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Associating parallel automata network dynamics and strictly one-way cellular automata

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Abstract

Automata networks are often conceived as a finite generalization of cellular automata. In this paper, we prove that the limit dynamics of any finite automata network under the parallel update schedule correspond exactly to the fixed points of so-called strictly one-way cellular automata. This correspondence is proven to be exact, as any strictly one-way cellular automata can be transformed into a corresponding automata network, where the attractors of the latter correspond exactly to the fixed points of the former. This transformation is easy to operate by using output functions which have been developed in the author's previous works.

1 Introduction

Automata networks are used to model gene regulatory networks [6, 12, 8, 1, 2]. In these applications the dynamics of automata networks are used to understand how the biological systems might evolve. As such, there is motivation in improving our computation and characterisation of automata network dynamics. Rather than considering the problem in general, we look for families or properties which allow for simpler dynamics that we might be able to characterise [3, 4].

To help in that effort, we developed the formalism of modules [9] which have inputs. By focusing on modules with acyclic interaction digraph, we proved that the set of attractors was determined (up to the renaming of configurations) by so called output functions [10]. By focusing on output function sets rather than automata networks, we allow for the automatic removal of many specificities of the networks which do not determine the limit dynamics.

In this paper, we take the output function set, compose it into a single output function on vectors, and then use this so-called global output function as the local function of a cellular automata. One of these properties of this cellular automata is that its fixed points correspond to the attractors of any automata networks which realises the initial set of output functions. Finally, we use this transformation to generalise a result about intertwined fixed points in strictly one-way cellular automata, which allows for the decomposition of the dynamics of the corresponding automata networks.

The demonstration of both results is available in the appendix.

2 Definitions

2.1 Automata networks

ANs are composed of a set S of automata. Each automaton in S, or node, is at any time in a state in Λ. Gathering those isolated states into a vector of dimension $|S|$ provides us with a configuration of the network. More formally, a *configuration* of S over Λ is a vector in Λ^S. The state of every automaton evolves as a function of the configuration of the entire network. Each node has a unique function, called a local function, that is predefined and does not change over time. A *local function* is thus a function f defined as $f : \Lambda^S \to \Lambda$. Formally, an AN F is a set that assigns a local function f_s over S for every $s \in S$.

Example 1. *Let $S = \{a, b, c\}$. Let F be the Boolean AN with local functions $f_a(x) = x_c$, $f_b(x) = x_a$ and $f_c(x) = \neg x_a \vee x_b$.*

In the scope of this paper, ANs (and modules) are udpated according to the parallel update schedule. Formally, for F an AN and x a configuration of F, the update of x under F is denoted by configuration $F(x)$, and defined as for all s in S, $F(x)_s = f_s(x)$.

2.2 Interaction digraph

ANs are usually represented by the influence that automata hold on each other. As such the visual representation of an AN is a digraph, called an interaction digraph, whose nodes are the automata of the network, and arcs are the influences that link the different automata. Formally, *s influences s'* if and only if there exist two configurations x, x' such that $f_{s'}(x) \neq f_{s'}(x')$ and for all r in S, $r \neq s$ if and only if $x_r = x'_r$.

The interaction digraph of the network detailed in Example 1 is illustrated in Figure 1.

2.3 Modules

Modules were first introduced in [9]. A module M is an AN with added inputs. It is defined on two sets: S a set of automata, and I a set of inputs, with $S \cap I = \varnothing$. Similarly to standard ANs, we can define configurations

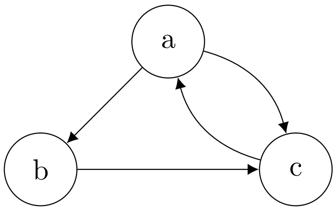

Figure 1: Interaction digraph of network F described in Example 1.

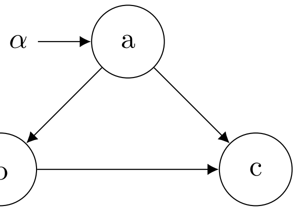

Figure 2: Interaction digraph of module M described in Example 2.

as vectors in Λ^S, and we define input configurations as vectors in Λ^I. A local function of a module updates itself based on a configuration x and an input configuration i, concatenated into one configuration. Formally, a local function is defined from $\Lambda^{S \cup I}$ to Λ. The module M defines a local function for every node s in S.

