Evaluating Polynomials Over the Unit Disk and the Unit Ball

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Abstract

We investigate the use of orthonormal polynomials over the unit disk $\mathbb{B}_2$ in $\mathbb{R}^2$ and the unit ball $\mathbb{B}_3$ in $\mathbb{R}^3$. An efficient evaluation of an orthonormal polynomial basis is given, and it is used in evaluating general polynomials over $\mathbb{B}_2$ and $\mathbb{B}_3$. The least squares approximation of a function f on the unit disk by polynomials of a given degree is investigated, including how to write a polynomial using the orthonormal basis. Matlab codes are given.

1 Introduction

A standard way to write a multivariate polynomial of degree n over $\mathbb{R}^2$ is

$$p(x,y) = \sum_{j=0}^{n} \sum_{k=0}^{j} a_{j,k} x^j y^{j-k}.$$

The space of all such polynomials is denoted by Π_n. We consider here the alternative formulation

$$p(x,y) = \sum_{j=0}^{n} \sum_{k=0}^{j} b_{j,k} \varphi_{j,k}(x,y) \tag{1}$$

with $\{\varphi_{j,k} \mid 0 \le k \le j,\ 0 \le j \le n\}$ an orthonormal basis of the set of Π_n over the closed unit disk $\mathbb{B}_2$, for each $n \ge 0$. There is a large literature on such orthonormal polynomials; and in contrast to the univariate case, there are many possible choices for this basis. See Dunkl and Xu [8] and Xu [14] for an investigation of such multivariate orthonormal polynomials and a number of particular examples.

To use (1), it is important to be able to evaluate the orthonormal polynomials $\{\varphi_{j,k}\}$ efficiently, just as is true with univariate polynomials. We consider a particularly good set of such polynomials in Section 2, one that seems much superior to other choices. In the univariate case, the best choices are based on using the triple recursion relation of the particular family $\{\varphi_n\}$ being used. This extends to the multivariate case. We investigate a particular choice of an orthonormal basis for Π_n that leads to an efficient way to evaluate the expression (1) by making use of the triple recursion relation it satisfies. Following that, in Section 3, we also consider the calculation of the least squares approximation over Π_n of a given function $f(x,y)$. In Section 4, these results are extended to polynomials over the unit ball. Finally, in Section 5, MATLAB codes are given for all of the problems being discussed.

2 Evaluating an orthonormal polynomial basis

We review some notation and results from Dunkl and Xu [8] and Xu [14]. For convenience, we initially denote a point in the unit disk by $x = (x_1, x_2)$, and later we revert to the more standard use of (x, y). We consider only the standard L^2 inner product

$$(p,q) = \int_{\mathbb{B}_2} p(x)\, q(x)\, dx. \tag{2}$$

Define

$$\mathcal{V}_n = \{p \in \Pi_n \mid (p,q) = 0,\ \forall q \in \Pi_{n-1}\}, \quad n \geq 1,$$

and let $\mathcal{V}_0$ denote the one dimensional space of constant functions. Thus

$$\Pi_n = \mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_n$$

is an orthogonal decomposition of Π_n. It is standard to give an orthonormal basis for each space $\mathcal{V}_n$ as the way to give an orthonormal basis of Π_n. The dimension of $\mathcal{V}_n$ equals $n+1$, and the dimension of Π_n equals

$$N_n = \frac{1}{2}(n+1)(n+2). \tag{3}$$

Introduce

$$\mathbb{P}_n = \left[Q_n^0, Q_n^1, \ldots, Q_n^n\right]^{\mathrm{T}}, \qquad n \geq 0,$$

with $\left\{Q_n^0, Q_n^1, \ldots, Q_n^n\right\}$ an orthonormal basis of $\mathcal{V}_m$. The triple recursion relation for $\{\mathbb{P}_m\}$ is given by

$$x_i \mathbb{P}_n(x) = A_{n,i}\mathbb{P}_{n+1}(x) + B_{n,i}\mathbb{P}_n(x) + A_{n-1,i}^{\mathrm{T}}\mathbb{P}_{n-1}(x), \qquad i = 1,2, \quad n \geq 1 \tag{4}$$

The matrices $A_{n,i}$ and $B_{n,i}$ are $(n+1) \times (n+2)$ and $(n+1) \times (n+1)$, respectively, and they are defined as follows:

$$A_{n,i} = \int_{\mathbb{B}_2} x_i \mathbb{P}_n(x)\, \mathbb{P}_{n+1}^{\mathrm{T}}(x)\, dx$$

$$B_{n,i} = \int_{\mathbb{B}_2} x_i \mathbb{P}_n(x)\, \mathbb{P}_n^{\mathrm{T}}(x)\, dx$$

For additional details, see Xu [14, Thm. 2.1]. One wants to use the relation (4) to solve for $\mathbb{P}_{n+1}(x)$. This amounts to solving an overdetermined system of $2(n+1)$ equations for the $n+2$ components of $\mathbb{P}_{n+1}(x)$. The expense of this will depend on the structure of the matrices $A_{n,i}$ and $B_{n,i}$. There is a well-known choice that leads, fortunately, to the matrices $B_{n,i}$ being zero and the matrices $A_{n,i}$ being very sparse.

To define this choice, begin by recalling the Gegenbauer polynomials $\left\{C_n^\lambda(t)\right\}$. They can be obtained using the following generating function:

$$\left(1-2rt+r^2\right)^{-\lambda}=\sum_{n=0}^{\infty} C_n^\lambda(t)\, r^n, \qquad |r|<1, \quad |t|\le 1$$

For particular cases,

$$C_0^\lambda(t)\equiv 1, \qquad C_1^\lambda(t)=2\lambda t, \qquad C_2^\lambda(t)=\lambda\left(2(\lambda+1)t^2-1\right),$$

$$C_3^\lambda(t)=\frac{2}{3}\lambda(\lambda+1)t\left((2\lambda+4)t^2-3\right).$$

Their triple recursion relation is given by

$$C_{n+1}^\lambda(t)=\frac{2(n+\lambda)}{n+1}tC_n^\lambda(t)-\frac{n+2\lambda-1}{n+1}C_{n-1}^\lambda(t), \qquad n\ge 1.$$

These polynomials are orthogonal over $(-1,1)$ with respect to the inner product

$$(f,g)=\int_{-1}^{1}\left(1-t^2\right)^{\lambda-\frac{1}{2}} f(t)\, g(t)\; dt,$$

and for $\lambda=\frac{1}{2}$ they are the Legendre polynomials. For additional information on the Gegenbauer polynomials, see [11, Chap. 18].

