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Non-separation Method-Based Global Stability Criteria for Takagi–Sugeno Fuzzy Quaternion-Valued BAM Delayed Neural Networks Using Quaternion-valued Auxiliary Function-Based Integral Inequality

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Abstract
This paper focuses on the global asymptotic stability (GAS) problem for Takagi–Sugeno (T-S) fuzzy quaternion-valued bidirectional associative memory neural networks (QVBAMNNs) with discrete, distributed and leakage delays by using non-separation method. By applying T-S fuzzy model, we first consider a general form of T-S fuzzy QVBAMNNs with time delays. Then, by constructing appropriate Lyapunov–Krasovskii functionals and employing quaternion-valued integral inequalities and homeomorphism theory, several delay-dependent sufficient conditions are obtained to guarantee the existence and GAS of the considered neural networks (NNs). In addition, these theoretical results are presented in the form of quaternion-valued linear matrix inequalities (LMIs), which can be verified numerically using the effective YALMIP toolbox in MATLAB. Finally, two numerical illustrations are presented along with their simulations to demonstrate the validity of the theoretical analysis.

Keywords Quaternion-valued neural network · BAM neural network · Global asymptotic stability · Lyapunov–Krasovskii functional · Takagi–Sugeno fuzzy model

1 Introduction

In recent years, NNs have become increasingly popular among researchers due to their potential applications in a variety of domains such as secure communications, parallel computing, image processing, optimization, and others [1–6]. In 1987, Kosko devised a two-layered hetero associative memory network called bidirectional associative memory neural networks (BAMNNs). Compared with other NNs, BAMNNs have recently attracted increased attention due to their superior features in pattern recognition, automatic control, associate memory, and

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others [5–10]. Therefore, various theoretical studies have been published on the dynamics of BAMNNs using LKFs and LMIs [7–11].

Recently, real-valued neural networks (RVNNs) and complex-valued neural networks (CVNNs) have been successfully implemented in automatic control, parallel computing, signal processing, pattern recognition, optoelectronics, computer vision, remote sensing, and others [3–12]. However, RVNNs and CVNNs have some restrictions, especially when it comes to high-dimensional data such as color images, four dimensional signals and body images [11–13]. Meanwhile, quaternion numbers provide a natural and elegant solution for high-dimensional data representation. Therefore, quaternion-valued neural networks (QVNNs) have become a powerful modelling tool for quaternion-valued data processing and they have many advantages over usual RVNNs and CVNNs [14–20]. Recently, several researchers have studied QVNNs more closely with some interesting results [21–25]. For example, by employing homeomorphism theory and inequality technique, several sufficient criteria for global μ-stabilization of quaternion-valued inertial BAMNNs via impulsive control are obtained in [23]. Based on the nonlinear measure approach, a set of new stability conditions for quaternion-valued inertial BAMNNs has been established in [24]. By considering two different types of activation functions, LMI-based sufficient conditions are derived to ensure the global dissipativity of QVBAMNNs by plural decomposition method in [25]. There are some other results pertaining to QVNNs dynamics that can be found in [26–29].

As is known to all, mathematical representation of physical systems is quite challenging. It has been demonstrated that fuzzy logic theory is an effective algorithm for modeling complex nonlinear systems. In [30], T-S fuzzy model is represented by fuzzy IF-THEN rules that express local input–output relations of a nonlinear system, which has been successfully used in complex nonlinear systems [30, 31]. The overall fuzzy model of a system is obtained by fuzzy blending of the linear system models. Based on the good approximation property of T-S fuzzy systems, T-S fuzzy NNs models are regarded as an important means to estimate complex nonlinear systems. Recently, by incorporating fuzzy logic in NNs, there are several stability conditions for T-S fuzzy NNs have been presented by using LKFs and LMIs [33–36]. For illustration, by employing suitable LKFs and matrix inequality technique, the authors of [35] have determined the exponential convergence for T-S fuzzy CVNNs including impulsive effects and time delays. By decomposing the original Clifford-valued NNs into $2^m n$-dimensional RVNNs, the authors of [36] have derived the GAS of T-S fuzzy Clifford-valued NNs with time-varying delays and impulses.

On the other hand, time delay is inevitable in the process of signal propagation of NNs describing very large scale integration circuits because of the finite transmission speed of information. In addition, NNs have a special nature because they consist of many parallel pathways of different axon lengths and sizes, they can be modeled with distributed delays [37–41]. Similar to usual time delays, leakage delays also has a significant impact on the dynamics of NNs [42–44]. These time delays can result in undesirable system behaviors, such as oscillations, instability, and bifurcation, and others. Therefore, it is essential to study how delays affect the system's dynamics. Recently, several research papers regarding the dynamics of NNs involving different time delays have been published [45–49]. On the other hand, recent studies have used various integral inequalities to deal with integral terms in the real domain. However, only Jensen's inequality has been utilized since the beginning to deal with integral terms in the quaternion domain. To fill such gaps, a new quaternion-valued integral inequality has been developed in this paper, which includes the famous auxiliary function-based integral inequality (AFBII).

In recent years, several papers have been published on the stability of QVBAMNNs with time delays; however, the T-S fuzzy QVBAMNNs have not been fully explored and are not receiving much attention. As far as we know, there have been no papers published on the existence and GAS of T-S fuzzy QVBAMNNs with discrete, distributed, and leakage delays. Therefore, this study aims to fulfill this research gap by exploring the existence and GAS of T-S fuzzy QVBAMNNs with discrete, distributed, and leakage delays by using non-separation method. The following are the main merits of this paper: (1) To represent more realistic dynamical behaviors of QVNNs, we considered a general form of T-S fuzzy QVBAMNNs with discrete, distributed and leakage delays. (2) We proposed and proved a new quaternion-valued AFBII, which provides a novel analytical pattern and helps to address the mathematical challenges associated with system decomposition method. (3) By constructing appropriate LKFs and employing quaternion-valued AFBII and homeomorphism theory, several sufficient conditions have been derived in the form of simplified quaternion-valued LMIs to ensure the existence and GAS of the considered NNs.

This paper is structured as follows: Sect. 2 provides the problem model, definitions of GAS, assumptions about activation functions and time-varying delays, and some helpful lemmas. Section 3 presents the main results of the paper: Theorem (5) presents sufficient criteria for the existence and uniqueness of the equilibrium point; Theorem (6) provides sufficient criteria for the GAS of the considered NNs. In Corollary (7) and (8), the results of stability criteria are discussed in a special case. Section 4 gives numerical illustrations that demonstrates the validity of the results. Section 5 presents the conclusion of this paper.

2 Problem Formulation and Preliminaries

2.1 Notations

Let us denote the quaternion, complex, and real numbers by $\mathbf{H}$, $\mathbf{C}$ and $\mathbf{R}$, respectively. The n-dimensional quaternion, complex and real vectors are denoted by $\mathbf{H}^n$, $\mathbf{C}^n$ and $\mathbf{R}^n$, respectively. The quaternion, complex and real matrices of size $n \times n$ are denoted by $\mathbf{H}^{n\times n}$, $\mathbf{C}^{n\times n}$ and $\mathbf{R}^{n\times n}$, respectively. Let the matrix $\mathcal{P} < 0$ ($\mathcal{P} > 0$) means $\mathcal{P}$ is negative (positive) definite matrix. $\mathcal{P}^T$ and $\mathcal{P}^*$ denote the transpose and Hermitian transpose of matrix $\mathcal{P}$. $\mathcal{I}_n$ denotes the identity matrix of dimension n, and the block diagonal matrix is shown by $diag\{\cdot\}$, while the symmetric term in a matrix is showed by $\star$.

2.2 Quaternion Algebra

The quaternion $\mathbf{H}$ consists of a four-dimensional vector space over $\mathbf{R}$ with an ordered basis, represented by i, j and k. The real quaternion can be written as follows:

$$\mathbf{z} = \mathbf{z}^R + i\mathbf{z}^I + j\mathbf{z}^J + k\mathbf{z}^K \in \mathbf{H},$$

where $\mathbf{z}^R, \mathbf{z}^I, \mathbf{z}^J, \mathbf{z}^K \in \mathbf{R}$, and i, j, k are the quaternion basis which subjects to Hamilton's multiplication rules as follows:

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \; ki = -ik = j, \; ij = -ji = k.$$

Some basic definitions, rules of operation, and some essential aspects of quaternions and quaternion matrices are given as follows [14–18].

(1) The conjugate of the quaternion as follows:

$$\bar{\mathbf{z}} = \mathbf{z}^R - i\mathbf{z}^I - j\mathbf{z}^J - k\mathbf{z}^K \in \mathbf{H}.$$

(2) The modulus of the quaternion as follows:

$$|\mathbf{z}| = \sqrt{\mathbf{z}\bar{\mathbf{z}}} = \sqrt{(\mathbf{z}^R)^2 + (\mathbf{z}^I)^2 + (\mathbf{z}^J)^2 + (\mathbf{z}^K)^2}.$$

(3) Let $\mathbf{x} = \mathbf{x}^R + i\mathbf{x}^I + j\mathbf{x}^J + k\mathbf{x}^K \in \mathbf{H}$ and $\mathbf{y} = \mathbf{y}^R + i\mathbf{y}^I + j\mathbf{y}^J + k\mathbf{y}^K \in \mathbf{H}$. The addition and multiplication of two quaternions can be accomplished as follows:

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (\mathbf{x}^R + \mathbf{y}^R) + i(\mathbf{x}^I + \mathbf{y}^I) + j(\mathbf{x}^J + \mathbf{y}^J) + k(\mathbf{x}^K + \mathbf{y}^K),\\
\mathbf{x}\mathbf{y} &= \big(\mathbf{x}^R\mathbf{y}^R - \mathbf{x}^I\mathbf{y}^I - \mathbf{x}^J\mathbf{y}^J - \mathbf{x}^K\mathbf{y}^K\big)\\
&\quad + i\big(\mathbf{x}^R\mathbf{y}^I + \mathbf{x}^I\mathbf{y}^R + \mathbf{x}^J\mathbf{y}^K - \mathbf{x}^K\mathbf{y}^J\big)\\
&\quad + j\big(\mathbf{x}^R\mathbf{y}^J + \mathbf{x}^J\mathbf{y}^R - \mathbf{x}^I\mathbf{y}^K + \mathbf{x}^K\mathbf{y}^I\big)\\
&\quad + k\big(\mathbf{x}^R\mathbf{y}^K + \mathbf{x}^K\mathbf{y}^I + \mathbf{x}^I\mathbf{y}^J - \mathbf{x}^J\mathbf{y}^I\big).
\end{aligned}$$

(4) Following are some other properties of quaternions: Let $\alpha, \beta \in \mathbf{H}$; $\mathcal{M}, \mathcal{N} \in \mathbf{H}^{n\times n}$, then (i) $|\alpha + \beta| \le |\alpha| + |\beta|$ and $|\alpha\beta| = |\beta\alpha| = |\alpha||\beta|$, (ii) $(\overline{\mathcal{M}})^T = \overline{(\mathcal{M}^T)}$, (iii) $(\mathcal{M}\mathcal{N})^* = \mathcal{N}^*\mathcal{M}^*$, (iv) $(\mathcal{M}\mathcal{N})^{-1} = \mathcal{N}^{-1}\mathcal{M}^{-1}$ if $\mathcal{M}$ and $\mathcal{N}$ are invertible, (v) $(\mathcal{M}^*)^{-1} = \big(\mathcal{M}^{-1}\big)^*$ if $\mathcal{M}$ is invertible, (vi) For any quaternion α can be formulated uniquely as $\alpha = \gamma_1 + \gamma_2 j$, where $\gamma_1, \gamma_2 \in \mathbf{C}$, (vii) $j\gamma = \overline{\gamma} j$ or $j\gamma j^* = \overline{\gamma}$, $\forall \gamma \in \mathbf{C}$.

(5) For $\mathcal{M}, \mathcal{N} \in \mathbf{H}^{n\times n}$; $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2 \in \mathbf{C}^{n\times n}$ and $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 j, \mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2 j$. Then (i) $\mathcal{M}^* = \mathcal{M}_1^* - \mathcal{M}_2^T j$, (ii) $\mathcal{M}\mathcal{N} = (\mathcal{M}_1\mathcal{N}_1 - \mathcal{M}_2\overline{\mathcal{N}}_2) + (\mathcal{M}_1\mathcal{N}_2 + \mathcal{M}_2\overline{\mathcal{N}}_1)j$.

2.3 Problem Formulation

In this paper, we consider the following QVBAMNNs with discrete, distributed and leakage delays

$$\begin{cases}
\dot{\mathbf{p}}(t) = & -\mathcal{D}_1\mathbf{p}(t-\delta) + \mathcal{A}_1\mathbf{f}_1(\mathbf{q}(t)) + \mathcal{B}_1\mathbf{g}_1(\mathbf{q}(t-\ell(t))) + \mathcal{C}_1\displaystyle\int_{-\infty}^{t} K_1(t-s)\mathbf{h}_1(\mathbf{q}(s))ds + \mathcal{J}_1,\\
\mathbf{p}(t) = & \varphi_1(t),\ t \in [-\sigma, 0],\\
\dot{\mathbf{q}}(t) = & -\mathcal{D}_2\mathbf{q}(t-\delta) + \mathcal{A}_2\mathbf{f}_2(\mathbf{p}(t)) + \mathcal{B}_2\mathbf{g}_2(\mathbf{p}(t-\ell(t))) + \mathcal{C}_2\displaystyle\int_{-\infty}^{t} K_2(t-s)\mathbf{h}_2(\mathbf{p}(s))ds + \mathcal{J}_2,\\
\mathbf{q}(t) = & \varphi_2(t),\ t \in [-\sigma, 0],
\end{cases} \tag{1}$$

where $\mathbf{p}(t) = [\mathbf{p}_1(t), ..., \mathbf{p}_n(t)]^T \in \mathbf{H}^n$, $\mathbf{q}(t) = [\mathbf{q}_1(t), ..., \mathbf{q}_m(t)]^T \in \mathbf{H}^m$ are the state vectors; $\mathcal{D}_1 = diag\{d_{11}, ..., d_{1n}\} \in \mathbf{R}^{n\times n}$, $\mathcal{D}_2 = diag\{d_{21}, ..., d_{2m}\} \in \mathbf{R}^{m\times m}$ are the self-feedback connection weight matrices with each $d_{1r} > 0$, $r = 1, 2, ..., n$, $d_{2s} > 0$, $s = 1, 2, ..., m$. $\mathcal{A}_1 \in \mathbf{H}^{n\times m}, \mathcal{B}_1 \in \mathbf{H}^{n\times m}$, $\mathcal{C}_1 \in \mathbf{H}^{n\times m}$, $\mathcal{A}_2 \in \mathbf{H}^{m\times n}$, $\mathcal{B}_2 \in \mathbf{H}^{m\times n}$, $\mathcal{C}_2 \in \mathbf{H}^{m\times n}$ are the interconnection weight matrices; $\mathbf{f}_1(\mathbf{q}(\cdot)) = [\mathbf{f}_{11}(\mathbf{q}_1(\cdot)), ..., \mathbf{f}_{1m}(\mathbf{q}_m(\cdot))]^T \in \mathbf{H}^m$, $\mathbf{g}_1(\mathbf{q}(\cdot)) = [\mathbf{g}_{11}(\mathbf{q}_1(\cdot)), ..., \mathbf{g}_{1m}(\mathbf{q}_m(\cdot))]^T \in \mathbf{H}^m$, $\mathbf{h}_1(\mathbf{q}(\cdot)) = [\mathbf{h}_{11}(\mathbf{q}_1(\cdot)), ..., \mathbf{h}_{1m}(\mathbf{q}_m(\cdot))]^T \in \mathbf{H}^m$, $\mathbf{f}_2(\mathbf{p}(\cdot)) = [\mathbf{f}_{21}(\mathbf{p}_1(\cdot)), ..., \mathbf{f}_{2n}(\mathbf{p}_n(\cdot))]^T \in \mathbf{H}^n$, $\mathbf{g}_2(\mathbf{p}(\cdot)) = [\mathbf{g}_{21}(\mathbf{p}_1(\cdot)), ..., \mathbf{g}_{2n}(\mathbf{p}_n(\cdot))]^T \in \mathbf{H}^n$, $\mathbf{h}_2(\mathbf{p}(\cdot)) = [\mathbf{h}_{21}(\mathbf{p}_1(\cdot)), ..., \mathbf{h}_{2n}(\mathbf{p}_n(\cdot))]^T \in \mathbf{H}^n$ are the neuron activation functions; $\mathcal{J}_1 = [J_{11}, ..., J_{1n}]^T \in \mathbf{H}^n$, $\mathcal{J}_2 = [J_{21}, ..., J_{2m}]^T \in \mathbf{H}^m$ are the external input vectors; $K_1(\cdot), K_2(\cdot) : [0, +\infty) \to [0, +\infty)$ are the delay kernels; $0 < \delta$ and $\ell(t)$ are the leakage delay and discrete time-varying delays, respectively; $\varphi_1 \in \mathscr{C}([-\sigma, 0], \mathbf{H}^n)$, $\varphi_2 \in \mathscr{C}([-\sigma, 0], \mathbf{H}^m)$ are the initial conditions, where $\sigma = max\{\delta,\ \ell\}$.

Assumption: 1 [49] The discrete delay $\ell(t) : \mathbf{R} \to \mathbf{R}$ is differentiable and bounded which satisfies $0 \le \ell(t) \le \ell$, $\dot{\ell}(t) \le \mu$, where ℓ and μ are real numbers.

Assumption: 2 [49] The activation functions $\mathbf{f}_{1s}(\cdot)$, $\mathbf{g}_{1s}(\cdot)$, $\mathbf{h}_{1s}(\cdot)$, $\mathbf{f}_{2r}(\cdot)$, $\mathbf{g}_{2r}(\cdot)$, $\mathbf{h}_{2r}(\cdot)$ are satisfy the Lipschitz continuous; that is, there exist positive constants $l_s^{\mathbf{f}_1}$, $l_s^{\mathbf{g}_1}$, $l_s^{\mathbf{h}_1}$, $l_r^{\mathbf{f}_2}$, $l_r^{\mathbf{g}_2}$, $l_r^{\mathbf{h}_2} \in \mathbf{R}$, such that $r = 1, 2, ..., n$ and $s = 1, 2..., m$

$$\begin{aligned}
|\mathbf{f}_{1s}(x) - \mathbf{f}_{1s}(y)| &\le l_s^{\mathbf{f}_1}|x - y|,\\
|\mathbf{g}_{1s}(x) - \mathbf{g}_{1s}(y)| &\le l_s^{\mathbf{g}_1}|x - y|,\\
|\mathbf{h}_{1s}(x) - \mathbf{h}_{1s}(y)| &\le l_s^{\mathbf{h}_1}|x - y|,\\
|\mathbf{f}_{2r}(x) - \mathbf{f}_{2r}(y)| &\le l_r^{\mathbf{f}_2}|x - y|,\\
|\mathbf{g}_{2r}(x) - \mathbf{g}_{2s}(y)| &\le l_r^{\mathbf{g}_2}|x - y|,\\
|\mathbf{h}_{2r}(x) - \mathbf{h}_{2s}(y)| &\le l_r^{\mathbf{h}_2}|x - y|,
\end{aligned}$$

for any $x, y \in \mathbf{H}$. Furthermore, we define $\mathcal{L}_{\mathbf{f}_1} = diag\{l_1^{\mathbf{f}_1}, ..., l_m^{\mathbf{f}_1}\}$, $\mathcal{L}_{\mathbf{g}_1} = diag\{l_1^{\mathbf{g}_1}, ..., l_m^{\mathbf{g}_1}\}$, $\mathcal{L}_{\mathbf{h}_1} = diag\{l_1^{\mathbf{h}_1}, ..., l_m^{\mathbf{h}_1}\}$, $\mathcal{L}_{\mathbf{f}_2} = diag\{l_1^{\mathbf{f}_2}, ..., l_n^{\mathbf{f}_2}\}$, $\mathcal{L}_{\mathbf{g}_2} = diag\{l_1^{\mathbf{g}_2}, ..., l_n^{\mathbf{g}_2}\}$, $\mathcal{L}_{\mathbf{h}_2} = diag\{l_1^{\mathbf{h}_2}, ..., l_n^{\mathbf{h}_2}\}$.

Assumption: 3 [29] The delay kernels $K_1(\cdot)$ and $K_2(\cdot)$ are some real value non-negative continuous functions defined in $[0, +\infty)$ and satisfy $\int_0^{+\infty} K_1(s)ds = 1$, $\int_0^{+\infty} K_2(s)ds = 1$.

2.4 Preliminaries

To achieve the main results, the following definitions and lemmas are used.

Definition 1 The NNs (1) with initial conditions $\varphi_1 \in \mathscr{C}([-\sigma, 0], \mathbf{H}^n)$ and $\varphi_2 \in \mathscr{C}([-\sigma, 0], \mathbf{H}^m)$, the trivial solution is called GAS if $\lim_{t\to+\infty}\left\{\|\mathbf{p}(t, \varphi_1)\|^2 + \|\mathbf{q}(t, \varphi_2)\|^2\right\} = 0$, where $\mathbf{p}(t, \varphi_1)$ and $\mathbf{q}(t, \varphi_2)$ are the solutions of NNs (1) at time t under the initial conditions φ_1 and φ_2, respectively.