We represent modules with an interaction digraph, in the same way as for ANs. The interaction digraph of a module has added arrows that represent the influence of the inputs over the nodes; for every node s and every input α, the node s of the interaction digraph has an ingoing arrow labelled α if and only if α influences s, that is, there exists two input configurations i, i' such that for all β in I, $\beta \neq \alpha$ if and only if $i_\beta = {i'}_\beta$, and x a configuration such that $f_s(x \cdot i) \neq f_s(x \cdot i')$, where $\cdot$ denotes the concatenation operator.

A module is *acyclic* if and only if its interaction digraph is cycle-free.

Example 2. *Consider set S from Example 1, and the set of inputs $I = \{\alpha\}$. Let M be the module with local functions $f'_a(x, i) = i_\alpha$, $f'_b(x, i) = x_a$, $f'_c(x, i) = \neg x_a \vee x_b$. The module M is acyclic. The interaction digraph of M is represented in Figure 2.*

2.4 Recursive wirings

A recursive wiring over a module M is defined by a partial function $\omega : I \nrightarrow S$. The result of such a wiring is denoted $\circlearrowright_\omega M$, a module defined over sets S and $I \setminus \mathrm{dom}(\omega)$, in which the local function of node s is denoted f'_s and defined as

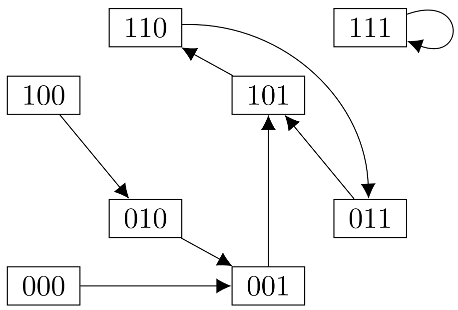

Figure 3: Dynamics of AN F as described in Example 1.

$$\forall x \in \mathbb{B}^{S \cup I},\ f'_s(x) = f_s(x \circ \hat{\omega}),\ \text{with}\ \hat{\omega}(i) = \left\{ \begin{array}{ll} \omega(i) & \text{if } i \in \mathrm{dom}(\omega) \\ i & \text{if } i \in I \setminus \mathrm{dom}(\omega) \end{array} \right. .$$

In the case where ω is a total function, then the resulting module $\circlearrowleft_\omega M$ has no remaining inputs and can be considered as an AN. We can then consider the *output set* of M defined by ω as the image set of ω, denoted $\mathrm{img}(\omega)$. Such wiring ω is called a total recursive wiring.

Example 3. *Consider F as defined in Example 1, and M as defined in Example 2. Let ω be the recursive wiring such that $\omega(\alpha) = c$. We observe that $\circlearrowleft_\omega M = F$, and that the output set of M defined by ω is* $\mathrm{img}(\omega) = \{c\}$.

2.5 Dynamics

We define the *dynamics* of an AN F as the digraph with Λ^S as its set of vertices. There exists an edge from x to y if and only if $F(x) = y$. The dynamics of BAN F as described in Example 1 is represented in Figure 3.

The *limit dynamics* of an AN F is the digraph defined as the subgraph of its dynamics, restricted to the configurations that are parts of cycles. Such limit dynamics are often called the *attractors* of F.

2.6 Output functions

For I a set of inputs, an output function is a function which depends on variables taken in I indexed with a positive integer. An output function is meant to be computed on a history of past inputs fed to a module. In that sense, the positive integer is called delay, and for any $\alpha \in I$, α_1 represents the last value fed to the input α in that history. Variable α_2 represents the value that was before that, and so on. The precedence rises with the index.

It has been shown that output functions can be used to predict the value of any node of an acyclic module from a long enough history of inputs [10].

Example 4. *Consider module M as developed in example 2. As M is acyclic, we can define for it the following output functions:*

$$\begin{aligned}O_a(J) &= \alpha_1\\ O_b(J) &= \alpha_2\\ O_c(J) &= \neg\alpha_2 \vee \alpha_3,\end{aligned}$$

where J is a long enough sequence of inputs. For example, let us consider $J = (0, 1, 1)$ which means that $\alpha_1 = 1, \alpha_2 = 1$ and $\alpha_3 = 0$. Then, $O_a(J) = 1$, $O_b(J) = 1$ and $O_c(J) = \neg 1 \vee 0 = 0$. These are the states of a, b and c respectively after any set of updates of M, from any configuration, that ends by affecting $J = (0, 1, 1)$ to the input α.

