

# Improvements on the infinity norm bound for the inverse of Nekrasov matrices

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## Abstract

We focus on the estimating problem of the infinity norm of the inverse of Nekrasov matrices, give new bounds which involve a parameter, and then determine the optimal value of the parameter such that the new bounds are better than those in L. Cvetković et al. (2013) [5]. Numerical examples are given to illustrate the corresponding results.

*Keywords:* Infinity norm; Nekrasov matrices; H-matrices

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## 1. Introduction

A matrix  $A = (a_{ij}) \in C^{m,n}$  is called an  $H$ -matrix if its comparison matrix  $\langle A \rangle = [m_{ij}]$  defined by

$$\langle A \rangle = [m_{ij}] \in C^{m,n}, m_{ij} = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is an  $M$ -matrix, i.e.,  $\langle A \rangle^{-1} \geq 0$  [1, 3, 4].  $H$ -matrices has a large number of applications. One special interest problem among them is to find upper bounds of the infinity norm of  $H$ -matrices, since it can be used for proving the convergence of matrix splitting and matrix multisplitting iteration methods for solving large sparse systems of linear equations, see [1, 5, 8, 9]. Many researchers gave some well-known bounds. In 1975, J.M. Varah[13] provided the following upper bound for strictly diagonally dominant (SDD) matrices as

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one most important subclass of  $H$ -matrices. Here a matrix  $A = [a_{ij}] \in C^{n,n}$  is called SDD if for each  $i \in N = \{1, 2, \dots, n\}$ ,

$$|a_{ii}| > r_i(A),$$

where  $r_i(A) = \sum_{j \neq i} |a_{ij}|$ .

**Theorem 1.** [13] *Let  $A = [a_{ij}] \in C^{n,n}$  be SDD. Then*

$$\|A^{-1}\|_{\infty} \leq \frac{1}{\min_{i \in N} (|a_{ii}| - r_i(A))}.$$

We call the bound in Theorem 1 the Varah's bound. As Cvetković et al. [5] said, the Varah's bound works only for SDD matrices, and even then it is not always good enough. Hence, it can be useful to obtain new upper bounds for a wider class of matrices which sometimes works better in the SDD case. In [5], Cvetković et al. study the class of Nekrasov matrices which contains SDD matrices and is a subclass of  $H$ -matrices, and give the following bounds (see Theorem 2).

**Definition 1.** [4, 5] A matrix  $A = [a_{ij}] \in C^{n,n}$  is called a Nekrasov matrix if for each  $i \in N$ ,

$$|a_{ii}| > h_i(A),$$

where  $h_1(A) = r_1(A) = \sum_{j \neq 1} |a_{1j}|$  and  $h_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j(A) + \sum_{j=i+1}^n |a_{ij}|$ ,  $i = 2, 3, \dots, n$ .

**Theorem 2.** [5, Theorem 2] *Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix. Then*

$$\|A^{-1}\|_{\infty} \leq \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}, \quad (1)$$

and

$$\|A^{-1}\|_{\infty} \leq \frac{\max_{i \in N} z_i(A)}{\min_{i \in N} (|a_{ii}| - h_i(A))}, \quad (2)$$

where  $z_1(A) = 1$  and  $z_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A) + 1$ ,  $i = 2, 3, \dots, n$ .

Since an SDD matrix is a Nekrasov matrices [4, 10], the bounds (1) and (2) can be also applied to SDD matrices. However, the Varah's bound can not be used to estimate the infinity norm of the inverse of Nekrasov matrices. Furthermore, when we use both bounds to estimate the infinity norm of the inverse of SDD matrices, the bound (1) or (2) works better than the Varah's bound in some cases (for details, see [5]).

In this paper, we also focus on the estimating problem of the infinity norm of the inverse of Nekrasov matrices, and give new bounds which involve a parameter  $\mu$  based on the bounds in Theorem 2, and then determine the optimal value of  $\mu$  such that the new bounds are better than those in Theorem 2 (Theorem 2 in [5]). Numerical examples are given to illustrate the corresponding results.

## 2. New bounds for the infinity norm of the inverse of Nekrasov matrices

First, some lemmas and notations are listed. Given a matrix  $A = [a_{ij}]$ , by  $A = D - L - U$  we denote the standard splitting of  $A$  into its diagonal ( $D$ ), strictly lower ( $-L$ ) and strictly upper ( $-U$ ) triangular parts. And by  $[A]_{ij}$  we denote the  $(i, j)$ -entry of  $A$ , that is,  $[A]_{ij} = a_{ij}$ .

**Lemma 3.** [2] *Let  $A = [a_{ij}] \in C^{n,n}$  be a nonsingular  $H$ -matrix. Then*

$$|A^{-1}| \leq \langle A \rangle^{-1}.$$

**Lemma 4.** [11] *Given any matrix  $A = [a_{ij}] \in C^{n,n}$ ,  $n \geq 2$ , with  $a_{ii} \neq 0$  for all  $i \in N$ , then*

$$h_i(A) = |a_{ii}| [ (|D| - |L|)^{-1} |U| e ]_i,$$

where  $e \in C^{n,n}$  is the vector with all components equal to 1.