Return to the use of (x,y) in place of (x_1,x_2). Using the Gegenbauer polynomials, introduce

$$Q_n^k(x,y)=\frac{1}{h_{k,n}}C_{n-k}^{k+1}(x)\left(1-x^2\right)^{\frac{k}{2}} C_k^{\frac{1}{2}}\left(\frac{y}{\sqrt{1-x^2}}\right), \qquad (x,y)\in\mathbb{B}_2, \qquad (5)$$

for $n=0,1.\ldots$ and $k=0,1,\ldots,n$. See Dunkl and Xu [8, p. 88]. Note that

$$x^2+y^2<1 \quad\Longrightarrow\quad \frac{|y|}{\sqrt{1-x^2}}<1$$

The lead constant $h_{k,n}$ is given by

$$h_{k,n}^2=\frac{\pi}{4^k}\frac{(n+k+1)!}{(n+1)(2k+1)(k!)^2(n-k)!}$$

and $h_{0,0}^2 = \pi$. The set $\left\{Q_m^k \mid 0 \le k \le m\right\}$ is an orthonormal basis of $\mathcal{V}_m$, and $\left\{Q_m^k \mid 0 \le k \le m,\ 0 \le m \le n\right\}$ is an orthonormal basis of Π_n, using the inner product of (2). Here are the Q_m^k of degrees 0,1,2,3.

$$Q_0^0(x,y) = \frac{1}{\sqrt{\pi}}, \quad Q_1^0(x,y) = \frac{2x}{\sqrt{\pi}}, \quad Q_1^1(x,y) = \frac{2y}{\sqrt{\pi}} \tag{6}$$

$$Q_2^0(x,y) = \frac{1}{\sqrt{\pi}}\left(4x^2-1\right), \quad Q_2^1(x,y) = \sqrt{\frac{24}{\pi}}xy, \quad Q_2^2(x,y) = \sqrt{\frac{2}{\pi}}\left(3y^2+x^2-1\right) \tag{7}$$

$$\begin{aligned} Q_3^0(x,y) &= \frac{4}{\sqrt{\pi}}x\left(2x^2-1\right) & Q_3^1(x,y) &= \frac{4}{\sqrt{5\pi}}y\left(6x^2-1\right), \\ Q_3^2(x,y) &= \frac{4}{\sqrt{\pi}}x\left(3y^2+x^2-1\right) & Q_3^3(x,y) &= \frac{4}{\sqrt{5\pi}}y\left(5y^2-3+3x^2\right). \end{aligned} \tag{8}$$

Because the formula (5) is not well-defined at $x = \pm 1$, we use

$$\lim_{(x,y)\to(\pm 1,0)} \left(1-x^2\right)^{\frac{k}{2}} C_k^{\frac{1}{2}}\left(\frac{y}{\sqrt{1-x^2}}\right) = \begin{cases} 0, & k > 0 \\ 1, & k = 0 \end{cases}$$

when evaluating (5).

Applying (4) to this choice of orthonormal polynomials leads to

$$x_i \mathbb{P}_n(x_1,x_2) = A_{n,i}\mathbb{P}_{n+1}(x_1,x_2) + A_{n-1,i}^{\mathrm{T}}\mathbb{P}_{n-1}(x_1,x_2), \qquad i = 1,2, \quad n \ge 1. \tag{9}$$

The coefficient matrices are given by

$$A_{n,1} = \begin{bmatrix} a_{0,n} & 0 & \cdots & 0 & 0 \\ 0 & a_{1,n} & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n,n} & 0 \end{bmatrix}$$

$$A_{n,2} = \begin{bmatrix} 0 & d_{0,n} & 0 & \cdots & 0 & 0 \\ c_{1,n} & 0 & d_{1,n} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & c_{n-1,n} & 0 & d_{n-1,n} & 0 \\ 0 & \cdots & 0 & c_{n,n} & 0 & d_{n,n} \end{bmatrix}$$

$$a_{k,n} = \frac{1}{2}\sqrt{\frac{(n-k+1)(n+k+2)}{(n+1)(n+2)}},$$

$$d_{k,n} = \frac{k+1}{2}\sqrt{\frac{(n+k+3)(n+k+2)}{(2k+1)(2k+3)(n+1)(n+2)}},$$

$$c_{k,n} = -\frac{k}{2}\sqrt{\frac{(n-k+1)(n-k+2)}{(n+1)(n+2)(2k-1)(2k+1)}}.$$

These results are taken from Dunkl and Xu [8, p. 88] (in the formula for $c_{k,n}$, change $n+k+1$ to $n-k+1$).

From the first triple recursion relation in (9),

$$x_1 \begin{bmatrix} Q_n^0 \\ Q_n^1 \\ \vdots \\ Q_n^n \end{bmatrix} = \begin{bmatrix} a_{0,n} & 0 & \cdots & 0 & 0 \\ 0 & a_{1,n} & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n,n} & 0 \end{bmatrix} \begin{bmatrix} Q_{n+1}^0 \\ Q_{n+1}^1 \\ \vdots \\ Q_{n+1}^n \\ Q_{n+1}^{n+1} \end{bmatrix} + \begin{bmatrix} a_{0,n-1} & 0 & \cdots & 0 \\ 0 & a_{1,n-1} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & a_{n-1,n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} Q_{n-1}^0 \\ Q_{n-1}^1 \\ \vdots \\ Q_{n-1}^{n-1} \end{bmatrix}$$

$$\begin{aligned} x_1 Q_n^i &= a_{i,n} Q_{n+1}^i + a_{i,n-1} Q_{n-1}^i, \qquad i = 0, 1, \ldots, n-1 \\ x_1 Q_n^n &= a_{n,n} Q_{n+1}^n \end{aligned}$$

This allows us to solve for $\left\{Q_{n+1}^0, \ldots, Q_{n+1}^n\right\}$. The second triple recursion relation in (9) yields

$$x_2 \begin{bmatrix} Q_n^0 \\ Q_n^1 \\ \vdots \\ Q_n^n \end{bmatrix} = \begin{bmatrix} 0 & d_{0,n} & 0 & \cdots & 0 & 0 \\ c_{1,n} & 0 & d_{1,n} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & c_{n-1,n} & 0 & d_{n-1,n} & 0 \\ 0 & \cdots & 0 & c_{n,n} & 0 & d_{n,n} \end{bmatrix} \begin{bmatrix} Q_{n+1}^0 \\ Q_{n+1}^1 \\ \vdots \\ Q_{n+1}^n \\ Q_{n+1}^{n+1} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & c_{1,n-1} & 0 & \cdots & & 0 \\ d_{0,n-1} & 0 & c_{2,n-1} & 0 & \cdots & 0 \\ 0 & d_{1,n-1} & 0 & c_{3,n-1} & & 0 \\ 0 & 0 & d_{2,n-1} & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots & c_{n-1,n-1} \\ 0 & & & & d_{n-2,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & d_{n-1,n-1} \end{bmatrix} \begin{bmatrix} Q_{n-1}^0 \\ Q_{n-1}^1 \\ \vdots \\ Q_{n-1}^{n-1} \end{bmatrix}$$

