Dynamical Systems and Numerical Analysis: the Study of Measures generated by Uncountable I.F.S.

G.Mantica*

Abstract

Measures generated by Iterated Function Systems composed of uncountably many one-dimensional affine maps are studied. We present numerical techniques as well as rigorous results that establish whether these measures are absolutely or singular continuous.

keywords: Iterated Function Systems, Singular Measures, Fourier Transform, Invariant Measures

AMS subject classification 47B38, 28A80, 60B10

1 Introduction

In this paper I want to describe an example of the fruitful interplay between the theory of dynamical systems and numerical analysis: I want to show how theoretical questions such as singularity (or continuity) with respect to Lebesgue of a dynamical measure can be attacked from a numerical point of view, and viceversa how particular features in the numerical analysis of measures can be properly explained by concepts in the theory of dynamical systems.

Let us consider the iteration of maps $\phi_{\lambda} : X \to X$ from a compact metric space X to itself, labelled by the variable λ which belongs to a measure space Λ , on which the probability measure σ is given. This process gives rise to what is called an Iterated Function System, or I.F.S., [10, 1] with invariant measure μ , which can be defined as follows. Consider the transfer operator T on the space C(X) of continuous functions on X, via

$$(Th)(x) = \int d\sigma(\lambda)(h \circ \phi_{\lambda})(x).$$
(1)

Then, let T^* be the adjoint operator in the space of regular Borel measures on X. An invariant measure of the I.F.S. is the fixed point of T^* ,

$$T^*(\mu) = \mu. \tag{2}$$

^{*}International Center for Non-linear and Complex Systems, Università dell'Insubria, Via Vallegio 11, Como, and CNISM, unità di Como, I.N.F.N. sezione di Milano, Italy E-mail: <giorgio@uninsubria.it>.

Suitable hypotheses can be formulated in order for μ to be unique, given the choice of the set of maps ϕ_{λ} and of the distribution σ [18]. We are interested in the nature of μ : is it of pure type? If so, is it pure point, singular, or absolutely continuous? What characteristics of ϕ_{λ} and σ have importance in this regard? This problem belongs to a classical topic of research in dynamical systems, that looks for *a.c.i.m.*, that is, absolutely continuous invariant measures. Indeed, it is my opinion that *singular continuous* measures are equally, if not more interesting, in many respects.

The plan of this paper is to study this problem in a class of maps of the real line. We will derive algorithms to reduce the quest for an invariant measure to a fixed point problem. If in addition there exists an attractive fixed point in a suitable space of densities, absolute continuity of the measure will follow. We will also describe the Fourier transform of the measure μ , its Mellin transform and its Sobolev dimension, that will also lead to numerical and theoretical methods to determine absolute continuity or singularity of the invariant measure.

2 Infinite Affine Iterated Function Systems

Iterated Function Systems have been originally introduced and studied in [10, 1], although in some form their history goes much further back in time, see [21]. They have become a versatile mathematical tool with applications to image compression [2, 12], quantum dynamics [15] and much more [5]. In its simplest form, an I.F.S. is a finite collection of contractive maps of a space X into itself: when iterated randomly, these maps produce a stochastic process in X with invariant measure μ . Interesting results and applications are found for affine one dimensional maps of the kind

$$\phi_{\delta,\beta}(x) = \delta(x-\beta) + \beta. \tag{3}$$

Each of these maps has the fixed point β and contractions ratio $0 < \delta < 1$. When dealing with finite collections of affine one-dimensional maps, the problem of constructing their Jacobi matrix, and therefore of computing integrals with respect to these measures, has been solved in [14, 7, 13].

Rather than working with a finite set of maps associated with the pairs (δ, β) , as done usually, in this paper we adopt a generalization by which δ is fixed to a positive constant value, strictly less than one, while β is free to vary into a finite interval of the real line, that, without loss of generality, we may understand to be [-1, 1]. This choice is called an infinite, homogeneous I.F.S.. It has been originally investigated by Elton and Yan [6].

Using the maps $\phi_{\delta,\beta}$ we can define a stochastic process in X = [-1, 1] via the following rule: given an initial point $x \in [-1, 1]$, choose a value of β at random in [-1, 1], according to a distribution $\sigma(\beta)$ (whose support may contain an infinity of points) and apply $\phi_{\delta,\beta}$ to map x into $\phi_{\delta,\beta}(x)$. Iterate the procedure. A general theorem, due to Mendivil [18], guarantees that there exists a unique invariant measure μ for this stochastic process. In addition, this measure can be found, probability one, by the Cesaro average of atomic measures at the points x_j of a trajectory of the process: $\frac{1}{n} \sum \delta_{x_j} \to \mu$. In a rather pictorial view, we may describe the process above by saying that the point x is the location of a predator in a rather peculiar *chase* of the prey, located at the point β : the predator moves towards the prey, but as soon as its distance from β is reduced by a factor δ the prey disappears, to reappear instantaneously at a new location, and the process is repeated. Therefore, σ is the distribution of the position of the prey, that, along with the value of δ yields the distribution μ of the predator's positions. The measure μ can be equivalently defined by eq. (2): its approximation properties and the related inverse problems have been discussed in [3, 9, 8, 16].