Lemma 1 *[48] For any vectors $p, q \in \mathbf{H}^n$ and a scalar $\epsilon > 0$, then the following inequality holds: $p^*q + q^*p \le \epsilon p^*p + \epsilon^{-1}q^*q$.*

Lemma 2 *[48] Let $\mathcal{H}(\mathbf{p}, \mathbf{q}) : \mathbf{H}^{n+m} \to \mathbf{H}^{n+m}$ is a continuous map that satisfies the following criteria:*
(i) $\mathcal{H}(\mathbf{p}, \mathbf{q})$ is injective on $\mathbf{H}^{n+m}$,
(ii) $\|\mathcal{H}(\mathbf{p}, \mathbf{q})\| \to \infty$ as $\|(\mathbf{p}, \mathbf{q})\| \to \infty$, then $\mathcal{H}$ is homeomorphism of $\mathbf{H}^{n+m}$ onto itself.

Lemma 3 *[49] A Hermitian matrix $\Pi = \Pi^R + i\Pi^I + j\Pi^J + k\Pi^K \in \mathbf{H}^{n\times n}$, then $\Pi < 0$ is equivalent to*

$$\begin{bmatrix}
\Pi^R & -\Pi^J & -\Pi^I & \Pi^K\\
\Pi^J & \Pi^R & \Pi^K & \Pi^I\\
\Pi^I & -\Pi^K & \Pi^R & -\Pi^J\\
-\Pi^K & -\Pi^I & \Pi^J & \Pi^R
\end{bmatrix} < 0,$$

where $\Pi^R = Re(\Pi)$, $\Pi^I = Im(\Pi)$, $\Pi^J = Im(\Pi)$ and $\Pi^K = Im(\Pi)$.

Lemma 4 (Quaternion-valued AFBII) *For every differentiable function $w : [a, b] \to \mathbf{H}^n$ and Hermitian matrix $0 < \Pi = \Pi^R + i\Pi^I + j\Pi^J + k\Pi^K \in \mathbf{H}^{n\times n}$, the following inequality holds:*

$$\int_a^b w^*(s)\Pi w(s)ds \geq \frac{1}{b-a}\begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{bmatrix}^*\begin{bmatrix}\Pi & 0 & 0\\ 0 & 3\Pi & 0\\ 0 & 0 & 5\Pi\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{bmatrix},$$

where

$$\begin{aligned}\zeta_1 &= \int_a^b w(s)ds,\\ \zeta_2 &= \int_a^b w(s)ds - \frac{2}{b-a}\int_a^b\int_u^b w(s)dsdu,\\ \zeta_3 &= \int_a^b w(s)ds - \frac{6}{b-a}\int_a^b\int_u^b w(s)dsdu + \frac{12}{(b-a)^2}\int_a^b\int_v^b\int_u^b w(s)dsdudv.\end{aligned}$$

Proof: Let $w(s) = w^R(s)+iw^I(s)+jw^J(s)+kw^K(s) \in \mathbf{H}$, $\zeta_o = \zeta_o^R+i\zeta_o^I+j\zeta_o^J+k\zeta_o^K \in \mathbf{H}$, $o = 1, 2, 3$, $\Pi = \Pi^R + i\Pi^I + j\Pi^J + k\Pi^K \in \mathbf{H}^{n\times n}$, where $\Pi^* = \Pi \Leftrightarrow (\Pi^R)^T = \Pi^R$, $-(\Pi^I)^T = \Pi^I$, $-(\Pi^J)^T = \Pi^J$, $(\Pi^K)^T = -\Pi^K$. Using, AFBII [50], we get

$$\int_a^b w^*(s)\Pi w(s)ds = \int_a^b \begin{bmatrix}w^R(s)\\ w^I(s)\\ w^J(s)\\ w^K(s)\end{bmatrix}^T\begin{bmatrix}\Pi^R & -\Pi^J & -\Pi^I & \Pi^K\\ \Pi^J & \Pi^R & \Pi^K & \Pi^I\\ \Pi^I & -\Pi^K & \Pi^R & -\Pi^J\\ -\Pi^K & -\Pi^I & \Pi^J & \Pi^R\end{bmatrix}\begin{bmatrix}w^R(s)\\ w^I(s)\\ w^J(s)\\ w^K(s)\end{bmatrix}ds,$$

$$\geq \frac{1}{b-a}\begin{bmatrix}\zeta_1^R\\ \zeta_1^I\\ \zeta_1^J\\ \zeta_1^K\\ \zeta_2^R\\ \zeta_2^I\\ \zeta_2^J\\ \zeta_2^K\\ \zeta_3^R\\ \zeta_3^I\\ \zeta_3^J\\ \zeta_3^K\end{bmatrix}^T\begin{bmatrix}\Pi^R & -\Pi^J & -\Pi^I & \Pi^K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \Pi^J & \Pi^R & \Pi^K & \Pi^I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \Pi^I & -\Pi^K & \Pi^R & -\Pi^J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -\Pi^K & -\Pi^I & \Pi^J & \Pi^R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 3\Pi^R & -3\Pi^J & -3\Pi^I & 3\Pi^K & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 3\Pi^J & 3\Pi^R & 3\Pi^K & 3\Pi^I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 3\Pi^I & -3\Pi^K & 3\Pi^R & -3\Pi^J & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -3\Pi^K & -3\Pi^I & 3\Pi^J & 3\Pi^R & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\Pi^R & -5\Pi^J & -5\Pi^I & 5\Pi^K\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\Pi^J & 5\Pi^R & 5\Pi^K & 5\Pi^I\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\Pi^I & -5\Pi^K & 5\Pi^R & -5\Pi^J\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\Pi^K & -5\Pi^I & 5\Pi^J & 5\Pi^R\end{bmatrix}\begin{bmatrix}\zeta_1^R\\ \zeta_1^I\\ \zeta_1^J\\ \zeta_1^K\\ \zeta_2^R\\ \zeta_2^I\\ \zeta_2^J\\ \zeta_2^K\\ \zeta_3^R\\ \zeta_3^I\\ \zeta_3^J\\ \zeta_3^K\end{bmatrix},$$

$$= \frac{1}{b-a}\begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{bmatrix}^*\begin{bmatrix}\Pi & 0 & 0\\ 0 & 3\Pi & 0\\ 0 & 0 & 5\Pi\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{bmatrix}.$$

3 Main Results

In this section, we provide new sufficient criteria for ensuring GAS of the considered NNs (1) using quaternion-valued AFBII with LKFs and LMI methods.

3.1 Existence and Uniqueness of the Equilibrium Point

Firstly, we obtain sufficient criteria that guarantee the existence and uniqueness of the equilibrium point for NNs (1) under Assumption 2.

Theorem 5 *If Assumption 1-3 are fulfilled, the NNs (1) has a unique equilibrium point if there exist Hermitian matrices $0 < \mathcal{P}_1$, $0 < \mathcal{P}_2$, and diagonal matrices $0 < \mathcal{O}_1$, $0 < \mathcal{O}_2$, $0 < \mathcal{O}_3$, $0 < \mathcal{O}_4$, $0 < \mathcal{O}_5$, $0 < \mathcal{O}_6$ such that the following LMIs hold:*

$$\begin{bmatrix} \Theta_1 & \mathcal{P}_1\mathcal{A}_1 & \mathcal{P}_1\mathcal{B}_1 & \mathcal{P}_1\mathcal{C}_1 \\ \star & -\mathcal{O}_1 & 0 & 0 \\ \star & \star & -\mathcal{O}_3 & 0 \\ \star & \star & \star & -\mathcal{O}_5 \end{bmatrix} < 0, \tag{2}$$

$$\begin{bmatrix} \Theta_2 & \mathcal{P}_2\mathcal{A}_2 & \mathcal{P}_2\mathcal{B}_2 & \mathcal{P}_2\mathcal{C}_2 \\ \star & -\mathcal{O}_2 & 0 & 0 \\ \star & \star & -\mathcal{O}_4 & 0 \\ \star & \star & \star & -\mathcal{O}_6 \end{bmatrix} < 0, \tag{3}$$

where $\Theta_1 = -\mathcal{P}_1\mathcal{D}_1 - \mathcal{D}_1\mathcal{P}_1 + \mathcal{L}^*_{\mathbf{f}_2}\mathcal{O}_2\mathcal{L}_{\mathbf{f}_2} + \mathcal{L}^*_{\mathbf{g}_2}\mathcal{O}_4\mathcal{L}_{\mathbf{g}_2} + \mathcal{L}^*_{\mathbf{h}_2}\mathcal{O}_6\mathcal{L}_{\mathbf{h}_2}$ *and* $\Theta_2 = -\mathcal{P}_2\mathcal{D}_2 - \mathcal{D}_2\mathcal{P}_2 + \mathcal{L}^*_{\mathbf{f}_1}\mathcal{O}_1\mathcal{L}_{\mathbf{f}_1} + \mathcal{L}^*_{\mathbf{g}_1}\mathcal{O}_3\mathcal{L}_{\mathbf{g}_1} + \mathcal{L}^*_{\mathbf{h}_1}\mathcal{O}_5\mathcal{L}_{\mathbf{h}_1}$.

Proof Define the function $\mathcal{H}(\mathbf{p}, \mathbf{q}) : \mathbf{H}^{n+m} \to \mathbf{H}^{n+m}$ by

$$\begin{aligned} \mathcal{H}(\mathbf{p}, \mathbf{q}) = & -\begin{bmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{bmatrix}\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} + \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix}\begin{bmatrix} \mathbf{f}_1(\mathbf{q}) \\ \mathbf{f}_2(\mathbf{p}) \end{bmatrix} + \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{bmatrix}\begin{bmatrix} \mathbf{g}_1(\mathbf{q}) \\ \mathbf{g}_2(\mathbf{p}) \end{bmatrix} \\ & + \begin{bmatrix} \mathcal{C}_1 & 0 \\ 0 & \mathcal{C}_2 \end{bmatrix}\begin{bmatrix} \mathbf{h}_1(\mathbf{q}) \\ \mathbf{h}_2(\mathbf{p}) \end{bmatrix} + \begin{bmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \end{bmatrix}. \end{aligned} \tag{4}$$

We start by proving that $\mathcal{H}(\mathbf{p}, \mathbf{q})$ is injective. Assume by contradiction that there exist $\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}$, $\begin{bmatrix} \mathbf{p}' \\ \mathbf{q}' \end{bmatrix} \in \mathbf{H}^{n+m}$, $\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} \neq \begin{bmatrix} \mathbf{p}' \\ \mathbf{q}' \end{bmatrix}$, such that $\mathcal{H}(\mathbf{p}, \mathbf{q}) = \mathcal{H}(\mathbf{p}', \mathbf{q}')$, or equivalently

$$\begin{aligned} \mathcal{H}(\mathbf{p}, \mathbf{q}) - \mathcal{H}(\mathbf{p}', \mathbf{q}') = & -\begin{bmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{bmatrix}\begin{bmatrix} \mathbf{p} - \mathbf{p}' \\ \mathbf{q} - \mathbf{q}' \end{bmatrix} + \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix}\begin{bmatrix} \mathbf{f}_1(\mathbf{q}) - \mathbf{f}_1(\mathbf{q}') \\ \mathbf{f}_2(\mathbf{p}) - \mathbf{f}_2(\mathbf{p}') \end{bmatrix} \\ & + \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{bmatrix}\begin{bmatrix} \mathbf{g}_1(\mathbf{q}) - \mathbf{g}_1(\mathbf{q}') \\ \mathbf{g}_2(\mathbf{p}) - \mathbf{g}_2(\mathbf{p}') \end{bmatrix} + \begin{bmatrix} \mathcal{C}_1 & 0 \\ 0 & \mathcal{C}_2 \end{bmatrix}\begin{bmatrix} \mathbf{h}_1(\mathbf{q}) - \mathbf{h}_1(\mathbf{q}') \\ \mathbf{h}_2(\mathbf{p}) - \mathbf{h}_2(\mathbf{p}') \end{bmatrix} = 0. \end{aligned} \tag{5}$$

Pre multiplication on both sides of (5) with $\begin{bmatrix} \mathbf{p} - \mathbf{p}' \\ \mathbf{q} - \mathbf{q}' \end{bmatrix}^* \begin{bmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{bmatrix}$, we get

$$\begin{aligned} & \begin{bmatrix} \mathbf{p} - \mathbf{p}' \\ \mathbf{q} - \mathbf{q}' \end{bmatrix}^* \begin{bmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{bmatrix}\Bigg(-\begin{bmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{bmatrix}\begin{bmatrix} \mathbf{p} - \mathbf{p}' \\ \mathbf{q} - \mathbf{q}' \end{bmatrix} + \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix}\begin{bmatrix} \mathbf{f}_1(\mathbf{q}) - \mathbf{f}_1(\mathbf{q}') \\ \mathbf{f}_2(\mathbf{p}) - \mathbf{f}_2(\mathbf{p}') \end{bmatrix} \\ & + \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{bmatrix}\begin{bmatrix} \mathbf{g}_1(\mathbf{q}) - \mathbf{g}_1(\mathbf{q}') \\ \mathbf{g}_2(\mathbf{p}) - \mathbf{g}_2(\mathbf{p}') \end{bmatrix} + \begin{bmatrix} \mathcal{C}_1 & 0 \\ 0 & \mathcal{C}_2 \end{bmatrix}\begin{bmatrix} \mathbf{h}_1(\mathbf{q}) - \mathbf{h}_1(\mathbf{q}') \\ \mathbf{h}_2(\mathbf{p}) - \mathbf{h}_2(\mathbf{p}') \end{bmatrix}\Bigg) = 0, \end{aligned} \tag{6}$$

that is

$$\begin{aligned} & -(\mathbf{p} - \mathbf{p}')^*\mathcal{P}_1\mathcal{D}_1(\mathbf{p} - \mathbf{p}') - (\mathbf{q} - \mathbf{q}')^*\mathcal{P}_2\mathcal{D}_2(\mathbf{q} - \mathbf{q}') + (\mathbf{p} - \mathbf{p}')^*\mathcal{P}_1\mathcal{A}_1(\mathbf{f}_1(\mathbf{q}) - \mathbf{f}_1(\mathbf{q}')) \\ & + (\mathbf{q} - \mathbf{q}')^*\mathcal{P}_2\mathcal{A}_2(\mathbf{f}_2(\mathbf{p}) - \mathbf{f}_2(\mathbf{p}')) + (\mathbf{p} - \mathbf{p}')^*\mathcal{P}_1\mathcal{B}_1(\mathbf{g}_1(\mathbf{q}) - \mathbf{g}_1(\mathbf{q}')) + (\mathbf{q} - \mathbf{q}')^*\mathcal{P}_2\mathcal{B}_2 \\ & \times (\mathbf{g}_2(\mathbf{p}) - \mathbf{g}_2(\mathbf{p}')) + (\mathbf{p} - \mathbf{p}')^*\mathcal{P}_1\mathcal{C}_1(\mathbf{h}_1(\mathbf{q}) - \mathbf{h}_1(\mathbf{q}')) + (\mathbf{q} - \mathbf{q}')^*\mathcal{P}_2\mathcal{C}_2 \\ & \times (\mathbf{h}_2(\mathbf{p}) - \mathbf{h}_2(\mathbf{p}')) = 0. \end{aligned} \tag{7}$$

Applying the complex conjugate, we get

$$\begin{aligned}&-(\mathbf{p}-\mathbf{p}')^*\mathcal{D}_1\mathcal{P}_1(\mathbf{p}-\mathbf{p}')-(\mathbf{q}-\mathbf{q}')^*\mathcal{D}_2\mathcal{P}_2(\mathbf{q}-\mathbf{q}')+(\mathbf{f}_1(\mathbf{q})-\mathbf{f}_1(\mathbf{q}'))^*\mathcal{A}_1^*\mathcal{P}_1(\mathbf{p}-\mathbf{p}')\\&+(\mathbf{f}_2(\mathbf{p})-\mathbf{f}_2(\mathbf{p}'))^*\mathcal{A}_2^*\mathcal{P}_2(\mathbf{q}-\mathbf{q}')+(\mathbf{g}_1(\mathbf{q})-\mathbf{g}_1(\mathbf{q}'))^*\mathcal{B}_1^*\mathcal{P}_1(\mathbf{p}-\mathbf{p}')+(\mathbf{g}_2(\mathbf{p})-\mathbf{g}_2(\mathbf{p}'))^*\\&\times\mathcal{B}_2^*\mathcal{P}_2(\mathbf{q}-\mathbf{q}')+(\mathbf{h}_1(\mathbf{q})-\mathbf{h}_1(\mathbf{q}'))^*\mathcal{C}_1^*\mathcal{P}_1(\mathbf{p}-\mathbf{p}')+(\mathbf{h}_2(\mathbf{p})-\mathbf{h}_2(\mathbf{p}'))^*\mathcal{C}_2^*\mathcal{P}_2\\&\times(\mathbf{q}-\mathbf{q}')=0.\end{aligned}\tag{8}$$

By combining (7) and (8), we get

$$\begin{aligned}0=&(\mathbf{p}-\mathbf{p}')^*(-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1)(\mathbf{p}-\mathbf{p}')+(\mathbf{q}-\mathbf{q}')^*(-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2)(\mathbf{q}-\mathbf{q}')\\&+2(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{A}_1(\mathbf{f}_1(\mathbf{q})-\mathbf{f}_1(\mathbf{q}'))+2(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{A}_2(\mathbf{f}_2(\mathbf{p})-\mathbf{f}_2(\mathbf{p}'))\\&+2(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{B}_1(\mathbf{g}_1(\mathbf{q})-\mathbf{g}_1(\mathbf{q}'))+2(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{B}_2(\mathbf{g}_2(\mathbf{p})-\mathbf{g}_2(\mathbf{p}'))\\&+2(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{C}_1(\mathbf{h}_1(\mathbf{q})-\mathbf{h}_1(\mathbf{q}'))+2(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{C}_2(\mathbf{h}_2(\mathbf{p})-\mathbf{h}_2(\mathbf{p}')).\end{aligned}\tag{9}$$

By Lemma (1) and Assumption 2, there exist diagonal matrices $0<\mathcal{O}_1$, $0<\mathcal{O}_2$, $0<\mathcal{O}_3$, $0<\mathcal{O}_4$, $0<\mathcal{O}_5$, $0<\mathcal{O}_6$, yields

$$\begin{aligned}2(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{A}_1(\mathbf{f}_1(\mathbf{q})-\mathbf{f}_1(\mathbf{q}'))\le&(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{A}_1\mathcal{O}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1(\mathbf{p}-\mathbf{p}')\\&+(\mathbf{q}-\mathbf{q}')^*\mathcal{L}_{\mathbf{f}_1}^*\mathcal{O}_1\mathcal{L}_{\mathbf{f}_1}(\mathbf{q}-\mathbf{q}'),\end{aligned}\tag{10}$$

$$\begin{aligned}2(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{A}_2(\mathbf{f}_2(\mathbf{p})-\mathbf{f}_2(\mathbf{p}'))\le&(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{A}_2\mathcal{O}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2(\mathbf{q}-\mathbf{q}')\\&+(\mathbf{p}-\mathbf{p}')^*\mathcal{L}_{\mathbf{f}_2}^*\mathcal{O}_2\mathcal{L}_{\mathbf{f}_2}(\mathbf{p}-\mathbf{p}'),\end{aligned}\tag{11}$$

$$\begin{aligned}2(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{B}_1(\mathbf{g}_1(\mathbf{q})-\mathbf{g}_1(\mathbf{q}'))\le&(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{B}_1\mathcal{O}_3^{-1}\mathcal{B}_1^*\mathcal{P}_1(\mathbf{p}-\mathbf{p}')\\&+(\mathbf{q}-\mathbf{q}')^*\mathcal{L}_{\mathbf{g}_1}^*\mathcal{O}_3\mathcal{L}_{\mathbf{g}_1}(\mathbf{q}-\mathbf{q}'),\end{aligned}\tag{12}$$

$$\begin{aligned}2(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{B}_2(\mathbf{g}_2(\mathbf{p})-\mathbf{g}_2(\mathbf{p}'))\le&(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{B}_2\mathcal{O}_4^{-1}\mathcal{B}_2^*\mathcal{P}_2(\mathbf{q}-\mathbf{q}')\\&+(\mathbf{p}-\mathbf{p}')^*\mathcal{L}_{\mathbf{g}_2}^*\mathcal{O}_4\mathcal{L}_{\mathbf{g}_2}(\mathbf{p}-\mathbf{p}'),\end{aligned}\tag{13}$$

$$\begin{aligned}2(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{C}_1(\mathbf{h}_1(\mathbf{q})-\mathbf{h}_1(\mathbf{q}'))\le&(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{C}_1\mathcal{O}_5^{-1}\mathcal{C}_1^*\mathcal{P}_1(\mathbf{p}-\mathbf{p}')\\&+(\mathbf{q}-\mathbf{q}')^*\mathcal{L}_{\mathbf{h}_1}^*\mathcal{O}_5\mathcal{L}_{\mathbf{h}_1}(\mathbf{q}-\mathbf{q}'),\end{aligned}\tag{14}$$

$$\begin{aligned}2(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{C}_2(\mathbf{h}_2(\mathbf{p})-\mathbf{h}_2(\mathbf{p}'))\le&(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{C}_2\mathcal{O}_6^{-1}\mathcal{C}_2^*\mathcal{P}_2(\mathbf{q}-\mathbf{q}')\\&+(\mathbf{p}-\mathbf{p}')^*\mathcal{L}_{\mathbf{h}_2}^*\mathcal{O}_6\mathcal{L}_{\mathbf{h}_2}(\mathbf{p}-\mathbf{p}').\end{aligned}\tag{15}$$