Its last equation is

$$x_2 Q_n^n = c_{n,n} Q_{n+1}^{n-1} + d_{n,n} Q_{n+1}^{n+1} + d_{n-1,n-1} Q_{n-1}^{n-1}$$

and from it we can calculate Q_{n+1}^{n+1}. Thus,

$$Q_{n+1}^{i} = \frac{x_1 Q_n^i - a_{i,n-1} Q_{n-1}^i}{a_{i,n}}, \qquad i = 0, 1, \ldots, n-1 \tag{10}$$

$$Q_{n+1}^{n} = \frac{x_1 Q_n^n}{a_{n,n}} \tag{11}$$

$$Q_{n+1}^{n+1} = \frac{x_2 Q_n^n - c_{n,n} Q_{n+1}^{n-1} - d_{n-1,n-1} Q_{n-1}^{n-1}}{d_{n,n}} \tag{12}$$

2.1 Computational cost

What is the cost of using this to evaluate the orthonormal basis

$$\mathcal{B}_n \equiv \left\{ Q_m^k \mid 0 \le k \le m,\ 0 \le m \le n \right\}?$$

Assume the coefficients $\{a_{i,n}, c_{i,n}, d_{i,n}\}$ have been computed. Apply (10)-(12) to the computation of $\left\{ Q_m^k \mid 0 \le k \le m \right\}$, assuming the lower degree polynomials of degrees $m-1$ and $m-2$ are known. This requires $4\,(m+1)$ arithmetic operations. The evaluation of $\{Q_0^0, Q_1^0, Q_1^1\}$ from (6) requires 2 arithmetic operations for each choice of $(x, y) = (x_1, x_2)$. Thus the calculation of $\mathcal{B}_n$ requires

$$2 + 4(3 + 4 + \cdots + (n+1)) = 2\left(n^2 + 3n - 3\right) \tag{13}$$

arithmetic operations. Recall (3) that the dimension of Π_n is approximately $\frac{1}{2}n^2$, and thus the cost of evaluating $\mathcal{B}_n$ is only approximately 4 times the dimension of Π_n. Qualitatively this is the same as in the univariate case. To evaluate a polynomial

$$p\,(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{j} b_{j,k} Q_j^k\,(x, y) \tag{14}$$

for which $\{b_{j,k}\}$ are given, we use

$$2\left(n^2 + 3n - 3\right) + (n+1)\,(n+2) \approx 3n^2$$

arithmetic operations, approximately 6 times the dimension N_n of Π_n.

There are other known choices of an orthonormal basis for Π_n; see Dunkl and Xu [8, §2.3.2] and Xu [14, §1.2]. In a number of previous papers (see [2], [4], [6], [7]) we have used the 'ridge polynomials' of [10], in large part because of their simple analytic form that is based on Chebyshev polynomials of the second kind. However, we have calculated experimentally the matrices $A_{i,n}$ and have found them to be dense for low order cases, leading us to believe the same is true for larger values of n. For that reason, solving the triple recursion relation (4) would be much more costly than $\mathcal{O}\,(n)$ operations, making the choice (5) preferable in computational cost. As a particular example of the lack of sparsity in the coefficient matrices $\{A_{n,i}\}$ for the ridge polynomials,

$$A_{2,1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{8} + \frac{\sqrt{6}}{12} & -\frac{\sqrt{3}}{12} & -\frac{\sqrt{2}}{8} + \frac{\sqrt{6}}{12} \\ 0 & \frac{\sqrt{2}}{8} - \frac{\sqrt{6}}{12} & \frac{\sqrt{3}}{12} & -\frac{\sqrt{2}}{8} - \frac{\sqrt{6}}{12} \end{bmatrix}$$

$$A_{2,2} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{6} & -\frac{1}{6} & \frac{\sqrt{2}}{6} \\ -\frac{\sqrt{3}}{12} & \frac{\sqrt{2}}{24} + \frac{\sqrt{6}}{12} & \frac{1}{3} & \frac{\sqrt{2}}{24} - \frac{\sqrt{6}}{12} \\ \frac{\sqrt{3}}{12} & \frac{\sqrt{2}}{24} - \frac{\sqrt{6}}{12} & \frac{1}{3} & \frac{\sqrt{2}}{24} + \frac{\sqrt{6}}{12} \end{bmatrix}$$

2.2 Evaluating derivatives

First derivatives of the orthonormal polynomials are required when implementing the spectral methods of [2], [4], [6], [7]). From (6), (7),

$$\begin{aligned} \frac{\partial Q_0^0}{\partial x_1} &= 0, & \frac{\partial Q_0^0}{\partial x_2} &= 0, \\ \frac{\partial Q_1^0}{\partial x_1} &= \frac{2}{\sqrt{\pi}}, & \frac{\partial Q_1^0}{\partial x_2} &= 0 \\ \frac{\partial Q_1^1}{\partial x_1} &= 0, & \frac{\partial Q_1^1}{\partial x_2} &= \frac{2}{\sqrt{\pi}} \end{aligned}$$

To obtain the first derivatives of the higher degree polynomials, we differentiate the triple recursion relations of (10)-(12). In particular,

$$\begin{aligned} \frac{\partial Q_{n+1}^i}{\partial x_1} &= \frac{1}{a_{i,n}} \left\{ Q_n^i + x_1 \frac{\partial Q_n^i}{\partial x_1} - a_{i,n-1} \frac{\partial Q_{n-1}^i}{\partial x_1} \right\}, \qquad i = 0, 1, \ldots, n-1 \\ \frac{\partial Q_{n+1}^n}{\partial x_1} &= \frac{1}{a_{i,n}} \left\{ Q_n^n + x_1 \frac{\partial Q_n^n}{\partial x_1} \right\} \\ \frac{\partial Q_{n+1}^{n+1}}{\partial x_1} &= \frac{1}{d_{n,n}} \left\{ x_2 \frac{\partial Q_n^n}{\partial x_1} - c_{n,n} \frac{\partial Q_{n+1}^{n-1}}{\partial x_1} - d_{n-1,n-1} \frac{\partial Q_{n-1}^{n-1}}{\partial x_1} \right\} \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial Q_{n+1}^i}{\partial x_2} &= \frac{1}{a_{i,n}} \left\{ x_1 \frac{\partial Q_n^i}{\partial x_2} - a_{i,n-1} \frac{\partial Q_{n-1}^i}{\partial x_2} \right\}, \qquad i = 0, 1, \ldots, n-1 \\ \frac{\partial Q_{n+1}^n}{\partial x_2} &= \frac{x_1}{a_{i,n}} \frac{\partial Q_n^n}{\partial x_2} \\ \frac{\partial Q_{n+1}^{n+1}}{\partial x_2} &= \frac{1}{d_{n,n}} \left\{ Q_n^n + x_2 \frac{\partial Q_n^n}{\partial x_2} - c_{n,n} \frac{\partial Q_{n+1}^{n-1}}{\partial x_2} - d_{n-1,n-1} \frac{\partial Q_{n-1}^{n-1}}{\partial x_2} \right\} \end{aligned} \tag{16}$$