When considering some acyclic module M paired with some total recursive wiring ω, it has been shown in [10] that the attractors of $\circlearrowright_\omega M$ (modulo the names of the configurations) only depend on the output functions of the nodes in the output set $\mathrm{img}(\omega)$ (modulo the names of inputs in I, but not their delay).

Example 5. *Consider module M as developed in Example 2, and the total recursive wiring ω developed in Example 3. As $\mathrm{img}(\omega) = \{c\}$, the output function $O_c(J) = \neg\alpha_2 \vee \alpha_3$ alone implies the attractors of $\circlearrowright_\omega M$. That is, for any module M' and total recursive wiring ω' such that $\mathrm{img}(\omega') = \{s'\}$, if $O_s(J) = \neg\alpha'_2 \vee \alpha'_3$ for some $\alpha' \in I'$, then $\circlearrowright_\omega M$ and $\circlearrowright_{\omega'} M'$ have isomorphic attractors (as per application of [10]).*

An acyclic module can have more than one node in $\mathrm{img}(\omega)$, and as such its attractors can be characterised by more than one output functions. If such is the case, consider that it is always possible to consider a set of output functions as one global output function, at the cost of expanding the alphabet Λ. This shorthand will be useful later on.

Example 6. *Let $O_a(J) = \alpha_1 \vee \beta_2$ and $O_b(J) = \neg\alpha_2 \wedge \beta_1$ two output functions. Without loss of generality we consider $O(J)$ the global output function defined as*

$$O(J) = \begin{pmatrix} \alpha_1 \vee \beta_2 \\ \neg\alpha_2 \wedge \beta_1 \end{pmatrix}.$$

As such, the output functions of any acyclic module that compute the attractors of related automata network can be considered as one global output function on vectors. This will be useful later on, when using said global output function as the local function of a cellular automaton.

2.7 Strictly one-way CA

For Λ an alphabet and $r \in \mathbb{N}$, a *cellular automaton* (CA) is defined by a local rule $f : \Lambda^{\{-r, 1-r, \ldots, r\}} \to \Lambda$. The number r is the radius of the CA, and its global rule is defined as $F : \Lambda^{\mathbb{Z}} \to \Lambda^{\mathbb{Z}}$, with $F(x)_k = f(x_{k-r}, x_{k+1-r}, \dots, x_{k+r})$. A given CA is said to be *strictly one-way* if its local function is defined over $f : \Lambda^{\{1,2,\dots,r\}} \to \Lambda$, and its global rule as $F(x)_k = f(x_{k+1}, x_{k+2}, \dots, x_{k+r})$ instead.

Example 7. *Let $\Lambda = \{0, 1, 2, 3\}$. The local rule $f(x) = x_1 \oplus x_2$ has radius 2 and defines a strictly one-way cellular automaton.*

One-way cellular automata (whose local functions depend on indexes from 0 to r) are an object of throughout study [11, 5, 7]. Strictly one-way cellular automata are, as far as we know, a sub-class of one-way cellular automata that have not been studied yet. Being strictly one-way is a very restrictive property, as we can characterise the fixed points of such cellular automata thanks to automata networks. As a consequence of Theorem 1, a strictly one-way cellular automata has a finite set of fixed points which are all periodic.

3 Associating global output functions and local rules

Using the notion of global output function defined in Section 2.6, let us consider what happens when considering the global output function O of a module M with total recursive wiring ω as the local rule of a cellular automaton.

To make this transformation possible, we consider the alphabet Λ' of the resulting cellular automaton as the set $\Lambda' = \Lambda^I$. Then, the local rule f' is defined as O in which any variable of the form α_k is substituted for the term $(x_k)_\alpha$. That is, to find the evaluation of input α with delay k, we look for the value behind index α in the vector contained in the neighboring cell of distance k. The obtained CA is called the associated CA of the global output function.

