ON THE LEADING COEFFICIENT OF POLYNOMIALS ORTHOGONAL OVER DOMAINS WITH CORNERS

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ABSTRACT. Let G be the interior domain of a piecewise analytic Jordan curve without cusps. Let $\{p_n\}_{n=0}^{\infty}$ be the sequence of polynomials that are orthonormal over G with respect to the area measure, with each p_n having leading coefficient $\lambda_n > 0$. It has been proven in [9] that the asymptotic behavior of λ_n as $n \to \infty$ is given by

$$\frac{n+1}{\pi}\frac{\gamma^{2n+2}}{\lambda_n^2} = 1 - \alpha_n$$

where $\alpha_n = O(1/n)$ as $n \to \infty$ and γ is the reciprocal of the logarithmic capacity of the boundary ∂G . In this paper, we prove that the O(1/n) estimate for the error term α_n is, in general, best possible, by exhibiting an example for which

$$\liminf_{n \to \infty} n\alpha_n > 0$$

The proof makes use of the Faber polynomials, about which a conjecture is formulated.

1. INTRODUCTION

Let L be a Jordan curve in the complex plane \mathbb{C} . The bounded and unbounded components of $\overline{\mathbb{C}} \setminus L$ will be denoted by G and Ω , respectively. Let $\{p_n\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to the area measure over G. That is, each $p_n(z) = \lambda_n z^n + \cdots$ is a polynomial of degree n, having positive leading coefficient λ_n , and for every pair of non-negative integers m, n, we have

$$\int_{G} p_n(z) \overline{p_m(z)} dx dy = \delta_{n,m}$$

The asymptotic behavior as $n \to \infty$ of these polynomials has been thoroughly investigated when L is an analytic Jordan curve in [1, 2, 3, 4, 7], while for L having some degree of smoothness, strong asymptotics for p_n , outside and on the curve itself, were obtained by Suetin [10].

For L a piecewise analytic Jordan curve, investigations on the *n*th-root asymptotics and zero distribution of the polynomials p_n have been carried out in [5, 6, 8]. More recently, finer results have been obtained by N. Stylianopoulos in [9] with the use of some tools from quasiconformal mapping theory. For instance, it is proven in [9, Thm. 1.1] that if L is a piecewise analytic curve without cusps, then the leading coefficients λ_n satisfy the asymptotic formula

$$\frac{n+1}{\pi}\frac{\gamma^{2n+2}}{\lambda_n^2} = 1 - \alpha_n$$

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where $\alpha_n = O(1/n)$ as $n \to \infty$, and γ is the reciprocal of the logarithmic capacity of L. This quantity γ can be introduced in this context via the conformal map

$$\phi: \Omega \to \{w: |w| > 1\}$$

of Ω onto the exterior of the unit circle, uniquely determined by the conditions $\phi(\infty) = \infty$ and $\phi'(\infty) := \lim_{z \to \infty} \phi(z)/z > 0$. This limit is precisely the value of γ .

We notice that [9] also establishes a strong asymptotic formula for p_n on the exterior of L, and several other important estimates and relations that we do not mention here.

In this paper, we prove that the O(1/n) estimate for the error term α_n is, in general, best possible, by exhibiting an example of a curve L for which

$$\liminf_{n \to \infty} n\alpha_n > 0$$

For each integer $n \ge 0$, the Faber polynomial F_n associated to L [11] is defined to be the polynomial part of the Laurent expansion at ∞ of ϕ^n . The polynomials F_n and the functions

$$E_n(z) := \phi^n(z) - F_n(z), \qquad z \in \Omega, \quad n \ge 0,$$

play an important role in the estimation of α_n , since, as proven in [9, Lem. 2.4],

$$\alpha_n = \frac{(n+1)}{\pi} \left\| \frac{F'_{n+1}}{n+1} - \frac{\gamma^{n+1}}{\lambda_n} p_n \right\|_{L^2(G)}^2 + \frac{\|E'_{n+1}\|_{L^2(\Omega)}^2}{\pi(n+1)}.$$
 (1)