**Lemma 5.** [12] *A matrix  $A = [a_{ij}] \in C^{n,n}$ ,  $n \geq 2$  is a Nekrasov matrix if and only if*

$$(|D| - |L|)^{-1} |U| e < e,$$

*i.e., if and only if  $E - (|D| - |L|)^{-1} |U|$  is an SDD matrix, where  $E$  is the identity matrix.*

Let

$$C = E - (|D| - |L|)^{-1} |U| = [c_{ij}]$$

and

$$B = |D|C = |D| - |D|(|D| - |L|)^{-1}|U| = [b_{ij}]$$

and Then from Lemma 5,  $B$  and  $C$  are SDD when  $A$  is a Nekrasov matrix. Note that  $c_{11} = 1$ ,  $c_{k1} = 0$ ,  $k = 2, 3, \dots, n$ , and  $c_{1k} = -\frac{|a_{1k}|}{|a_{11}|}$ ,  $k = 2, 3, \dots, n$ , and that  $b_{11} = |a_{11}|$ ,  $b_{k1} = 0$ ,  $k = 2, 3, \dots, n$ , and  $b_{1k} = -|a_{1k}|$ ,  $k = 2, 3, \dots, n$ , which lead to the following lemma.

**Lemma 6.** *Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix,*

$$C(\mu) = CD(\mu) = (E - (|D| - |L|)^{-1}|U|) D(\mu), \quad (3)$$

and

$$B(\mu) = BD(\mu) = (|D| - |D|(|D| - |L|)^{-1}|U|) D(\mu), \quad (4)$$

where  $D(\mu) = \text{diag}(\mu, 1, \dots, 1)$  and  $\mu > \frac{r_1(A)}{|a_{11}|}$ . Then  $C(\mu)$  and  $B(\mu)$  are SDD,

$$\|C(\mu)^{-1}\|_{\infty} \leq \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\}, \quad (5)$$

and

$$\|B(\mu)^{-1}\|_{\infty} \leq \frac{1}{\min \left\{ \mu|a_{11}| - h_1(A), \min_{i \neq 1} (|a_{ii}| - h_i(A)) \right\}}. \quad (6)$$

*Proof.* We first prove (5) holds. It is not difficult from (3) to see that  $[C(\mu)]_{k1} = \mu c_{k1}$  for all  $k \in N$  and  $[C(\mu)]_{kj} = c_{kj}$  for all  $k \in N$  and  $j \neq 1$ . Hence

$$[C(\mu)]_{11} = \mu, \quad r_1(C(\mu)) = r_1(C) = \frac{r_1(A)}{|a_{11}|}$$

and for  $i = 2, \dots, n$ ,

$$[C(\mu)]_{ii} = c_{ii}, \quad r_i(C(\mu)) = r_i(C).$$

From  $C$  is SDD and  $\mu > \frac{r_1(A)}{|a_{11}|}$ , we have that  $C(\mu)$  is SDD.

Moreover, by applying the Varah's bound to estimate the infinity norm of its inverse matrix, we can obtain

$$\|C(\mu)^{-1}\|_{\infty} \leq \max_{i \in N} \frac{1}{|[C(\mu)]_{ii}| - r_i(C(\mu))} = \max \left\{ \frac{1}{\mu - r_1(C)}, \max_{i \neq 1} \frac{1}{|c_{ii}| - r_i(C)} \right\}.$$

Note that  $C = E - (|D| - |L|)^{-1}|U| = [c_{ij}]$  and all diagonal entries of matrix  $(|D| - |L|)^{-1}|U|$  are less than 1. Then we have that for  $i \in N$ ,  $i \neq 1$ ,

$$|c_{ii}| = 1 - [ (|D| - |L|)^{-1}|U| ]_{ii}$$

and that for each  $i \in N$ ,

$$r_i(C) = \sum_{k \neq i} [ (|D| - |L|)^{-1}|U| ]_{ik}.$$

These lead to that (also see the proof of Theorem 2 in [5]) for  $i \in N$ ,  $i \neq 1$ ,

$$|c_{ii}| - r_i(C) = 1 - \sum_{k \in N} [ (|D| - |L|)^{-1}|U| ]_{ik} = 1 - [ (|D| - |L|)^{-1}|U|e ]_i = 1 - \frac{h_i(A)}{|a_{ii}|}.$$

Since  $r_1(C) = \frac{r_1(A)}{|a_{11}|} = \frac{h_1(A)}{|a_{11}|}$ , we have

$$\begin{aligned} \|C(\mu)^{-1}\|_\infty &\leq \max \left\{ \frac{1}{\mu - r_1(C)}, \max_{i \neq 1} \frac{1}{|c_{ii}| - r_i(C)} \right\} \\ &= \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \max_{i \neq 1} \frac{1}{1 - \frac{h_i(A)}{|a_{ii}|}} \right\} \\ &= \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\}. \end{aligned}$$