Example 8. *Consider the global output function O_c developed in Example 5. Its associated CA is defined on the alphabet $\mathbb{B}$ and its local rule f is defined as*

$$f(x) = \neg x_2 \vee x_3.$$

Example 9. *Consider the global output function O developed in Example 6. Its associated CA is defined on the alphabet $\{0, 1\}^{\{\alpha,\beta\}}$ and its local rule f is defined as*

$$f(x) = \begin{pmatrix} (x_1)_\alpha \vee (x_2)_\beta \\ \neg(x_2)_\alpha \wedge (x_1)_\beta \end{pmatrix}.$$

As the delay of any variable in an output function is a positive integer, any cellular automaton constructed that way is necesserarily strictly one-way. Starting from the other side and considering any strictly one-way

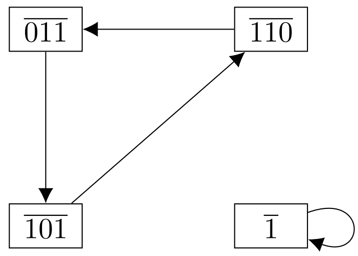

Figure 4: Fixed points shift dynamics of the CA developed in Example 8. The name $\overline{011}$ corresponds to the periodic bi-infinite configuration composed of the repetition of the word 011, starting at the index 0.

cellular automaton, it is straight-forward to construct a corresponding global output function of which it is the associated CA.

Example 10. *Let us consider the local rule f developed in Example 7, and the set of inputs $\{\alpha\}$. Taking the alphabet $\{0, 1, 2, 3\}$, let O be a global output function such that*

$$O(J) = \alpha_1 \oplus \alpha_2.$$

If we assume some implicit bijection between $\{0, 1, 2, 3\}$ and $\{0, 1\}^{\{\alpha,\beta\}}$, then taking set of inputs $\{\alpha, \beta\}$ and the alphabet $\{0, 1\}$, we can define O' a global output function such that

$$O'(J) = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \oplus \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}.$$

For both O and O', f defines an associated cellular automaton.

For x a bi-infinite configuration of a cellular automaton, we define $T(x)$ as the right-shift of x. That is, for any $i \in \mathbb{Z}$, $T(x)_i = x_{i+1}$.

For our final definition before the main result of this paper, let us use the fixed points of a strictly one-way cellular automaton to construct a graph. For F some cellular automaton, we define the *fixed points shift dynamics* as the graph which takes the fixed points of F as nodes and any (x, y) as an edge if and only if $y = T(x)$. As an example, the fixed point shift dynamics of the CA developed in Example 8 is represented in Figure 4.

With this representation in hand, we are ready for the main result.

Theorem 1. *Consider M an acyclic module with ω a total recursive wiring. Let F be an associated cellular automaton of M. The fixed points shift dynamics of F and the limit dynamics of $\circlearrowright_\omega M$ are isomorphic.*

As an application of this theorem, we can observe that the limit dynamics contained in the dynamics in Figure 3 and the fixed points shift dynamics represented in Figure 4 are isomorphic using the homorphism $\hat{h}$ such that $\hat{h}(110) = \overline{110}$, $\hat{h}(011) = \overline{011}$, $\hat{h}(101) = \overline{101}$ and $\hat{h}(111) = \overline{1}$.

This result proposes an interesting correspondence between cellular automata and automata networks, as any automata network can be turned into an associated CA and any CA is the associate of an automata network. As such, this result states that computing the attractors of an automata network under the parallel update schedule and computing the fixed points of a strictly one-way cellular automaton is the same exact computational problem.

As an example of the usefulness of this correspondence, we propose a slight generalisation of a known result about automata neworks [4]: if the length of all the cycles in the interaction digraph of an AN are divisible by some factor d greater than 1, then this AN is reducible into a smaller AN, from which its attractors can be derived. This theorem has an intuitive correspondence in strictly one-way cellular automata. Before stating the result, some extra formalism is needed.

Let us define that for $f : A^B \to C$ some function and $b \in B$, the variable of index b is said to influence f if and only if there exist $x, x' \in A^B$ such that $x|_{B\setminus\{b\}} = x'|_{B\setminus\{b\}}$, $x_b \neq x'_b$ and $f(x) \neq f(x')$.

For some positive integers r and d such that $d|r$, let $\mu_d : \Lambda^{\{1,\dots,\frac{r}{d}\}} \to \Lambda^{\{1,\dots,r\}}$ the function defined by $\mu_d(x)_k = x_{d\times k}$.

Let Σ be a set of bi-infinite words, and d a positive integer. The d-*interlacing* of Σ is the set of bi-infinite words Σ^d such that $w' \in \Sigma^d$ if and only if there exists some sequence of words $\{w^1, w^2, \dots, w^d\}$ in Σ such that for all $k \in \mathbb{Z}$, $w'_k = w^a_b$, for a and b the reminder and quotient of the division of k by d respectively. This word is also denoted $w' = w^1 \sim w^2 \sim \dots \sim w.^d$.