3 Least squares approximation

When given a function $f \in C(\mathbb{B}_2)$, we are interested in obtaining the least squares approximation to f from the polynomial subspace Π_n. When given the basis $\mathcal{B}_n$, this approximation is given by the truncated Fourier expansion

$$\mathcal{Q}_n f(x,y) \equiv P_n(x,y) = \sum_{j=0}^{n} \sum_{k=0}^{j} \left(f, Q_m^k\right) Q_m^k(x,y). \tag{17}$$

The linear operator $\mathcal{Q}_n$ is the orthogonal projection of $L^2(\mathbb{B}_2)$ onto Π_n. As an operator on $L^2(\mathbb{B}_2)$, it has norm 1. As an operator on $C(\mathbb{B}_2)$ with the uniform norm $\|\cdot\|_\infty$, $\mathcal{Q}_n$ has norm $\mathcal{O}(n)$; see [13].

The Fourier coefficients $\left(f, Q_m^k\right)$ must be evaluated numerically, and we review a standard quadrature scheme to do so. Use the formula

$$\int_{\mathbb{B}_2} g(x,y)\,dx\,dy \approx \sum_{l=0}^{q} \sum_{m=0}^{2q} g\left(r_l, \frac{2\pi\, m}{2q+1}\right) \omega_l \frac{2\pi}{2q+1} r_l \tag{18}$$

Here the numbers r_l and ω_l are the nodes and weights, respectively, of the $(q+1)$-point Gauss-Legendre quadrature formula on $[0,1]$. Note that

$$\int_0^1 p(x)dx = \sum_{l=0}^{q} p(r_l)\omega_l,$$

for all single-variable polynomials $p(x)$ with $\deg(p) \le 2q+1$. The formula (18) uses the trapezoidal rule with $2q+1$ subdivisions for the integration over $\mathbb{B}_2$ in the azimuthal variable. This quadrature is exact for all polynomials $g \in \Pi_{2q}$. For functions $f, g \in C(\mathbb{B}_2)$, let $(f,g)_q$ denote the approximation of (f,g) by the scheme (18).

Our discrete approximation to (17) is

$$\widetilde{P}_{n,q}(x,y) = \sum_{j=0}^{n} \sum_{k=0}^{j} \left(f, Q_m^k\right)_q Q_m^k(x,y) \tag{19}$$

When $q = n$, this approximation is known as the 'discrete orthogonal projection of f onto Π_n', 'hyperinterpolation of f by Π_n', or the 'discrete least squares approximation'. We denote it by

$$\widetilde{\mathcal{Q}}_n f(x,y) \equiv \widetilde{P}_{n,n}(x,y) \equiv \widetilde{P}_n(x,y)$$

In applying this numerical integration to the coefficients $\left(f, Q_m^k\right)$, we always require $q \ge n$ in order to force the formula (17) to reproduce all polynomials $f \in \Pi_n$. With this requirement,

$$f \in \Pi_n \quad \Rightarrow \quad \widetilde{P}_{n,q} = f.$$

The operator $\widetilde{\mathcal{Q}}_n$ is a discrete orthogonal projection of $C(\mathbb{B}_2)$ onto Π_n. For this specific case of approximation over $\mathbb{B}_2$, see the discussion in [9]. In particular,

$$\left\|\widetilde{\mathcal{Q}}_n\right\|_{C\to C} = \mathcal{O}(n \log n).$$

3.1 Cost of the discrete least squares approximation

The main computational cost in (19) is the evaluation of the coefficients $\left\{\left(f, Q_m^k\right)_q\right\}$. We begin with the evaluation of the basis $\mathcal{B}_n$ at the points used in (18), of which there are

$$(q+1)(2q+1).$$

The cost to evaluate $\mathcal{B}_n$ will be

$$2\left(n^2+3n-3\right) \times (q+1)(2q+1) \approx 4n^2q^2 \tag{20}$$

arithmetic operations. For comparison, recall that the dimension of Π_n is approximately $\frac{1}{2}n^2$. The evaluation of the function f at these same nodes is

$$(q+1)(2q+1)\,N_f, \tag{21}$$

with N_f the cost of an individual evaluation of the function f. The subsequent evaluations of the coefficients $\left\{\left(f, Q_m^k\right)_q\right\}$ involves an additional

$$\frac{1}{2}(n+1)(n+2) \times (q+1)(2q+1) \tag{22}$$

arithmetic operations. Having the coefficients $\left\{\left(f, Q_m^k\right)_q\right\}$, the polynomial (19) then requires

$$4\left(n^2+3n-3\right) \tag{23}$$

arithmetic operations for each evaluation point (x, y).

In the case $q = n$, the evaluation of $\widetilde{\mathcal{Q}}_n f$ is dominated by (20) and (22), approximately $5n^4$ arithmetic operations. If we then evaluate $\widetilde{\mathcal{Q}}_n f\,(x, y)$ at the points used in the quadrature formula (18), then the cost is an additional $8n^4$ operations, approximately.

3.2 Convergence of least squares approximation

Because the polynomials are dense in $L^2(\mathbb{B}_2)$, we have

$$\|f - P_n\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty.$$

For convergence in $C(\mathbb{B}_2)$, we refer to the presentation in [3, §4.3.3, §5.7.1]. In particular,

$$\|f - \mathcal{Q}_n f\|_\infty \le (1 + \|\mathcal{Q}_n\|)\, E_{n,\infty}(f) \tag{24}$$

$$\left\|f - \widetilde{\mathcal{Q}}_n f\right\|_\infty \le \left(1 + \left\|\widetilde{\mathcal{Q}}_n\right\|\right) E_{n,\infty}(f) \tag{25}$$

where

$$E_{n,\infty}(f) = \min_{f \in \Pi_n} \|f - p\|_\infty\,,$$

the minimax error in the approximation of f by polynomials from Π_n.