We prove easily that (6) holds in an analogous way. The proof is completed.  $\square$

The main result of this paper is the following theorem:

**Theorem 7.** *Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix. Then for  $\mu > \frac{r_1(A)}{|a_{11}|}$ ,*

$$\|A^{-1}\|_\infty \leq \max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\}, \quad (7)$$

and

$$\|A^{-1}\|_\infty \leq \frac{\max\{\mu, 1\} \max_{i \in N} z_i(A)}{\min \left\{ \mu |a_{11}| - h_1(A), \min_{i \neq 1} (|a_{ii}| - h_i(A)) \right\}}. \quad (8)$$

*Proof.* We only prove that (7) holds, and in an analogous way, (8) is proved easily. Let  $C(\mu) = CD(\mu) = (E - (|D| - |L|)^{-1}|U|)D(\mu)$ , where  $D(\mu) = \text{diag}(\mu, 1, \dots, 1)$ . From (3), we have

$$C(\mu) = (E - (|D| - |L|)^{-1}|U|)D(\mu) = (|D| - |L|)^{-1} \langle A \rangle D(\mu),$$

which implies that

$$\langle A \rangle = (|D| - |L|)C(\mu)D(\mu)^{-1}.$$

Furthermore, since a Nekrasov matrix is an  $H$ -matrix, we have from Lemma 3,

$$\|A^{-1}\|_{\infty} \leq \|\langle A \rangle^{-1}\|_{\infty} \leq \|D(\mu)\|_{\infty} \|C(\mu)^{-1}\|_{\infty} \|( |D| - |L| )^{-1}\|_{\infty}. \quad (9)$$

Note that  $|D| - |L|$  is an  $M$ -matrix, and then similar to the proof of Theorem 2 in [5], we can easily obtain

$$\|( |D| - |L| )^{-1}\|_{\infty} = \|y\|_{\infty} = \max_{i \in n} \frac{z_i(A)}{|a_{ii}|}, \quad (10)$$

where  $y = (|D| - |L|)^{-1}e = [y_1, y_2, \dots, y_n]^T$  and  $z_i(A) = |a_{ii}|y_i$ , i.e.,

$$z_1(A) = 1, \text{ and } z_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A) + 1, i = 2, \dots, n.$$

From (5), (9), (10) and the fact that  $\|D(\mu)\|_{\infty} = \max\{\mu, 1\}$ , we have

$$\|A^{-1}\|_{\infty} \leq \max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\}.$$

The conclusions follows. □

**Example 1.** Consider the Nekrasov matrix  $A_1$  in [5], where

$$A_1 = \begin{bmatrix} -7 & 1 & -0.2 & 2 \\ 7 & 88 & 2 & -3 \\ 2 & 0.5 & 13 & -2 \\ 0.5 & 3.0 & 1 & 6 \end{bmatrix}.$$

By computation,  $h_1(A) = 3.2000$ ,  $h_2(A) = 8.2000$ ,  $h_3(A) = 2.9609$ ,  $h_4(A) = 0.7359$ ,  $z_1(A) = 1$ ,  $z_2(A) = 2$ ,  $z_3(A) = 1.2971$  and  $z_4(A) = 1.2394$ . By Theorem 2 (Theorem 2 in [5]), we have

$$\|A_1^{-1}\|_\infty \leq 0.3805, \text{ (The bound (1) of Theorem 2)}$$

and

$$\|A_1^{-1}\|_\infty \leq 0.5263. \text{ (The bound (2) of Theorem 2)}$$

By the bound (7) of Theorem 7, we have

$$\begin{aligned} \|A_1^{-1}\|_\infty &\leq 4.8198 && \text{(Taking } \mu = 0.5), \\ \|A_1^{-1}\|_\infty &\leq 0.6025 && \text{(Taking } \mu = 0.8), \\ \|A_1^{-1}\|_\infty &\leq 0.3535 && \text{(Taking } \mu = 1.1), \\ \|A_1^{-1}\|_\infty &\leq 0.3745 && \text{(Taking } \mu = 1.4), \\ \|A_1^{-1}\|_\infty &\leq 0.4547 && \text{(Taking } \mu = 1.7), \end{aligned}$$

and by the bound (8) of Theorem 7, we have

$$\begin{aligned} \|A_1^{-1}\|_\infty &\leq 2.0000 && \text{(Taking } \mu = 0.6), \\ \|A_1^{-1}\|_\infty &\leq 0.6452 && \text{(Taking } \mu = 0.9), \\ \|A_1^{-1}\|_\infty &\leq 0.4615 && \text{(Taking } \mu = 1.2), \\ \|A_1^{-1}\|_\infty &\leq 0.5699 && \text{(Taking } \mu = 1.5), \\ \|A_1^{-1}\|_\infty &\leq 0.6839 && \text{(Taking } \mu = 1.8). \end{aligned}$$