Substituting (10)–(15) in (9), the right side of (9) can be bounded as

$$\begin{aligned}&(\mathbf{p}-\mathbf{p}')^*(-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1)(\mathbf{p}-\mathbf{p}')+(\mathbf{q}-\mathbf{q}')^*(-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2)(\mathbf{q}-\mathbf{q}')\\&\quad+2(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{A}_1(\mathbf{f}_1(\mathbf{q})-\mathbf{f}_1(\mathbf{q}'))+2(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{A}_2(\mathbf{f}_2(\mathbf{p})-\mathbf{f}_2(\mathbf{p}'))\\&\quad+2(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{B}_1(\mathbf{g}_1(\mathbf{q})-\mathbf{g}_1(\mathbf{q}'))+2(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{B}_2(\mathbf{g}_2(\mathbf{p})-\mathbf{g}_2(\mathbf{p}'))\\&\quad+2(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{C}_1(\mathbf{h}_1(\mathbf{q})-\mathbf{h}_1(\mathbf{q}'))+2(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{C}_2(\mathbf{h}_2(\mathbf{p})-\mathbf{h}_2(\mathbf{p}'))\\&\le(\mathbf{p}-\mathbf{p}')^*(-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1)(\mathbf{p}-\mathbf{p}')+(\mathbf{q}-\mathbf{q}')^*(-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2)(\mathbf{q}-\mathbf{q}')\\&\quad+(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{A}_1\mathcal{O}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1(\mathbf{p}-\mathbf{p}')+(\mathbf{q}-\mathbf{q}')^*\mathcal{L}_{\mathbf{f}_1}^*\mathcal{O}_1\mathcal{L}_{\mathbf{f}_1}(\mathbf{q}-\mathbf{q}')\\&\quad+(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{A}_2\mathcal{O}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2(\mathbf{q}-\mathbf{q}')+(\mathbf{p}-\mathbf{p}')^*\mathcal{L}_{\mathbf{f}_2}^*\mathcal{O}_2\mathcal{L}_{\mathbf{f}_2}(\mathbf{p}-\mathbf{p}')\\&\quad+(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{B}_1\mathcal{O}_3^{-1}\mathcal{B}_1^*\mathcal{P}_1(\mathbf{p}-\mathbf{p}')+(\mathbf{q}-\mathbf{q}')^*\mathcal{L}_{\mathbf{g}_1}^*\mathcal{O}_3\mathcal{L}_{\mathbf{g}_1}(\mathbf{q}-\mathbf{q}')\\&\quad+(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{B}_2\mathcal{O}_4^{-1}\mathcal{B}_2^*\mathcal{P}_2(\mathbf{q}-\mathbf{q}')+(\mathbf{p}-\mathbf{p}')^*\mathcal{L}_{\mathbf{g}_2}^*\mathcal{O}_4\mathcal{L}_{\mathbf{g}_2}(\mathbf{p}-\mathbf{p}')\\&\quad+(\mathbf{p}-\mathbf{p}')^*\mathcal{P}_1\mathcal{C}_1\mathcal{O}_5^{-1}\mathcal{C}_1^*\mathcal{P}_1(\mathbf{p}-\mathbf{p}')+(\mathbf{q}-\mathbf{q}')^*\mathcal{L}_{\mathbf{h}_1}^*\mathcal{O}_5\mathcal{L}_{\mathbf{h}_1}(\mathbf{q}-\mathbf{q}')\\&\quad+(\mathbf{q}-\mathbf{q}')^*\mathcal{P}_2\mathcal{C}_2\mathcal{O}_6^{-1}\mathcal{C}_2^*\mathcal{P}_2(\mathbf{q}-\mathbf{q}')+(\mathbf{p}-\mathbf{p}')^*\mathcal{L}_{\mathbf{h}_2}^*\mathcal{O}_6\mathcal{L}_{\mathbf{h}_2}(\mathbf{p}-\mathbf{p}')\end{aligned}$$

$$
\begin{aligned}
&\leq (\mathbf{p}-\mathbf{p}')^*(-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1+\mathcal{P}_1\mathcal{A}_1\mathcal{O}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{f}_2}^*\mathcal{O}_2\mathcal{L}_{\mathbf{f}_2}+\mathcal{P}_1\mathcal{B}_1\mathcal{O}_3^{-1}\mathcal{B}_1^*\mathcal{P}_1\\
&\quad+\mathcal{L}_{\mathbf{g}_2}^*\mathcal{O}_4\mathcal{L}_{\mathbf{g}_2}+\mathcal{P}_1\mathcal{C}_1\mathcal{O}_5^{-1}\mathcal{C}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{h}_2}^*\mathcal{O}_6\mathcal{L}_{\mathbf{h}_2})(\mathbf{p}-\mathbf{p}')\\
&\quad+(\mathbf{q}-\mathbf{q}')^*(-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2+\mathcal{P}_2\mathcal{A}_2\mathcal{O}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{f}_1}^*\mathcal{O}_1\mathcal{L}_{\mathbf{f}_1}+\mathcal{P}_2\mathcal{B}_2\mathcal{O}_4^{-1}\mathcal{B}_2^*\mathcal{P}_2\\
&\quad+\mathcal{L}_{\mathbf{g}_1}^*\mathcal{O}_3\mathcal{L}_{\mathbf{g}_1}+\mathcal{P}_2\mathcal{C}_2\mathcal{O}_6^{-1}\mathcal{C}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{h}_1}^*\mathcal{O}_5\mathcal{L}_{\mathbf{h}_1})(\mathbf{q}-\mathbf{q}'). \qquad (16)
\end{aligned}
$$

If (2) and (3) hold, by Schur complement, we have

$$
\begin{aligned}
&-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1+\mathcal{P}_1\mathcal{A}_1\mathcal{O}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{f}_2}^*\mathcal{O}_2\mathcal{L}_{\mathbf{f}_2}+\mathcal{P}_1\mathcal{B}_1\mathcal{O}_3^{-1}\mathcal{B}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{g}_2}^*\mathcal{O}_4\mathcal{L}_{\mathbf{g}_2}\\
&\quad+\mathcal{P}_1\mathcal{C}_1\mathcal{O}_5^{-1}\mathcal{C}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{h}_2}^*\mathcal{O}_6\mathcal{L}_{\mathbf{h}_2}<0, \qquad (17)\\
&-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2+\mathcal{P}_2\mathcal{A}_2\mathcal{O}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{f}_1}^*\mathcal{O}_1\mathcal{L}_{\mathbf{f}_1}+\mathcal{P}_2\mathcal{B}_2\mathcal{O}_4^{-1}\mathcal{B}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{g}_1}^*\mathcal{O}_3\mathcal{L}_{\mathbf{g}_1}\\
&\quad+\mathcal{P}_2\mathcal{C}_2\mathcal{O}_6^{-1}\mathcal{C}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{h}_1}^*\mathcal{O}_5\mathcal{L}_{\mathbf{h}_1}<0. \qquad (18)
\end{aligned}
$$

and thus, that is to say the right side of (16) is negative, which is a contradiction. Therefore, the function $\mathcal{H}(\mathbf{p}, \mathbf{q})$ is injective.

Now, we shall show that $\|\mathcal{H}(\mathbf{p}, \mathbf{q})\| \to \infty$ as $\|(\mathbf{p}, \mathbf{q})\| \to \infty$. We infer that from (17), (18) and small constant $\epsilon > 0$ exist, such that

$$
\begin{aligned}
&-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1+\mathcal{P}_1\mathcal{A}_1\mathcal{O}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{f}_2}^*\mathcal{O}_2\mathcal{L}_{\mathbf{f}_2}+\mathcal{P}_1\mathcal{B}_1\mathcal{O}_3^{-1}\mathcal{B}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{g}_2}^*\mathcal{O}_4\mathcal{L}_{\mathbf{g}_2}\\
&\quad+\mathcal{P}_1\mathcal{C}_1\mathcal{O}_5^{-1}\mathcal{C}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{h}_2}^*\mathcal{O}_6\mathcal{L}_{\mathbf{h}_2}<-\epsilon\mathcal{I}_n,\\
&-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2+\mathcal{P}_2\mathcal{A}_2\mathcal{O}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{f}_1}^*\mathcal{O}_1\mathcal{L}_{\mathbf{f}_1}+\mathcal{P}_2\mathcal{B}_2\mathcal{O}_4^{-1}\mathcal{B}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{g}_1}^*\mathcal{O}_3\mathcal{L}_{\mathbf{g}_1}\\
&\quad+\mathcal{P}_2\mathcal{C}_2\mathcal{O}_6^{-1}\mathcal{C}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{h}_1}^*\mathcal{O}_5\mathcal{L}_{\mathbf{h}_1}<-\epsilon\mathcal{I}_m.
\end{aligned}
$$

Taking $(\mathbf{p}', \mathbf{q}') = (0, 0)$ and using (16) and the above relations, we have

$$
\begin{aligned}
&\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix}^* \begin{bmatrix}\mathcal{P}_1 & 0\\ 0 & \mathcal{P}_2\end{bmatrix}(\mathcal{H}(\mathbf{p}, \mathbf{q})-\mathcal{H}(0, 0))\\
&\quad\leq \mathbf{p}^*(-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1+\mathcal{P}_1\mathcal{A}_1\mathcal{O}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{f}_2}^*\mathcal{O}_2\mathcal{L}_{\mathbf{f}_2}+\mathcal{P}_1\mathcal{B}_1\mathcal{O}_3^{-1}\mathcal{B}_1^*\mathcal{P}_1\\
&\qquad+\mathcal{L}_{\mathbf{g}_2}^*\mathcal{O}_4\mathcal{L}_{\mathbf{g}_2}+\mathcal{P}_1\mathcal{C}_1\mathcal{O}_5^{-1}\mathcal{C}_1^*\mathcal{P}_1+\mathcal{L}_{\mathbf{h}_2}^*\mathcal{O}_6\mathcal{L}_{\mathbf{h}_2})\mathbf{p}+\mathbf{q}^*(-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2\\
&\qquad+\mathcal{P}_2\mathcal{A}_2\mathcal{O}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{f}_1}^*\mathcal{O}_1\mathcal{L}_{\mathbf{f}_1}+\mathcal{P}_2\mathcal{B}_2\mathcal{O}_4^{-1}\mathcal{B}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{g}_1}^*\mathcal{O}_3\mathcal{L}_{\mathbf{g}_1}\\
&\qquad+\mathcal{P}_2\mathcal{C}_2\mathcal{O}_6^{-1}\mathcal{C}_2^*\mathcal{P}_2+\mathcal{L}_{\mathbf{h}_1}^*\mathcal{O}_5\mathcal{L}_{\mathbf{h}_1})\mathbf{q}\\
&\quad\leq -\epsilon(\|\mathbf{p}\|^2+\|\mathbf{q}\|^2). \qquad (19)
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, then (19) becomes:

$$
\epsilon(\|\mathbf{p}\|^2+\|\mathbf{q}\|^2) \leq 2\|(\mathbf{p}, \mathbf{q})\|\,\|\mathcal{P}_1\|\,\|\mathcal{P}_2\|\,(\|\mathcal{H}(\mathbf{p}, \mathbf{q})\|+\|\mathcal{H}(0, 0)\|), \qquad (20)
$$

which gives result that $\|\mathcal{H}(\mathbf{p}, \mathbf{q})\| \to \infty$ as $\|(\mathbf{p}, \mathbf{q})\| \to \infty$. Hence, the map $\mathcal{H}(\mathbf{p}, \mathbf{q})$ satisfies all conditions in Lemma (2) and is homeomorphism of $\mathbf{H}^{n+m}$ onto itself. Then, there exist $(\mathbf{p}^*, \mathbf{q}^*)$ such that $\mathcal{H}(\mathbf{p}^*, \mathbf{q}^*) = 0$, that is, NN (1) has a unique equilibrium point $(\mathbf{p}^*, \mathbf{q}^*)$.

Let $\mathbf{u}(t)=\mathbf{p}(t)-\mathbf{p}^*$, $\mathbf{v}(t)=\mathbf{q}(t)-\mathbf{q}^*$, we can get

$$
\begin{cases}
\dot{\mathbf{u}}(t)= & -\mathcal{D}_1\mathbf{u}(t-\delta)+\mathcal{A}_1\hat{\mathbf{f}}_1(\mathbf{v}(t))+\mathcal{B}_1\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t)))+\mathcal{C}_1\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds,\\
\mathbf{u}(t)= & \hat{\varphi}_1(t),\ t\in[-\sigma,0],\\
\dot{\mathbf{v}}(t)= & -\mathcal{D}_2\mathbf{v}(t-\delta)+\mathcal{A}_2\hat{\mathbf{f}}_2(\mathbf{u}(t))+\mathcal{B}_2\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t)))+\mathcal{C}_2\int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds,\\
\mathbf{v}(t)= & \hat{\varphi}_2(t),\ t\in[-\sigma,0],
\end{cases}
\tag{21}
$$

where $\hat{\mathbf{f}}_1(\mathbf{v}(\cdot))=\mathbf{f}_1(\mathbf{q}(\cdot)+\mathbf{q}^*+\mathcal{J}_1)-\mathbf{f}_1(\mathbf{q}^*+\mathcal{J}_1)$, $\hat{\mathbf{g}}_1(\mathbf{v}(\cdot))=\mathbf{g}_1(\mathbf{q}(\cdot)+\mathbf{q}^*+\mathcal{J}_1)-\mathbf{g}_1(\mathbf{q}^*+\mathcal{J}_1)$, $\hat{\mathbf{h}}_1(\mathbf{v}(\cdot))=\mathbf{h}_1(\mathbf{q}(\cdot)+\mathbf{q}^*+\mathcal{J}_1)-\mathbf{h}_1(\mathbf{q}^*+\mathcal{J}_1)$, $\hat{\mathbf{f}}_2(\mathbf{u}(\cdot))=\mathbf{f}_2(\mathbf{p}(\cdot)+\mathbf{p}^*+\mathcal{J}_2)-\mathbf{f}_2(\mathbf{p}^*+\mathcal{J}_2)$, $\hat{\mathbf{g}}_2(\mathbf{u}(\cdot))=\mathbf{g}_2(\mathbf{p}(\cdot)+\mathbf{p}^*+\mathcal{J}_2)-\mathbf{g}_2(\mathbf{p}^*+\mathcal{J}_2)$, $\hat{\mathbf{h}}_2(\mathbf{u}(\cdot))=\mathbf{h}_2(\mathbf{p}(\cdot)+\mathbf{p}^*+\mathcal{J}_2)-\mathbf{h}_2(\mathbf{p}^*+\mathcal{J}_2)$, $\hat{\varphi}_1=\varphi_1-\mathbf{p}^*$, $\hat{\varphi}_2=\varphi_2-\mathbf{q}^*$ and $\hat{\varphi}_1\in\mathscr{C}([-\sigma,0],\mathbf{H}^n)$, $\hat{\varphi}_2\in\mathscr{C}([-\sigma,0],\mathbf{H}^m)$.

3.2 Quaternion-Valued T-S Fuzzy BAM Neural Networks

To describe a nonlinear system, the continuous fuzzy system was introduced in [30] and this concept well discussed in [31, 32]. As shown in [33–36], the T-S fuzzy QVBAMNNs with time delays can be described as bellow.

Plant Rule z:

If $\vartheta_1(t)$ is η_1^z and $\vartheta_2(t)$ is η_2^z and ... and $\vartheta_g(t)$ is η_g^z, Then

$$
\begin{cases}
\dot{\mathbf{u}}(t)= & -\mathcal{D}_1^z\mathbf{u}(t-\delta)+\mathcal{A}_1^z\hat{\mathbf{f}}_1(\mathbf{v}(t))+\mathcal{B}_1^z\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t)))+\mathcal{C}_1^z\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds,\\
\mathbf{u}(t)= & \hat{\varphi}_1(t),\ t\in[-\sigma,0],\\
\dot{\mathbf{v}}(t)= & -\mathcal{D}_2^z\mathbf{v}(t-\delta)+\mathcal{A}_2^z\hat{\mathbf{f}}_2(\mathbf{u}(t))+\mathcal{B}_2^z\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t)))+\mathcal{C}_2^z\int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds,\\
\mathbf{v}(t)= & \hat{\varphi}_2(t),\ t\in[-\sigma,0],
\end{cases}
\tag{22}
$$

where the premise variables are $\vartheta_c(t),\ c=1,...,g$, the fuzzy sets are $\eta_c^z,\ z=1,...,m$; $c=1,...,g$ and m is the number of If-Then rules.

The final output of T-S fuzzy QVBAMNNs can be derived from the fuzzy models as follows:

$$
\begin{cases}
\dot{\mathbf{u}}(t)=\dfrac{\sum_{z=1}^{m}w_z(\vartheta(t))\Big\{-\mathcal{D}_1^z\mathbf{u}(t-\delta)+\mathcal{A}_1^z\hat{\mathbf{f}}_1(\mathbf{v}(t))+\mathcal{B}_1^z\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t)))+\mathcal{C}_1^z\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds\Big\}}{\sum_{z=1}^{m}w_z(\vartheta(t))},\\
\mathbf{u}(t)=\hat{\varphi}_1(t),\ t\in[-\sigma,0],\\
\dot{\mathbf{v}}(t)=\dfrac{\sum_{z=1}^{m}w_z(\vartheta(t))\Big\{-\mathcal{D}_2^z\mathbf{v}(t-\delta)+\mathcal{A}_2^z\hat{\mathbf{f}}_2(\mathbf{u}(t))++\mathcal{B}_2^z\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t)))+\mathcal{C}_2^z\int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds\Big\}}{\sum_{z=1}^{m}w_z(\vartheta(t))},\\
\mathbf{v}(t)=\hat{\varphi}_2(t),\ t\in[-\sigma,0],
\end{cases}
\tag{23}
$$

or equivalently

$$\begin{cases} \dot{\mathbf{u}}(t) &= \sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{ -\mathcal{D}_1^z\mathbf{u}(t-\delta)+\mathcal{A}_1^z\hat{\mathbf{f}}_1(\mathbf{v}(t))+\mathcal{B}_1^z\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t))) \\ &\quad +\mathcal{C}_1^z\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds\Big\}, \\ \mathbf{u}(t) &= \hat{\varphi}_1(t),\ t\in[-\sigma,0], \\ \dot{\mathbf{v}}(t) &= \sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{ -\mathcal{D}_2^z\mathbf{v}(t-\delta)+\mathcal{A}_2^z\hat{\mathbf{f}}_2(\mathbf{u}(t))+\mathcal{B}_2^z\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t))) \\ &\quad +\mathcal{C}_2^z\int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds\Big\}, \\ \mathbf{v}(t) &= \hat{\varphi}_2(t),\ t\in[-\sigma,0], \end{cases} \tag{24}$$

where $\vartheta(t)=(\vartheta_1(t),...,\vartheta_g(t))^T$, $\chi_z(\vartheta(t))=\frac{w_z(\vartheta(t))}{\sum_{z=1}^{m}w_z(\vartheta(t))}$ and $w_z(\vartheta(t))=\prod_{c=1}^{g}\eta_c^z(\vartheta(t))$. The term $\eta_c^z(\vartheta(t))$ is the grade membership of $\vartheta_c(t)$ in η_c^z. It is stated that $w_z(\vartheta(t))\geq 0,\ z=1,...,m$ and $\sum_{z=1}^{m}w_z(\vartheta(t))>0$ for all $t\geq 0$. From the fuzzy set theory, we have $\chi_z(\vartheta(t))\geq 0,\ z=1,...,m$ and $\sum_{z=1}^{m}\chi_z(\vartheta(t))=1$ for all $t\geq 0$.

