Adaptive mesh point selection for the efficient solution of scalar IVPs [1]

Bolesław Kacewicz [2]

Abstract

We discuss adaptive mesh point selection for the solution of scalar IVPs. We consider a method that is optimal in the sense of the speed of convergence, and aim at minimizing the local errors. Although the speed of convergence cannot be improved by using the adaptive mesh points compared to the equidistant points, we show that the factor in the error expression can be significantly reduced. We obtain formulas specifying the gain achieved in terms of the number of discretization subintervals, as well as in terms of the prescribed level of the local error. Both nonconstructive and constructive versions of the adaptive mesh selection are shown, and a numerical example is given.

Mathematics Subject Classification: 65L05, 65L50, 65L70

[1] This research was partly supported by the Polish Ministry of Science and Higher Education

[2] *AGH University of Science and Technology, Faculty of Applied Mathematics,*
Al. Mickiewicza 30, paw. A3/A4, III p., pok. 301,
30-059 Krakow, Poland
E-mail: kacewicz@agh.edu.pl

1 Introduction

We deal with the question how much an adaptive choice of mesh points pays off in the solution of initial-value problems

$$z'(t) = f(z(t)), \quad t \in [a, b], \qquad z(a) = \eta, \tag{1}$$

where $a < b$, $f : \mathbf{R} \to \mathbf{R}$ is a C^r function and $\eta \in \mathbf{R}$.
Numerical analysts have been using adaptive techniques in numerical codes for solving various problems with considerable success. Adaption is a standard tool in numerical packages, see e.g. the package QUADPACK [8] for numerical integration, or, among many others, the well known solver DIFSUB by C.W. Gear or the library ODEPACK by A. Hindmarsh devoted to various types of ODEs. A measure of practical efficiency of an adaptive strategy is usually the performance for a number of computational examples. A method is considered 'good' if it works well for large number of problems, and fails in small number of cases. Many papers have reported the advantage of adaption over nonadaption in that sense, to mention only as samples the old paper [6], more recent one [2], or [7], where a mesh selection strategy is discussed for Runge-Kutta methods. Such an approach obviously gives us a considerable practical knowledge, but it is not complete. Many step size control strategies are not supported by a theoretical analysis. For instance, an important question how a particular strategy influences the cost of the process most often remains open. Recently, advantages of adaptive selection of mesh points were rigorously studied for problems with singularities, see e.g. [5], [10].
In this paper we present results explaining potential gain of adaptive mesh point selection for a regular problem (1). In particular, we rigorously discuss the accuracy and cost of an adaptive process for a well precised class of problems, not only for a number of computational examples. For the integration of scalar C^4 functions similar questions have recently been addressed for the Simpson rule in [9], where it is shown that the adaptive mesh selection allows us to reduce the error by reducing the asymptotic constant of the method. Adaptive mesh points for the approximation of univariate $W^{2,\infty}$ functions is discussed in [1] .
We consider in this work the C^r right-hand side functions f in (1). It is well known that in the worst-case or asymptotic settings, with $m + 1$ mesh points one can achieve local errors of order $m^{-(r+1)}$ as $m \to \infty$. This by standard means translates to the global error $O(m^{-r})$. The exponent $-r$ has been shown best possible, for details see e.g. [3]. Furthermore, the best speed of convergence as $m \to \infty$ can be achieved by using the equidistant mesh points. We have to add that for systems of IVPs the *information* about

f that gives us the global error $O(m^{-r})$ must itself be adaptive, in spite of the fact that the mesh points can be equidistant, see [4] for explanation of a difference between adaptive mesh and adaptive information.

In these results a constant in the $'O'$- notation depends on a class of functions f in the worst-case setting, and on a particular f in the asymptotic setting. The size of the constant is not controlled; it depends on a *global* behavior of derivatives of f in the domain. In order to reduce local errors, in the next sections we will include the constant in the $'O'$- notation to the analysis. To study possible advantages of adaptive selection of mesh points, we consider one of the methods with best convergence $O(m^{-r})$, given by (7). We show formulas for the local error of the method, which will serve us to define mesh points with asymptotically minimal maximum local error. The selected points are adapted to a *local* behavior of f. We express the local errors in terms of m, or, alternatively, for a given $\varepsilon > 0$, we ask what m should be to achieve local errors proportional to ε.

The formulas obtained for the optimal mesh points are not constructive. We next show how the method can be modified to get computable mesh points and approximations. Compared to the 'ideal' result, the local error bound and the cost bound are in this version increased by (known) factors dependent only on r (but not on f). It turns out that the adaptive choice of the mesh points allows us to achieve the maximum local error $((b-a)/m)^{r+1}\, S(m)$. The factor $S(m)$ is bounded from above and below by positive constants dependent on f, so that it does not improve the rate of convergence. However, the advantage of using adaptive mesh points is hidden in $S(m)$, since the value of $S(m)$ can be much smaller for adaptive than nonadaptive points.

The paper is organized as follows. In Section 2 we define the class of functions f and precise the aim of the paper. Section 3 presents the method under consideration and a convergence result. In Section 4 we give local error expressions which are used in Section 5 to define optimal (nonconstructive) mesh points. Section 6 is devoted to a constructive modification of the method which is finally described in the algorithm ADMESH. The error and cost properties of ADMESH are shown in Theorem 1 which summarizes the results of the paper. In Remark 3 we shortly comment on generalization of the results to systems of IVPs. The behavior of the algorithm ADMESH is illustrated in Section 7 by a numerical example. The experiment shows how much one can gain using adaptive mesh over the equidistant mesh for a right hand side function with derivatives of varying magnitude in parts of the domain.