In fact,  $\|A_1^{-1}\|_\infty = 0.1921$ ,

**Remark 1.** Example 1 shows that by choosing the value of  $\mu$ , the bound (7) ((8), resp.) of Theorem 7 is better than the bound (1) ((2), resp.) of Theorem 2 in some cases. We further observe the bounds in Theorem 7 by Figures 1 and 2, and find that there is an interval such that for any  $\mu$  in this interval, the bound (7) ((8), resp.) of Theorem 7 for the matrix  $A_1$  is always smaller than the bound (1) ((2), resp.) of Theorem 2. An interesting problem arises: whether there is an interval of  $\mu$  such that the bound (7) ((8), resp.) of Theorem 7 for any Nekrasov matrix is smaller than the bound (1) ((2), resp.) of Theorem 2? In the following section, we will study this problem.

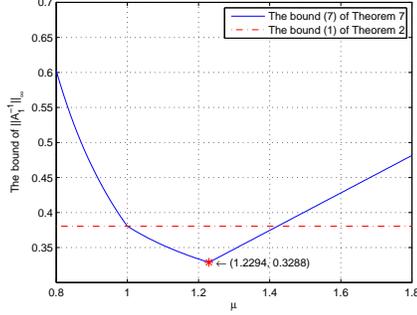


Figure 1: The bounds (1) and (7)

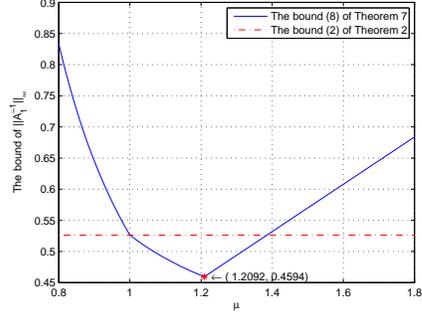


Figure 2: The bounds (2) and (8)

### 3. The choice of $\mu$

In this section, we determine the value of  $\mu$  such that our bounds for  $\|A^{-1}\|_{\infty}$  are less or equal to those of [5].

#### 3.1. the optimal value of $\mu$ for the bound (7)

First, we consider the Nekrasov matrix  $A = [a_{ij}] \in C^{m,n}$  with

$$\frac{h_1(A)}{|a_{11}|} > \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}, \quad (11)$$

and give the following lemma.

**Lemma 8.** Let  $A = [a_{ij}] \in C^{m,n}$  be a Nekrasov matrix with

$$\frac{h_1(A)}{|a_{11}|} > \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}.$$

Then

$$1 < 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} < \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}}. \quad (12)$$

*Proof.* Obviously, the first Inequality in (12) holds. We only prove that the second holds. From Inequality (11), we have that

$$\frac{h_1(A)}{|a_{11}|} \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} - \left( \frac{r_1(A)}{|a_{11}|} \right)^2 < 0.$$

Equivalently,

$$1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} + \frac{h_1(A)}{|a_{11}|} - \frac{h_1(A)}{|a_{11}|} + \frac{h_1(A)}{|a_{11}|} \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} - \left( \frac{h_1(A)}{|a_{11}|} \right)^2 < 1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|},$$

i.e.,

$$\left( 1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} + \frac{h_1(A)}{|a_{11}|} \right) \left( 1 - \frac{h_1(A)}{|a_{11}|} \right) < 1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}.$$

Note that  $1 - \frac{h_1(A)}{|a_{11}|} > 0$ , then

$$1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} + \frac{h_1(A)}{|a_{11}|} < \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}}.$$

The conclusion follows.  $\square$

We now give an interval of  $\mu$  such that the bound (7) of Theorem 7 is less than the bound (1) of Theorem 2.

**Lemma 9.** *Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix with*

$$\frac{h_1(A)}{|a_{11}|} > \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}.$$

Then for each  $\mu \in \left( 1, \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}} \right)$ ,

$$\begin{aligned} \|A^{-1}\|_\infty &\leq \max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} \\ &< \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}. \end{aligned}$$

*Proof.* From Lemma 8, we have

$$\mu \in \left( 1, 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} \right] \cup \left[ 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}, \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}} \right).$$

and  $\max\{\mu, 1\} = \mu$ .