3.3 Stability Analysis

This subsection presents new sufficient criteria for ensuring GAS for the NNs (24). In order to simplify, we have used the following notations:

$$\begin{aligned} \xi(t)=\Big[&\mathbf{u}^*(t)\ \mathbf{u}^*(t-\ell(t))\ \mathbf{u}^*(t-\ell)\ \mathbf{u}^*(t-\delta)\ \hat{\mathbf{f}}_2^*(\mathbf{u}(t))\ \hat{\mathbf{g}}_2^*(\mathbf{u}(t-\ell(t))) \\ &\int_{t-\ell}^{t}\mathbf{u}^*(s)ds\ \frac{1}{\ell}\int_{t-\ell}^{t}\int_{s}^{t}\mathbf{u}^*(u)duds\ \frac{1}{\ell^2}\int_{t-\ell}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{u}^*(v)dvduds \\ &\int_{t-\delta}^{t}\mathbf{u}^*(s)ds\ \frac{1}{\delta}\int_{t-\delta}^{t}\int_{s}^{t}\mathbf{u}^*(u)duds\ \frac{1}{\delta^2}\int_{t-\delta}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{u}^*(v)dvduds \\ &\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_2^*(\mathbf{u}(s))ds\ \mathbf{v}^*(t)\ \mathbf{v}^*(t-\ell(t))\ \mathbf{v}^*(t-\ell)\ \mathbf{v}^*(t-\delta) \\ &\hat{\mathbf{f}}_1^*(\mathbf{v}(t))\ \hat{\mathbf{g}}_1^*(\mathbf{v}(t-\ell(t)))\ \int_{t-\ell}^{t}\mathbf{v}^*(s)ds\ \frac{1}{\ell}\int_{t-\ell}^{t}\int_{s}^{t}\mathbf{v}^*(u)duds \\ &\frac{1}{\ell^2}\int_{t-\ell}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{v}^*(v)dvduds\ \int_{t-\delta}^{t}\mathbf{v}^*(s)ds\ \frac{1}{\delta}\int_{t-\delta}^{t}\int_{s}^{t}\mathbf{v}^*(u)duds \\ &\frac{1}{\delta^2}\int_{t-\delta}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{v}^*(v)dvduds\ \int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_1^*(\mathbf{v}(s))ds\Big]^*, \end{aligned}$$

$$\mathbf{e}_r=[0_{n\times(r-1)n}\ \mathcal{I}_{n\times n}\ 0_{26\times(r-1)n}],\ r=1,2,...,26,$$

$$\Omega_1^z=\big[\mathbf{e}_1-\mathcal{D}_1^z\mathbf{e}_{10}\big]^*\mathcal{P}_1\big[-\mathcal{D}_1^z\mathbf{e}_1+\mathcal{A}_1^z\mathbf{e}_{18}+\mathcal{B}_1^z\mathbf{e}_{19}+\mathcal{C}_1^z\mathbf{e}_{26}\big]$$

$$
\begin{aligned}
&+\big[-\mathcal{D}_1^z\mathbf{e}_1+\mathcal{A}_1^z\mathbf{e}_{18}+\mathcal{B}_1^z\mathbf{e}_{19}+\mathcal{C}_1^z\mathbf{e}_{26}\big]^*\mathcal{P}_1\big[\mathbf{e}_1-\mathcal{D}_1^z\mathbf{e}_{10}\big]\\
&+\big[\mathbf{e}_{14}-\mathcal{D}_2^z\mathbf{e}_{23}\big]^*\mathcal{P}_2\big[-\mathcal{D}_2^z\mathbf{e}_{14}+\mathcal{A}_2^z\mathbf{e}_5+\mathcal{B}_2^z\mathbf{e}_6+\mathcal{C}_2^z\mathbf{e}_{13}\big]\\
&+\big[-\mathcal{D}_2^z\mathbf{e}_{14}+\mathcal{A}_2^z\mathbf{e}_5+\mathcal{B}_2^z\mathbf{e}_6+\mathcal{C}_2^z\mathbf{e}_{13}\big]^*\mathcal{P}_2\big[\mathbf{e}_{14}-\mathcal{D}_2^z\mathbf{e}_{23}\big],
\end{aligned}
$$

$$
\begin{aligned}
\Omega_2 =\ &\mathbf{e}_1^*[\mathcal{Q}_1+\mathcal{Q}_3+\mathcal{Q}_5]\mathbf{e}_1-\mathbf{e}_3^*\mathcal{Q}_1\mathbf{e}_3+\mathbf{e}_{14}^*[\mathcal{Q}_2+\mathcal{Q}_4+\mathcal{Q}_6]\mathbf{e}_{14}-\mathbf{e}_{16}^*\mathcal{Q}_2\mathbf{e}_{16}\\
&-\mathbf{e}_4^*\mathcal{Q}_3\mathbf{e}_4-\mathbf{e}_{17}^*\mathcal{Q}_4\mathbf{e}_{17}-(1-\mu)\mathbf{e}_2^*\mathcal{Q}_5\mathbf{e}_2-(1-\mu)\mathbf{e}_{15}^*\mathcal{Q}_6\mathbf{e}_{15},
\end{aligned}
$$

$$
\begin{aligned}
\Omega_3 =\ &\mathbf{e}_1^*\big[\ell^2\mathcal{R}_1+\delta^2\mathcal{R}_3\big]\mathbf{e}_1+\mathbf{e}_{14}^*\big[\ell^2\mathcal{R}_2+\delta^2\mathcal{R}_4\big]\mathbf{e}_{14}\\
&-\begin{bmatrix}\mathbf{e}_7\\ \mathbf{e}_7-\frac{2}{\ell}\mathbf{e}_8\\ \mathbf{e}_7-\frac{6}{\ell}\mathbf{e}_8+\frac{12}{\ell^2}\mathbf{e}_9\end{bmatrix}^*\begin{bmatrix}\mathcal{R}_1&0&0\\0&3\mathcal{R}_1&0\\0&0&5\mathcal{R}_1\end{bmatrix}\begin{bmatrix}\mathbf{e}_7\\ \mathbf{e}_7-\frac{2}{\ell}\mathbf{e}_8\\ \mathbf{e}_7-\frac{6}{\ell}\mathbf{e}_8+\frac{12}{\ell^2}\mathbf{e}_9\end{bmatrix}\\
&-\begin{bmatrix}\mathbf{e}_{20}\\ \mathbf{e}_{20}-\frac{2}{\ell}\mathbf{e}_{21}\\ \mathbf{e}_{20}-\frac{6}{\ell}\mathbf{e}_{21}+\frac{12}{\ell^2}\mathbf{e}_{22}\end{bmatrix}^*\begin{bmatrix}\mathcal{R}_2&0&0\\0&3\mathcal{R}_2&0\\0&0&5\mathcal{R}_2\end{bmatrix}\begin{bmatrix}\mathbf{e}_{20}\\ \mathbf{e}_{20}-\frac{2}{\ell}\mathbf{e}_{21}\\ \mathbf{e}_{20}-\frac{6}{\ell}\mathbf{e}_{21}+\frac{12}{\ell^2}\mathbf{e}_{22}\end{bmatrix}\\
&-\begin{bmatrix}\mathbf{e}_{10}\\ \mathbf{e}_{10}-\frac{2}{\ell}\mathbf{e}_{11}\\ \mathbf{e}_{10}-\frac{6}{\ell}\mathbf{e}_{11}+\frac{12}{\ell^2}\mathbf{e}_{12}\end{bmatrix}^*\begin{bmatrix}\mathcal{R}_3&0&0\\0&3\mathcal{R}_3&0\\0&0&5\mathcal{R}_3\end{bmatrix}\begin{bmatrix}\mathbf{e}_{10}\\ \mathbf{e}_{10}-\frac{2}{\ell}\mathbf{e}_{11}\\ \mathbf{e}_{10}-\frac{6}{\ell}\mathbf{e}_{11}+\frac{12}{\ell^2}\mathbf{e}_{12}\end{bmatrix}\\
&-\begin{bmatrix}\mathbf{e}_{23}\\ \mathbf{e}_{23}-\frac{2}{\ell}\mathbf{e}_{24}\\ \mathbf{e}_{23}-\frac{6}{\ell}\mathbf{e}_{24}+\frac{12}{\ell^2}\mathbf{e}_{25}\end{bmatrix}^*\begin{bmatrix}\mathcal{R}_4&0&0\\0&3\mathcal{R}_4&0\\0&0&5\mathcal{R}_4\end{bmatrix}\begin{bmatrix}\mathbf{e}_{23}\\ \mathbf{e}_{23}-\frac{2}{\ell}\mathbf{e}_{24}\\ \mathbf{e}_{23}-\frac{6}{\ell}\mathbf{e}_{24}+\frac{12}{\ell^2}\mathbf{e}_{25}\end{bmatrix},
\end{aligned}
$$

$$
\Omega_4 = \mathbf{e}_{14}^*\mathcal{L}_{\mathbf{h}_1}^*\mathcal{X}\mathcal{L}_{\mathbf{h}_1}\mathbf{e}_{14}-\mathbf{e}_{26}^*\mathcal{X}\mathbf{e}_{26}+\mathbf{e}_1^*\mathcal{L}_{\mathbf{h}_2}^*\mathcal{Y}\mathcal{L}_{\mathbf{h}_2}\mathbf{e}_1-\mathbf{e}_{13}^*\mathcal{Y}\mathbf{e}_{13},
$$

$$
\begin{aligned}
\Omega_5 =\ &\mathbf{e}_1^*\mathcal{L}_{\mathbf{f}_2}^*\mathcal{G}_1\mathcal{L}_{\mathbf{f}_2}\mathbf{e}_1-\mathbf{e}_5^*\mathcal{G}_1\mathbf{e}_5+\mathbf{e}_2^*\mathcal{L}_{\mathbf{g}_2}^*\mathcal{G}_2\mathcal{L}_{\mathbf{g}_2}\mathbf{e}_2-\mathbf{e}_6^*\mathcal{G}_2\mathbf{e}_6+\mathbf{e}_{14}^*\mathcal{L}_{\mathbf{f}_1}^*\mathcal{G}_3\mathcal{L}_{\mathbf{f}_1}\mathbf{e}_{14}\\
&-\mathbf{e}_{18}^*\mathcal{G}_3\mathbf{e}_{18}+\mathbf{e}_{15}^*\mathcal{L}_{\mathbf{g}_1}^*\mathcal{G}_4\mathcal{L}_{\mathbf{g}_1}\mathbf{e}_{15}-\mathbf{e}_{19}^*\mathcal{G}_4\mathbf{e}_{19}.
\end{aligned}
$$

Theorem 6 *If Assumption 1-3 are fulfilled, if there exist Hermitian matrices $0<\mathcal{P}_1$, $0<\mathcal{P}_2$, $0<\mathcal{Q}_1$, $0<\mathcal{Q}_2$, $0<\mathcal{Q}_3$, $0<\mathcal{Q}_4$, $0<\mathcal{Q}_5$, $0<\mathcal{Q}_6$, $0<\mathcal{R}_1$, $0<\mathcal{R}_2$, $0<\mathcal{R}_3$, $0<\mathcal{R}_4$, and diagonal matrices $0<\mathcal{X}$, $0<\mathcal{Y}$, $0<\mathcal{G}_1$, $0<\mathcal{G}_2$, $0<\mathcal{G}_3$, $0<\mathcal{G}_4$ such that the following LMI holds for all $z=1,2,\ldots,m$*

$$
\Omega_1^z+\Omega_2+\Omega_3+\Omega_4+\Omega_5<0,\ z=1,2,\ldots,m, \tag{25}
$$

then the equilibrium point of NN model (24) is GAS.

Proof Consider the following LKFs (26) for NNs (24) described by

$$
\begin{aligned}
\mathcal{V}(t,\mathbf{u}(t),\mathbf{v}(t),z) =\ &\mathcal{V}_1(t,\mathbf{u}(t),\mathbf{v}(t),z)+\mathcal{V}_2(t,\mathbf{u}(t),\mathbf{v}(t),z)+\mathcal{V}_3(t,\mathbf{u}(t),\mathbf{v}(t),z)\\
&+\mathcal{V}_4(t,\mathbf{u}(t),\mathbf{v}(t),z),
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
\mathcal{V}_1(t,\mathbf{u}(t),\mathbf{v}(t),z) =\ &\left[\mathbf{u}(t)-\mathcal{D}_1^z\int_{t-\delta}^t\mathbf{u}(s)ds\right]^*\mathcal{P}_1\left[\mathbf{u}(t)-\mathcal{D}_1^z\int_{t-\delta}^t\mathbf{u}(s)ds\right]\\
&+\left[\mathbf{v}(t)-\mathcal{D}_2^z\int_{t-\delta}^t\mathbf{v}(s)ds\right]^*\mathcal{P}_2\left[\mathbf{v}(t)-\mathcal{D}_2^z\int_{t-\delta}^t\mathbf{v}(s)ds\right],\\
\mathcal{V}_2(t,\mathbf{u}(t),\mathbf{v}(t),z) =\ &\int_{t-\ell}^t\mathbf{u}^*(s)\mathcal{Q}_1\mathbf{u}(s)ds+\int_{t-\ell}^t\mathbf{v}^*(s)\mathcal{Q}_2\mathbf{v}(s)ds\\
&+\int_{t-\delta}^t\mathbf{u}^*(s)\mathcal{Q}_3\mathbf{u}(s)ds+\int_{t-\delta}^t\mathbf{v}^*(s)\mathcal{Q}_4\mathbf{v}(s)ds
\end{aligned}
$$

$$+\int_{t-\ell(t)}^{t}\mathbf{u}^*(s)\mathcal{Q}_5\mathbf{u}(s)ds+\int_{t-\ell(t)}^{t}\mathbf{v}^*(s)\mathcal{Q}_6\mathbf{v}(s)ds,$$

$$\mathcal{V}_3(t,\mathbf{u}(t),\mathbf{v}(t),z)=\ell\int_{t-\ell}^{t}\int_{s}^{t}\mathbf{u}^*(u)\mathcal{R}_1\mathbf{u}(u)duds+\ell\int_{t-\ell}^{t}\int_{s}^{t}\mathbf{v}^*(u)\mathcal{R}_2\mathbf{v}(u)duds$$
$$+\delta\int_{t-\delta}^{t}\int_{s}^{t}\mathbf{u}^*(u)\mathcal{R}_3\mathbf{u}(u)duds+\delta\int_{t-\delta}^{t}\int_{s}^{t}\mathbf{v}^*(u)\mathcal{R}_4\mathbf{v}(u)duds,$$

$$\mathcal{V}_4(t,\mathbf{u}(t),\mathbf{v}(t),z)=\sum_{s=1}^{m}x_s\int_0^{+\infty}K_1(s)\int_{t-s}^{t}\hat{\mathbf{h}}_1^*(\mathbf{v}(u))\hat{\mathbf{h}}_1(\mathbf{v}(u))duds$$
$$+\sum_{r=1}^{n}y_r\int_0^{+\infty}K_2(s)\int_{t-s}^{t}\hat{\mathbf{h}}_2^*(\mathbf{u}(u))\hat{\mathbf{h}}_2(\mathbf{u}(u))duds.$$

The time-derivative of $\mathcal{V}(t,\mathbf{u}(t),\mathbf{v}(t),z)$ can be obtained as follows:

$$\dot{\mathcal{V}}(t,\mathbf{u}(t),\mathbf{v}(t),z)=\dot{\mathcal{V}}_1(t,\mathbf{u}(t),\mathbf{v}(t),z)+\dot{\mathcal{V}}_2(t,\mathbf{u}(t),\mathbf{v}(t),z)+\dot{\mathcal{V}}_3(t,\mathbf{u}(t),\mathbf{v}(t),z)$$
$$+\dot{\mathcal{V}}_4(t,\mathbf{u}(t),\mathbf{v}(t),z),\tag{27}$$

where

$$\dot{\mathcal{V}}_1(t,\mathbf{u}(t),\mathbf{v}(t),z)=\Big[\mathbf{u}(t)-\mathcal{D}_1^z\int_{t-\delta}^{t}\mathbf{u}(s)ds\Big]^*\mathcal{P}_1\Big[\dot{\mathbf{u}}(t)-\mathcal{D}_1^z\mathbf{u}(t)+\mathcal{D}_1^z\mathbf{u}(t-\delta)\Big]$$
$$+\Big[\dot{\mathbf{u}}(t)-\mathcal{D}_1^z\mathbf{u}(t)+\mathcal{D}_1^z\mathbf{u}(t-\delta)\Big]^*\mathcal{P}_1\Big[\mathbf{u}(t)-\mathcal{D}_1^z\int_{t-\delta}^{t}\mathbf{u}(s)ds\Big]$$
$$+\Big[\mathbf{v}(t)-\mathcal{D}_2^z\int_{t-\delta}^{t}\mathbf{v}(s)ds\Big]^*\mathcal{P}_2\Big[\dot{\mathbf{v}}(t)-\mathcal{D}_2^z\mathbf{v}(t)+\mathcal{D}_2^z\mathbf{v}(t-\delta)\Big]$$
$$+\Big[\dot{\mathbf{v}}(t)-\mathcal{D}_2^z\mathbf{v}(t)+\mathcal{D}_1^z\mathbf{v}(t-\delta)\Big]^*\mathcal{P}_2\Big[\mathbf{v}(t)-\mathcal{D}_2^z\int_{t-\delta}^{t}\mathbf{v}(s)ds\Big]$$
$$=\Big[\mathbf{u}(t)-\mathcal{D}_1^z\int_{t-\delta}^{t}\mathbf{u}(s)ds\Big]^*\mathcal{P}_1\Big[\sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{-\mathcal{D}_1^z\mathbf{u}(t)+\mathcal{A}_1^z\hat{\mathbf{f}}_1(\mathbf{v}(t))$$
$$+\mathcal{B}_1^z\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t)))+\mathcal{C}_1^z\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds\Big\}\Big]$$
$$+\Big[\sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{-\mathcal{D}_1^z\mathbf{u}(t)+\mathcal{A}_1^z\hat{\mathbf{f}}_1(\mathbf{v}(t))+\mathcal{B}_1^z\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t)))$$
$$+\mathcal{C}_1^z\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds\Big\}\Big]^*\mathcal{P}_1\Big[\mathbf{u}(t)-\mathcal{D}_1^z\int_{t-\delta}^{t}\mathbf{u}(s)ds\Big]$$
$$+\Big[\mathbf{v}(t)-\mathcal{D}_2^z\int_{t-\delta}^{t}\mathbf{v}(s)ds\Big]^*\mathcal{P}_2\Big[\sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{-\mathcal{D}_2^z\mathbf{v}(t)+\mathcal{A}_2^z\hat{\mathbf{f}}_2(\mathbf{u}(t))$$
$$+\mathcal{B}_2^z\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t)))+\mathcal{C}_2^z\int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds\Big\}\Big]$$
$$+\Big[\sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{-\mathcal{D}_2^z\mathbf{v}(t)+\mathcal{A}_2^z\hat{\mathbf{f}}_2(\mathbf{u}(t))+\mathcal{B}_2^z\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t)))$$
$$+\mathcal{C}_2^z\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds\Big\}\Big]^*\mathcal{P}_2\Big[\mathbf{v}(t)-\mathcal{D}_2^z\int_{t-\delta}^{t}\mathbf{v}(s)ds\Big]$$

$$= \sum_{z=1}^{m} \chi_z(\vartheta(t)) \Big\{ \xi^*(t)\Omega_1^z \xi(t) \Big\}, \tag{28}$$

$$\begin{aligned}
\dot{\mathcal{V}}_2(t, \mathbf{u}(t), \mathbf{v}(t), z) \leq\ & \mathbf{u}^*(t)\mathcal{Q}_1\mathbf{u}(t) - \mathbf{u}^*(t-\ell)\mathcal{Q}_1\mathbf{u}(t-\ell) + \mathbf{v}^*(t)\mathcal{Q}_2\mathbf{v}(t) \\
& - \mathbf{v}^*(t-\ell)\mathcal{Q}_2\mathbf{v}(t-\ell) + \mathbf{u}^*(t)\mathcal{Q}_3\mathbf{u}(t) - \mathbf{u}^*(t-\delta)\mathcal{Q}_3\mathbf{u}(t-\delta) \\
& + \mathbf{v}^*(t)\mathcal{Q}_4\mathbf{v}(t) - \mathbf{v}^*(t-\delta)\mathcal{Q}_4\mathbf{v}(t-\delta) + \mathbf{u}^*(t)\mathcal{Q}_5\mathbf{u}(t) \\
& - (1-\mu)\mathbf{u}^*(t-\ell(t))\mathcal{Q}_5\mathbf{u}(t-\ell(t)) + \mathbf{v}^*(t)\mathcal{Q}_6\mathbf{v}(t) \\
& - (1-\mu)\mathbf{v}^*(t-\ell(t))\mathcal{Q}_6\mathbf{v}(t-\ell(t)) \\
=\ & \sum_{z=1}^{m} \chi_z(\vartheta(t)) \Big\{ \xi^*(t)\Omega_2 \xi(t) \Big\},
\end{aligned} \tag{29}$$

$$\begin{aligned}
\dot{\mathcal{V}}_3(t, \mathbf{u}(t), \mathbf{v}(t), z) =\ & \ell^2\mathbf{u}^*(t)\mathcal{R}_1\mathbf{u}(t) - \ell\int_{t-\ell}^{t} \mathbf{u}^*(s)\mathcal{R}_1\mathbf{u}(s)ds + \ell^2\mathbf{v}^*(t)\mathcal{R}_2\mathbf{v}(t) \\
& - \ell\int_{t-\ell}^{t} \mathbf{v}^*(s)\mathcal{R}_2\mathbf{v}(s)ds + \delta^2\mathbf{u}^*(t)\mathcal{R}_3\mathbf{u}(t) - \delta\int_{t-\delta}^{t} \mathbf{u}^*(s)\mathcal{R}_3\mathbf{u}(s)ds \\
& + \delta^2\mathbf{v}^*(t)\mathcal{R}_4\mathbf{v}(t) - \delta\int_{t-\delta}^{t} \mathbf{v}^*(s)\mathcal{R}_4\mathbf{v}(s)ds.
\end{aligned} \tag{30}$$