The transfer operator T for infinite, affine homogeneous I.F.S. takes the following form:

$$(Tf)(x) = \int d\sigma(\beta) \ f \circ \phi_{\delta,\beta}(x) = \int d\sigma(\beta) f(\delta x + \bar{\delta}\beta), \tag{4}$$

where f is any continuous function, and where, to simplify the notation, we have introduced the symbol $\bar{\delta} := 1 - \delta$, to be used throughout the paper. Eq. (2) now becomes:

$$\int d(T^*\mu)(x)f(x) := \int d\mu(x)(Tf)(x) = \int d\mu(x) \int d\sigma(\beta)f(\delta x + \bar{\delta}\beta).$$
(5)

This is equivalent to say that, if μ is invariant, then the equality

$$\int d\mu(x)f(x) = \int d\mu(x)(Tf)(x) \tag{6}$$

holds for any continuous function f. In this setting, our problem can therefore be formulated by asking what characteristics of the constant δ and of the measure σ influence the nature of μ . We start attacking this question in the next section, in the case when σ is the Lebesgue measure.

3 Test Case I: σ is the Lebesgue Measure on [-1,1]

In this section, we consider a case that can be treated analytically as well as numerically. It is defined by letting σ be the Lebesque measure on [-1, 1]. By proving theoretically the right answer for the spectral type of μ , we shall be able to use it as a test of the numerical techniques that we shall introduce in the sequel. We first prove a general result

Lemma 3.1 Suppose that σ is absolutely continuous with a bounded density. Then, so is μ .

Proof. Apply the balance relation (6) to $f(x) = \chi_{B_{\epsilon}(s)}(x)$, the characteristic function of $B_{\epsilon}(s)$, the ball of radius ϵ centered at s (technically, eq. (2) holds

also for summable functions), to get:

$$\mu(B_{\epsilon}(s)) = \int d\mu(x)\sigma(B_{\epsilon/\bar{\delta}}(\frac{s-\delta x}{\bar{\delta}})).$$
(7)

If σ has a bounded density, this means that the following limit exists and is equal to $\rho(\sigma; y)$, the density of the measure σ at the point y:

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \sigma(B_{\epsilon}(y)) = \rho(\sigma; y).$$
(8)

Using this definition, we can now divide both sides of eq. (7) by 2ϵ and take the limit for $\epsilon \to 0$. Thanks to the dominated convergence theorem the limit can be taken inside the integral at r.h.s. to obtain

$$\rho(\mu; s) = \frac{1}{\overline{\delta}} \int d\mu(x) \rho(\sigma; \frac{s - \delta x}{\overline{\delta}}).$$
(9)

From this last relation the thesis easily follows.

We now specialize the theory to the case when σ is the Lebesgue measure, that is to say, all possible values of β in [-1, 1] are equally probable.

Theorem 1 Let $0 < \delta < 1$ and let σ be given by $\rho(\sigma; \beta) = \frac{1}{2}\chi_{[-1,1]}(\beta)$. Then, the invariant measure μ is absolutely continuous, with a density $\rho(\mu; x)$ that is infinitely differentiable and non-analytic.

Proof. The fact that μ has a bounded density follows from Lemma 3.1. Furthermore, inserting the value of $\rho(\sigma; \beta)$ into eq. (9), we obtain

$$\rho(\mu; x) = \frac{1}{2\bar{\delta}} \mu([\frac{x-\bar{\delta}}{\delta}, \frac{x+\bar{\delta}}{\delta}]), \tag{10}$$

and, taking the derivative with respect to x,

$$\rho'(\mu; x) = \frac{1}{2\delta\bar{\delta}} \left[\rho(\mu; \frac{x - \bar{\delta}}{\delta}) - \rho(\mu; \frac{x + \bar{\delta}}{\delta})\right]. \tag{11}$$

This equation can be iterated to show the existence of derivatives of $\rho(\mu; x)$ of all orders. Furthermore, observe that, when taking x = -1, both points $(\frac{-1-\bar{\delta}}{\delta})$ and $(\frac{-1+\bar{\delta}}{\delta})$ are to the left of -1, that is, outside of the support of μ , so that all derivatives of $\rho(\mu; x)$ are null in x = -1. Therefore, $\rho(\mu; x)$ cannot be analytic.

In the next section, we will describe a numerical technique to compute the density of this invariant measure.

