_2} h_{j_2} = 0$, and hence $n|a_{j_2}$ ($o(h_{j_2}) = d^k n$ and n is odd), which is a contradiction since $d^k a_{j_2} \not\equiv 0 \pmod{d^k n}$. Therefore, $I_x \neq \emptyset$ or $I_y \neq \emptyset$.

If $I_x \neq I_y$, then there is a new A-weighted zero-sum subsequence of S and therefore $N_{A,0}(S) > 2^{|S|-r}$, which is a contradiction. Now, suppose that $I_x = I_y$. Consider $g_l | \prod_{i\in I_{j_1}\cap I_{j_2}} g_i$ (observe that $d^k c_l \not\equiv 0 \pmod{d^k n}$ and $d^k d_l \not\equiv 0 \pmod{d^k n}$ in (4.4) and (4.5)) and take $T' = \left(\prod_{i=1}^{r+1} g_i\right) g_l^{-1}$, where $g_{r+1} = h_{j_2}$. If T' is not an extremal A-weighted zero-sum free, then there is $\bar{I}_{j_2} \subset [1, r+1] \setminus \{l\}$ such that $z_{j_2} h_{j_2} = \sum_{i\in \bar{I}_{j_2}} s_i g_i$, i.e., we can obtain a new A-weighted zero-sum subsequence of S and thus $N_{A,0}(S) > 2^{|S|-r}$, which is a contradiction. If T' is an extremal A-weighted zero-sum free, then by Corollary 3.5 we have $\bar{I}_{j_1} \subset [1, r+1] \setminus \{l\}$ such that $v_{j_1} h_{j_1} = \sum_{i\in \bar{I}_{j_1}} u_i g_i$, i.e., we can obtain a new A-weighted zero-sum subsequence of S. Therefore, we have $N_{A,0}(S) > 2^{|S|-r}$ again, which is a contradiction.

We observe that if $\nu > r$, then there are I_{j_1} and I_{j_2} with $j_1 \neq j_2$, such that $I_{j_1} \cap I_{j_2} \neq \emptyset$. Therefore, $N_{A,0}(S) > 2^{|S|-r}$. Thus, $r \leq |S| \leq 2r$. □

The example below shows a case that is not covered by hypotheses of Theorem 4.4. We believe that it is possible to obtain a similar theorem that covers this case.

Example 4.5. *Let $S = e_1 e_2 e_3 (2e_2)(2e_3)(3e_2)(3e_3)$ be a sequence over $G = C_2 \oplus C_4^2$, where $\{e_1, e_2, e_3\}$ is the canonical basis of G. Note that $|S| = 7 = D_A(G) + 1$ and $N_{A,0}(S) = 2^{|S|-D_A(G)+1} = 2^2 = 4$, where $D_A(G) = 6$, with $A = \{1, 3\}$, by Proposition 3.3. In this case, $T = e_1 e_2 e_3 (2e_2)(2e_3)$ is an extremal A-weighted zero-sum free.*

References

[1] S.D. Adhikari and P. Rath, Davenport constant with weights and some related questions, *Integers* **6** (2006), p. A30.

[2] S.D. Adhikari, D.J. Grynkiewicz and Z.W. Sun, On weighted zero-sum sequences. *Adv. Appl. Math.*, 48 (2012), p. 506–527.

[3] E. Balandraud, An addition theorem and maximal zero-sum free set in $\mathbb{Z}/p\mathbb{Z}$, *Israel J. Math.* **188** (2012), 405–429.

[4] A. Bialostocki and M. Lotspeich, Some developments of the Erdős-Ginzburg-Ziv Theorem I, Sets, Graphs and Numbers, *Coll. Math. Soc. J. Bolyai* **60** (1992), 97–117.

[5] H.Q. Cao and Z.W. Sun, On the number of zero-sum subsequences, *Discrete Math.* **307** (2007), 1687–1691.

[6] G.J. Chang, S. Chen, Y. Qu, G. Wang and H. Zhang, On the number of subsequence with a give sum in finite abelian group, *Electron. J. Combin.* **18** (2011), no. 1, paper 133.

[7] H. Davenport, On the addition of residue classes, *J. Lond. Math. Soc.* **10** (1935), 30–32.
[8] P. Erdős, A. Ginzburg and A. Ziv, Theorem in the additive number theory, *Bulletim Research Council Israel* **10F** (1961), 41–43.
[9] Z. Füredi and D.J. Kleitman, The minimal number of zero-sums, Combinatorics, Paul Erdős is Eighty, *J. Bolyai Math. Soc.* (1993), 159–172.
[10] W.D. Gao, On the number of zero-sum subsequences, *Discrete Math.* **163** (1997), 267–273.
[11] W.D. Gao, On the number of subsequences with given sum, *Discrete Math.* **195** (1999), 127–138.
[12] W.D. Gao and A. Geroldinger, On the number of subsequences with given sum of sequences over finite abelian p-groups, *Rocky Mountain J. Math.* **37** (2007), 1541– 1550.
[13] W.D. Gao and J.T. Peng, On the number of zero-sum subsequences of restricted size, *Integers* **9** (2009), 537–554.
[14] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations, Combinatorial and Analytic Theory*, Pure and Applied Mathematics **278**, Chapman & Hall/CRC, 2006.
[15] A. Geroldinger, Additive group theory and non-unique factorizations, Combinatorial Number Theory and Additive Group Theory, *Advanced Courses in Mathematics*, CRM Barcelona, Birkhauser, (2009), 1–86.
[16] D.J. Grynkiewicz, On the number of m-term zero-sum subsequences, *Acta Arith.* **121** (2006), 275–298.
[17] D.J. Grynkiewicz, E. Marchan and O. Ordaz, Representation of finite abelian group elements by subsequence sums, *Journal de Théorie des Nombres de Bordeaux* **21** (2009), 559–587.
[18] D.R. Guichard, Two theorems on the addition residue classes, *Discrete Math.* **81** (1990), 11–18.
[19] M. Kisin, The number of zero sums modulo m in a sequence of length n, *Mathematica* **41** (1994), 149–163.
[20] A. Lemos, A. de Oliveira Moura, A.T. Silva and B.K. Moriya, On the number of fully weighted zero-sum subsequences, *International Journal of Number Theory*, **15(05)** (2019), 1051–1057.
[21] L.E. Marchan, O. Ordaz and W.A. Schmid, Remarks on the plus-minus weighted Davenport constant, *International Journal of Number Theory* **10** (05) (2014), 1219–1239.
[22] J.E. Olson. A combinatorial problem on finite abelian groups I, *Jornal of Number Theory* **1** (1969), 8–10.
[23] J.E. Olson. A combinatorial problem on finite abelian groups II, *Jornal of Number Theory* **1** (1969), 195–199.

Departamento de Matemática, Universidade Federal de Viçosa, Viçosa-MG, Brazil
Email address: `abiliolemos@ufv.br, bhavinkumar@ufv.br, allan.moura@ufv.br`
Email address: `anderson.tiago@ufv.br`