The proof that $\alpha_n = O(1/n)$ is then accomplished by showing that [9, Thm. 2.1] the first summand in the right-hand side of (1) is a big O of the second one, while for the second summand [9, Thm. 2.4] we have

$$||E'_{n+1}||^2_{L^2(\Omega)} = O(1), \qquad (n \to \infty).$$
(2)

Summarizing, there exists some constant C independent of n for which

$$\frac{\|E'_{n+1}\|_{L^2(\Omega)}^2}{\pi(n+1)} \le \alpha_n \le C \frac{\|E'_{n+1}\|_{L^2(\Omega)}^2}{\pi(n+1)}, \quad n \ge 0.$$
(3)

Consider the circles $C_1 = \{z : |z - i| = \sqrt{2}\}$ and $C_2 = \{z : |z + i| = \sqrt{2}\}$, which intersect at the points ± 1 . Let us take L to be the curve consisting of the two arcs of these circles that lie exterior to each other, that is,

$$L := \{ z \in C_1 : \Im(z) \ge 0 \} \cup \{ z \in C_2 : \Im(z) \le 0 \}.$$
(4)

This is a piecewise analytic curve with corners at ± 1 and exterior angles $\pi/2$. We shall prove the following result.

Theorem 1.1. For the curve L defined by (4), we have

$$\liminf_{n \to \infty} n\alpha_n \ge \frac{1}{\pi} \lim_{n \to \infty} \|E'_{n+1}\|_{L^2(\Omega)}^2 = \frac{1}{2\pi^2}.$$
(5)

The inequality in (5) is, of course, a consequence of (3), but it takes some effort to establish the existence and value of the limit in (5).

Based on Theorem 1.1, we find plausible that the following conjecture be true.

Conjecture 1.2. For an arbitrary piecewise analytic Jordan curve L having at least one corner with exterior angle different from $0, \pi$, and 2π , we have

$$\lim_{n \to \infty} \|E'_{n+1}\|_{L^2(\Omega)}^2 > 0.$$

The weaker thesis that $\liminf_{n\to\infty} \|E'_{n+1}\|^2_{L^2(\Omega)} > 0$ would be enough to guarantee that the O(1/n) estimate for α_n is sharp for every such curve. If the conjecture were true, it would be interesting to determine whether the value of the limit is, indeed, independent of the curve L, and therefore equal to $(2\pi)^{-1}$.

2. Proof of Theorem 1.1

Hereafter, L will denote the curve in (4). The other two arcs of the circles C_1 and C_2 also form a piecewise analytic Jordan curve that we denote by

$$\mathcal{L} := \{ z \in C_1 : \Im(z) \le 0 \} \cup \{ z \in C_2 : \Im(z) \ge 0 \}.$$

The exterior of L will be denoted by Ω , while the interior of \mathcal{L} will be denoted by \mathcal{R} . It is easy to verify that

$$\mathcal{R} = \{1/z : z \in \Omega\},\$$

and that the Zhoukowsky transformation $\phi(z) = 2^{-1} (z + 1/z)$ maps Ω conformally onto $\{w : |w| > 1\}$. This same function ϕ takes both L and \mathcal{L} onto the unit circle.