4 A Density Mapping

Let us suppose that the invariant measure μ of an infinite, affine, homogeneous I.F.S. is absolutely continuous, with density $\rho(\mu; x)$. We want to derive a numerical technique to compute this density. In the case of Bernoulli I.F.S., to be

discussed later in section 5, this idea has been proposed in [4], without quantitative tests of convergence. Observe for starters that the action of the adjoint operator T^* can be transferred on densities. Use $f(x) = \chi_{B_{\epsilon}(s)}(x)$ in eq. (5) to get, for any probability measure ν :

$$(T^*\nu)(B_{\epsilon}(x)) = \int d\sigma(\beta)\nu(B_{\epsilon/\delta}(\frac{x-\bar{\delta}\beta}{\delta})).$$
(12)

Clearly, a result quite similar to Lemma 3.1 holds:

Lemma 4.1 Suppose that ν is absolutely continuous with a bounded density. Then, so is $T^*\nu$.

In fact, operating as in the proof of Lemma 3.1 of the previous section, one also finds the density of $T^*\nu$:

$$\rho(T^*\nu;x) = \frac{1}{\delta} \int d\sigma(\beta) \rho(\nu; \frac{x - \bar{\delta}\beta}{\delta}).$$
(13)

Eq. (13) is particularly suited for analytical computations. It can also be implemented numerically, whenever a convenient representation of ρ is found, and integration with respect to σ is numerically feasible. In this perspective, it is best to consider a Fourier space representation. Take therefore $f(x) = e^{-iyx}$ in eq. (5) and use the notation

$$\hat{\nu}(y) := \int d\nu(x) e^{-iyx} \tag{14}$$

to indicate the Fourier transform of an arbitrary measure ν , to get

$$\widetilde{T^*\mu}(y) = \hat{\mu}(\delta y) \cdot \hat{\sigma}(\bar{\delta} y).$$
(15)

This is a rather crucial relation, already discussed in [6], that links the Fourier transforms of σ and μ . We shall make use of it repeatedly. Observe also that when ν is absolutely continuous, that is, $d\nu(x) = \rho(\nu; x)dx$, $\hat{\nu}(y)$ in eq. (14) can equally well be seen as the Fourier transform of the density $\rho(\nu; x)$. Furthermore, since the support of μ is enclosed in [-1, 1], we consider the Fourier coefficients of $\rho(\mu; x)$ over the basis set $e^{-i\pi kx}$, with k integer. Let us call them $c_k(\mu)$:

$$c_k(\mu) := \hat{\mu}(\pi k) := \int d\mu(x) e^{-i\pi kx} = \int_{-1}^1 e^{-i\pi kx} \rho(\mu; x) dx.$$
(16)

Setting $y = \pi k$ yields the Fourier coefficients of $T^*\mu$ at l.h.s. of eq. (15). Simple formal manipulations of the right hand side prove the Lemma:

Lemma 4.2 The Fourier coefficients of $T^*\mu$ depend linearly on those of μ via the following operator

$$c_k(T^*\mu) = \hat{\sigma}(\bar{\delta}\pi k) \sum_{j=-\infty}^{\infty} \operatorname{sinc}\left(\pi(j-k\delta)\right) c_j(\mu).$$
(17)

This lemma is theoretically inspiring, and, at the same time, it can be turned into a computational procedure. From the theoretical side, consider the following argument. Suppose that we start from an initial distribution μ_0 whose Fourier coefficients decay extremely fast for |k| larger than some value, say k_0 . As it is well known, this means that μ_0 is absolutely continuous, with a very regular density. We observe from eq. (17) that this distribution of coefficients may spread under the action of T^* . In fact, two phenomena are competing, in this regard: On the one hand, $c_k(T^*\mu_0)$ receives a contribution from $c_i(\mu_0)$ that is maximum for $j \sim k\delta$, and therefore tends to "populate" higher frequencies, that is, to decrease the regularity of the densities. On the other hand, this contribution is multiplied by $\hat{\sigma}(\delta \pi k)$, which tends to zero as k grows, if σ is sufficiently regular, and therefore its effect is diminished. On this basis, a theoretical estimate can be carried out, in line with the project described in the introduction: use numerical analysis to inspire theoretical proofs. This estimate will be presented elsewhere. Instead, we now proceed with numerical techniques and experimentations, that illustrate this point.

We use eq. (17) as in a fixed point method: that is, we define a sequence of measures $\mu_n := T^{*n}\mu_0$, with μ_0 an arbitrary starting measure and we iterate eq. (17). This leads to

Algorithm 1.

- 0 Fix an integer size M and a threshold η and compute the vector of Fourier coefficients $\hat{\sigma}(\bar{\delta}\pi k)$, for $k = -M, \ldots, M$.
- 1 Initialization: Put n = 0. Choose a suitable density to define the initial measure μ_0 and its Fourier coefficients $c_k(\mu_0)$, for for $k = -M, \ldots, M$.
- 2 Iteration: given the Fourier coefficients of μ_n , compute $c_k(T^*\mu_n) = c_k(\mu_{n+1})$ via eq. (17), restricting the summation over j to the range [-M, M].
- 3 Control and termination. Compute a distance (chosen among various possibilities, see later) between μ_n and μ_{n+1} . If difference is less than the threshold η , stop. Otherwise, augment n to n + 1 and loop back to 1.