Let $f \in C^{k,\alpha}(\mathbb{B}_2)$, functions that are $k-$times continuously differentiable and whose k^{th} derivatives are Hölder continuous with exponent $\alpha \in (0,1]$ Then

$$E_{n,\infty}(f) \leq \frac{c_{k,\alpha}(f)}{n^{k+\alpha}}, \qquad n \geq 1. \tag{26}$$

Combining these results with (24)-(25) gives uniform convergence of both $\mathcal{Q}_n f$ and $\widetilde{\mathcal{Q}}_n f$ to f for all $f \in C^{k,\alpha}(\mathbb{B}_2)$ with $k \geq 1$.

4 Triple recursion relation over the unit ball

In this section we repeat for the three dimensional case some of the results from the two dimensional case of Sections 2 and 3. The orthonormal polynomials in this case are again taken from [8, Proposition 2.3.2]. Here we first derive the coefficients of the three term recursion relation in (9).

4.1 The recursion coefficients and the three term recurrence

The orthonormal polynomials for the three dimensional unit ball are given by

$$\begin{aligned} Q_n^{j,k}(x,y,z) = \frac{1}{h_{j,k}} C_{n-j-k}^{j+k+3/2}(x)(1-x^2)^{j/2} \\ \cdot C_j^{k+1}\left(\frac{y}{\sqrt{1-x^2}}\right)(1-x^2-y^2)^{k/2} C_k^{1/2}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right) \end{aligned} \tag{27}$$

where $j+k \leq n$, and $n \in \mathbb{N}$ is the degree of the polynomial $Q_n^{j,k}$. The normalization constant $h_{j,k}$ will be derived further below. We introduce the vector of all orthonormal polynomials $\mathbb{P}_n$ of degree n:

$$\mathbb{P}_n = [Q_n^{0,0}, \ldots, Q_n^{0,n}, Q_n^{1,0}, \ldots, Q_n^{1,n-1}, Q_n^{2,0}, \ldots, Q_n^{2,n-2}, \ldots, Q_n^{n,0}]^T, \quad n \geq 0. \tag{28}$$

Here we have $\binom{n+2}{2}$ polynomials of degree n and the space Π_n has dimension $\binom{n+3}{3}$, see [8]. In formula (9) we have matrices $A_{n,i}$, $i = 1,2,3$, of dimension $\binom{n+2}{2} \times \binom{n+3}{2}$. First we derive the normalization constant $h_{j,k}$ with a calculation which is typical for calculations involved in the calculation of the coefficients of the matrices $A_{n,i}$. By definition we have

$$\begin{aligned} h_{j,k}^2 = \int_{-1}^{1} (C_{n-j-k}^{j+k+3/2}(x))^2 (1-x^2)^j \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (C_j^{k+1}(\frac{y}{\sqrt{1-x^2}}))^2 (1-x^2-y^2)^k \\ \cdot \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (C_k^{1/2}(\frac{z}{\sqrt{1-x^2-y^2}}))^2 dz\, dy\, dx \end{aligned} \tag{29}$$

Using the substitution

$$u := \frac{z}{\sqrt{1-x^2-y^2}}$$
$$dz = \sqrt{1-x^2-y^2}\; du$$

we get

$$h_{j,k}^2 = \int_{-1}^{1} (C_{n-j-k}^{j+k+3/2}(x))^2 (1-x^2)^j \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (C_j^{k+1}(\frac{y}{\sqrt{1-x^2}}))^2 (1-x^2-y^2)^{k+1/2}$$
$$\cdot \int_{-1}^{1} (C_k^{1/2}(u))^2 du\; dy\; dx$$
$$= N_k^{[1/2]} \int_{-1}^{1} (C_{n-j-k}^{j+k+3/2}(x))^2 (1-x^2)^j$$
$$\cdot \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (C_j^{k+1}(\frac{y}{\sqrt{1-x^2}}))^2 (1-x^2-y^2)^{k+1/2}\; dy\; dx$$

where we defined

$$(N_k^{[\mu]})^2 := \int_{-1}^{1} (C_k^{\mu}(x))^2 (1-x^2)^{\mu-1/2} dx$$
$$= \frac{\pi\Gamma(2\mu+k)}{2^{2\mu-1} k! (\mu+k)\Gamma^2(\mu)} \tag{30}$$

see [1]. Now we use the substitution

$$u := \frac{y}{\sqrt{1-x^2}}$$
$$dz = \sqrt{1-x^2}\; du$$
$$(1-x^2-y^2) = (1-x^2-(1-x^2)u^2)$$
$$= (1-x^2)(1-u^2)$$

to obtain

$$h_{j,k}^2 = N_k^{[1/2]} \int_{-1}^{1} (C_{n-j-k}^{j+k+3/2}(x))^2 (1-x^2)^{j+k+1}$$
$$\cdot \int_{-1}^{1} (C_j^{k+1}(u))^2 (1-u^2)^{k+1/2} du\; dx$$
$$= N_k^{[1/2]} N_j^{[k+1]} N_{n-j-k}^{[j+k+3/2]}$$

If we denote the coefficients of the matrices $A_{n,i}$ by $a^{[n,i]}_{j,k;j',k'}$ $j+k \leq n$ and $j'+k' \leq n+1$ we get

$$
\begin{aligned}
a^{[n,1]}_{j,k;j',k'} &= \int_{\mathbb{B}_3} x Q_n^{j,k}(x,y,z) Q_{n+1}^{j',k'}(x,y,z)\, d(x,y,z) \\
a^{[n,2]}_{j,k;j',k'} &= \int_{\mathbb{B}_3} y Q_n^{j,k}(x,y,z) Q_{n+1}^{j',k'}(x,y,z)\, d(x,y,z) \\
a^{[n,3]}_{j,k;j',k'} &= \int_{\mathbb{B}_3} z Q_n^{j,k}(x,y,z) Q_{n+1}^{j',k'}(x,y,z)\, d(x,y,z)
\end{aligned}
$$

Each of the integrals can be written in the same way as the integral in (29) and then the two above substitutions together with the orthonormal property of the Gegenbauer polynomials allows us to calculate the coefficients of $A_{n,i}$, $i = 1,2,3$. Again we obtain very sparsely populated matrices. Equation (9) takes on the following form:

$$
xQ_n^{j,k} = a^{[n,1]}_{j,k;j,k} Q_{n+1}^{j,k} + a^{[n-1,1]}_{j,k;j,k} Q_{n-1}^{j,k}, : j+k \leq n \tag{31}
$$

where

$$
a^{[n,1]}_{j,k;j,k} = \frac{1}{2}\Big(\frac{(j+k+n+3)(n+1-j-k)}{(n+5/2)(n+3/2)}\Big)^{1/2} \tag{32}
$$

and the term $a^{[n-1,1]}_{j,k;j,k}$ has to be replaced by 0 if $j+k=n$. Furthermore we get