With the notation we previously introduced for the Faber polynomials and related quantities, we then have for the curve L that

$$\phi^{n}(z) = \frac{1}{2^{n}} \left(z + \frac{1}{z} \right)^{n} = F_{n}(z) + E_{n}(z), \tag{6}$$

where F_n is the polynomial part of ϕ^n , so that if we define

$$G_n(z) := F_n(z) - F_n(0), \quad n \ge 0,$$

then $G_0(z) \equiv 0$ and

$$G_n(z) = \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} {n \choose j} z^{n-2j}, \quad n \ge 1.$$
(7)

Since ϕ is invariant under $z \mapsto 1/z$, we get from (6) that

$$E_{n+1}(z) = G_n(1/z),$$
 (8)

and that G_n satisfies the recurrence relation

$$G_{n+1}(z) = \phi(z)G_n(z) + \frac{za_n}{2} - \frac{a_{n+1}}{2}, \quad n \ge 0,$$
(9)

where

$$a_n := F_n(0) = \begin{cases} 0, & n \text{ odd,} \\ 2^{-n} \binom{n}{n/2}, & n \text{ even,} \end{cases} \quad n \ge 0.$$

From this explicit expression for a_n one can easily verify that

$$a_n = \frac{n-1}{n} a_{n-2}, \quad n \ge 2.$$
(10)

At some point, we will need to deal with the quantities

$$b_n := \int_{-1}^1 \frac{G_{n+1}(x)}{x} dx \quad n \ge 0.$$

If n is odd, G_{n+1} is even and so $b_n = 0$. If n is even, then (7) yields

$$b_n = \frac{1}{2^n} \sum_{j=0}^{n/2} \frac{\binom{n+1}{j}}{n+1-2j}, \quad n = 2k, \quad k \ge 0.$$

From this last expression, it is not difficult to see that

$$b_n = \frac{n}{n+1}b_{n-2} + \frac{a_n}{n+1}, \quad n \ge 2.$$
(11)

Combining (10) and (11), we find $(n+2)a_{n+2}b_n - na_nb_{n-2} = a_n^2$, so that

$$\sum_{j=0}^{k} a_{2j}^2 = (2k+1)a_{2k}b_{2k}, \quad k \ge 0.$$
(12)

We now have everything we need to give the proof of Theorem 1.1.

It follows from (8), and the fact that $z \mapsto 1/z$ takes Ω onto the region \mathcal{R} , that

$$||E'_{n+1}||^2_{L^2(\Omega)} = \int_{\mathcal{R}} |G'_{n+1}(z)|^2 dx dy.$$

Let \mathcal{L}_1 denote the part of \mathcal{L} lying in the closed upper half plane. Since $G_n(\overline{z}) = \overline{G_n(z)}$, and since $G'_{n+1}G_{n+1}$ is an odd function, the complex version of Green's formula yields

$$\|E'_{n+1}\|^2_{L^2(\Omega)} = I_{n+1,n+1}, \tag{13}$$

where

$$I_{n,k} := \frac{1}{i} \int_{\mathcal{L}_1} [G_k(z)\phi^{n-k}(z)]' \overline{G_k(z)\phi^{n-k}(z)} dz, \quad n \ge 0, \ 0 \le k \le n.$$

Notice that $I_{n,0} = 0$.

Using the recurrence relation (9), we find that

$$I_{n+1,k+1} = I_{n+1,k} + A_{n,k} + B_{n,k} + C_{n,k},$$
(14)

where

$$A_{n,k} := \frac{1}{2i} \int_{\mathcal{L}_1} [\phi^{n+1-k}(z)G_k(z)]' \overline{\phi^{n-k}(z)} \left[\overline{z}a_k - a_{k+1}\right] dz, \tag{15}$$

$$B_{n,k} := \frac{1}{2i} \int_{\mathcal{L}_1} \overline{\phi^{n+1-k}(z)G_k(z)} \left(\phi^{n-k}(z) \left[za_k - a_{k+1}\right]\right)' dz, \tag{16}$$

and

$$C_{n,k} := \frac{1}{4i} \int_{\mathcal{L}_1} \overline{\phi^{n-k}(z)} \left[\overline{z} a_k - a_{k+1} \right] \left(\phi^{n-k}(z) \left[z a_k - a_{k+1} \right] \right)' dz \,. \tag{17}$$