In numerical experimentations we use as initial measure μ_0 either the uniform density of [-1, 1], given by $\rho(\mu_0; x) = \frac{1}{2}\chi_{[-1,1]}(x)$, whose Fourier transform is readily computed, or the Fourier transform of a Gaussian distribution centered at zero, of variance $\sqrt{2}/\omega$, with $\omega < 1$.

To test this technique we apply it first to the case where σ is the Lebesgue measure on [-1, 1], discussed in section 3, for which we have a precise theoretical result. In Fig. 1 we plot the density of the measures obtained by the first five iterations of the method, $\rho(\mu_n; x)$, $n = 1, \ldots, 5$, when and $\delta = \frac{2}{5}$ and M = 100. The last two curves are practically undistinguishable, indicating convergence. Indeed, a distance function is required to assess this fact with rigour. Various choices are possible. In Fig. 2 we plot all of the following distances, in the same



Figure 1: Densities $\rho(\mu_n; x)$ in the iteration of Algorithm 1, for $n = 1, \ldots, 5$ when σ is the Lebesgue measure. The initial density $\rho(\mu_0; x)$ is constant.

case of Fig. 1, but on a larger number of iterations.

$$d_{l}(\rho(\mu_{n};x),\rho(\mu_{n-1};x)) = \begin{cases} \sum_{j=0}^{M} |c_{j}(\mu_{n}) - c_{j}(\mu_{n-1})|, & l = 1\\ \|\rho(\mu_{n};x) - \rho(\mu_{n-1};x)\|_{1}, & l = 2\\ \|\rho(\mu_{n};x) - \rho(\mu_{n-1};x)\|_{\infty}, & l = 3\\ \|\rho(\mu_{n};x) - \rho(\mu_{n-1};x)\|_{BV}, & l = 4 \end{cases}$$
(18)

In all cases we observe exponential decrease of $d_l(\rho(\mu_n; x), \rho(\mu_{n-1}; x))$ with n, which implies convergence to a density $\rho_M(\mu; x)$ that is an approximation of $\rho(\mu; x)$. Validity of the approximation can be checked by increasing the value of M, see the next Section and Sect. 6, where a drastically different case is encountered.

5 Test Case II: σ is a two-atoms Bernoulli Measure

Opposite to the previous choice of the measure σ , lies the Bernoulli measure, that is, an atomic measure composed of two atoms. This is the first non-trivial case, because one atom alone leads to the equality $\mu = \sigma$. Recall the predator-previnterpretation: if the prev at β does not move, the predator x converges to it, and the invariant distribution μ is an atomic measure located at the same point. Let therefore σ be defined as the sum of two atomic measures located at 1 and -1: $\sigma(\beta) = \frac{1}{2}(\Delta_{-1}(\beta) + \Delta_{1}(\beta))$, or $\int f(\beta) d\sigma(\beta) = \frac{1}{2}(f(1) + f(-1))$, for any continuous f.

The prey appears with equal probability (this symmetry can be easily changed, though) at either end of the interval. Contrary to what it might seem, this problem is far from being trivial. It has a rich and classical history, going back at



Figure 2: Distances $d_l(\rho(\mu_n; x), \rho(\mu_{n-1}; x))$ between successive densities in the iteration of Algorithm 1. Symbols for l = 4 are red lines, crosses; l = 3 blue lines, stars; l = 2 green lines, stars; and l = 1 magenta lines, squares.

least to Wiener, Wintner, Erdös and others in the 1930's. Present knowledge can be summed up in the

Theorem 2 Let $\sigma(\beta) = \frac{1}{2}(\Delta_{-1}(\beta) + \Delta_1(\beta))$. For any δ , μ is of pure type. It is singular continuous, supported on a Cantor set, if $\delta < \frac{1}{2}$. When $\delta = \frac{1}{2}$, μ is the Lebesque measure on [-1, 1]. There exist two constructive, countable sets of values of $\delta > \frac{1}{2}$ for which μ is singular continuous, and absolutely continuous, respectively. For almost all $\delta > \frac{1}{2}$, μ is absolutely continuous.

It is just the case to remark here that this theorem collect significant results obtained along more than seventy years of research, see [4, 19] for lists of references. It is easy to understand what happens in the case $\delta < \frac{1}{2}$. Consider eq. (13) and let $\rho(\mu_0)$ be the uniform density on [-1, 1]. It follows by direct computation that $\rho(\mu_n)$ is a piece–wise constant function for any n, which takes values $2^{-n-1}\delta^{-n}$ on a set of 2^n disjoint intervals of equal length $2\delta^n$, and zero otherwise. These intervals constitute the usual generations in the hierarchical construction of a Cantor set (take $\delta = \frac{1}{3}$ to obtain the classical, ternary Cantor set). Clearly, the sequence $\rho(\mu_n)$ does not tend to any function and this is a remarkable example where measures converge while densities do not.