Using (4), tighter bounds were obtained for integral terms in (30):

$$\begin{aligned}
& -\ell\int_{t-\ell}^{t} \mathbf{u}^*(s)\mathcal{R}_1\mathbf{u}(s)ds \\
& \leq - \begin{bmatrix} \int_{t-\ell}^{t} \mathbf{u}(s)ds \\ \int_{t-\ell}^{t} \mathbf{u}(s)ds - \frac{2}{\ell}\int_{t-\ell}^{t}\int_{s}^{t} \mathbf{u}(u)duds \\ \int_{t-\ell}^{t} \mathbf{u}(s)ds - \frac{6}{\ell}\int_{t-\ell}^{t}\int_{s}^{t} \mathbf{u}(u)duds + \frac{12}{\ell^2}\int_{t-\ell}^{t}\int_{s}^{t}\int_{u}^{t} \mathbf{u}(v)dvduds \end{bmatrix}^* \\
& \quad \times \begin{bmatrix} \mathcal{R}_1 & 0 & 0 \\ 0 & 3\mathcal{R}_1 & 0 \\ 0 & 0 & 5\mathcal{R}_1 \end{bmatrix} \begin{bmatrix} \int_{t-\ell}^{t} \mathbf{u}(s)ds \\ \int_{t-\ell}^{t} \mathbf{u}(s)ds - \frac{2}{\ell}\int_{t-\ell}^{t}\int_{s}^{t} \mathbf{u}(u)duds \\ \int_{t-\ell}^{t} \mathbf{u}(s)ds - \frac{6}{\ell}\int_{t-\ell}^{t}\int_{s}^{t} \mathbf{u}(u)duds + \frac{12}{\ell^2}\int_{t-\ell}^{t}\int_{s}^{t}\int_{u}^{t} \mathbf{u}(v)dvduds \end{bmatrix} \\
& = - \begin{bmatrix} \mathbf{e}_7 \\ \mathbf{e}_7 - \frac{2}{\ell}\mathbf{e}_8 \\ \mathbf{e}_7 - \frac{6}{\ell}\mathbf{e}_8 + \frac{12}{\ell^2}\mathbf{e}_9 \end{bmatrix}^* \begin{bmatrix} \mathcal{R}_1 & 0 & 0 \\ 0 & 3\mathcal{R}_1 & 0 \\ 0 & 0 & 5\mathcal{R}_1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_7 \\ \mathbf{e}_7 - \frac{2}{\ell}\mathbf{e}_8 \\ \mathbf{e}_7 - \frac{6}{\ell}\mathbf{e}_8 + \frac{12}{\ell^2}\mathbf{e}_9 \end{bmatrix}.
\end{aligned} \tag{31}$$

Similarly we can prove that

$$-\ell\int_{t-\ell}^{t} \mathbf{v}^*(s)\mathcal{R}_2\mathbf{v}(s)ds \leq - \begin{bmatrix} \mathbf{e}_{20} \\ \mathbf{e}_{20} - \frac{2}{\ell}\mathbf{e}_{21} \\ \mathbf{e}_{20} - \frac{6}{\ell}\mathbf{e}_{21} + \frac{12}{\ell^2}\mathbf{e}_{22} \end{bmatrix}^* \begin{bmatrix} \mathcal{R}_2 & 0 & 0 \\ 0 & 3\mathcal{R}_2 & 0 \\ 0 & 0 & 5\mathcal{R}_2 \end{bmatrix} \begin{bmatrix} \mathbf{e}_{20} \\ \mathbf{e}_{20} - \frac{2}{\ell}\mathbf{e}_{21} \\ \mathbf{e}_{20} - \frac{6}{\ell}\mathbf{e}_{21} + \frac{12}{\ell^2}\mathbf{e}_{22} \end{bmatrix}, \tag{32}$$

$$-\delta\int_{t-\delta}^{t} \mathbf{u}^*(s)\mathcal{R}_3\mathbf{u}(s)ds \leq - \begin{bmatrix} \mathbf{e}_{10} \\ \mathbf{e}_{10} - \frac{2}{\ell}\mathbf{e}_{11} \\ \mathbf{e}_{10} - \frac{6}{\ell}\mathbf{e}_{11} + \frac{12}{\ell^2}\mathbf{e}_{12} \end{bmatrix}^* \begin{bmatrix} \mathcal{R}_3 & 0 & 0 \\ 0 & 3\mathcal{R}_3 & 0 \\ 0 & 0 & 5\mathcal{R}_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_{10} \\ \mathbf{e}_{10} - \frac{2}{\ell}\mathbf{e}_{11} \\ \mathbf{e}_{10} - \frac{6}{\ell}\mathbf{e}_{11} + \frac{12}{\ell^2}\mathbf{e}_{12} \end{bmatrix}, \tag{33}$$

$$-\delta\int_{t-\delta}^{t}\mathbf{v}^*(s)\mathcal{R}_4\mathbf{v}(s)ds \le -\begin{bmatrix}\mathbf{e}_{23}\\ \mathbf{e}_{23}-\frac{2}{\ell}\mathbf{e}_{24}\\ \mathbf{e}_{23}-\frac{6}{\ell}\mathbf{e}_{24}+\frac{12}{\ell^2}\mathbf{e}_{25}\end{bmatrix}^*\begin{bmatrix}\mathcal{R}_4 & 0 & 0\\ 0 & 3\mathcal{R}_4 & 0\\ 0 & 0 & 5\mathcal{R}_4\end{bmatrix}\begin{bmatrix}\mathbf{e}_{23}\\ \mathbf{e}_{23}-\frac{2}{\ell}\mathbf{e}_{24}\\ \mathbf{e}_{23}-\frac{6}{\ell}\mathbf{e}_{24}+\frac{12}{\ell^2}\mathbf{e}_{25}\end{bmatrix}. \tag{34}$$

Combining from (32)–(34), we get

$$\dot{\mathcal{V}}_3(t,\mathbf{u}(t),\mathbf{v}(t),z) \le \sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{\xi^*(t)\Omega_3\xi(t)\Big\}, \tag{35}$$

$$\begin{aligned}
\dot{\mathcal{V}}_4(t,\mathbf{u}(t),\mathbf{v}(t),z) &= \sum_{s=1}^{m}x_s\int_0^{+\infty}K_1(s)\hat{\mathbf{h}}_1^*(\mathbf{v}(t))\hat{\mathbf{h}}_1(\mathbf{v}(t))ds\\
&\quad-\sum_{s=1}^{m}x_s\int_0^{+\infty}K_1(s)\hat{\mathbf{h}}_1^*(\mathbf{v}(t-s))\hat{\mathbf{h}}_1(\mathbf{v}(t-s))ds\\
&\quad+\sum_{r=1}^{n}y_r\int_0^{+\infty}K_2(s)\hat{\mathbf{h}}_2^*(\mathbf{u}(t))\hat{\mathbf{h}}_2(\mathbf{u}(t))ds\\
&\quad-\sum_{r=1}^{n}y_r\int_0^{+\infty}K_2(s)\hat{\mathbf{h}}_2^*(\mathbf{u}(t-s))\hat{\mathbf{h}}_2(\mathbf{u}(t-s))ds\\
&= \hat{\mathbf{h}}_1^*(\mathbf{v}(t))\mathcal{X}\hat{\mathbf{h}}_1(\mathbf{v}(t))-\sum_{s=1}^{m}x_s\int_0^{+\infty}K_1(s)\hat{\mathbf{h}}_1^*(\mathbf{v}(t-s))\hat{\mathbf{h}}_1(\mathbf{v}(t-s))ds\\
&\quad+\hat{\mathbf{h}}_2^*(\mathbf{u}(t))\mathcal{Y}\hat{\mathbf{h}}_2(\mathbf{u}(t))-\sum_{r=1}^{n}y_r\int_0^{+\infty}K_2(s)\hat{\mathbf{h}}_2^*(\mathbf{u}(t-s))\hat{\mathbf{h}}_2(\mathbf{u}(t-s))ds\\
&\le \mathbf{v}^*(t)\mathcal{L}_{\mathbf{h}_1}^*\mathcal{X}\mathcal{L}_{\mathbf{h}_1}\mathbf{v}(t)-\left[\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_1^*(\mathbf{v}(s))ds\right]^*\mathcal{X}\\
&\quad\times\left[\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_1^*(\mathbf{v}(s))ds\right]+\mathbf{u}^*(t)\mathcal{L}_{\mathbf{h}_2}^*\mathcal{Y}\mathcal{L}_{\mathbf{h}_2}\mathbf{u}(t)\\
&\quad-\left[\int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_2^*(\mathbf{u}(s))ds\right]^*\mathcal{Y}\left[\int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_2^*(\mathbf{u}(s))ds\right]\\
&= \sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{\xi^*(t)\Omega_4\xi(t)\Big\}.
\end{aligned} \tag{36}$$

From Assumption 2, there exist diagonal matrices $0<\mathcal{G}_1, 0<\mathcal{G}_2, 0<\mathcal{G}_3, 0<\mathcal{G}_4$ such that

$$0 \le \mathbf{u}^*(t)\mathcal{L}_{\mathbf{f}_2}^*\mathcal{G}_1\mathcal{L}_{\mathbf{f}_2}\mathbf{u}(t)-\hat{\mathbf{f}}_2^*(\mathbf{u}(t))\mathcal{G}_1\hat{\mathbf{f}}_2(\mathbf{u}(t)), \tag{37}$$

$$0 \le \mathbf{u}^*(t-\ell(t))\mathcal{L}_{\mathbf{g}_2}^*\mathcal{G}_2\mathcal{L}_{\mathbf{g}_2}\mathbf{u}(t-\ell(t))-\hat{\mathbf{g}}_2^*(\mathbf{u}(t-\ell(t)))\mathcal{G}_2\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t))), \tag{38}$$

$$0 \le \mathbf{v}^*(t)\mathcal{L}_{\mathbf{f}_1}^*\mathcal{G}_3\mathcal{L}_{\mathbf{f}_1}\mathbf{v}(t)-\hat{\mathbf{f}}_1^*(\mathbf{v}(t))\mathcal{G}_3\hat{\mathbf{f}}_1(\mathbf{v}(t)), \tag{39}$$

$$0 \le \mathbf{v}^*(t-\ell(t))\mathcal{L}_{\mathbf{g}_1}^*\mathcal{G}_4\mathcal{L}_{\mathbf{g}_1}\mathbf{v}(t-\ell(t))-\hat{\mathbf{g}}_1^*(\mathbf{v}(t-\ell(t)))\mathcal{G}_4\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t))). \tag{40}$$

From (37)–(40), we obtain

$$0 \le \sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{\xi^*(t)\Omega_5\xi(t)\Big\}. \tag{41}$$

By combining (28)–(41), we obtain

$$\dot{\mathcal{V}}(t,\mathbf{u}(t),\mathbf{v}(t),z) \le \sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{\xi^*(t)(\Omega_1^z+\Omega_2+\Omega_3+\Omega_4+\Omega_5)\xi(t)\Big\}. \tag{42}$$

It is obvious that for $\Omega_1^z+\Omega_2+\Omega_3+\Omega_4+\Omega_5<0$, $z=1,2,...,m$, it shows that the NN (24) is GAS according to the Lyapunov stability theory. This completes the proof. □

In the following, we show how our results can be specialized to different cases.

Remark 1 When there is no leakage delay, the NN model (24) becomes:

$$\begin{cases}\dot{\mathbf{u}}(t)= & \sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{-\mathcal{D}_1^z\mathbf{u}(t)+\mathcal{A}_1^z\hat{\mathbf{f}}_1(\mathbf{v}(t))+\mathcal{B}_1^z\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t)))+\mathcal{C}_1^z\int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds\Big\},\\ \mathbf{u}(t)= & \hat{\varphi}_1(t),\ t\in[-\ell,0],\\ \dot{\mathbf{v}}(t)= & \sum_{z=1}^{m}\chi_z(\vartheta(t))\Big\{-\mathcal{D}_2^z\mathbf{v}(t)+\mathcal{A}_2^z\hat{\mathbf{f}}_2(\mathbf{u}(t))+\mathcal{B}_2^z\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t)))+\mathcal{C}_2^z\int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds\Big\},\\ \mathbf{v}(t)= & \hat{\varphi}_2(t),\ t\in[-\ell,0].\end{cases} \tag{43}$$

The following Corollary (7) gives the GAS criterion for the NNs (43). The following notations are used to simplify:

$$\begin{aligned}\bar{\xi}(t) =& \Big[\mathbf{u}^*(t)\ \ \mathbf{u}^*(t-\ell(t))\ \ \mathbf{u}^*(t-\ell)\ \ \hat{\mathbf{f}}_2^*(\mathbf{u}(t))\ \ \hat{\mathbf{g}}_2^*(\mathbf{u}(t-\ell(t)))\ \ \int_{t-\ell}^{t}\mathbf{u}^*(s)ds\\ & \frac{1}{\ell}\int_{t-\ell}^{t}\int_{s}^{t}\mathbf{u}^*(u)duds\ \ \frac{1}{\ell^2}\int_{t-\ell}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{u}^*(v)dvduds\ \ \int_{-\infty}^{t}K_1(t-s)\hat{\mathbf{h}}_2^*(\mathbf{u}(s))ds\\ & \mathbf{v}^*(t)\ \ \mathbf{v}^*(t-\ell(t))\ \ \mathbf{v}^*(t-\ell)\ \ \hat{\mathbf{f}}_1^*(\mathbf{v}(t))\ \ \hat{\mathbf{g}}_1^*(\mathbf{v}(t-\ell(t)))\ \ \int_{t-\ell}^{t}\mathbf{v}^*(s)ds\\ & \frac{1}{\ell}\int_{t-\ell}^{t}\int_{s}^{t}\mathbf{v}^*(u)duds\ \ \frac{1}{\ell^2}\int_{t-\ell}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{v}^*(v)dvduds\ \ \int_{-\infty}^{t}K_2(t-s)\hat{\mathbf{h}}_1^*(\mathbf{v}(s))ds\Big]^*,\\ \bar{\mathbf{e}}_r =& [0_{n\times(r-1)n}\ \ \mathcal{I}_{n\times n}\ \ 0_{18\times(r-1)n}],\ r=1,2,...,18,\\ \bar{\Omega}_1^z =& \bar{\mathbf{e}}_1^*\mathcal{P}_1\big[-\mathcal{D}_1^z\bar{\mathbf{e}}_1+\mathcal{A}_1^z\bar{\mathbf{e}}_{13}+\mathcal{B}_1^z\bar{\mathbf{e}}_{14}+\mathcal{C}_1^z\bar{\mathbf{e}}_{18}\big]\\ &+\big[-\mathcal{D}_1^z\bar{\mathbf{e}}_1+\mathcal{A}_1^z\bar{\mathbf{e}}_{13}+\mathcal{B}_1^z\bar{\mathbf{e}}_{14}+\mathcal{C}_1^z\bar{\mathbf{e}}_{18}\big]^*\mathcal{P}_1\bar{\mathbf{e}}_1\\ &+\bar{\mathbf{e}}_{10}^*\mathcal{P}_2\big[-\mathcal{D}_2^z\bar{\mathbf{e}}_{10}+\mathcal{A}_2^z\bar{\mathbf{e}}_4+\mathcal{B}_2^z\bar{\mathbf{e}}_5+\mathcal{C}_2^z\bar{\mathbf{e}}_9\big]\\ &+\big[-\mathcal{D}_2^z\bar{\mathbf{e}}_{10}+\mathcal{A}_2^z\bar{\mathbf{e}}_4+\mathcal{B}_2^z\bar{\mathbf{e}}_5+\mathcal{C}_2^z\bar{\mathbf{e}}_9\big]^*\mathcal{P}_2\bar{\mathbf{e}}_{10},\\ \bar{\Omega}_2 =& \bar{\mathbf{e}}_1^*[\mathcal{Q}_1+\mathcal{Q}_5]\bar{\mathbf{e}}_1-\bar{\mathbf{e}}_3^*\mathcal{Q}_1\bar{\mathbf{e}}_3+\bar{\mathbf{e}}_{10}^*[\mathcal{Q}_2+\mathcal{Q}_6]\bar{\mathbf{e}}_{10}\\ &-\bar{\mathbf{e}}_{12}^*\mathcal{Q}_2\bar{\mathbf{e}}_{12}-(1-\mu)\bar{\mathbf{e}}_2^*\mathcal{Q}_5\bar{\mathbf{e}}_2-(1-\mu)\bar{\mathbf{e}}_{11}^*\mathcal{Q}_6\bar{\mathbf{e}}_{11},\\ \bar{\Omega}_3 =& \bar{\mathbf{e}}_1^*\ell^2\mathcal{R}_1\bar{\mathbf{e}}_1+\bar{\mathbf{e}}_{10}^*\ell^2\mathcal{R}_2\bar{\mathbf{e}}_{10}\\ &-\begin{bmatrix}\bar{\mathbf{e}}_6\\ \bar{\mathbf{e}}_6-\frac{2}{\ell}\bar{\mathbf{e}}_7\\ \bar{\mathbf{e}}_6-\frac{6}{\ell}\bar{\mathbf{e}}_7+\frac{12}{\ell^2}\bar{\mathbf{e}}_8\end{bmatrix}^*\begin{bmatrix}\mathcal{R}_1&0&0\\0&3\mathcal{R}_1&0\\0&0&5\mathcal{R}_1\end{bmatrix}\begin{bmatrix}\bar{\mathbf{e}}_6\\ \bar{\mathbf{e}}_6-\frac{2}{\ell}\bar{\mathbf{e}}_7\\ \bar{\mathbf{e}}_6-\frac{6}{\ell}\bar{\mathbf{e}}_7+\frac{12}{\ell^2}\bar{\mathbf{e}}_8\end{bmatrix}\\ &-\begin{bmatrix}\bar{\mathbf{e}}_{15}\\ \bar{\mathbf{e}}_{15}-\frac{2}{\ell}\bar{\mathbf{e}}_{16}\\ \bar{\mathbf{e}}_{15}-\frac{6}{\ell}\bar{\mathbf{e}}_{16}+\frac{12}{\ell^2}\bar{\mathbf{e}}_{17}\end{bmatrix}^*\begin{bmatrix}\mathcal{R}_2&0&0\\0&3\mathcal{R}_2&0\\0&0&5\mathcal{R}_2\end{bmatrix}\begin{bmatrix}\bar{\mathbf{e}}_{15}\\ \bar{\mathbf{e}}_{15}-\frac{2}{\ell}\bar{\mathbf{e}}_{16}\\ \bar{\mathbf{e}}_{15}-\frac{6}{\ell}\bar{\mathbf{e}}_{16}+\frac{12}{\ell^2}\bar{\mathbf{e}}_{17}\end{bmatrix},\\ \bar{\Omega}_4 =& \bar{\mathbf{e}}_{10}^*\mathcal{L}_{\mathbf{h}_1}^*\mathcal{X}\mathcal{L}_{\mathbf{h}_1}\bar{\mathbf{e}}_{10}-\bar{\mathbf{e}}_{18}^*\mathcal{X}\bar{\mathbf{e}}_{18}+\bar{\mathbf{e}}_1^*\mathcal{L}_{\mathbf{h}_2}^*\mathcal{Y}\mathcal{L}_{\mathbf{h}_2}\bar{\mathbf{e}}_1-\bar{\mathbf{e}}_9^*\mathcal{Y}\bar{\mathbf{e}}_9,\end{aligned}$$

$$\begin{aligned}\bar{\Omega}_5 = &\ \bar{\mathbf{e}}_1^*\mathcal{L}_{\mathbf{f}_2}^*\mathcal{G}_1\mathcal{L}_{\mathbf{f}_2}\bar{\mathbf{e}}_1 - \bar{\mathbf{e}}_4^*\mathcal{G}_1\bar{\mathbf{e}}_4 + \bar{\mathbf{e}}_2^*\mathcal{L}_{\mathbf{g}_2}^*\mathcal{G}_2\mathcal{L}_{\mathbf{g}_2}\bar{\mathbf{e}}_2 - \bar{\mathbf{e}}_5^*\mathcal{G}_2\bar{\mathbf{e}}_5\\ &+\bar{\mathbf{e}}_{10}^*\mathcal{L}_{\mathbf{f}_1}^*\mathcal{G}_3\mathcal{L}_{\mathbf{f}_1}\bar{\mathbf{e}}_{10} - \bar{\mathbf{e}}_{13}^*\mathcal{G}_3\bar{\mathbf{e}}_{13} + \bar{\mathbf{e}}_{11}^*\mathcal{L}_{\mathbf{g}_1}^*\mathcal{G}_4\mathcal{L}_{\mathbf{g}_1}\bar{\mathbf{e}}_{11} - \bar{\mathbf{e}}_{14}^*\mathcal{G}_4\bar{\mathbf{e}}_{14}.\end{aligned}$$

Corollary 7 *If Assumption 1-3 are fulfilled, if there exist Hermitian matrices* $0 < \mathcal{P}_1$, $0 < \mathcal{P}_2$, $0 < \mathcal{Q}_1$, $0 < \mathcal{Q}_2$, $0 < \mathcal{Q}_5$, $0 < \mathcal{Q}_6$, $0 < \mathcal{R}_1$, $0 < \mathcal{R}_2$, *and diagonal matrices* $0 < \mathcal{X}$, $0 < \mathcal{Y}$, $0 < \mathcal{G}_1$, $0 < \mathcal{G}_2$, $0 < \mathcal{G}_3$, $0 < \mathcal{G}_4$ *such that the following LMI holds for all* $z = 1, 2, ..., m$

$$\bar{\Omega}_1^z + \bar{\Omega}_2 + \bar{\Omega}_3 + \bar{\Omega}_4 + \bar{\Omega}_5 < 0,\ z = 1, 2, ..., m, \tag{44}$$

then the equilibrium point of NN model (44) *is GAS.*

Proof Replacing $\mathcal{V}_1(t, \mathbf{u}(t), \mathbf{v}(t), z) = \mathbf{u}^*(t)\mathcal{P}_1\mathbf{u}(t) + \mathbf{v}^*(t)\mathcal{P}_2\mathbf{v}(t)$, and $\mathcal{Q}_3 = \mathcal{Q}_4 = \mathcal{R}_3 = \mathcal{R}_4 = 0$ in LKF (26). The remaining proof is similar to that in Theorem (6), and so it is omitted.