(I) For  $\mu \in \left(1, 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}\right]$ , then

$$\mu - \frac{h_1(A)}{|a_{11}|} \leq 1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|},$$

that is,

$$\frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}} \geq \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}.$$

Therefore,

$$\max\{\mu, 1\} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} = \frac{\mu}{\mu - \frac{h_1(A)}{|a_{11}|}}.$$

Consider the function  $f(x) = \frac{x}{x - \frac{h_1(A)}{|a_{11}|}}$ ,  $x \in \left[1, 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}\right]$ . It is easy from  $\frac{h_1(A)}{|a_{11}|} < 1$  to prove that  $f(x)$  is a monotonically decreasing function of  $x$ . Hence, for any  $\mu \in \left(1, 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}\right]$ ,

$$f(\mu) < f(1),$$

i.e.,

$$\frac{\mu}{\mu - \frac{h_1(A)}{|a_{11}|}} < \frac{1}{1 - \frac{h_1(A)}{|a_{11}|}} = \frac{1}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}},$$

which implies that

$$\frac{\mu \max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{\mu - \frac{h_1(A)}{|a_{11}|}} < \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}.$$

Hence,

$$\max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} < \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}.$$

(II) For  $\mu \in \left[ 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}, \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}} \right)$ , then

$$\mu - \frac{h_1(A)}{|a_{11}|} \geq 1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|},$$

that is,

$$\frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}} \leq \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}.$$

Therefore,

$$\max\{\mu, 1\} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} = \frac{\mu}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}.$$

Consider the function  $g(x) = \frac{x}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}$ ,  $x \in \left[ 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}, \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}} \right]$ .

Obviously,  $g(x)$  is a monotonically increasing function of  $x$ . Hence, for any

$$\mu \in \left[ 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}, \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}} \right),$$

$$g(\mu) < g \left( \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}} \right),$$

that is,

$$\frac{\mu}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} < \frac{1}{1 - \frac{h_1(A)}{|a_{11}|}} = \frac{1}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}},$$

which implies that

$$\frac{\mu \max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} < \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}.$$

Hence,

$$\max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} < \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}.$$

The conclusion follows from (I) and (II).  $\square$

Lemma 9 provides an interval of  $\mu$  such that the bound (7) in Theorem 7 is better than the bound (1) in Theorem 2. Moreover, we can determine the optimal value of  $\mu$  by the following theorem.

**Theorem 10.** *Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix with*

$$\frac{h_1(A)}{|a_{11}|} > \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}.$$

Then

$$\begin{aligned} \min \left\{ \max\{\mu, 1\} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} : \mu \in \left( 1, \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}} \right) \right\} \\ = \frac{1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}. \end{aligned} \quad (13)$$

Furthermore,

$$\|A^{-1}\|_\infty \leq \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \left( 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} \right)}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} < \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}. \quad (14)$$

*Proof.* From the proof of Lemma 9, we have that

$$f(x) = \frac{x}{x - \frac{h_1(A)}{|a_{11}|}}, \quad x \in \left[ 1, 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} \right]$$

is decreasing, and that

$$g(x) = \frac{x}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}, \quad x \in \left[ 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}, \frac{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \frac{h_1(A)}{|a_{11}|}} \right]$$

is increasing. Therefore, the minimum of  $f(x)$ , which is equal to that of  $g(x)$ , is

$$f \left( 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} \right) = g \left( 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} \right) = \frac{1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}},$$

which implies that (13) holds. Again by Lemma 9, (14) follows easily.  $\square$

**Remark 2.** Theorem 10 provides a method to determine the optimal value of  $\mu$  for a Nekrasov matrix  $A = [a_{ij}] \in C^{n,n}$  with

$$\frac{h_1(A)}{|a_{11}|} > \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}.$$

Also consider the matrix  $A_1$ . By computation, we get

$$\frac{h_1(A_1)}{|a_{11}|} = 0.4571 > 0.2278 = \max_{i \neq 1} \frac{h_i(A_1)}{|a_{ii}|}.$$

Hence, by Theorem 10, we can obtain that the bound (7) in Theorem 7 reaches its minimum

$$\frac{\max_{i \in N} \frac{z_i(A_1)}{|a_{ii}|} \left( 1 + \frac{h_1(A_1)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A_1)}{|a_{ii}|} \right)}{1 - \max_{i \neq 1} \frac{h_i(A_1)}{|a_{ii}|}} = 0.3288$$

at  $\mu = 1 + \frac{h_1(A_1)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A_1)}{|a_{ii}|} = 1.2294$  (also see Figure 1).

Next, we study the bound in Theorem 7 for the Nekrasov matrix  $A = [a_{ij}] \in C^{n,n}$  with

$$\frac{h_1(A)}{|a_{11}|} \leq \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}.$$

**Theorem 11.** Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix with

$$\frac{h_1(A)}{|a_{11}|} \leq \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}.$$

Then we can take  $\mu = 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}$  such that

$$\begin{aligned} \|A^{-1}\|_\infty &\leq \max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} \\ &= \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}. \end{aligned}$$

*Proof.* Since  $\frac{h_1(A)}{|a_{11}|} \leq \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}$ , we have  $\mu = 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|} \leq 1$ ,  $\max\{\mu, 1\} = 1$  and

$$\max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} = \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} = \frac{1}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}.$$

Hence,

$$\max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} = \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}.$$

The proof is completed.  $\square$

### 3.2. the optimal value of $\mu$ for the bound (8)

First, we consider the Nekrasov matrix  $A = [a_{ij}] \in C^{n,n}$  with

$$|a_{11}| - h_1(A) < \min_{i \neq 1} (|a_{ii}| - h_i(A)),$$

and give the following lemmas.