Theorem 2. *Let f be the local rule of a strictly one-way CA. If there exists some integer $d > 1$ that divides all integers in $\{k \mid x_k$ influences $f\}$, then the set of fixed points of the CA with local rule f is the d-interlacing of the set of fixed points of the CA with local rule $f \circ \mu_d$.*

Example 11. *Let us consider the Boolean local rule $f(x) = \neg x_2 \vee x_4$. As all the indexes of the variables that influence f are divisible by $d = 2$, the fixed points of the CA with local rule f are therefore the* 2-interlacing *of the fixed points of the CA with local rule $f \circ \mu_2$. This rule is defined as $f \circ \mu_2(x) = \neg x_1 \vee x_2$, and the fixed points of the related CA are $\overline{01}$ and $\overline{1}$, and their shifted equivalents. Let us name those fixed points a and b respectively. By application of the* 2-interlacing*, the fixed points of the CA with local rule f are $a \sim a = \overline{0011}$, $a \sim b = \overline{0111}$, $b \sim a = \overline{1011} \equiv \overline{0111}$ and $b \sim b = \overline{11} = \overline{1}$, for a total of 3 distinct fixed points up to shifting.*

By applying both theorems, and given a global output function O for which there exists some integer $d > 1$ that divides the delay of all the input variables that influence O, then the attractors of any automata network that realises O can be deduced from a copy of O in which all delays have been divided by d.

4 Final words

To us, the main interest of Theorem 1 is to show that the intricate task of describing the limit behavior of automata networks, which are diverse in both interaction graphs and local functions, can actually be done by describing the fixed points of cellular automata, with their uniform structure and local behavior. This certainly speaks more to the complexity of the latter than the simplicity of the former; however, we feel more confident studying this combinatorial problem in the shape of a cellular automaton, and hope that this shift in perspective yields interesting characterisations in the future.

References

[1] M.I. Davidich and S. Bornholdt. Boolean network model predicts cell cycle sequence of fission yeast. *PLoS One*, 3:e1672, 2008.

[2] J. Demongeot, E. Goles, M. Morvan, M. Noual, and S. Sené. Attraction basins as gauges of robustness against boundary conditions in biological complex systems. *PLoS One*, 5:e11793, 2010.

[3] J. Demongeot, M. Noual, and S. Sené. Combinatorics of Boolean automata circuits dynamics. *Discr. Appl. Math.*, 160:398–415, 2012.

[4] Z. Gao, X. Chen, and T. Başar. Stability structures of conjunctive boolean networks. *Automatica*, 89:8–20, 2018.

[5] J. Kari. Theory of cellular automata: A survey. *Theoretical computer science*, 334(1-3):3–33, 2005.

[6] S. A. Kauffman. Metabolic stability and epigenesis in randomly constructed genetic nets. *J. Theor. Biol.*, 22:437–467, 1969.

[7] M. Kutrib. Complexity of one-way cellular automata. In *International Workshop on Cellular Automata and Discrete Complex Systems*, pages 3–18. Springer, 2014.

[8] L. Mendoza and E. R. Alvarez-Buylla. Dynamics of the genetic regulatory network for *Arabidopsis thaliana* flower morphogenesis. *J. Theor. Biol.*, 193:307–319, 1998.

[9] K. Perrot, P. Perrotin, and S. Sené. A framework for (de)composing with Boolean automata networks. In *Proc. of MCU'18*, volume 10881 of *LNCS*, pages 121–136, 2018.

[10] K. Perrot, P. Perrotin, and S. Sené. On the complexity of acyclic modules in automata networks. In *Proceedings of TAMC'20*, 2020. Accepted, arXiv:1910.07299.

[11] P Sarkar. A brief history of cellular automata. *Acm computing surveys (csur)*, 32(1):80–107, 2000.

[12] R. Thomas. Boolean formalization of genetic control circuits. *J. Theor. Biol.*, 42:563–585, 1973.

A Proofs

First, let us state that we will consider strictly one-way cellular automata as defined on local rules of the form $f : \Lambda^{\{-1,\dots,-r\}} \to \Lambda$, without loss of generality. This inversion allows us to consider sequences both for cellular automata and automata network attractors without any further reversing. For any global output function O, the associated local rule f is defined as a copy of O in which any variable α_k is substitued for the term $(x_{-k})_\alpha$. This inversion also implies the use of the left-shift instead of the right-shift in the definition of the fixed points shift dynamics.