Remark 2 When there is no distributed delay, the NN model (24) becomes:

$$\begin{cases}\dot{\mathbf{u}}(t) = \sum\limits_{z=1}^{m}\chi_z(\vartheta(t))\Big\{ - \mathcal{D}_1^z\mathbf{u}(t-\delta) + \mathcal{A}_1^z\hat{\mathbf{f}}_1(\mathbf{v}(t)) + \mathcal{B}_1^z\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t)))\Big\},\\ \mathbf{u}(t) = \hat{\varphi}_1(t),\ t\in[-\sigma, 0],\\ \dot{\mathbf{v}}(t) = \sum\limits_{z=1}^{m}\chi_z(\vartheta(t))\Big\{ - \mathcal{D}_2^z\mathbf{v}(t-\delta) + \mathcal{A}_2^z\hat{\mathbf{f}}_2(\mathbf{u}(t)) + \mathcal{B}_2^z\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t)))\Big\},\\ \mathbf{v}(t) = \hat{\varphi}_2(t),\ t\in[-\sigma, 0].\end{cases} \tag{45}$$

The following Corollary (8) gives the GAS criterion for the NNs (45). For simplicity, we define the notations as follows:

$$\begin{aligned}\tilde{\xi}(t) = \Bigg[&\mathbf{u}^*(t)\ \ \mathbf{u}^*(t-\ell(t))\ \ \mathbf{u}^*(t-\ell)\ \ \mathbf{u}^*(t-\delta)\ \hat{\mathbf{f}}_2^*(\mathbf{u}(t))\ \ \hat{\mathbf{g}}_2^*(\mathbf{u}(t-\ell(t)))\\ &\int_{t-\ell}^{t}\mathbf{u}^*(s)ds\ \ \frac{1}{\ell}\int_{t-\ell}^{t}\int_{s}^{t}\mathbf{u}^*(u)duds\ \ \frac{1}{\ell^2}\int_{t-\ell}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{u}^*(v)dvduds\\ &\int_{t-\delta}^{t}\mathbf{u}^*(s)ds\ \ \frac{1}{\delta}\int_{t-\delta}^{t}\int_{s}^{t}\mathbf{u}^*(u)duds\ \ \frac{1}{\delta^2}\int_{t-\delta}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{u}^*(v)dvduds\\ &\mathbf{v}^*(t)\ \ \mathbf{v}^*(t-\ell(t))\ \ \mathbf{v}^*(t-\ell)\ \ \mathbf{v}^*(t-\delta)\ \ \hat{\mathbf{f}}_1^*(\mathbf{v}(t))\ \ \hat{\mathbf{g}}_1^*(\mathbf{v}(t-\ell(t)))\\ &\int_{t-\ell}^{t}\mathbf{v}^*(s)ds\ \ \frac{1}{\ell}\int_{t-\ell}^{t}\int_{s}^{t}\mathbf{v}^*(u)duds\ \ \frac{1}{\ell^2}\int_{t-\ell}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{v}^*(v)dvduds\\ &\int_{t-\delta}^{t}\mathbf{v}^*(s)ds\ \ \frac{1}{\delta}\int_{t-\delta}^{t}\int_{s}^{t}\mathbf{v}^*(u)duds\ \ \frac{1}{\delta^2}\int_{t-\delta}^{t}\int_{s}^{t}\int_{u}^{t}\mathbf{v}^*(v)dvduds\Bigg]^*,\\ \tilde{\mathbf{e}}_r =&\ [0_{n\times(r-1)n}\ \ \mathcal{I}_{n\times n}\ \ 0_{24\times(r-1)n}],\ r = 1, 2, ..., 24,\\ \tilde{\Omega}_1^z =&\ \big[\tilde{\mathbf{e}}_1 - \mathcal{D}_1^z\tilde{\mathbf{e}}_{10}\big]^*\mathcal{P}_1\big[-\mathcal{D}_1^z\tilde{\mathbf{e}}_1 + \mathcal{A}_1^z\tilde{\mathbf{e}}_{17} + \mathcal{B}_1^z\tilde{\mathbf{e}}_{18}\big]\\ &+\big[-\mathcal{D}_1^z\tilde{\mathbf{e}}_1 + \mathcal{A}_1^z\tilde{\mathbf{e}}_{17} + \mathcal{B}_1^z\tilde{\mathbf{e}}_{18}\big]^*\mathcal{P}_1\big[\tilde{\mathbf{e}}_1 - \mathcal{D}_1^z\tilde{\mathbf{e}}_{10}\big]\\ &+\big[\tilde{\mathbf{e}}_{13} - \mathcal{D}_2^z\tilde{\mathbf{e}}_{22}\big]^*\mathcal{P}_2\big[-\mathcal{D}_2^z\tilde{\mathbf{e}}_{13} + \mathcal{A}_2^z\tilde{\mathbf{e}}_5 + \mathcal{B}_2^z\tilde{\mathbf{e}}_6\big]\\ &+\big[-\mathcal{D}_2^z\tilde{\mathbf{e}}_{13} + \mathcal{A}_2^z\tilde{\mathbf{e}}_5 + \mathcal{B}_2^z\tilde{\mathbf{e}}_6\big]^*\mathcal{P}_2\big[\tilde{\mathbf{e}}_{13} - \mathcal{D}_2^z\tilde{\mathbf{e}}_{22}\big],\\ \tilde{\Omega}_2 =&\ \tilde{\mathbf{e}}_1^*\big[\mathcal{Q}_1 + \mathcal{Q}_3 + \mathcal{Q}_5\big]\tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_3^*\mathcal{Q}_1\tilde{\mathbf{e}}_3 + \tilde{\mathbf{e}}_{13}^*\big[\mathcal{Q}_2 + \mathcal{Q}_4 + \mathcal{Q}_6\big]\tilde{\mathbf{e}}_{13} - \tilde{\mathbf{e}}_{15}^*\mathcal{Q}_2\tilde{\mathbf{e}}_{15}\\ &-\tilde{\mathbf{e}}_4^*\mathcal{Q}_3\tilde{\mathbf{e}}_4 - \tilde{\mathbf{e}}_{16}^*\mathcal{Q}_4\tilde{\mathbf{e}}_{16} - (1-\mu)\tilde{\mathbf{e}}_2^*\mathcal{Q}_5\tilde{\mathbf{e}}_2 - (1-\mu)\tilde{\mathbf{e}}_{14}^*\mathcal{Q}_6\tilde{\mathbf{e}}_{14},\end{aligned}$$

$$\begin{aligned}
\tilde{\Omega}_3 &= \tilde{\mathbf{e}}_1^*\big[\ell^2\mathcal{R}_1 + \delta^2\mathcal{R}_3\big]\tilde{\mathbf{e}}_1 + \tilde{\mathbf{e}}_{13}^*\big[\ell^2\mathcal{R}_2 + \delta^2\mathcal{R}_4\big]\tilde{\mathbf{e}}_{13} \\
&\quad - \begin{bmatrix} \tilde{\mathbf{e}}_7 \\ \tilde{\mathbf{e}}_7 - \frac{2}{\ell}\tilde{\mathbf{e}}_8 \\ \tilde{\mathbf{e}}_7 - \frac{6}{\ell}\tilde{\mathbf{e}}_8 + \frac{12}{\ell^2}\tilde{\mathbf{e}}_9 \end{bmatrix}^* \begin{bmatrix} \mathcal{R}_1 & 0 & 0 \\ 0 & 3\mathcal{R}_1 & 0 \\ 0 & 0 & 5\mathcal{R}_1 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{e}}_7 \\ \tilde{\mathbf{e}}_7 - \frac{2}{\ell}\tilde{\mathbf{e}}_8 \\ \tilde{\mathbf{e}}_7 - \frac{6}{\ell}\tilde{\mathbf{e}}_8 + \frac{12}{\ell^2}\tilde{\mathbf{e}}_9 \end{bmatrix} \\
&\quad - \begin{bmatrix} \tilde{\mathbf{e}}_{19} \\ \tilde{\mathbf{e}}_{19} - \frac{2}{\ell}\tilde{\mathbf{e}}_{20} \\ \tilde{\mathbf{e}}_{19} - \frac{6}{\ell}\tilde{\mathbf{e}}_{20} + \frac{12}{\ell^2}\tilde{\mathbf{e}}_{21} \end{bmatrix}^* \begin{bmatrix} \mathcal{R}_2 & 0 & 0 \\ 0 & 3\mathcal{R}_2 & 0 \\ 0 & 0 & 5\mathcal{R}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{e}}_{19} \\ \tilde{\mathbf{e}}_{19} - \frac{2}{\ell}\tilde{\mathbf{e}}_{20} \\ \tilde{\mathbf{e}}_{19} - \frac{6}{\ell}\tilde{\mathbf{e}}_{20} + \frac{12}{\ell^2}\tilde{\mathbf{e}}_{21} \end{bmatrix} \\
&\quad - \begin{bmatrix} \tilde{\mathbf{e}}_{10} \\ \tilde{\mathbf{e}}_{10} - \frac{2}{\ell}\tilde{\mathbf{e}}_{11} \\ \tilde{\mathbf{e}}_{10} - \frac{6}{\ell}\tilde{\mathbf{e}}_{11} + \frac{12}{\ell^2}\tilde{\mathbf{e}}_{12} \end{bmatrix}^* \begin{bmatrix} \mathcal{R}_3 & 0 & 0 \\ 0 & 3\mathcal{R}_3 & 0 \\ 0 & 0 & 5\mathcal{R}_3 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{e}}_{10} \\ \tilde{\mathbf{e}}_{10} - \frac{2}{\ell}\tilde{\mathbf{e}}_{11} \\ \tilde{\mathbf{e}}_{10} - \frac{6}{\ell}\tilde{\mathbf{e}}_{11} + \frac{12}{\ell^2}\tilde{\mathbf{e}}_{12} \end{bmatrix} \\
&\quad - \begin{bmatrix} \tilde{\mathbf{e}}_{22} \\ \tilde{\mathbf{e}}_{22} - \frac{2}{\ell}\tilde{\mathbf{e}}_{23} \\ \tilde{\mathbf{e}}_{22} - \frac{6}{\ell}\tilde{\mathbf{e}}_{23} + \frac{12}{\ell^2}\tilde{\mathbf{e}}_{24} \end{bmatrix}^* \begin{bmatrix} \mathcal{R}_4 & 0 & 0 \\ 0 & 3\mathcal{R}_4 & 0 \\ 0 & 0 & 5\mathcal{R}_4 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{e}}_{22} \\ \tilde{\mathbf{e}}_{22} - \frac{2}{\ell}\tilde{\mathbf{e}}_{23} \\ \tilde{\mathbf{e}}_{22} - \frac{6}{\ell}\tilde{\mathbf{e}}_{23} + \frac{12}{\ell^2}\tilde{\mathbf{e}}_{24} \end{bmatrix}, \\
\tilde{\Omega}_4 &= \tilde{\mathbf{e}}_1^*\mathcal{L}_{\mathbf{f}_2}^*\mathcal{G}_1\mathcal{L}_{\mathbf{f}_2}\tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_5^*\mathcal{G}_1\tilde{\mathbf{e}}_5 + \tilde{\mathbf{e}}_2^*\mathcal{L}_{\mathbf{g}_2}^*\mathcal{G}_2\mathcal{L}_{\mathbf{g}_2}\tilde{\mathbf{e}}_2 - \tilde{\mathbf{e}}_6^*\mathcal{G}_2\tilde{\mathbf{e}}_6 \\
&\quad + \tilde{\mathbf{e}}_{13}^*\mathcal{L}_{\mathbf{f}_1}^*\mathcal{G}_3\mathcal{L}_{\mathbf{f}_1}\tilde{\mathbf{e}}_{13} - \tilde{\mathbf{e}}_{17}^*\mathcal{G}_3\tilde{\mathbf{e}}_{17} + \tilde{\mathbf{e}}_{14}^*\mathcal{L}_{\mathbf{g}_1}^*\mathcal{G}_4\mathcal{L}_{\mathbf{g}_1}\tilde{\mathbf{e}}_{14} - \tilde{\mathbf{e}}_{18}^*\mathcal{G}_4\tilde{\mathbf{e}}_{18}.
\end{aligned}$$

Corollary 8 *If Assumption 1-2 are fulfilled, if there exist Hermitian matrices* $0 < \mathcal{P}_1$, $0 < \mathcal{P}_2$, $0 < \mathcal{Q}_1$, $0 < \mathcal{Q}_2$, $0 < \mathcal{Q}_3$, $0 < \mathcal{Q}_4$, $0 < \mathcal{Q}_5$, $0 < \mathcal{Q}_6$, $0 < \mathcal{R}_1$, $0 < \mathcal{R}_2$, $0 < \mathcal{R}_3$, $0 < \mathcal{R}_4$, *and diagonal matrices* $0 < \mathcal{G}_1$, $0 < \mathcal{G}_2$, $0 < \mathcal{G}_3$, $0 < \mathcal{G}_4$ *such that the following LMI holds for all* $z = 1, 2, ..., m$

$$\tilde{\Omega}_1^z + \tilde{\Omega}_2 + \tilde{\Omega}_3 + \tilde{\Omega}_4 < 0, \; z = 1, 2, ..., m, \tag{46}$$

then the equilibrium point of NN model (46) is GAS.

Proof Take $\mathcal{V}_1(t, \mathbf{u}(t), \mathbf{v}(t), z)$, $\mathcal{V}_2(t, \mathbf{u}(t), \mathbf{v}(t), z)$, $\mathcal{V}_3(t, \mathbf{u}(t), \mathbf{v}(t), z)$ same as in LKF (26) and $\mathcal{V}_4(t, \mathbf{u}(t), \mathbf{v}(t), z), = 0$ in LKF (26). The remaining proof is similar to that in Theorem (6), and so it is omitted.

Remark 3 It is a special case of the NNs (24) when $m = 1$. For simplicity, we deleted the superscript 1.

$$\begin{cases}
\dot{\mathbf{u}}(t) = & -\mathcal{D}_1\mathbf{u}(t-\delta) + \mathcal{A}_1\hat{\mathbf{f}}_1(\mathbf{v}(t)) + \mathcal{B}_1\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t))) + \mathcal{C}_1\displaystyle\int_{-\infty}^{t} K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds, \\
\mathbf{u}(t) = & \hat{\varphi}_1(t), \; t \in [-\sigma, 0], \\
\dot{\mathbf{v}}(t) = & -\mathcal{D}_2\mathbf{v}(t-\delta) + \mathcal{A}_2\hat{\mathbf{f}}_2(\mathbf{u}(t)) + \mathcal{B}_2\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t))) + \mathcal{C}_2\displaystyle\int_{-\infty}^{t} K_2(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds, \\
\mathbf{v}(t) = & \hat{\varphi}_2(t), \; t \in [-\sigma, 0].
\end{cases} \tag{47}$$

Remark 4 In [16–29], the authors used several methods to study the dynamics of delayed QVNNs. As an example, (i) The real-valued decomposition method [20]; (ii) The complex-valued decomposition method [25]; (iii) The direct quaternion method [19]. In general, real-valued and complex-valued decomposition methods have two problems. The decomposition method increases the size of the systems and which makes mathematical challenges, and also the decomposition method leads to the complexity of theoretical analysis. Therefore, this paper uses the non-separation method to resolve this issue.

Remark 5 In recent years, there have been several notable studies have been conducted on the stability issues of various QVNNs. For example, in [17], the authors examined fractional-order QVNNs with impulses and derived some sufficient conditions for global Mittag-Leffler stability by using direct quaternion method. In [26], the authors considered QVNNs with uncertain time-delayed impulses, and stability and stabilization analysis was conducted based on direct quaternion method. In [27], the authors examined QVNNs with inertial term and time-varying delay, and derived some sufficient conditions for global exponential and asymptotic synchronization direct quaternion method. In [28], the authors considered QVNNs with time-varying delays and studied their global μ-stability and power stability issues. It is important to note that there have been no studies conducted on T-S fuzzy QVBAMNNs with discrete, distributed and leakage delays by using non-separation method. Therefore, this paper aims to fill such gap by considering T-S fuzzy QVBAMNNs with discrete, distributed and leakage delays.

Remark 6 The authors of [44] used WBII to investigate the global μ-stability of neutral-type impulsive complex-valued BAMNNs with leakage delay and unbounded time delays by non-separation method. In comparison to [44], we extended the AFBII into the quaternion domain and its proof has been presented for the first time. Furthermore, the GAS criteria for T-S fuzzy QVBAMNNs are established by using new quaternion-valued AFBII and non-separation method.

4 Numerical Evaluations

This section provides two numerical evaluations to emphasize the applicability of the theoretical analysis.
Example 1: Consider the following two neuron QVBAMNNs

$$\begin{cases} \dot{\mathbf{p}}(t) = & -\mathcal{D}_1\mathbf{p}(t-\delta) + \mathcal{A}_1\mathbf{f}_1(\mathbf{q}(t)) + \mathcal{B}_1\mathbf{g}_1(\mathbf{q}(t-\ell(t))) + \mathcal{C}_1\int_{-\infty}^{t} K_1(t-s)\mathbf{h}_1(\mathbf{q}(s))ds + \mathcal{J}_1, \\ \mathbf{p}(t) = & \varphi_1(t),\ t\in[-\sigma,0], \\ \dot{\mathbf{q}}(t) = & -\mathcal{D}_2\mathbf{q}(t-\delta) + \mathcal{A}_2\mathbf{f}_2(\mathbf{p}(t)) + \mathcal{B}_2\mathbf{g}_2(\mathbf{p}(t-\ell(t))) + \mathcal{C}_2\int_{-\infty}^{t} K_2(t-s)\mathbf{h}_2(\mathbf{p}(s))ds + \mathcal{J}_2, \\ \mathbf{q}(t) = & \varphi_2(t),\ t\in[-\sigma,0], \end{cases} \tag{48}$$

where

$$\mathcal{D}_1 = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix},\ \mathcal{D}_2 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix},$$

$$\mathcal{A}_1 = \begin{bmatrix} 0.4-0.1i+0.3j+0.5k & 0.5+0.1i+0.2j+0.3k \\ 0.1-0.2i+0.2j+0.2k & 0.2+0.3i+0.1j+0.1k \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0.2+0.3i+0.2j+0.1k & 0.2-0.2i+0.3j+0.5k \\ 0.3+0.4i+0.1j+0.7k & 0.3+0.3i+0.2j+0.1k \end{bmatrix},$$

$$\mathcal{B}_1 = \begin{bmatrix} 0.3+0.2i+0.4j+0.1k & 0.3+0.3i+0.2j+0.1k \\ 0.2+0.1i+0.2j+0.5k & 0.4+0.3i+0.1j+0.7k \end{bmatrix},$$

$$\mathcal{B}_2 = \begin{bmatrix} 0.3-0.2i+0.4j+0.2k & 0.3+0.3i+0.4j+0.1k \\ 0.2+0.1i+0.4j+0.5k & 0.2+0.1i+0.2j+0.1k \end{bmatrix},$$

$$\mathcal{C}_1 = \begin{bmatrix} 0.2+0.1i-0.1j+0.1k & 0.1-0.1i-0.1j-0.1k \\ -0.2+0.1i-0.1j+0.1k & 0.1+0.1i-0.2j-0.2k \end{bmatrix},$$
$$\mathcal{C}_2 = \begin{bmatrix} 0.3+0.1i-0.1j-0.1k & 0.1+0.2i+0.3j-0.1kk \\ 0.3-0.4i-0.2j-0.1k & 0.4+0.1i-0.1j-0.2k \end{bmatrix},$$
$$\mathcal{J}_1 = \begin{bmatrix} 0.2+0.1i-0.2j-0.1k \\ 0.1+0.2i-0.1j-0.1k \end{bmatrix}, \ \mathcal{J}_2 = \begin{bmatrix} 0.2+0.2i-0.1j-0.2k \\ 0.3+0.1i-0.1j-0.2k \end{bmatrix}.$$

The activation functions are taken as $\mathbf{f}_{1s}(\mathbf{q}_s(\cdot)) = \mathbf{g}_{1s}(\mathbf{q}_s(\cdot)) = \mathbf{h}_{1s}(\mathbf{q}_s(\cdot)) = 0.5\ tanh(\mathbf{q}_s(\cdot)) + 0.5\ tanh(\mathbf{q}_s(\cdot))i + 0.5\ tanh(\mathbf{q}_s(\cdot))j + 0.5\ tanh(\mathbf{q}_s(\cdot))k$, $\mathbf{f}_{2r}(\mathbf{p}_r(\cdot)) = \mathbf{g}_{2r}(\mathbf{p}_r(\cdot)) = \mathbf{h}_{2r}(\mathbf{p}_r(\cdot)) = 0.5\ tanh(\mathbf{p}_r(\cdot)) + 0.5\ tanh(\mathbf{p}_r(\cdot))i + 0.5\ tanh(\mathbf{p}_r(\cdot))j + 0.5\ tanh(\mathbf{p}_r(\cdot))k$ $(s, r = 1, 2)$. Obviously, they satisfy Assumption 2 with $l_s^{\mathbf{f}_1} = l_r^{\mathbf{f}_2} = l_s^{\mathbf{g}_1} = l_r^{\mathbf{g}_2} = l_s^{\mathbf{h}_1} = l_r^{\mathbf{h}_2} = 0.25$ $(s, r = 1, 2)$. The discrete delay $\ell(t)$ is regarded as $\ell(t) = 0.1 + 0.2\ sin(t)$, implying that the maximum permissible upper bound is $\ell = 0.3$. It is observable that $0 \leq \dot{\ell}(t) \leq \mu = 0 \leq 0.2\ cos(t) \leq 0.2$, and distributed delays $K_1(t) = K_2(t) = e^{-t}$.