The bounded variation norm is particularly suited to illustrate this point: we have that $\|\rho(\mu_n)\|_{BV} = \delta^{-n}$ and $\|\rho(\mu_n) - \rho(\mu_{n-1})\|_{BV} = 2\delta^{-n}(\frac{1}{2\delta} - 1)$. Both quantities diverge as $n \to \infty$. This provides a second test for algorithm I, that should *not* converge in this case. In numerical experiments we indeed observe an exponential increase in the bounded variation norm $\|\rho(\mu_n)\|_{BV}$, that only saturates because of the finite cardinality 2M + 1 of the basis set. Quite obviously, increasing M pushes the saturation point to the right: higher and higher frequencies are required to describe the densities $\rho(\mu_n)$. This much for Algorithm I: a better technique will be described momentarily.

We need therefore to face numerically the possibility of μ not being absolutely continuous, with slowly decaying (or not decaying at all!) Fourier coefficients. In the next section we tackle this problem, first theoretically and then numerically.

6 Fourier Transforms and Sobolev Dimension

Let us therefore develop techniques for the case when μ may be singular continuous. A few general results must be quoted at this point. Consider the Mellin transform of a function h defined on $[1, \infty]$:

$$M_1(h;z) := \int_1^\infty y^{z-1} h(y) \, dy.$$
(19)

The integral may diverge if Re(z) is too large. The supremum of the set of values of Re(z) for which one has convergence is called the *divergence abscissa* of the Mellin transform. Define the Sobolev dimension of a measure μ as the divergence abscissa of the Mellin transform of $h(y) = |\hat{\mu}(y)|^2$:

$$d_s(\mu) := \sup\{s \in \mathbf{R} \ s.t. \ M_1(|\hat{\mu}(\cdot)|^2; s) < \infty\}.$$
(20)

Clearly, $d_s(\mu) \geq 0$. When $d_s(\mu)$ is less than, or equal to one, it coincides with $D_2(\mu)$, the correlation dimension of the measure μ , a common quantity in the multifractal analysis of measures [20]. In this case, the leading asymptotic behavior for large y of the Cesaro average of $h(y) = |\hat{\mu}(y)|^2$ is y^{-D_2} , see [11, 17] for a discussion of this and other asymptotic behaviors of singular continuous measures. To the contrary, when $d_s(\mu) > 1$, one knows that μ is absolutely continuous and its density $\rho(\mu)$ has fractional derivative of order (d-1)/2 in $L^2(\mathbf{R})$ for all $1 < d < d_s(\mu)$. If in addition $d_s(\mu) > 2$, then $\rho(\mu; x)$ is a continuous function, the larger d_s the higher its regularity. To sum up, the regularity of a measure can be assessed by the study of the asymptotic behavior of its Fourier transform.

To do this in our case, observe that equation (15) can be iterated, to get

$$\widehat{\mu_n}(y) := \widehat{T^{*n}\mu_0}(y) = \widehat{\mu_0}(\delta^n y) \prod_{j=0}^{n-1} \widehat{\sigma}(\delta^j \overline{\delta} y).$$
(21)

Observe furthermore that, as n tends to infinity, $\hat{\mu}_0(\delta^n y)$ tends to $\hat{\mu}_0(0) = 1$ for any y, so that this proves the

Lemma 6.1 The Fourier transform of μ_n can be computed according to eq. (21), and that of μ in the form of the infinite product

$$\widehat{\mu}(y) = \prod_{j=0}^{\infty} \widehat{\sigma}(\delta^j \overline{\delta} y).$$
(22)

Therefore, μ is an infinite convolution product of rescaled copies of the measure σ , a fact well known when σ is the Bernoulli measure and $\hat{\mu}(y)$ is an infinite product of trigonometric functions. This observation is the basis of many theoretical investigations. It also entails a numerical technique to compute either $\hat{\mu}_n(y)$ (just use eq. (21)) or $\hat{\mu}(y)$:

Algorithm 2.

- 0 Fix a threshold $\eta > 0$ and $y \in \mathbf{R}$.
- 1 Initialization: Put n = 0 and let $\phi_0 = 1$.
- 2 Iteration: compute $\psi_n = \hat{\sigma}(\delta^n \bar{\delta} y)$. Update $\phi_{n+1} = \phi_n \psi_n$
- 3 Control and termination. Compute $|\psi_n 1|$. If this difference is less than the threshold η , stop. Otherwise, augment n to n + 1 and loop back to 2.

At this point, we can outline a two-fold strategy for detecting numerically the continuity properties of a measure μ generated by affine I.F.S.. In the first approach, we use Algorithm 2 to compute the Fourier coefficients of μ , and we look at their asymptotic behavior.