Let us now re-define some important notations. For F an automata network with nodes S and x a configuration, we write $F(x)$ the configuration obtained by updating x under the parallel update schedule, which verifies $F(x)_s = f_s(x)$, for f_s the local function of node s. We also write $F^k(x)$ the configuration obtained by chaining k updates of x.

For M a module, x a configuration and i an input configuration, we denote $M(x, i)$ the configuration such that $M(x, i)_s = f_s(x, i)$. For J a sequence of input configurations, we denote $M(x, J)$ the configuration obtained by chaining $|J|$ updates, starting with x, and taking subsequent input configurations in J. Formally, $M(x, (i_1, i_2, \dots, i_k)) = M(M(x, i_1), (i_2, \dots, i_k))$ and $M(x, (\ )) = x$.

For O_s the output function of some node s of some module M, and for J some long enough sequence of input configuration, we have by definition that $O_s(J) = M(x, J)|_s$ for any configuration x. It follows that for ω a total recursive wiring over M, O the resulting global output function of M and J a long enough input sequence,

$$\forall x, O(J) = M(x, J)|_{\mathrm{img}(\omega).} \tag{1}$$

Let us know verify that if any acyclic module has a global output function, that the symmetric is verified and that any global output function is realised by some acyclic module.

Lemma 1. *Let O be a global output function defined on the alphabet Λ. There exists some acyclic module M, such that O is the global output function of M.*

Proof. Without loss of generality, we can consider O as operating over a set of inputs that is a singleton $\{\alpha\}$. As remarked in Section 3, any global output function taking multiple inputs in a set I can be considered as an output function taking one input evaluated over the set Λ^I.

Let D be the maximum delay such that α_D influences the computation of O. We will construct the module M with input set $\{\alpha\}$ and taking states in the alphabet Λ, with nodes set

$$S = \{s_o\} \cup \{s_k | 0 < k < D\}$$

,

And local functions

$$\begin{aligned} f_{s_1}(x,i) &= i_\alpha \\ \forall 1 < k < D, f_{s_k}(x,i) &= x_{s_{k-1}} \\ f_{s_o} &= O \circ \nu, \end{aligned}$$

where $\nu : \Lambda^{S\cup I} \to \Lambda^{\{1,\ldots,D\}}$ with $\nu(x,i)_k = x_{s_k}$. By construction, M is acyclic and $O_{s_o} = O$.

$\square$

Let us define the function $\lambda : \Lambda^{\mathrm{img}(\omega)} \to \Lambda^I$ such that $\lambda(x|_{\mathrm{img}(\omega)})_\alpha = x_{\omega(\alpha)}$. This function allows us to read how inputs in M would be evaluated so to imitate an update in $\circlearrowleft_\omega M$. It verifies

$$\circlearrowleft_\omega M(x) = M(x, \lambda(x|_{\mathrm{img}(\omega)})).$$

Theorem 1. *Consider M an acyclic module with ω a total recursive wiring. Let F be an associated cellular automata of M. The fixed points shift dynamics of F and the limit dynamics of $\circlearrowleft_\omega M$ are isomorphic.*

Proof. Let x be a configuration part of the limit dynamics of $\circlearrowleft_\omega M$. As x is part of an attractor, let us consider the sequence of configuration $X = (x^1, x^2, \ldots, x^k)$ that starts with $x^1 = x$, such that $\circlearrowleft_\omega M(x^i) = x^{i+1}$ for all $i < k$, and $\circlearrowleft_\omega M(x^k) = x$.

For every x a configuration part of an attractor $X = (x^1, x^2, \ldots, x^k)$, we define the input sequence $J_x = (\lambda(x^1|_{\mathrm{img}(\omega)}), \lambda(x^2|_{\mathrm{img}(\omega)}), \ldots, \lambda(x^k|_{\mathrm{img}(\omega)}))$.

Claim 1. *For every configuration x part of an attractor, $\overline{J_x}$ is a fixed point of F.*

Let us denote the cell at index $z \in \mathbb{Z}$ as c_z. To prove this fact, we only need to prove that the value of c_k in configuration $\overline{J_x}$ is stable. Indeed, as every configuration in the attractor X generates a different shift of the CA configuration $\overline{J_x}$, proving the stability of the configuration as a whole (and thus making it a fixed point) can be shown by proving the stability of the same cell over each shift of the CA configuration. Let us prove the cell c_0 is stable.