By employing MATLAB YALMIP toolbox, the LMIs (2) and (3) in Theorem (5) are verified and the feasible solutions are

$$\mathcal{P}_1 = \begin{bmatrix} 1.1742 & -0.0124-0.2270i+0.0452j-0.2277k \\ -0.0124+0.2270i-0.0452j+0.2277k & 1.3084 \end{bmatrix},$$
$$\mathcal{P}_2 = \begin{bmatrix} 1.0972 & 0.0266+0.0320i+0.1040j+0.0126k \\ 0.0266-0.0320i-0.1040j-0.0126k & 1.1269 \end{bmatrix},$$
$$\mathcal{O}_1 = \begin{bmatrix} 4.6913 & 0 \\ 0 & 4.1476 \end{bmatrix}, \ \mathcal{O}_2 = \begin{bmatrix} 2.3596 & 0 \\ 0 & 2.5704 \end{bmatrix}, \ \mathcal{O}_3 = \begin{bmatrix} 2.5674 & 0 \\ 0 & 2.9031 \end{bmatrix},$$
$$\mathcal{O}_4 = \begin{bmatrix} 1.3350 & 0 \\ 0 & 1.9042 \end{bmatrix}, \ \mathcal{O}_5 = \begin{bmatrix} 2.5683 & 0 \\ 0 & 2.9961 \end{bmatrix}, \ \mathcal{O}_6 = \begin{bmatrix} 3.6453 & 0 \\ 0 & 3.6224 \end{bmatrix}.$$

Based on this example, we conclude that all the conditions associated with Theorem (5) are fulfilled and the NN model (48) has a unique equilibrium point.

Example 2: Consider the following two neuron T-S fuzzy QVBAMNNs with $z = 1, 2$

$$\begin{cases} \dot{\mathbf{u}}(t) & = \sum_{z=1}^{2} \chi_z(\vartheta(t)) \Big\{ -\mathcal{D}_1^z \mathbf{u}(t-\delta) + \mathcal{A}_1^z \hat{\mathbf{f}}_1(\mathbf{v}(t)) + \mathcal{B}_1^z \hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t))) \\ & \quad + \mathcal{C}_1^z \int_{-\infty}^{t} K_1(t-s) \hat{\mathbf{h}}_1(\mathbf{v}(s)) ds \Big\}, \\ \mathbf{u}(t) & = \hat{\varphi}_1(t), \ t \in [-\sigma, 0], \\ \dot{\mathbf{v}}(t) & = \sum_{z=1}^{2} \chi_z(\vartheta(t)) \Big\{ -\mathcal{D}_2^z \mathbf{v}(t-\delta) + \mathcal{A}_2^z \hat{\mathbf{f}}_2(\mathbf{u}(t)) + \mathcal{B}_2^z \hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t))) \\ & \quad + \mathcal{C}_2^z \int_{-\infty}^{t} K_2(t-s) \hat{\mathbf{h}}_2(\mathbf{u}(s)) ds \Big\}, \\ \mathbf{v}(t) = & \hat{\varphi}_2(t), \ t \in [-\sigma, 0]. \end{cases} \tag{49}$$

Plant Rule 1: If $\vartheta_1(t)$ is η_1^1, Then

$$\begin{cases} \dot{\mathbf{u}}(t) = & -\mathcal{D}_1^1\mathbf{u}(t-\delta) + \mathcal{A}_1^1\hat{\mathbf{f}}_1(\mathbf{v}(t)) + \mathcal{B}_1^1\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t))) + \mathcal{C}_1^1\int_{-\infty}^t K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds, \\ \mathbf{u}(t) = & \hat{\varphi}_1(t),\ t\in[-\sigma,0], \\ \dot{\mathbf{v}}(t) = & -\mathcal{D}_2^1\mathbf{v}(t-\delta) + \mathcal{A}_2^1\hat{\mathbf{f}}_2(\mathbf{u}(t)) + \mathcal{B}_2^1\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t))) + \mathcal{C}_2^1\int_{-\infty}^t K_2(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds, \\ \mathbf{v}(t) = & \hat{\varphi}_2(t),\ t\in[-\sigma,0]. \end{cases}$$

Plant Rule 2: If $\vartheta_1(t)$ is η_1^2, Then

$$\begin{cases} \dot{\mathbf{u}}(t) = & -\mathcal{D}_1^2\mathbf{u}(t-\delta) + \mathcal{A}_1^2\hat{\mathbf{f}}_1(\mathbf{v}(t)) + \mathcal{B}_1^2\hat{\mathbf{g}}_1(\mathbf{v}(t-\ell(t))) + \mathcal{C}_1^2\int_{-\infty}^t K_1(t-s)\hat{\mathbf{h}}_1(\mathbf{v}(s))ds, \\ \mathbf{u}(t) = & \hat{\varphi}_1(t),\ t\in[-\sigma,0], \\ \dot{\mathbf{v}}(t) = & -\mathcal{D}_2^2\mathbf{v}(t-\delta) + \mathcal{A}_2^2\hat{\mathbf{f}}_2(\mathbf{u}(t)) + \mathcal{B}_2^2\hat{\mathbf{g}}_2(\mathbf{u}(t-\ell(t))) + \mathcal{C}_2^2\int_{-\infty}^t K_2(t-s)\hat{\mathbf{h}}_2(\mathbf{u}(s))ds, \\ \mathbf{v}(t) = & \hat{\varphi}_2(t),\ t\in[-\sigma,0]. \end{cases}$$

where

$$\mathcal{D}_1^1 = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix},\ \mathcal{D}_2^1 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix},\ \mathcal{D}_1^2 = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix},\ \mathcal{D}_2^2 = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix},$$

$$\mathcal{A}_1^1 = \begin{bmatrix} 1.2+i-1.5j-0.8k & 1+1.2i+1.3j-1.5k \\ 1.4-2.7i-2j-1.3k & 0.5+0.8i-1.4j-1.7k \end{bmatrix},$$

$$\mathcal{A}_2^1 = \begin{bmatrix} 1.1-1.4i-1.3j-1.2k & 2.1+1.3i-0.9j-1.1k \\ 1.3+1.2i-1.2j+1.1k & -1.5+i+1.2j+1.4k \end{bmatrix},$$

$$\mathcal{A}_1^2 = \begin{bmatrix} 1.2+i-1.5j-0.8k & 1+1.2i+1.3j-1.5k \\ 1.4-2.7i-2j-1.3k & 0.5+0.8i-1.4j-1.7k \end{bmatrix},$$

$$\mathcal{A}_2^2 = \begin{bmatrix} 1.1-1.4i-1.3j-1.2k & 2.1+1.3i-0.9j-1.1k \\ 1.3+1.2i-1.2j+1.1k & -1.5+i+1.2j+1.4k \end{bmatrix},$$

$$\mathcal{B}_1^1 = \begin{bmatrix} 1+0.8i-1.2j-0.6k & 0.9+i+1.5j-1.2k \\ 1.5-2.5i-1.8j-1.2k & 0.7+0.5i-1.2j-1.4k \end{bmatrix},$$

$$\mathcal{B}_2^1 = \begin{bmatrix} 1-1.3i-1.2j-1.1k & 2+1.2i-0.8j-k \\ 1.2+1.1i-1.1j+1.2k & -1.3+0.9i+1.1j+1.2k \end{bmatrix},$$

$$\mathcal{B}_1^2 = \begin{bmatrix} 1.3+1.1i-1.4j-0.9k & 1.2+1.1i+1.4j-1.3k \\ 1.2-2.5i-2.2j-1.3k & 0.9+0.7i-1.2j-1.5k \end{bmatrix},$$

$$\mathcal{B}_2^2 = \begin{bmatrix} 1.2-i-1.4j-k & 2.2+1.4i-1.2j-1.3k \\ 1.2+i-1.1j+k & 1.5+1.2i+j+1.3k \end{bmatrix},$$

$$\mathcal{C}_1^1 = \begin{bmatrix} 0.8+0.9i-1.1j-1.3k & 1.4+1.3i+1.2j-1.2k \\ 1.2-1.7i-1.5j-1.4k & 0.9+0.6i-1.1j-1.5k \end{bmatrix},$$

$$\mathcal{C}_2^1 = \begin{bmatrix} 1-1.2i-1.1j-1.3k & 2+1.2i-0.8j-1.2k \\ 1.2+1.1i-1.1j+1.2k & -1.7+1.2i+1.3j+1.5k \end{bmatrix},$$

$$\mathcal{C}_1^2 = \begin{bmatrix} 1+1.1i-1.2j-0.8k & 1.2+1.3i+1.4j-1.5k \\ 1.5-2.5i-2.1j-1.4k & 0.8+0.9i-1.1j-1.3k \end{bmatrix},$$

$$\mathcal{C}_2^2 = \begin{bmatrix} 1.4-1.2i-1.1j-1.2k & 2+i-1.2j-1.2k \\ 1.4+1.2i-1.3j+1.2k & -1.5+1.2i+1.4j+1.4k \end{bmatrix}.$$

Table 1 Calculated upper bound of ℓ for different values of μ and δ in Example 2

δ / μ	$\mu = 0.1$	$\mu = 0.3$	$\mu = 0.5$	$\mu = 0.7$	$\mu = 0.9$
$\delta = 0$	1.2418	1.1814	1.1036	0.9862	0.8253
$\delta = 0.08$	0.8540	0.8057	0.7801	0.6941	0.5712
$\delta = 0.13$	0.5831	0.4625	0.4072	0.3826	0.3065
$\delta \geq 0.14$	Infeasible	Infeasible	Infeasible	Infeasible	Infeasible

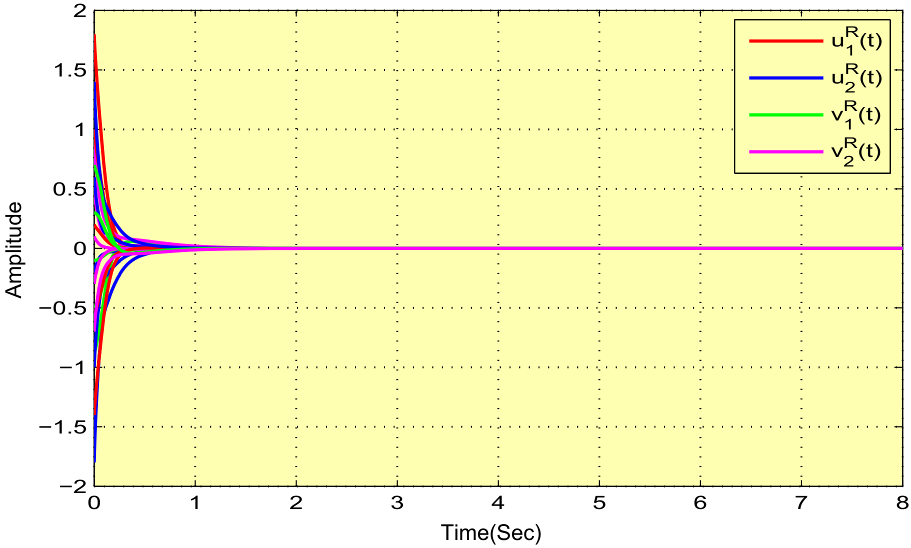

Fig. 1 Transient behaviors of the states $\mathbf{u}(t)^R$, $\mathbf{v}(t)^R$ of NNs (49) with $\delta = 0$

Let us consider the activation functions $\hat{\mathbf{f}}_{1_s}(\mathbf{v}_s(\cdot)) = \hat{\mathbf{g}}_{1_s}(\mathbf{v}_s(\cdot)) = \hat{\mathbf{h}}_{1_s}(\mathbf{v}_s(\cdot)) = 0.5\ tanh(\mathbf{v}_s(\cdot)) + 0.5\ tanh(\mathbf{v}_s(\cdot))i + 0.5\ tanh(\mathbf{v}_s(\cdot))j + 0.5\ tanh(\mathbf{v}_s(\cdot))k$, $\hat{\mathbf{f}}_{2_r}(\mathbf{u}_r(\cdot)) = \hat{\mathbf{g}}_{2_r}(\mathbf{u}_r(\cdot)) = \hat{\mathbf{h}}_{2_r}(\mathbf{u}_r(\cdot)) = 0.5\ tanh(\mathbf{u}_r(\cdot)) + 0.5\ tanh(\mathbf{u}_r(\cdot))i + 0.5\ tanh(\mathbf{u}_r(\cdot))j + 0.5\ tanh(\mathbf{u}_r(\cdot))k$ $(s, r = 1, 2)$. Obviously, they satisfy Assumption 2 with $l_s^{\mathbf{f}_1} = l_r^{\mathbf{f}_2} = l_s^{\mathbf{g}_1} = l_r^{\mathbf{g}_2} = l_s^{\mathbf{h}_1} = l_r^{\mathbf{h}_2} = 0.25$ $(s, r = 1, 2)$. The discrete delay $\ell(t)$ is regarded as $\ell(t) = 0.1625 + 0.3\ sin(t)$, which implies that $\ell = 0.4625$. It is observable that $0 \leq \dot{\ell}(t) \leq \mu = 0 \leq 0.3\,cos(t) \leq 0.3$, and distributed delays $K_1(t) = K_2(t) = e^{-t}$. Furthermore, the membership functions are considered as $\chi_1(\vartheta(t)) = \frac{1}{1+e^{-2t}}$, $\chi_2(\vartheta(t)) = 1 - \frac{1}{1+e^{-2t}}$.

By solving the LMI condition (25) in Theorem (6) with the above parameter values by using the MATLAB YALMIP toolbox, we obtain the maximum permissible upper bounds of ℓ for different values of μ and δ, which are listed in Table 1. Under the randomly selected 5 initial values, the time responses and phase diagrams of states $\mathbf{u}_1^R(t)$, $\mathbf{u}_1^I(t)$, $\mathbf{u}_1^J(t)$, $\mathbf{u}_1^K(t)$, $\mathbf{u}_2^R(t)$, $\mathbf{u}_2^I(t)$, $\mathbf{u}_2^J(t)$, $\mathbf{u}_2^K(t)$, $\mathbf{v}_1^R(t)$, $\mathbf{v}_1^I(t)$, $\mathbf{v}_1^J(t)$, $\mathbf{v}_1^K(t)$, $\mathbf{v}_2^R(t)$, $\mathbf{v}_2^I(t)$, $\mathbf{v}_2^J(t)$, $\mathbf{v}_2^K(t)$ of QVBAMNNs (49) are illustrated in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15 with various levels of leakage delay and fixed discrete delay. From this example, we can conclude that all the conditions associated with Theorem (6) are satisfied and the equilibrium point of NNs (24) is GAS.

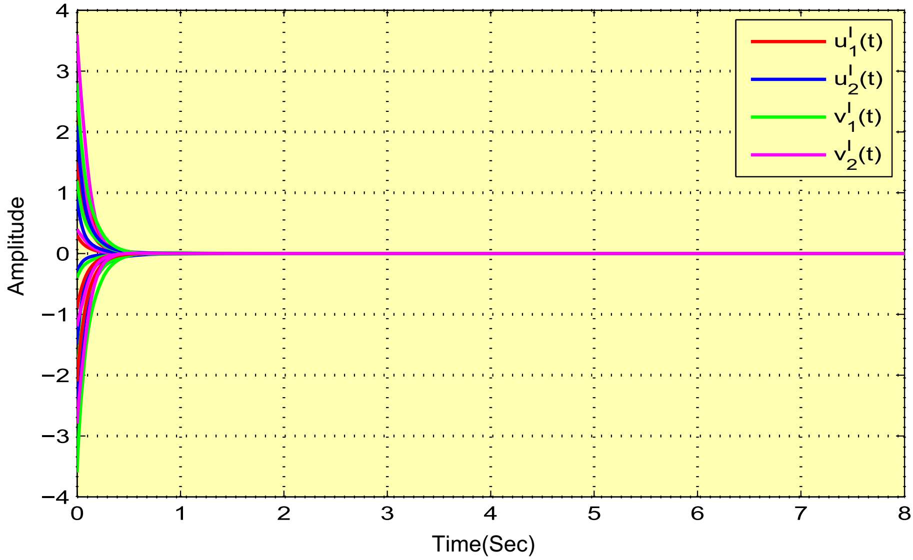

Fig. 2 Transient behaviors of the states $\mathbf{u}(t)^I$, $\mathbf{v}(t)^I$ of NNs (49) with $\delta = 0$

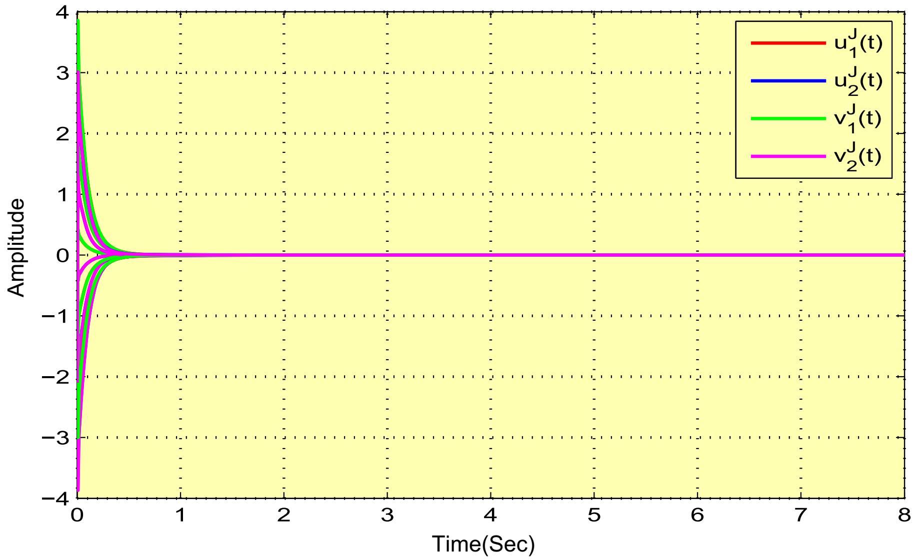

Fig. 3 Transient behaviors of the states $\mathbf{u}(t)^J$, $\mathbf{v}(t)^J$ of NNs (49) with $\delta = 0$

Remark 7 In general, leakage delay has a significant impact on the stability performance of the systems. For example, if we consider QVBAMNNs (49) with $\delta = 0$, then it becomes the well-known case which has been extensively studied by many authors over the years. In this case, the trajectories of the states of QVBAMNNs (49) converges to the equilibrium point $(0, 0)^T$ within a short period of time, as shown in Figs. 1, 2, 3, 4 and 5. When we take leakage delay $\delta = 0.13$, the trajectories of the states of QVBAMNNs (49) also converges to

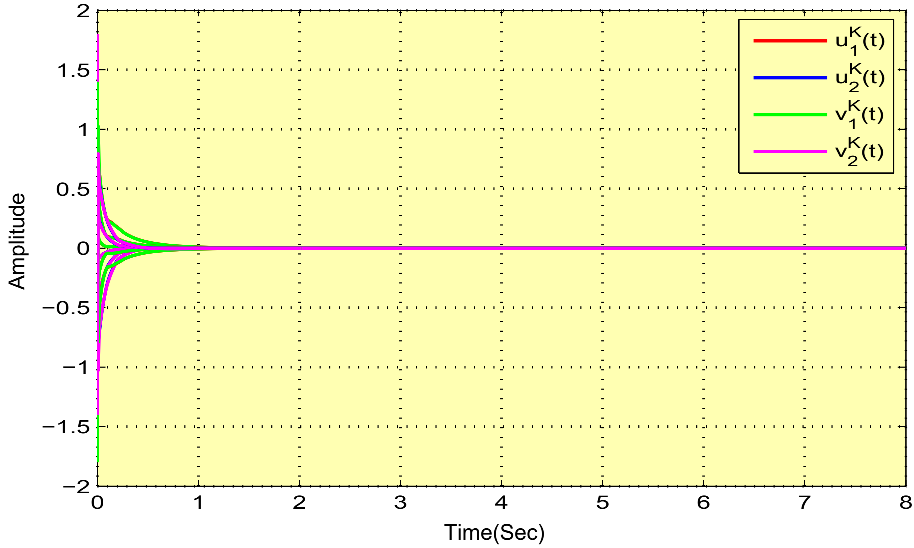

Fig. 4 Transient behaviors of the states $\mathbf{u}(t)^K$, $\mathbf{v}(t)^K$ of NNs (49) with $\delta = 0$

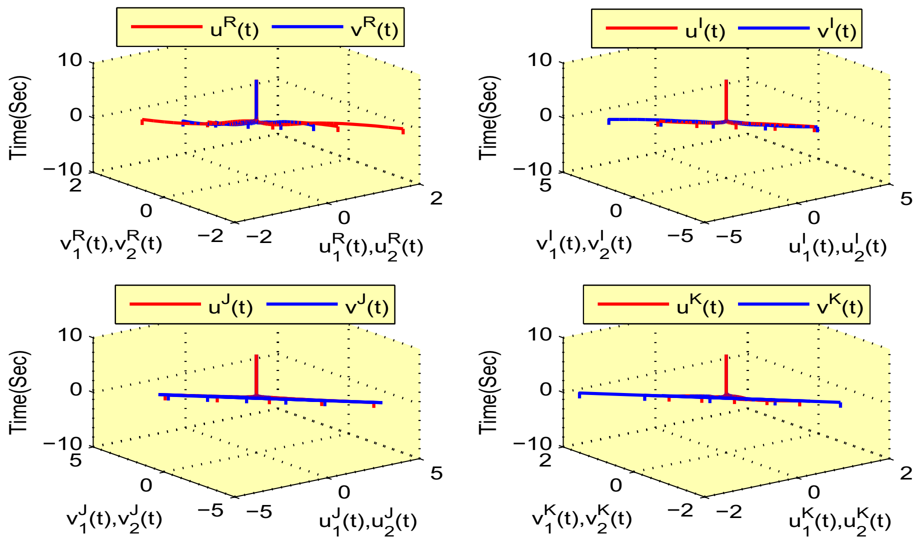

Fig. 5 Phase trajectories of the states $\mathbf{u}(t)^z$, $\mathbf{v}(t)^z$, $z = R, I, J, K$ of NNs (49) with $\delta = 0$

the equilibrium point $(0, 0)^T$ within a long period of time, as shown in Figs. 6, 7, 8, 9 and 10. In the above cases $0 \leq \delta \leq 0.13$, one can check that the LMI condition in Theorem (6) have feasible solutions via MATLAB YALMIP toolbox, which is listed in Table 1. However, in the case of leakage delay $\delta \geq 0.14$, one can check that the LMI condition in Theorem (6) does not have feasible solutions via MATLAB YALMIP toolbox, since the maximum permissible value of leakage delay is $\delta = 0.13$. In this case, the trajectories of the states of QVBAMNNs (49) does not converges to the equilibrium point $(0, 0)^T$, as shown in Figs. 11, 12, 13, 14 and 15. These simulation Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and