$$
\begin{aligned}
yQ_n^{j,k} &= a^{[n,2]}_{j,k;j+1,k} Q_{n+1}^{j+1,k} + a^{[n,2]}_{j,k;j-1,k} Q_{n+1}^{j-1,k} \\
&\quad + a^{[n+1,2]}_{j+1,k;j,k} Q_{n-1}^{j+1,k} + a^{[n-1,2]}_{j-1,k;j,k} Q_{n-1}^{j-1,k}, : j+k \leq n
\end{aligned} \tag{33}
$$

where the terms of the matrix $A_{n-1,2}$ and $A_{n,2}$ have to substituted by zero if $j-1+k<0$ or $j+1+k>n-1$ in the case of $A_{n-1,2}$. Here

$$
a^{[n,2]}_{j,k;j+1,k} = \frac{1}{4}\Big(\frac{(j+2k+2)(j+1)(j+k+n+4)(j+k+n+3)}{(j+k+1)(j+k+2)(n+5/2)(n+3/2)}\Big)^{1/2} \tag{34}
$$

$$
a^{[n,2]}_{j,k;j-1,k} = -\frac{1}{4}\Big(\frac{j(j+2k+1)(n+2-j-k)(n+1-j-k)}{(j+k+1)(j+k)(n+3/2)(n+5/2)}\Big)^{1/2} \tag{35}
$$

Finally we get

$$
\begin{aligned}
zQ_n^{j,k} &= a^{[n,3]}_{j,k;j,k-1} Q_{n+1}^{j,k-1} + a^{[n,3]}_{j,k;j+2,k-1} Q_{n+1}^{j+2,k-1} + a^{[n,3]}_{j,k;j,k+1} Q_{n+1}^{j,k+1} \\
&\quad + a^{[n,3]}_{j,k;j-2,k+1} Q_{n+1}^{j-2,k+1} + a^{[n-1,3]}_{j+2,k-1;j,k} Q_{n-1}^{j+2,k-1} + a^{[n-1,3]}_{j,k-1;j,k} Q_{n-1}^{j,k-1} \\
&\quad + a^{[n-1,3]}_{j-2,k+1;j,k} Q_{n-1}^{j-2,k+1} + a^{[n-1,3]}_{j,k+1;j,k} Q_{n-1}^{j,k+1}, : j+k \leq n
\end{aligned} \tag{36}
$$

where again the terms have to be replaced by zero if the indices are out of the range of the corresponding matrix. Here

$$a^{[n,3]}_{j,k;j,k-1} = -\frac{k}{8}\Big(\frac{(j+2k+1)(j+2k)(n+2-j-k)(n+1-j-k)}{(k+1/2)(k-1/2)(j+k+1)(j+k)(n+3/2)(n+5/2)}\Big)^{1/2} \tag{37}$$

$$a^{[n,3]}_{j,k;j+2,k-1} = -\frac{k}{8}\Big(\frac{(j+2)(j+1)(j+k+n+4)(j+k+n+3)}{(k+1/2)(k-1/2)(j+k+1)(j+k+2)(n+3/2)(n+5/2)}\Big)^{1/2} \tag{38}$$

$$a^{[n,3]}_{j,k;j,k+1} = \frac{k+1}{8}\Big(\frac{(j+2k+3)(j+2k+2)(j+k+n+4)(j+k+n+3)}{(k+1/2)(k+3/2)(j+k+1)(j+k+2)(n+3/2)(n+5/2)}\Big)^{1/2} \tag{39}$$

$$a^{[n,3]}_{j,k;j-2,k+1} = \frac{k+1}{8}\Big(\frac{(n+2-j-k)(n+1-j-k)j(j-1)}{(k+1/2)(k+3/2)(j+k)(j+k+1)(n+3/2)(n+5/2)}\Big)^{1/2} \tag{40}$$

The equations (31), (33), and (36) allow the calculation of all $Q^{j,k}_{n+1}$ in the following way. For $j+k \le n$ we can use (31) and solve for $Q^{j,k}_{n+1}$:

$$Q^{j,k}_{n+1} = \frac{xQ^{j,k}_n - a^{[n-1,1]}_{j,k;j,k} Q^{j,k}_{n-1}}{a^{[n,1]}_{j,k;j,k}} \tag{41}$$

Then we use (33) for the calculation of $Q^{j+1,n-j}_{n+1}$, $j = 0, \ldots, n$:

$$\begin{aligned} Q^{j+1,n-j}_{n+1} = \Big(&yQ^{j,n-j}_n - a^{[n,2]}_{j,n-j;j-1,n-j} Q^{j-1,n-j}_{n+1} \\ &- a^{[n-1,2]}_{j-1,n-j;j,n-j} Q^{j-1,n-j}_{n-1}\Big)\Big/ a^{[n,2]}_{j,n-j;j+1,n-j} \end{aligned} \tag{42}$$

Finally (36) allows us to calculate $Q^{0,n+1}_{n+1}$

$$\begin{aligned} Q^{0,n+1}_{n+1} = \Big(&zQ^{0,n}_n - a^{[n,3]}_{0,n;0,n-1} Q^{0,n-1}_{n+1} - a^{[n,3]}_{0,n;2,n-1} Q^{2,n-1}_{n+1} \\ &- a^{[n-1,3]}_{0,n-1;0,n} Q^{0,n-1}_{n-1}\Big)\Big/ a^{[n,3]}_{0,n;0,n+1} \end{aligned} \tag{43}$$

By taking partial derivatives in equation (41)–(43) we are able to derive recursion formulas for the partial derivatives of the orthonormal polynomials as in (15)–(16).

4.2 Least square approximation

Similar to Section 3, the least square approximation in $L^2(\mathbb{B}_3)$ for a function $f \in L^2(\mathbb{B}_3)$ is given by

$$\mathcal{Q}_n f(x,y,z) = P_n(x,y,z) = \sum_{m=0}^{n} \sum_{j+k\le m} (f, Q_m^{j,k}) Q_m^{j,k}(x,y,z) \tag{44}$$

where the inner product is given by

$$(f, Q_m^{j,k}) = \int_{\mathbb{B}_3} f(x,y,z) Q_m^{j,k}(x,y,z)\, d(x,y,z) \tag{45}$$

For practical calculations we have to replace the integral in (45) by a quadrature rule for $f \in C(\mathbb{B}_3)$. One choice is to use a quadrature rule which will integrate polynomials of degree smaller or equal to $2n$ exactly, so we have

$$\mathcal{Q}_n p(x,y,z) = p(x,y,z), \quad \forall p \in \Pi_n \tag{46}$$

We will use

$$\begin{aligned}\int_{\mathbb{B}_3} g(x,y,z)\, d(x,y,z) &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} \widetilde{g}(r,\theta,\phi)\, r^2 \sin(\phi)\, d\phi\, d\theta\, dr \approx Q_q[g] \\ Q_q[g] &:= \sum_{i=1}^{2q} \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{\pi}{q}\, \omega_j\, \nu_k \widetilde{g}\left(\frac{\zeta_k+1}{2}, \frac{\pi\, i}{2q}, \arccos(\xi_j)\right) \end{aligned} \tag{47}$$