To see that this claim is true, consider that $\overline{J_x}$ is the sequence generated by reading inputs over the attractor X, but can also generate it. By the definition of J_x, the cell c_0 is evaluated at $\lambda(x^1|_{\mathrm{img}(\omega)})$. And by the definition of the local rule of the CA, the next value of the cell c_0 is equal to $f(c_{-1}, c_{-2}, \ldots, c_{-r})$, which is equal to $\lambda(O(J_x^*)|_{\mathrm{img}(\omega)})$, where J_x^* is J_x repeated enough times so to exceed in length the maximum delay in O. To prove that the cell c_0 is stable, we need to show that $\lambda(x^1|_{\mathrm{img}(\omega)}) = \lambda(O(J_x^*))$, which is equivalent to show that $x^1|_{\mathrm{img}(\omega)} = O(J_x^*)$.

Let us consider m some large enough integer such that $O(J_x^m)$ is well defined. By Equation 1, and for y any configuration of M, $O(J_x^m) = M(x, J_x^m)|_{\mathrm{img}(\omega)}$. To evaluate this new term, let us consider that $J_{x,1} = \lambda(x^1|_{\mathrm{img}(\omega)})$. Thus, $M(x, J_{x,1}) = \circlearrowright_\omega M(x) = x^2$. By repeating this argument one can verify that $M(x, (J_{x,1}, \ldots, J_{x,k-1})) = x^k$, and that $M(x, J_x) = \circlearrowright_\omega M^k(x) = x$. It follows that $M(x, J_x^m)|_{\mathrm{img}(\omega)} = \circlearrowright_\omega M^{k\times m}(x)|_{\mathrm{img}(\omega)} = x|_{\mathrm{img}(\omega)}$, and thus $x^1|_{\mathrm{img}(\omega)} = O(J_x^*)$, which proves the CA configuration $\overline{J_x}$ is a fixed point. This concludes the proof of Claim 1.

Let us state and prove the symmetric of Claim 1. For J some sequence of inputs, we define $\circlearrowright_\omega M(J^*)$ the configuration obtained by using the output functions of M over the input configuration J^* (which is the repetition of J enough time so to overcome in length the maximum delay of any output function of M).

Claim 2. *Let $\overline{J}$ be a fixed point of F, M any acyclic module and ω a total recursive wiring such that F is the associated CA of M. Then $\circlearrowright_\omega M(J^*)$ is a configuration of $\circlearrowright_\omega M$ which is part of an attractor, and $\overline{J} = \overline{J_{\circlearrowright_\omega M(J^*)}}$.*

Without loss of generality, we consider that M has only one input α. Let O be the global output function defined by M and ω. Let us take m a large enough integer such that $O(J^m)$ is defined. As $\overline{J}$ is a fixed point, it follows that $\overline{J}_k = \lambda(O(\overline{J_{[k-r,\ldots,k-1]}}))$. In particular, for $n = m \times |J|$ and for all $1 \leq k \leq n$,

$$J_k^m = \lambda(O(J_{[k,\ldots,n]}^m \cdot J_{[1,\ldots,k-1]}^m)).$$

Let us define the function $\Theta : \Lambda^{\{1,\ldots,D\}} \to \Lambda^S$ with D the maximal delay between the output functions of M, such that for K a long enough input sequence, $\Theta(K)_s = O_s(K)$.

For m a long enough integer, we define X the sequence of configurations of length $|J|$, such that for $n = m \times |J|$, $X_k = \Theta(J_{[k,\ldots,n]}^m \cdot J_{[1,\ldots,k-1]}^m)$. This sequence is constructed explicitely so that for every k,

$$\lambda(X_k|_{\mathrm{img}(\omega)}) = J_k.$$

In other terms, the sequence X is a sequence of configurations of M, and reading the output of each element in sequence produces the sequence J.

Let us now show that X is an attractor of $\circlearrowright_\omega M$. That is, updating any element but the last gives us the next element, and updating the last element gives us the first.