**Lemma 12.** *Let  $a, b$  and  $c$  be positive real numbers, and  $0 < a - b < c$ . Then*

$$\frac{b+c}{a} < \frac{c}{a-b}.$$

*Proof.* we only prove that  $\frac{c}{a-b} - \frac{b+c}{a} > 0$ . In fact,

$$\begin{aligned} \frac{c}{a-b} - \frac{b+c}{a} &= \frac{ac - (a-b)(b+c)}{a(a-b)} \\ &= \frac{ac - (ab + ac - b^2 - bc)}{a(a-b)} \\ &= \frac{-ab + b^2 + bc}{a(a-b)} \\ &= \frac{b(c - (a-b))}{a(a-b)} > 0. \end{aligned}$$

The proof is completed.  $\square$

**Lemma 13.** Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix with

$$|a_{11}| - h_1(A) < \min_{i \neq 1} (|a_{ii}| - h_i(A)).$$

Then

$$1 < \frac{\min_{i \neq 1} (|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|} < \frac{\min_{i \neq 1} (|a_{ii}| - h_i(A))}{|a_{11}| - h_1(A)}. \quad (15)$$

*Proof.* Since  $A$  is a Nekrasov matrix, we have  $|a_{11}| - h_1(A) > 0$ , consequently, the first Inequality in (15) holds. Moreover, Let  $a = |a_{11}|$ ,  $b = h_1(A)$  and  $c = \min_{i \neq 1} (|a_{ii}| - h_i(A))$ . Then from Lemma 12, the second holds.  $\square$

We now give an interval of  $\mu$  such that the bound (8) of Theorem 7 is less than the bound (2) of Theorem 2.

**Lemma 14.** Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix with

$$|a_{11}| - h_1(A) < \min_{i \neq 1} (|a_{ii}| - h_i(A)).$$

Then for each  $\mu \in \left(1, \frac{\min_{i \neq 1} (|a_{ii}| - h_i(A))}{|a_{11}| - h_1(A)}\right)$ ,

$$\begin{aligned} \|A^{-1}\|_\infty &\leq \frac{\max\{\mu, 1\} \max_{i \in n} z_i(A)}{\min \left\{ \mu |a_{11}| - h_1(A), \min_{i \neq 1} (|a_{ii}| - h_i(A)) \right\}} \\ &< \frac{\max_{i \in N} z_i(A)}{\min_{i \in N} (|a_{ii}| - h_i(A))}. \end{aligned}$$

*Proof.* From Lemma 13, we have

$$\mu \in \left(1, \frac{\min_{i \neq 1} (|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}\right] \cup \left[\frac{\min_{i \neq 1} (|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}, \frac{\min_{i \neq 1} (|a_{ii}| - h_i(A))}{|a_{11}| - h_1(A)}\right).$$

and  $\max\{\mu, 1\} = \mu$ .

(I) For  $\mu \in \left(1, \frac{\min_{i \neq 1} (|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}\right]$ , then

$$\mu |a_{11}| \leq \min_{i \neq 1} (|a_{ii}| - h_i(A)) + h_1(A),$$

that is,

$$\mu|a_{11}| - h_1(A) \leq \min_{i \neq 1}(|a_{ii}| - h_i(A)).$$

Therefore,

$$\frac{\max\{\mu, 1\}}{\min\left\{\mu|a_{11}| - h_1(A), \min_{i \neq 1}(|a_{ii}| - h_i(A))\right\}} = \frac{\mu}{\mu|a_{11}| - h_1(A)}.$$

Consider the function  $f(x) = \frac{x}{|a_{11}|x - h_1(A)}$ ,  $x \in \left[1, \frac{\min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}\right]$ . It is easy to prove that  $f(x)$  is a monotonically decreasing function of  $x$ . Hence, for any  $\mu \in \left(1, \frac{\min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}\right]$ ,

$$f(\mu) < f(1),$$

i.e.,

$$\frac{\mu}{\mu|a_{11}| - h_1(A)} < \frac{1}{|a_{11}| - h_1(A)} = \frac{1}{\min_{i \in N}(|a_{ii}| - h_i(A))},$$

which implies that

$$\frac{\mu \max_{i \in N} z_i(A)}{\mu|a_{11}| - h_1(A)} < \frac{\max_{i \in N} z_i(A)}{\min_{i \in N}(|a_{ii}| - h_i(A))}.$$

Hence,

$$\frac{\max\{\mu, 1\} \max_{i \in n} z_i(A)}{\min\left\{\mu|a_{11}| - h_1(A), \min_{i \neq 1}(|a_{ii}| - h_i(A))\right\}} < \frac{\max_{i \in N} z_i(A)}{\min_{i \in N}(|a_{ii}| - h_i(A))}.$$