$q > n$. Here $\widetilde{g}(r,\theta,\phi) = g(x,y,z)$ is the representation of g in spherical coordinates. For the θ integration we use the trapezoidal rule, because the function is $2\pi-$periodic in θ. For the r direction we use the transformation

$$\begin{aligned}\int_0^1 r^2 v(r)\, dr &= \int_{-1}^1 \left(\frac{t+1}{2}\right)^2 v\left(\frac{t+1}{2}\right) \frac{dt}{2} \\ &= \frac{1}{8} \int_{-1}^1 (t+1)^2 v\left(\frac{t+1}{2}\right)\, dt \\ &\approx \sum_{k=1}^{q} \underbrace{\frac{1}{8}\nu_k'}_{=:\nu_k} v\left(\frac{\zeta_k+1}{2}\right)\end{aligned}$$

where the ν_k' and ζ_k are the weights and the nodes of the Gauss quadrature with q nodes on $[-1,1]$ with respect to the inner product

$$(v,w) = \int_{-1}^1 (1+t)^2 v(t) w(t)\, dt$$

The weights and nodes also depend on q but we omit this index. For the ϕ direction we use the transformation

$$\int_0^\pi \sin(\phi)v(\phi)\,d\phi = \int_{-1}^1 v(\arccos(\phi))\,d\phi \approx \sum_{j=1}^q \omega_j v(\arccos(\xi_j))$$

where the ω_j and ξ_j are the nodes and weights for the Gauss–Legendre quadrature on $[-1, 1]$. This quadrature rule has been used in our earlier articles, see [2]. For more information on this quadrature rule on the unit ball in $\mathbb{R}^3$, see [12]. For the complexity estimation in the next section we will assume that we use the smallest possible q to satisfy (46) which is $q = n + 1$. Although a little bit larger values of q might improve the approximation property of (44) in practice. With this value of q the quadrature formula (47) uses $2\,(n+1)^3 = 2n^3 + \mathcal{O}\left(n^2\right)$ points in the unit ball $\mathbb{B}_3$.

The discrete L^2 projection is now given by

$$\widetilde{\mathcal{Q}}_n f(x,y,z) = \widetilde{P}_n(x,y,z) = \sum_{m=0}^{n} \sum_{j+k\le m} Q_n[f \cdot Q_m^{j,k}]Q_m^{j,k}(x,y,z) \tag{48}$$

Regarding the convergence of the convergence of $\mathcal{Q}_n f$ towards f in $L^2(\mathbb{B}_3)$ and $L^\infty(\mathbb{B}_3)$ we have similar results to Section 3.2. Because the polynomials are dense we have convergence in $L^2(\mathbb{B}_3)$ and formulas (24) and (25) hold as before, and the same is true for the estimate for $E_{n,\infty}(f)$ in (26). But the Lebesgue constant for the projection $\mathcal{Q}_n$ in $L^\infty(\mathbb{B}_3)$ is larger,

$$\|\mathcal{Q}_n\|_{C\mapsto C} = \mathcal{O}_{n\to\infty}(n^{3/2}) \tag{49}$$

see [13]. Together with (26) we obtain the convergence in $C(\mathbb{B}_3)$ for functions which are in $C^{1,\alpha}(\mathbb{B}_3)$, $\alpha > 1/2$.

For the bound of $\|\widetilde{\mathcal{Q}}_n\|_{C\mapsto C}$ we can use the same arguments as in (2.10)–(2.18) of our previous article [9] together with the results about the reproducing kernel in [13]. This shows

$$\|\widetilde{\mathcal{Q}}_n\|_{C\mapsto C} = \mathcal{O}_{n\to\infty}(n^2)$$

and proves the convergence of the discrete L^2 approximation in the inifinity norm for functions which are in $C^{2,\alpha}(\mathbb{B}_3)$, $\alpha > 0$.

4.3 Computational cost

First we give here a brief analysis of the computational cost to evaluate all polynomials $Q_m^{j,k}$ in Π_n at a given point. We assume again, that all coefficients in (41)–(43) have been calculated. If we further assume that $Q_m^{j,k}$ and $Q_{m-1}^{j,k}$ have been calculated then (41), for $j + k \le m$, constitutes the dominant work for the calculation of $Q^{j,k}_{m+1}$, $j+k \leq m+1$. To evaluate (41) for $j+k \leq m$ requires $4\binom{m+2}{2} = 2m^2 + \mathcal{O}_{m\to\infty}(m)$ arithmetic operations. The evaluation of (42) and (43) will not change this asymptotic behavior. Adding these up for $m = 0, 1, \ldots n-1$ leads to a total number of arithmetic operations given by $\frac{2}{3}n^3 + \mathcal{O}_{n\to\infty}(n^2)$. If we further consider the problem to evaluate the polynomial

$$p(x,y,z) = \sum_{m=0}^{n} \sum_{j+k\leq m} b^{j,k}_m Q^{j,k}_m(x,y,z) \tag{50}$$

we have to add another $2\sum_{m=0}^{n}\binom{m+2}{2} = \frac{1}{3}n^3 + \mathcal{O}_{n\to\infty}(n^2)$ operations, which means that the evaluation of (50) requires a total $n^3 + \mathcal{O}_{n\to\infty}(n^2)$ operations, if the recursion coefficients are known. The set Π_n has $\frac{n^3}{6} + \mathcal{O}_{n\to\infty}(n^2)$ elements, so about 6 operations are needed in average per basis functions, exactly the same as in Section 2.

To calculate the discrete L^2 projection (48) we first need to evaluate f at the $\sim 2n^3$ quadrature points of Q_n, this requires an effort of $\sim 2n^3 N_f$, where N_f again measures the cost of an individual evaluation of f. Then we have to calculate all basis functions $Q^{j,k}_m$ in Π_n for all $2n^3$ points. This requires $\frac{4}{3}n^6 + \mathcal{O}_{n\to\infty}(n^5)$ operations. The calculation of a single $Q_n[f \cdot Q^{j,k}_m]$ requires $6n^3$ operations and we have to do this for all $\binom{n+3}{3}$ basis functions of Π_n which results in an additional $n^6 + \mathcal{O}_{n\to\infty}(n^5)$ operations. If we assume that the N_f is less than $\mathcal{O}(n^3)$ we see that the evaluation of the discrete inner products $Q_n[f \cdot Q^{j,k}_m]$ is the dominant term and the complexity of the calculation of (48) is given by $\frac{7}{3}n^6 + \mathcal{O}_{n\to\infty}(n^5)$.