We now observe that for $z \in \mathcal{L}_1$,

$$|\phi(z)| = 1, \quad [\phi(z)]^2 \overline{\phi'(z)} \,\overline{dz} = -\phi'(z) dz, \tag{18}$$

and

$$\overline{z} = \frac{1+iz}{z+i} = \frac{2z+i(z^2-1)}{z^2+1}, \quad \overline{dz} = -\frac{2dz}{(z+i)^2} = d\left(\frac{1+iz}{z+i}\right).$$
(19)

Hence, integration by parts in (15) gives

$$A_{n,k} = \frac{n-k}{2i} \int_{\mathcal{L}_1} G_k(z) \phi'(z) \left[\overline{z}a_k - a_{k+1} \right] dz + \frac{a_k}{i} \int_{\mathcal{L}_1} \frac{\phi(z)G_k(z)}{(z+i)^2} dz.$$

Similarly, by expanding the derivative in (16), multiplying, and using (18) and (19), we obtain that $B_{n,k} = \overline{A_{n,k}}$. Hence,

$$A_{n,k} + B_{n,k} = (n-k)\Im\left(\int_{\mathcal{L}_1} G_k(z)\phi'(z) \left[\overline{z}a_k - a_{k+1}\right] dz\right) + 2a_k\Im\left(\int_{\mathcal{L}_1} \frac{\phi(z)G_k(z)}{(z+i)^2} dz\right) = 2(n+1-k)a_k \int_{-1}^1 \frac{G_k(z)}{z^2+1} dz - \frac{(n-k)a_k}{2} \int_{-1}^1 \frac{G_k(z)(z^2+1)}{z^2} dz + \frac{\pi(n-k)a_{k+1}^2}{2}.$$
(20)

In the calculations leading to the equality of these two last expressions, we have used (19), that

$$\frac{(z^2-1)^2}{z^2(z^2+1)} = \frac{z^2+1}{z^2} - \frac{4}{z^2+1},$$

and that for k even

$$\int_{-1}^{1} \frac{G_k(z)}{z} \frac{(z^2 - 1)}{z^2 + 1} dz = 0,$$

while for k odd,

$$\int_{\mathcal{L}_1} \frac{G_k(z)(z^2 - 1)}{z^2} dz = -G'_k(0) \int_{\mathcal{L}_1} \frac{1}{z} dz = -i\pi a_{k+1}.$$

Applying the recurrence relation (9) one more time yields

$$\int_{-1}^{1} \frac{G_k(z)}{z^2 + 1} dz = \frac{b_{k-2}}{2} - \frac{\pi a_k}{4}, \qquad \int_{-1}^{1} \frac{G_k(z)(z^2 + 1)}{z^2} dz = 2b_k - 2a_k, \tag{21}$$

and since $G_0(z) \equiv 0$ and $a_1 = 0$, we get from (20), (21) and (10) that

$$\sum_{k=0}^{n} [A_{n,k} + B_{n,k}] = \sum_{k=0}^{n-1} (n-k-1)a_{k+2}b_k - \sum_{k=2}^{n} (n-k)a_kb_k + \sum_{k=1}^{n} (n-k)a_k^2$$
$$= na_0b_0 - (n+1)\sum_{k=0}^{n-1} \frac{a_kb_k}{k+2} + \sum_{k=1}^{n} (n-k)a_k^2.$$
(22)

We now consider (17) and compute

$$C_{n,k} = \frac{(n-k)a_k^2}{4i} \int_{\mathcal{L}_1} \overline{\phi(z)}\phi'(z)|z|^2 dz + \frac{(n-k)a_{k+1}^2}{4i} \int_{\mathcal{L}_1} \overline{\phi(z)}\phi'(z)dz + \frac{a_k^2}{4i} \int_{\mathcal{L}_1} \overline{z}dz \,.$$