The second means is a refined density mapping: we fix a measure defined by an initial density $\rho(\mu_0; x)$. Then, we compute the Fourier coefficients $c_k(\mu_n)$ via eq. (21). Of course, while $\mu_n \to \mu$ weakly because of Mendivil's theorem, there is no guarantee that the densities $\rho(\mu_n; x)$ will tend to a function. Therefore, we compute numerically the norms of $\rho(\mu_n; x)$ reconstructed from their Fourier coefficients, and the distances in eq. (18). We then look for either divergence of the bounded variation norm (the L^1 norm must be conserved), or geometric convergence of the distances. In the first case we conclude for singularity of the measure μ , in the second for absolute continuity.

We can test this approach by examining three cases of I.F.S. measures with two maps, $\sigma = \frac{1}{2}(\Delta_{-1} + \Delta_1)$, and $\delta > \frac{1}{2}$. The first two values of δ are chosen in the two denumerable sets for which the spectral type is rigorously known: absolutely continuous in the first case, a) $\delta = 2^{-1/2} \sim 0.7071067811865$ and singular continuous in the second case, b) $\delta = 1/p_1 \sim 0.7548776662467, p_1$ being a Pisot number. We also consider the case c) $\delta = 3/4 = .75$ which is pretty close to b). The last case is particularly interesting, since it is conjectured that for rational values of δ larger than one half the measure μ is absolutely continuous. In this specific case our analysis shows that the conjecture appears to be numerically validated.

Observe Fig. 3: the analysis of the L^1 distances $\|\rho(\mu_n) - \rho(\mu_{n-1})\|_1$ versus n shows exponential decay (hence convergence in virtue of Cauchy criterion) for the known absolutely continuous case b) and also for the conjectured case c). In case a) one observes to the contrary (as expected) divergence of the bounded variation norm of $\rho(\mu_n; x)$. Also notice that difference between cases b) and c) can only be appreciated after about ten iterations.

We also plot in fig. 4 the absolute values of the Fourier coefficients $c_k(\mu)$ of the three invariant measures, versus k, in double logarithmic scale. This



Figure 3: Bounded variation norms of densities ρ_n for the three I.F.S. described in the text: a) open squares, magenta; b) filled squares, light blue; c) open circles, yellow. Also plotted are the L^1 distances between successive densities: a) green crosses; b) red pluses; c) blue asterisks.

confirms the the results obtained by the analysis of the iterative algorithm. Coefficients for the case b) can also be computed analytically.

7 Test Case III: σ is a Singular Continuous Measure

We have seen so far two cases where σ is a discrete measure or an absolutely continuous one. We want now to consider a singular continuous measure σ . In order to be able to investigate this case numerically, we need a fast and reliable way to compute the Fourier transform of σ . But indeed we have already at hand a measure σ with these characteristics. Consider in fact the measure μ generated, for $\delta < \frac{1}{2}$, by the Bernoulli distribution, section 5 and 6. It is singular continuous and in addition its Fourier transform can be easily computed via Algorithm 2. Therefore, by taking now this measure to be the *new* distribution σ , we can apply Algorithm 2, in a recursive fashion.

The basic idea of this recursion is simple: suppose that the prey moves with distribution $\sigma(\beta) = \mu^{(0)}(\beta)$, this latter being an arbitrary measure. The predator is attracted to the prey according to the affine maps with contraction ratio δ_1 : this determines its invariant distribution $\mu^{(1)}$. In turn, this predator is hunted by a second species: for this last animal, $\sigma(\beta) = \mu^{(1)}(\beta)$ and δ_2 is the contraction ratio: its distribution is then $\mu^{(2)}$. Clearly, there is no limit to the number of species in this food chain: this fact has an interesting mathematical formulation that will be exploited elsewhere. For simplicity we investigate in this section only a simple case: $\mu^{(0)}$ is the Bernoulli measure, $\delta_1 < \frac{1}{2}$, and the



Figure 4: Fourier coefficients $|c_k(\mu)|$ vs. k for the three I.F.S. measures a) (red dots) b) (green dots) and c) (magenta dots) described in Sect. 6. Solid lines are drawn to mark the asymptotic behaviors $k^{-1/2}$ (borderline for assessing absolute continuity), k^{-1} and k^{-2} .

choice $\sigma(\beta) = \mu^{(1)}(\beta)$ leads to an I.F.S. with uncountably many maps, whose fixed points populate a Cantor set.