(II) For  $\mu \in \left[\frac{\min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}, \frac{\min_{i \neq 1}(|a_{ii}| - h_i(A))}{|a_{11}| - h_1(A)}\right)$ , then

$$\mu|a_{11}| \geq \min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A),$$

that is,

$$\mu|a_{11}| - h_1(A) \geq \min_{i \neq 1}(|a_{ii}| - h_i(A)).$$

Therefore,

$$\frac{\max\{\mu, 1\}}{\min\left\{\mu|a_{11}| - h_1(A), \min_{i \neq 1}(|a_{ii}| - h_i(A))\right\}} = \frac{\mu}{\min_{i \neq 1}(|a_{ii}| - h_i(A))}.$$

Consider the function  $g(x) = \frac{x}{\min_{i \neq 1}(|a_{ii}| - h_i(A))}$ ,  $x \in \left[\frac{\min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}, \frac{\min_{i \neq 1}(|a_{ii}| - h_i(A))}{|a_{11}| - h_1(A)}\right]$ .

Obviously,  $g(x)$  is a monotonically increasing function of  $x$ . Hence, for any  $\mu \in \left[\frac{\min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}, \frac{\min_{i \neq 1}(|a_{ii}| - h_i(A))}{|a_{11}| - h_1(A)}\right)$ ,

$$g(\mu) < g\left(\frac{\min_{i \neq 1}(|a_{ii}| - h_i(A))}{|a_{11}| - h_1(A)}\right),$$

that is,

$$\frac{\mu}{\min_{i \neq 1}(|a_{ii}| - h_i(A))} < \frac{1}{|a_{11}| - h_1(A)} = \frac{1}{\min_{i \in N}(|a_{ii}| - h_i(A))},$$

which implies that

$$\frac{\mu \max_{i \in N} z_i(A)}{\min_{i \neq 1}(|a_{ii}| - h_i(A))} < \frac{\max_{i \in N} z_i(A)}{\min_{i \in N}(|a_{ii}| - h_i(A))}.$$

Hence,

$$\frac{\max\{\mu, 1\} \max_{i \in n} z_i(A)}{\min\left\{\mu|a_{11}| - h_1(A), \min_{i \neq 1}(|a_{ii}| - h_i(A))\right\}} < \frac{\max_{i \in N} z_i(A)}{\min_{i \in N}(|a_{ii}| - h_i(A))}.$$

The conclusion follows from (I) and (II).  $\square$

Similar to the proof of Theorem 10, we can easily determine the optimal value of  $\mu$  by Lemma 14.

**Theorem 15.** *Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix with*

$$|a_{11}| - h_1(A) < \min_{i \neq 1}(|a_{ii}| - h_i(A)).$$

Then

$$\min \left\{ \frac{\max\{\mu, 1\}}{\min\left\{\mu|a_{11}| - h_1(A), \min_{i \neq 1}(|a_{ii}| - h_i(A))\right\}} : \mu \in \left(1, \frac{\min_{i \neq 1}(|a_{ii}| - h_i(A))}{|a_{11}| - h_1(A)}\right) \right\} \\ = \frac{\min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}| \min_{i \neq 1}(|a_{ii}| - h_i(A))}. \quad (16)$$

Furthermore,

$$\|A^{-1}\|_\infty \leq \frac{\max_{i \in N} z_i(A) \left( \min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A) \right)}{|a_{11}| \min_{i \neq 1}(|a_{ii}| - h_i(A))} < \frac{\max_{i \in N} z_i(A)}{\min_{i \in N}(|a_{ii}| - h_i(A))}. \quad (17)$$

**Remark 3.** Theorem 15 provides a method to determine the optimal value of  $\mu$  for a Nekrasov matrix  $A = [a_{ij}] \in C^{n,n}$  with

$$|a_{11}| - h_1(A) < \min_{i \neq 1}(|a_{ii}| - h_i(A)).$$

Also consider the matrix  $A_1$ . By computation, we get

$$|a_{11}| - h_1(A) = 3.8000 < 5.2641 = \min_{i \neq 1}(|a_{ii}| - h_i(A)).$$

Hence, by Theorem 15, we can obtain that the bound (8) in Theorem 7 reaches its minimum

$$\frac{\max_{i \in N} z_i(A) \min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}| \min_{i \neq 1}(|a_{ii}| - h_i(A))} = 0.4594$$

at  $\mu = \frac{\min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|} = 1.2092$  (also see Figure 2).