Using (18) again, and having in mind that ϕ takes \mathcal{L}_1 onto the lower half of the unit circle, we get

$$\int_{\mathcal{L}_1} \overline{\phi(z)} \phi'(z) dz = \int_{\mathcal{L}_1} \frac{\phi'(z)}{\phi(z)} dz = -i\pi,$$
$$\int_{\mathcal{L}_1} \overline{z} dz = -\int_{-1}^1 \frac{2z}{z^2 + 1} dz - i \int_{-1}^1 \frac{z^2 - 1}{z^2 + 1} dz = i(\pi - 2)$$

and since $|z|^2 = 1 + iz - i\overline{z}$ for $z \in \mathcal{L}_1$,

$$\int_{\mathcal{L}_1} \overline{\phi(z)} \phi'(z) |z|^2 dz = -i\pi + i \int_{\mathcal{L}_1} \frac{\phi'(z)}{\phi(z)} z dz + i \overline{\int_{\mathcal{L}_1} \frac{\phi'(z)}{\phi(z)} z dz}$$
$$= -i\pi - 2i \int_{-1}^1 \frac{z^2 - 1}{z^2 + 1} dz.$$

Hence,

$$C_{n,k} = \frac{\pi}{4} \left[(n+1-k)a_k^2 - (n-k)a_{k+1}^2 \right] - (n-k)a_k^2 - a_k^2/2.$$
(23)

Then, combining (14), (22) and (23), and since $a_0 = b_0 = 1$, we obtain

$$I_{n+1,n+1} = \sum_{k=0}^{n} \left(A_{n,k} + B_{n,k} + C_{n,k} \right) = (n+1) \left[\frac{\pi}{4} - \sum_{k=0}^{n-1} \frac{a_k b_k}{k+2} \right] - \frac{1}{2} \sum_{k=0}^{n} a_k^2.$$

Now, summation by parts gives

$$\frac{1}{2N+1}\sum_{k=0}^{N}a_{2k}^{2} = \sum_{k=0}^{N}\frac{a_{2k}^{2}}{2k+1} - \sum_{k=0}^{N-1}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)\sum_{j=0}^{k}a_{2j}^{2}.$$

Combining the two last equalities and (12), we get that for every integer $N \ge 0$, we have

$$\frac{I_{2N+1,2N+1}}{2N+1} = \frac{\pi}{4} - \frac{1}{2} \sum_{k=0}^{N} \frac{a_{2k}^2}{2k+1} - \sum_{k=0}^{N-1} \frac{\sum_{j=0}^{k} a_{2j}^2}{(2k+1)(2k+2)(2k+3)}, \quad (24)$$
$$I_{2N+2,2N+2} = \frac{(2N+2)}{(2N+1)} I_{2N+1,2N+1} - \frac{\sum_{j=0}^{N} a_{2j}^2}{(2N+1)}.$$

Since (2) tells us that $I_{n+1,n+1}$, which is defined by (13), is bounded above, the bracket in (24) must converge to zero as $n \to \infty$, and since

$$a_{2k} = \frac{1}{2^{2k}} \binom{2k}{k} = \frac{\Gamma(1/2)\Gamma(k+1/2)}{\pi\Gamma(k+1)} = \frac{1+O(1/k)}{\sqrt{\pi(k-1/2)}},$$

we arrive at

$$\begin{split} \frac{I_{2N+1,2N+1}}{2N+1} &= \frac{1}{2} \sum_{k=N+1}^{\infty} \frac{a_{2k}^2}{2k+1} + \sum_{k=N}^{\infty} \frac{\sum_{j=0}^k a_{2j}^2}{(2k+1)(2k+2)(2k+3)} \\ &= \frac{1}{2\pi} \sum_{k=N+1}^{\infty} \frac{2+O(1/k)}{(2k-1)(2k+1)} + \sum_{k=N}^{\infty} \frac{O(\sum_{j=0}^k \frac{1}{j})}{(2k+1)(2k+2)(2k+3)} \\ &= \frac{1}{2\pi(2N+1)} + O(\ln N/N^2), \end{split}$$

and Theorem 1.1 follows.

6

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