5 Numerical examples and Matlab programs

We present Matlab programs for using orthonormal polynomials over the unit disk. We compute the coefficients $\{a_{i,n}, c_{i,n}, d_{i,n}\}$, the basis $\mathcal{B}_n$, and the discrete least squares approximation (19) with $q \geq n$. The program `TripleRecurCoeff` is used to produce the needed coefficients $\{a_{i,n}, c_{i,n}, d_{i,n}\}$, the program `EvalOrthoPolys` is used to evaluate the polynomials in the basis $\mathcal{B}_n$, and the program `LeastSqCoeff` evaluates the coefficients in (19). The program `EvalLstSq` is used to evaluate $\widetilde{P}_{n,q}(x,y)$ at a selected set of nodes in $\mathbb{B}_2$; it also evaluates the error and produces various graphs of the error as the degree n is increased. The program `Test_EvalLstSq` is used to test the programs just listed.

Consider the function

$$f(x,y) = \frac{1+x}{1+x^2+y^2}\cos\left(6xy^2\right) \tag{51}$$

This was approximated using `Test_EvalLstSq` for degrees 1 through 30. Figure 1 shows $\widetilde{P}_{30,40}$ and Figure 2 shows its error. The error as it varies with the degree n is shown in Figure 3. This last graph suggests an exponential rate of convergence for $\widetilde{P}_{n,q}$ to f.

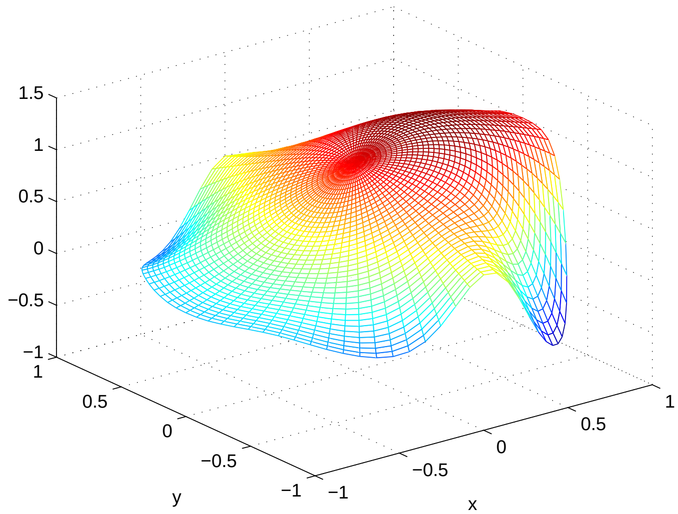

Figure 1: The approximation $\widetilde{P}_{n,q}(x,y)$ for (51), with $n = 30$ and $q = 40$

We have found often that the error $f(x,y) - \widetilde{P}_{n,q}(x,y)$ is slightly smaller than that of $f(x,y) - \widetilde{P}_{n,n}(x,y)$ if q is taken a small amount larger than n, say $q = n + 5$. However, the qualitative behaviour shown in Figure 3 is still valid for $f - \widetilde{P}_{n,n}$.

5.1 Additional comments

These programs can also be used for constructing approximations over other planar regions Ω. For example, the mapping

$$(x,y) \mapsto (\xi,\eta) = (ax, by), \qquad (x,y) \in \mathbb{B}_2,$$

with $a, b > 0$, can be used to create polynomial approximations to a function defined over the ellipse

$$\left(\frac{\xi}{a}\right)^2 + \left(\frac{\eta}{b}\right)^2 \le 1.$$

If polynomials are not required, only an approximating function, then mappings

$$(x,y) \mapsto (\xi,\eta) = \Phi(x,y), \qquad (x,y) \in \mathbb{B}_2$$

with Φ a 1-1 mapping can be used to convert an approximation problem over a planar region Ω to one over $\mathbb{B}_2$. The construction of such mappings Φ is explored in [5].

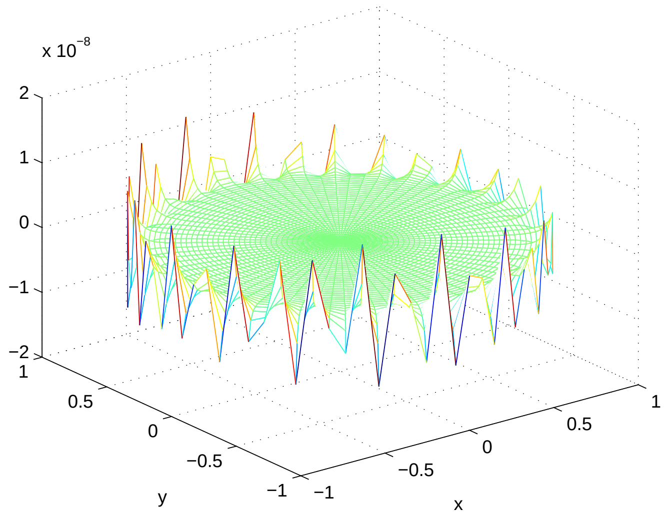

Figure 2: The error $f - \widetilde{P}_{n,q}$ for (51), with $n = 30$ and $q = 40$

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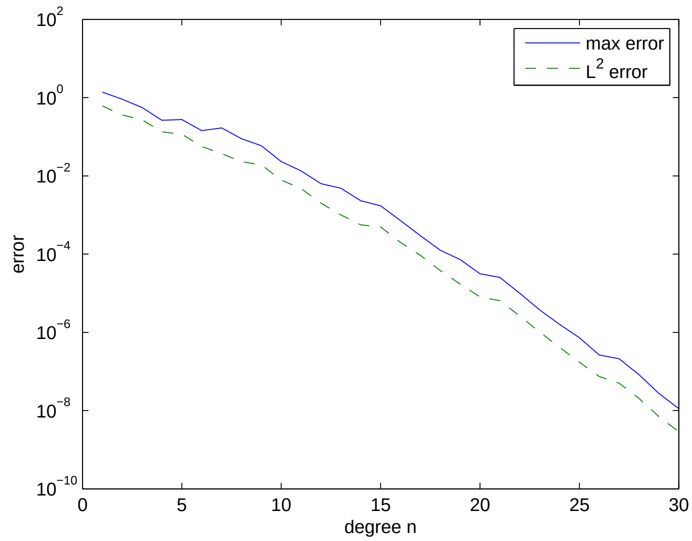

Figure 3: The error $f - \widetilde{P}_{n,q}$ for (51) with $q = 40$

012-9620-8, to appear. A preliminary version is available at http://arxiv.org/abs/1203.6709

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