The theory developed so far enables us to examine numerically the invariant measures of these I.F.S. As the typical case, we observe that the measure $\mu^{(2)}$ is smoother than $\mu^{(1)}$. It can indeed be absolutely continuous even when $\mu^{(1)}$ is singular continuous and supported on a Cantor set. This case is pictured in Fig. 5, in which $\mu^{(1)}$ is generated by an I.F.S. with $\mu^{(0)} = \frac{1}{2}(\Delta_{-1} + \Delta_1)$ and $\delta_1 = 0.4$, and we choose two values of the second contraction ratio: $\delta_2 = 0.1$ and $\delta_2 = 0.2$. In both cases we observe absolute continuity of the measure $\mu^{(2)}$, with a continuous density. The densities of the two measures $\mu^{(2)}$ are plotted in Fig. 5. Observe that the support of the measure $\mu^{(2)}$ includes that of $\mu^{(1)}$ [9]. Indeed, a stricter "inequality" formula between them can be derived. We also plot in the inset the geometric convergence of the BV and L^1 distance functions (18). The initial increase of the bounded variation distance is due to the fact that we have employed here an initial gaussian distribution (see sect. 4).

Should one be tempted to conjecture that $\mu^{(2)}$ is always absolutely continuous in this setting, in the next and final section we shall meet a case when $\mu^{(2)}$ is singular continuous.

8 A computer assisted proof of absolute continuity

In this final section, we show how a computer assisted proof of absolute continuity of the measure μ can be developed in a specific case.



Figure 5: Densities $\rho(\mu^{(2)}; x)$ of the two I.F.S. measures described in the text. Labels are: $\delta_2 = .1$ (green curve), $\delta_2 = .2$ (red curve). For symmetry, only half of the curves are shown. In the inset, bounded variation and L^1 distances $\|\rho(\mu_{n+1}) - \rho(\mu_n)\|$ versus iteration number *n*: labels are $\delta_2 = .1$, *BV* (green curve, crosses), L^1 (magenta, squares); $\delta_2 = .2$, *BV* (red, pluses), L^1 (blue, asterisks).

Lemma 8.1 Let the measure $\mu^{(0)}$ with support in [-1,1] be given, and let $\mu^{(1)}$ be the invariant measure generated by an affine, homogeneous I.F.S. with contraction ratio δ and with distributions of fixed points $\mu^{(0)}$. Let also $\mu^{(2)}$ be the invariant measure generated by a second affine, homogeneous I.F.S. with contraction ratio δ (the same as before) and with distributions of fixed points $\mu^{(1)}$. Then, for any N > 0 and for any t, the inequality holds:

$$\log |\widehat{\mu^{(2)}}(\delta^{-N}t)| \le N\{\log |\widehat{\mu^{(1)}}(\bar{\delta}t)| + \log |\widehat{\mu^{(0)}}(\delta^{-1}\bar{\delta}^{2}t)|\}$$
(23)

Proof. Use eq. (22) to express $\log \widehat{\mu^{(2)}}(y)$, and use the convenient notation $\Phi_j(y) := \log |\widehat{\mu^{(j)}}(y)|$. Gather terms (some algebra required) to obtain

$$\Phi_2(\delta^{-N}t) = \sum_{l=0}^{N-1} (l+1)\Phi_0(\delta^{l-N}\bar{\delta}^2 t) + N\Phi_1(\bar{\delta}t) + \Phi_2(t).$$
(24)

Observe that all functions Φ_j are less than, or equal to, zero so that by keeping only the l = N - 1 term in the summation at r.h.s. of eq. (24) and discarding also the term $\Phi_2(t)$ leads to the inequality (23). This is crucial for proving:

Theorem 3 Let $\mu^{(j)}$, for j = 0, 1, 2, be defined as in Lemma 8.1. Let also $\eta = -\sup\{\Phi_0(\delta^{-1}\bar{\delta}^2 t) + \Phi_1(\bar{\delta}t); t \in [1, \delta^{-1}]\}$. Then, if $\eta > \frac{1}{2}\log(\delta^{-1})$ the measure $\mu^{(2)}$ is absolutely continuous with an L^2 density, and if $\eta > \log(\delta^{-1})$ the density $\rho(\mu; x)$ is continuous.

Proof. From Lemma 8.1 it follows that for all N > 0, and for all $t \in [\delta^{-N}, \delta^{-N-1}]$ the function $\Phi_2(t)$ is less than $-\eta N$. One can therefore apply the Sobolev criterion described in Sect. 6.

This theorem quickly turns into a computer–assisted proof of absolute continuity. In fact, notice that the supremum required to compute η is indeed the maximum of a continuous function defined on a finite interval. As such, it is easily computable with sufficient precision to establish whether the inequalities in Thm. 3 are satisfied. For the systems we have been considering in this paper this is easily accomplished, by computing the Fourier transforms via algorithm 2. As just an instance of this procedure, we can prove

Theorem 4 Let $\mu^{(0)} = \frac{1}{2}(\Delta_{-1} + \Delta_1)$, and let $\mu^{(j)}$, j = 1, 2 defined as in Lemma 8.1. Then, if $\delta = \frac{2}{5}$ the measure $\mu^{(2)}$ is absolutely continuous with a continuous density, and if $\delta = \frac{3}{10}$ it is absolutely continuous with an L^2 density.

Proof. In the two cases quoted, one can easily compute that $\eta / \log(\delta^{-1}) \simeq 1.5035$ and $\eta / \log(\delta^{-1}) \simeq .96838$, respectively.