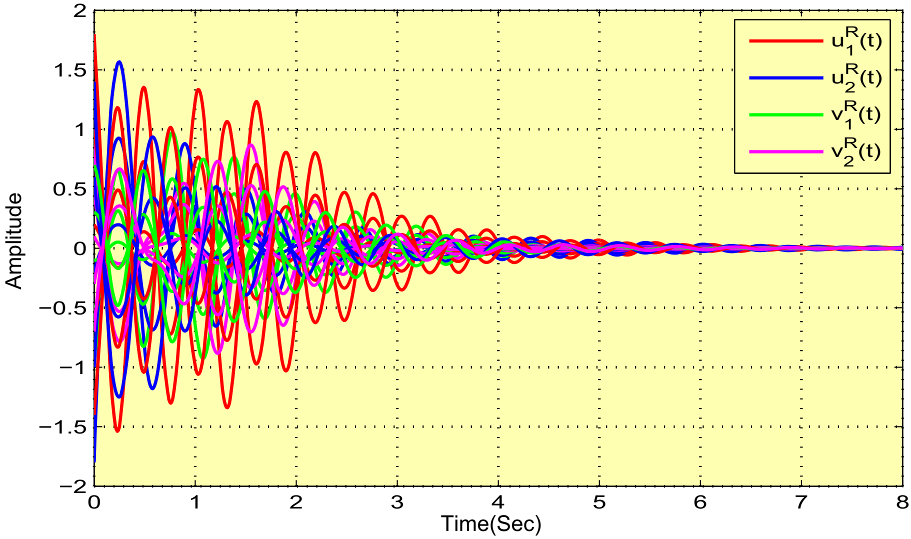

Fig. 6 Transient behaviors of the states $\mathbf{u}(t)^R$, $\mathbf{v}(t)^R$ of NNs (49) with $\delta = 0.13$

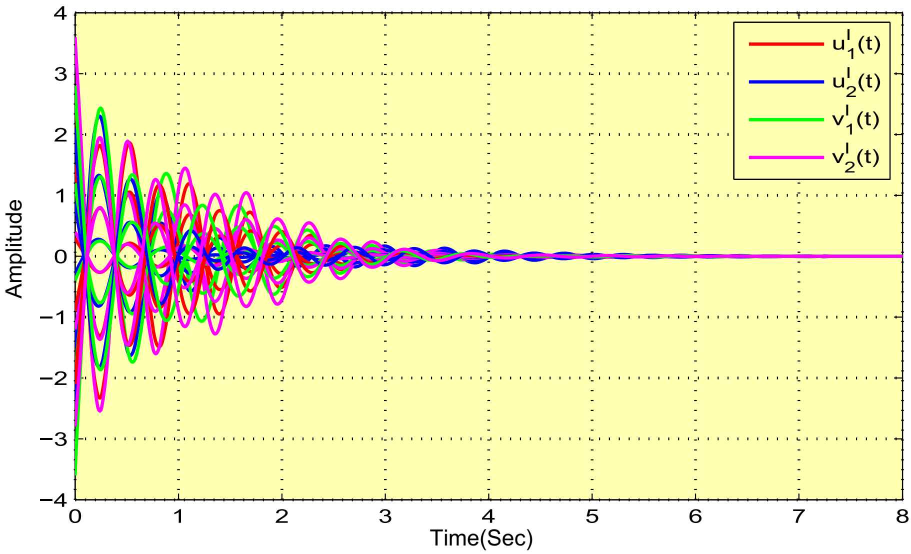

Fig. 7 Transient behaviors of the states $\mathbf{u}(t)^I$, $\mathbf{v}(t)^I$ of NNs (49) with $\delta = 0.13$

15 illustrate how leakage delay has a significant influence on the stability of QVBAMNNs (49) and it is evident that leakage delays always impact the stability of NNs. Therefore, it is essential that time delays should be taken into account when studying NN models.

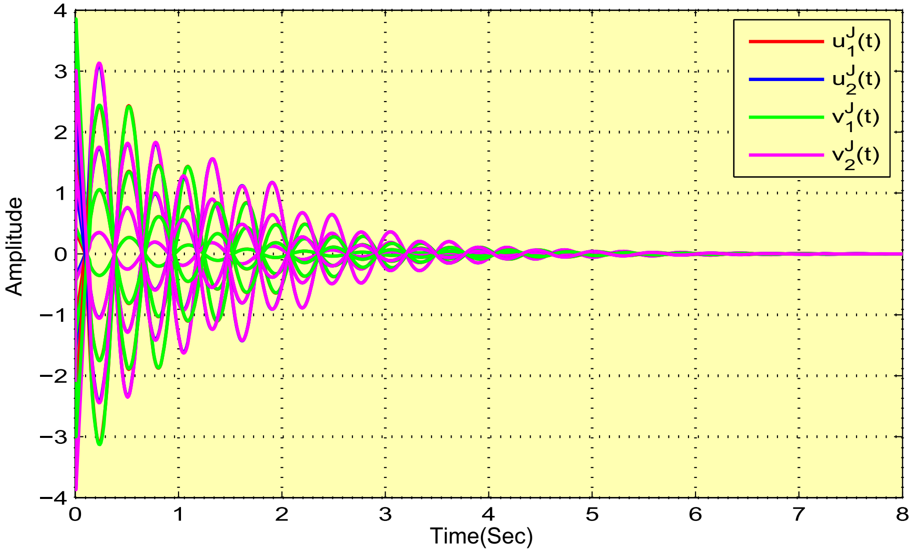

Fig. 8 Transient behaviors of the states $\mathbf{u}(t)^J$, $\mathbf{v}(t)^J$ of NNs (49) with $\delta = 0.13$

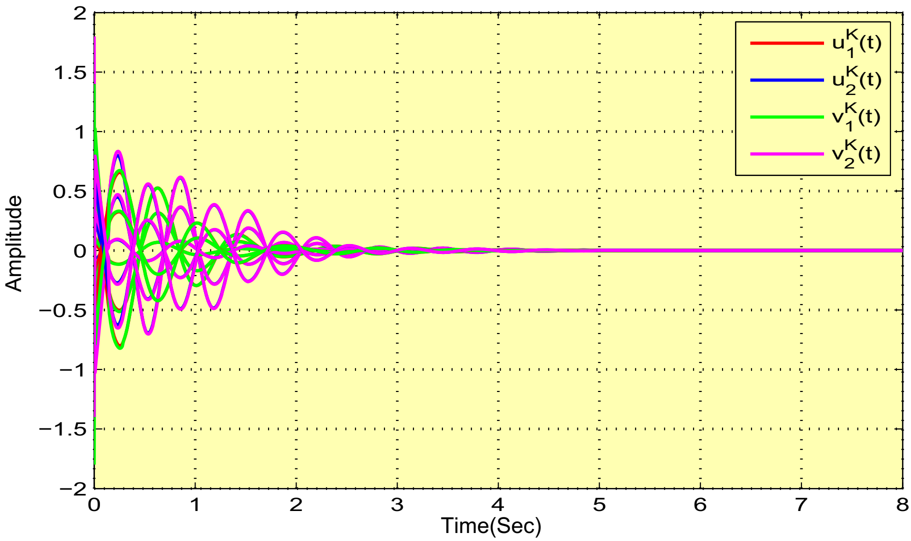

Fig. 9 Transient behaviors of the states $\mathbf{u}(t)^K$, $\mathbf{v}(t)^K$ of NNs (49) with $\delta = 0.13$

5 Conclusion

This paper studied the GAS problem for a class of T-S fuzzy QVBAMNNs with discrete, distributed and leakage delays using non-separation method. By applying T-S fuzzy model, we first considered a general form of T-S fuzzy QVBAMNNs with time delays. Then, by constructing appropriate LKFs and employing quaternion-valued integral inequalities and homeomorphism theory, several delay-dependent sufficient conditions are obtained to guar-

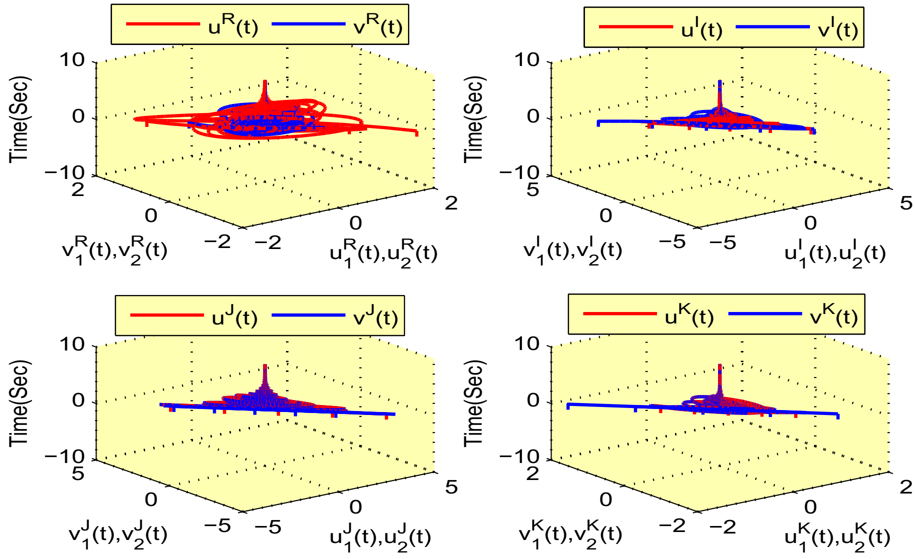

Fig. 10 Phase trajectories of the states $\mathbf{u}(t)^z$, $\mathbf{v}(t)^z$, $z = R, I, J, K$ of NNs (49) with $\delta = 0.13$

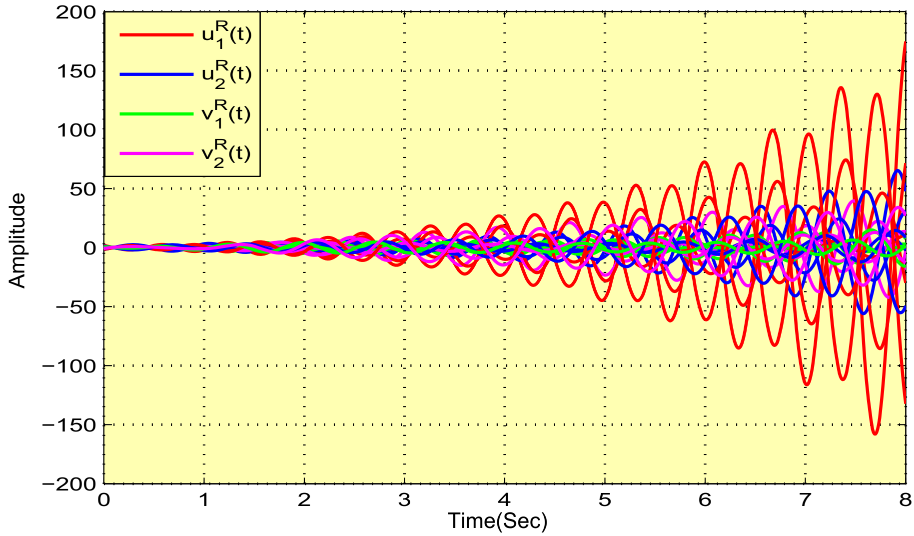

Fig. 11 Transient behaviors of the states $\mathbf{u}(t)^R$, $\mathbf{v}(t)^R$ of NNs (49) with $\delta = 0.18$

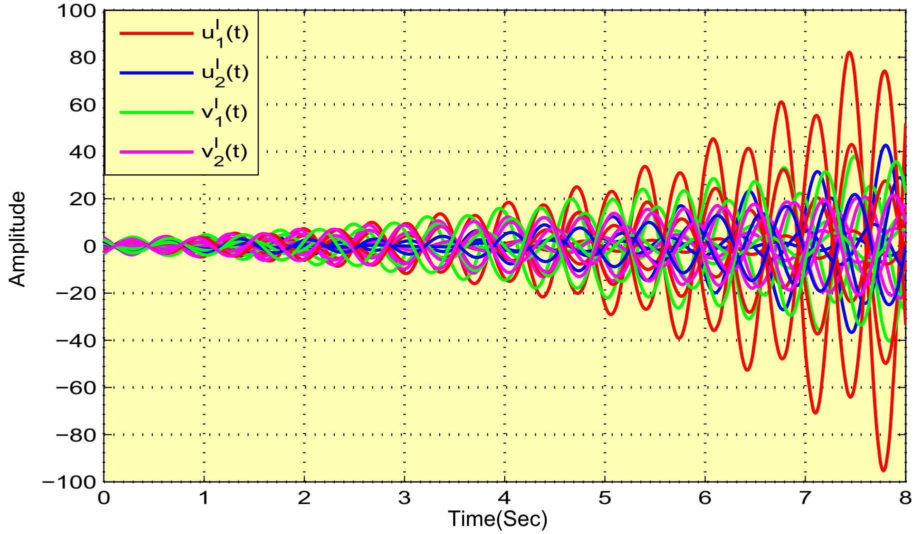

Fig. 12 Transient behaviors of the states $\mathbf{u}(t)^I$, $\mathbf{v}(t)^I$ of NNs (49) with $\delta = 0.18$

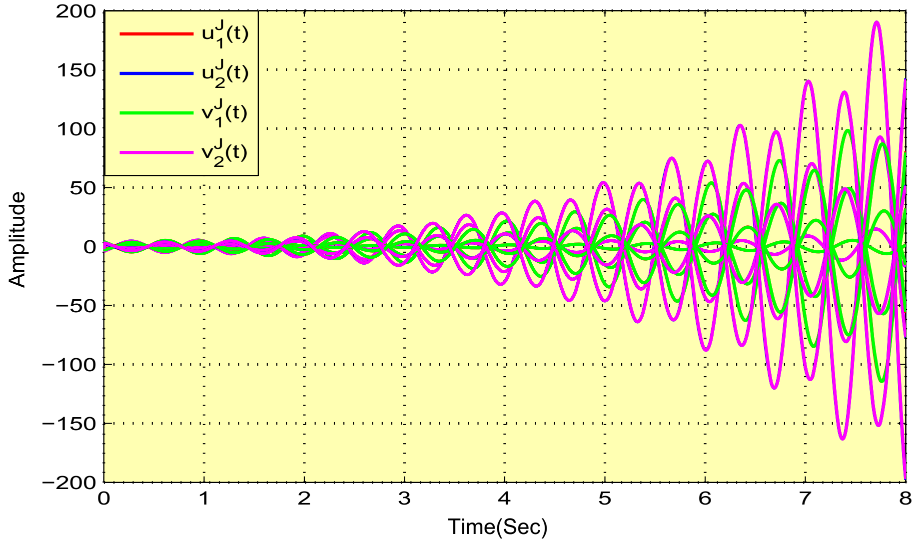

Fig. 13 Transient behaviors of the states $\mathbf{u}(t)^J$, $\mathbf{v}(t)^J$ of NNs (49) with $\delta = 0.18$

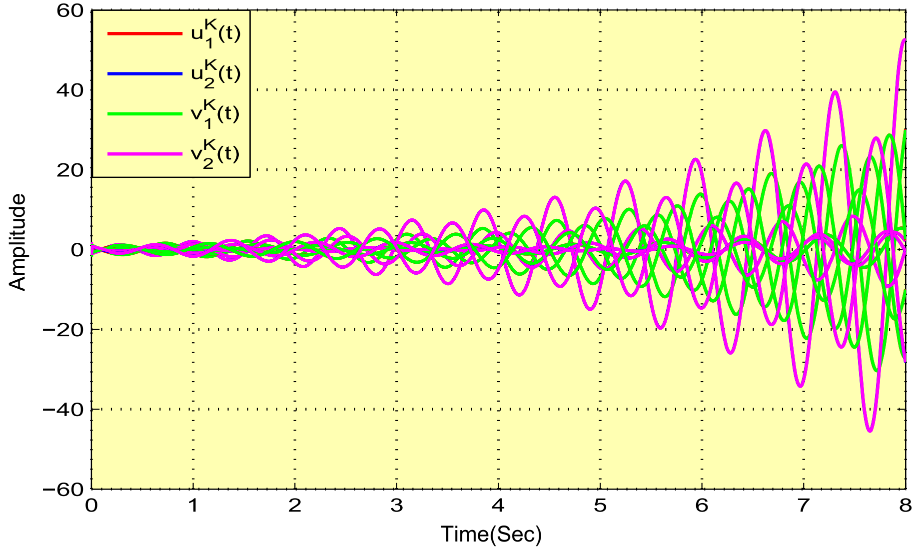

Fig. 14 Transient behaviors of the states $\mathbf{u}(t)^K$, $\mathbf{v}(t)^K$ of NNs (49) with $\delta = 0.18$

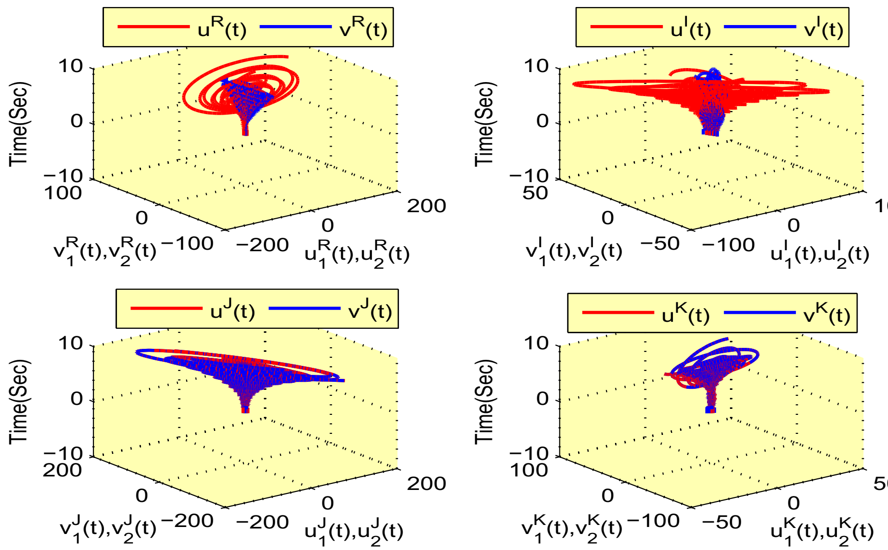

Fig. 15 Phase trajectories of the states $\mathbf{u}(t)^z$, $\mathbf{v}(t)^z$, $z = R, I, J, K$ of NNs (49) with $\delta = 0.18$

antee the existence and GAS of the considered NNs. In addition, these theoretical results are presented in the form of quaternion-valued LMIs, which can be verified numerically using the effective YALMIP toolbox in MATLAB. Finally, two numerical illustrations are presented along with their simulations to demonstrate the validity of the theoretical analysis.

By using the results of this paper, we can analyze various dynamics of T-S fuzzy QVBAMNNs including finite-time stability, synchronization, and others. There are certain advancements worth investigating further in this research area. Therefore, we will study finite-time stability for the following T-S fuzzy QVBAMNNs with time delays and impulses.

$$
\begin{cases}
\dot{\mathbf{u}}_r(t) = \sum\limits_{z=1}^{m} \chi_z(\vartheta(t))\Big\{ -d_{1}{}_r^z\mathbf{u}_r(t) + \sum\limits_{s=1}^{m} a_{1}{}_{rs}^z\mathbf{g}_{1s}(\mathbf{v}_s(t-\ell(t)))\Big\}, \ t \neq t_k, \\
\triangle\mathbf{u}_r(t_k) = \alpha_k(\mathbf{u}_r(t_k)), \ t = t_k, \ r = 1, 2, ..., n, \ k = 1, 2, ..., \\
\mathbf{u}_r(t) = \varphi_{1r}(t), \ t \in [-\ell, 0], \\
\dot{\mathbf{v}}_s(t) = \sum\limits_{z=1}^{m} \chi_z(\vartheta(t))\Big\{ -d_{2}{}_s^z\mathbf{v}_s(t) + \sum\limits_{r=1}^{n} a_{2}{}_{sr}^z\mathbf{g}_{2r}(\mathbf{u}_r(t-\ell(t)))\Big\}, \ t \neq t_k, \\
\triangle\mathbf{v}_s(t_k) = \beta_k(\mathbf{v}_s(t_k)), \ t = t_k, \ s = 1, 2, ..., m, \ k = 1, 2, ..., \\
\mathbf{v}_s(t) = \varphi_{2s}(t), \ t \in [-\ell, 0].
\end{cases}
$$

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Data Availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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