Next, we study the bound (8) in Theorem 7 for the Nekrasov matrix  $A = [a_{ij}] \in C^{n,n}$  with

$$|a_{11}| - h_1(A) \geq \min_{i \neq 1}(|a_{ii}| - h_i(A)).$$

**Theorem 16.** Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix with

$$|a_{11}| - h_1(A) \geq \min_{i \neq 1} (|a_{ii}| - h_i(A)).$$

Then we can take  $\mu = \frac{\min_{i \neq 1} (|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}$  such that

$$\begin{aligned} \|A^{-1}\|_\infty &\leq \frac{\max\{\mu, 1\} \max_{i \in n} z_i(A)}{\min \left\{ \mu |a_{11}| - h_1(A), \min_{i \neq 1} (|a_{ii}| - h_i(A)) \right\}} \\ &= \frac{\max_{i \in N} z_i(A)}{\min_{i \in N} (|a_{ii}| - h_i(A))}. \end{aligned}$$

*Proof.* since  $|a_{11}| - h_1(A) \geq \min_{i \neq 1} (|a_{ii}| - h_i(A))$ , we have

$$\mu = \frac{\min_{i \neq 1} (|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|} \leq 1,$$

$\max\{\mu, 1\} = 1$ , and

$$\frac{\max\{\mu, 1\}}{\min \left\{ \mu |a_{11}| - h_1(A), \min_{i \neq 1} (|a_{ii}| - h_i(A)) \right\}} = \frac{1}{\min_{i \in N} (|a_{ii}| - h_i(A))}.$$

Hence,

$$\frac{\max\{\mu, 1\} \max_{i \in n} z_i(A)}{\min \left\{ \mu |a_{11}| - h_1(A), \min_{i \neq 1} (|a_{ii}| - h_i(A)) \right\}} = \frac{\max_{i \in N} z_i(A)}{\min_{i \in N} (|a_{ii}| - h_i(A))}.$$

The proof is completed. □

**Remark 4.** (I) Theorems 10 and 11 provide the value of  $\mu$ , i.e.,

$$\mu = 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}$$

such that the bound (7) in Theorem 7 is not worse than the bound (1) in theorem 2 for a Nekrasov matrix  $A = [a_{ij}] \in C^{n,n}$ . In particular, for the

Nekrasov matrix  $A$  with  $\frac{h_1(A)}{|a_{11}|} > \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}$ , the bound (7) is better than the bound (1).

(II) Theorems 15 and 16 provide the value of  $\mu$ , i.e.,

$$\mu = \frac{\min_{i \neq 1} (|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}$$

such that the bound (8) in Theorem 7 is not worse than the bound (2) in theorem 2 for a Nekrasov matrix  $A = [a_{ij}] \in C^{n,n}$ . In particular, for the Nekrasov matrix  $A$  with  $|a_{11}| - h_1(A) < \min_{i \neq 1} (|a_{ii}| - h_i(A))$ , the bound (8) is better than the bound (2).

#### 4. Numerical Examples

**Example 2.** Consider the following five Nekrasov matrices in [5]:

$$A_2 = \begin{bmatrix} 8 & 1 & -0.2 & 3.3 \\ 7 & 13 & 2 & -3 \\ -1.3 & 6.7 & 13 & -2 \\ 0.5 & 3 & 1 & 6 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 21 & -9.1 & -4.2 & -2.1 \\ -0.7 & 9.1 & -4.2 & -2.1 \\ -0.7 & -0.7 & 4.9 & -2.1 \\ -0.7 & -0.7 & -0.7 & 2.8 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 5 & 1 & 0.2 & 2 \\ 1 & 21 & 1 & -3 \\ 2 & 0.5 & 6.4 & -2 \\ 0.5 & -1 & 1 & 9 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 6 & -3 & -2 \\ -1 & 11 & -8 \\ -7 & -3 & 10 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 8 & -0.5 & -0.5 & -0.5 \\ -9 & 16 & -5 & -5 \\ -6 & -4 & 15 & -3 \\ -4.9 & -0.9 & -0.9 & 6 \end{bmatrix}.$$

Obviously,  $A_2$ ,  $A_3$  and  $A_4$  are SDD. And it is not difficult to verify that  $A_4$ ,  $A_5$  satisfy the conditions in Theorems 10 and 15 and  $A_2$ ,  $A_3$ ,  $A_6$  satisfy the conditions in Theorems 11 and 16. We compute by Matlab 7.0 the upper bounds for the infinity norm of the inverse of  $A_i$ ,  $i = 2, \dots, 6$ , which are showed in Table 1. It is easy to see from Table 1 that this example illustrates Theorems 10, 11, 15 and 16,.

Matrix	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$
Exact $\ A^{-1}\ _\infty$	0.2390	0.8759	0.2707	1.1519	0.4474
Varah	1	1.4286	0.5556	–	–
The bound (1)	0.8848	1.8076	0.6200	1.4909	1.1557
Theorems 10 or 11	0.8848	1.8076	0.5270	1.4266	1.1557
The bound (2)	0.6885	0.9676	0.7937	2.4848	0.5702
Theorems 15 or 16	0.6885	0.9676	0.5895	1.5923	0.5702

Table 1. The upper bounds for  $\|A_i^{-1}\|_\infty$ ,  $i = 2, \dots, 6$ .

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