It appears numerically that in addition to being a rigorous inequality, formula (23) is a rather precise estimate of the asymptotic behavior of the Fourier transform of $\mu^{(2)}$. So, by letting now $\delta = \frac{2}{10}$ we find that the value $\eta/\log(\delta^{-1}) \simeq$ 0.2218 implies that the corresponding measure $\mu^{(2)}$ is singular continuous. This conclusion is also confirmed by the iterative analysis detailed in Sect. 6.

9 Conclusions

We have described numerical techniques to determine the continuity type of measures generated by affine, homogeneous I.F.S. with uncountably many maps. These techniques are inspired by the theory of dynamical systems. In turn, they suggest methods to obtain rigorous, and computer–assisted proofs: one should focus on the existence (or non–existence) of an attractive fixed point in a suitable space of densities, most conveniently under the bounded variation distance. Alternatively, one must consider the decay rate of Fourier coefficients, also obtained via fixed point procedures.

We have sketched examples of both techniques, that have helped us to clarify the fact that such I.F.S. measures can be of all continuity types. According to the language adopted in this paper, we have described the implications of the continuity of σ on those of μ in what we believe to be a rather detailed picture. We have also shown applications of our methods to the classical problem of infinite convolutions of Bernoulli measures, for which determining the continuity type of a specific measure whose parameter does not belong to a known countable set of values is still an open problem.

References

 M. F. Barnsley and S. G. Demko, Iterated function systems and the global construction of fractals. Proc. R. Soc. London A, 399:243–275, 1985.

- [2] M. F. Barnsley and L. P. Hurd, Fractal Image Compression. A.K. Peters Ltd, 1993.
- [3] D. Bessis and S. G. Demko, Stable recovery of fractal measures by polynomial sampling. *Physica D* 47:427-438, 1991.
- [4] J. Borwein, D. Bailey and R. Girgensohn, Experimentation in Mathematics: Computational Paths to Discovery. A.K. Peters Ltd, 2004.
- [5] P. Diaconis and D. Freedman, Iterated random functions. SIAM Rev., 41:45-76, 1999.
- [6] J. H. Elton and Z. Yan, Approximation of measures by Markov processes and homogeneous affine iterated function systems. *Constr. Appr.*, 5:69–87, 1989.
- [7] H. J. Fischer, On generating orthogonal polynomials for self-similar measures. In N. Papamichael, St. Ruscheweyh and E. B. Saff, editors, *Computational Methods and Function Theory 1997*, 191–201, World Scientific Publishing Co. Pte. Ltd. 1999.
- [8] B. Forte and E.R. Vrscay, Solving the inverse problem for measures using iterated function systems: a new approach. Adv. Appl. Prob., 27:800–820, 1995.
- [9] C. R. Handy and G. Mantica, Inverse problems in fractal construction: moment method solutions. *Physica D*, 43:17–36, 1990.
- [10] J. Hutchinson, Fractals and self-similarity. Indiana J. Math., 30:713–747, 1981.
- [11] P. Janardhan, D. Rosenblum and R. S. Strichartz, Numerical experiments in Fourier asymptotics of Cantor measures and wavelets. *Experiment. Math.*, 1:249–273, 1992.
- [12] D. La Torre, E.R. Vrscay, M. Ebrahimi, M. F. Barnsley, Measure-valued images, associated fractal transforms and the affine self-similarity of images, *SIAM Journal on Imaging Sciences*, 2:470–507, 2009.
- [13] D. Laurie and J. de Villiers, Orthogonal polynomials for refinable linear functionals. *Math. Comp.*, 75:1891–1903, 2006.
- [14] G. Mantica, A Stieltjes technique for computing Jacobi matrices associated with singular measures. *Constr. Appr.*, 12:509–530, 1996.
- [15] G. Mantica, Fourier–Bessel functions of singular continuous measures and their many asymptotics. E.T.N.A. 25:409–430, 2006.
- [16] G. Mantica, Fractal measures and polynomial sampling: I.F.S.-Gaussian integration. Numer. Algor. 45:269–281, 2007.

- [17] G. Mantica and D. Guzzetti, The asymptotic behaviour of the Fourier transform of orthogonal polynomials II: Iterated Function Systems and Quantum Mechanics. Ann. Henri Poincaré 8:301–336, 2007.
- [18] F. Mendivil, A generalization of IFS with probabilities to infinitely many maps. *Rocky Mountain J. Math.* 28:1043–1051, 1998.
- [19] Y. Peres, K. Simon and B. Solomyak, Absolute continuity for random iterated function systems with overlaps. J. London Math. Soc. 74:739–756, 2006.
- [20] Y. Pesin, Dimension Theory in Dynamical System: Contemporary Views and Applications. Univ. Chicago Press, 1996.
- [21] A. Zygmund, Trigonometric series, Vols. I, II. Cambridge University Press, 2002.