Let assume $k < |X|$. We observe that

$$\circlearrowright_\omega M(X_k) = M(X_k, \lambda(X_k|_{\mathrm{img}(\omega)}))$$
$$= M(X_k, J_k)$$
$$= M(\Theta(J^m_{[k,\ldots,n]} \cdot J^m_{[1,\ldots,k-1]}), J_k).$$

By definition of Θ, for K long enough and for some input configuration i, we state that $M(\Theta(K), i) = \Theta(K \cdot (i))$. This is verified by the nature of output functions. As Θ decides the value of any node in S over long enough input sequences, adding a an input configuration i to K is the same as updating $\Theta(K)$ with input configuration i. Applied to our equation, we obtain that

$$M(\Theta(J^m_{[k,\ldots,n]} \cdot J^m_{[1,\ldots,k-1]}), J_k) = \Theta(J^m_{[k,\ldots,n]} \cdot J^m_{[1,\ldots,k-1]} \cdot J_k)$$
$$= \Theta(J^m_{[k,\ldots,n]} \cdot J^m_{[1,\ldots,k]})$$

The resulting sequence of inputs $J^m_{[k,\ldots,n]} \cdot J^m_{[1,\ldots,k]}$ is of length $|J^m| + 1$. As J^m was already long enough to define the value of Θ, removing the first element in sequence does not change the evaluation of the underlying output functions. We thus obtain that

$$\Theta(J^m_{[k,\ldots,n]} \cdot J^m_{[1,\ldots,k]}) = \Theta(J^m_{[k+1,\ldots,n]} \cdot J^m_{[1,\ldots,k]})$$
$$= X_{k+1}$$

Thus $\circlearrowright_\omega M(X_k) = X_{k+1}$ for $k < |X|$. Let us know evaluate $\circlearrowright_\omega M(X_{|X|})$:

$$\circlearrowright_\omega M(X_{|X|}) = M(\Theta(J^m_{[|X|,\ldots,n]} \cdot J^m_{[1,\ldots,|X|-1]}), J_{|X|})$$
$$= \Theta(J^m_{[|X|+1,\ldots,n]} \cdot J^m_{[1,\ldots,|X|]}).$$

Let us notice that $|X| = |J|$. We thus obtain that

$$\Theta(J^m_{[|X|+1,\ldots,n]} \cdot J^m_{[1,\ldots,|X|]}) = \Theta(J^m_{[|J|+1,\ldots,n]} \cdot J)$$
$$= \Theta(J^m_{[1,\ldots,n-|J|]} \cdot J)$$
$$= \Theta(J^m_{[1,\ldots,n]})$$
$$= X_1,$$

Which proves that $\circlearrowright_\omega M(X_{|X|}) = X_1$, and altogether X is an attractor of $\circlearrowright_\omega M$. Furthermore, the definition of Θ verifies that $\circlearrowright_\omega M(J^*) = \Theta(J^*) = X_1$. It follows naturally that the sequence $\overline{J_{X_1}}$ is equal to $\overline{J}$, which concludes the proof of Claim 2.

Wrapping up, we can define an homomorphism $\hat{h}$ which associates to any configuration x part of an attractor of $\circlearrowleft_\omega M$ the fixed point $\overline{J_x}$. Claim 1 verifies this fixed point always exists, and Claim 2 verifies $\hat{h}$ is well defined : any fixed point is of the form $\overline{J_y}$, for y some unique configuration part of an attractor of $\circlearrowleft_\omega M$. To prove that $\hat{h}$ is an homomorphism, let us consider that for any recurrent configuration x, the fixed point $\overline{J_{\circlearrowleft_\omega M(x)}}$ is equal to the left-shifting of $\overline{J_x}$.

□

Theorem 2. *Let f be the local rule of a strictly one-way CA. If there exists some integer $d > 1$ that divides all integers in $\{k|x_k$ influences $f\}$, then the set of fixed points of the CA with local rule f is the d-interlacing of the set of fixed points of the CA with local rule $f \circ \mu_d$.*

Proof. Let C_z^1 be the set of cells which contains c_z and the sells that influence the update of cell c_z. We define C_z^{n+1} as the union of C_z^n and the set of cell that influence any cell in C_z^n. As there exists some factor d that divides the distance between any cell and its influences, this property is also true of any cell in C_z^ω. The same argument follows for the cells influenced by the cell at z.

Taking cells of index $a + d \times b$ for some a and all b, we obtain a band of cells that are independant from every other. That is, their value does not have influence over the value over the rest of the configuration, and the rest of the configuration does not influence their value; as such, we can consider every band (one defined by each $a < d$) as a different cellular automata with local rule $f \circ \mu_d$.

From this, the result is obtained by observing that any fixed point in the CA with local rule f is obtained by composing the independant fixed points of the d bands which can be simulated by the CA with local rule $f \circ \mu_d$. To compose all possible fixed points in the right shape we use the notion of d-interlacing, which intuitively constructs a valid fixed point for the CA with local rule f. □