2 Problem formulation

Let $m \in \mathbf{N}$. We wish to compute approximations to the solution z of (1) at $m+1$ points $a = x_{0,m} < x_{1,m} < \ldots < x_{m,m} = b$, that is, to find pairs $(x_{i,m}, y_{i,m})$, $i = 0, 1, \ldots, m$, where $y_{i,m}$ is a (good) approximation to $z(x_{i,m})$. Let $l(m)$ be any sequence convergent to 0 as $m \to \infty$. We consider for any f a class of partitions of $[a, b]$. We assume that there exist $K = K(f, a, b, \eta)$ and $k_0 = k_0(f, a, b, \eta)$ such that for all $m \geq k_0$ and any partition it holds

$$\max_{0 \leq i \leq m-1} (x_{i+1,m} - x_{i,m}) \leq K\, l(m). \tag{2}$$

Note that we always have $\max_{0 \leq i \leq m-1} (x_{i+1,m} - x_{i,m}) \geq (b-a)/m$ for $m \geq 1$. Thus, the condition (2) implies that $l(m)$ cannot go to zero faster than $1/m$. The convergence of $l(m)$ can be arbitrarily slow, and the constant K can be arbitrarily large.

To shorten the notation, we shall omit in the sequel the second subscript m, remembering that the choice of points x_i and y_i can be different for varying m.

We denote by z_i the solution of the local problem

$$z_i'(t) = f(z_i(t)), \quad t \in [x_i, x_{i+1}], \qquad z_i(x_i) = y_i. \tag{3}$$

If the pairs (x_i, y_i) are outputs of a certain method, then the local errors of the method are given by $|z_i(x_{i+1}) - y_{i+1}|$, $i = 0, 1, \ldots, m-1$. Our aim is to minimize the maximal local error

$$\max_{0 \leq i \leq m-1} |z_i(x_{i+1}) - y_{i+1}| \to \min \tag{4}$$

with respect to all possible choices of the mesh points $x_0, \ldots, x_m$, and to find minimizing (optimal) pairs (x_i^*, y_i^*).

The class of right-hand functions f under consideration is given as follows. For $r \in \mathbf{N}$,

$$F_r = \{f = 1/g : \ g \in C^r(\mathbf{R}), g \text{ and } g^{(r)} \text{ have constant sign in } \mathbf{R}, f \text{ is Lipschitz in } \mathbf{R}\}. \tag{5}$$

We denote the Lipschitz constant of f by L, and assume without loss of generality that f is a positive function. Regarding the constant sign of $g^{(r)}$, we note that the same assumption about constant sign of the fourth derivative ($r = 4$) of the integrand was essential in [9] in the analysis of adaptive integration of scalar C^4 functions .

In the next sections we aim at choosing a subdivision of $[a, b]$, possibly adapting it to a local behavior of f, in order to minimize the local errors. Our goal will be to propose a rigorous strategy of mesh point selection, keeping the cost of the process under control, and to establish possible gain of adaption.

3 The method under consideration

We shall use the identity

$$t - x_i = \int_{y_i}^{z_i(t)} \frac{1}{f(y)}\,dy = \int_{y_i}^{z_i(t)} g(y)\,dy, \quad t \in [x_i, x_{i+1}]. \tag{6}$$

For a positive f, the solutions z and z_i are increasing functions.

Let r be even. For a given interval $[y_i, y_{i+1}]$, let $\hat{g}_i$ be the Lagrange interpolation polynomial of degree $\leq r-2$ for g in $[y_i, y_{i+1}]$ based on $r-1$ equidistant nodes $p_0 = y_i, p_1, \dots, p_{r-3}, p_{r-2} = y_{i+1}$ for $r \geq 4$, and one node $p_0 = (y_{i+1} + y_i)/2$ for $r = 2$. For odd r, we define $p_0 = y_i, p_1, \dots, p_r = y_{i+1}$ as equidistant points in $[y_i, y_{i+1}]$. The interpolation polynomial $\hat{g}_i$ of degree $\leq r-1$ is now based on the nodes $p_0, \dots, p_{r-1}$.

Let $x_0 = a$, $y_0 = \eta$. We shall study the following method relating sequences $\{x_i\}$ and $\{y_i\}$:

$$x_{i+1} - x_i = \int_{y_i}^{y_{i+1}} \hat{g}_i(y)\,dy. \tag{7}$$

Note that the right-hand side is the Newton-Cotes type quadrature approximating $\int_{y_i}^{y_{i+1}} g(y)\,dy$, and it continuously depends on y_{i+1}.

We shall now derive convenient expressions for the remainder of the Newton-Cotes formulas for even and odd r. Denote $\hat{e}_i(y) = g(y) - \hat{g}_i(y)$. For even r, $r \geq 4$, we recall that the remainder of the Newton-Cotes quadrature is given by

$$\int_{y_i}^{y_{i+1}} \hat{e}_i(y)\,dy = \frac{g^{(r)}(\xi_i)}{r!} \int_{y_i}^{y_{i+1}} (y - p_0)^2 (y - p_1) \cdot \ldots \cdot (y - p_{r-2})\,dy, \tag{8}$$

where $\xi_i \in [y_i, y_{i+1}]$. Denoting $\Delta_i = y_{i+1} - y_i$ and changing variables $y = \Delta_i x + y_i$, $x \in [0, 1]$, we get for $r \geq 4$

$$\int_{y_i}^{y_{i+1}} \hat{e}_i(y)\,dy = \frac{g^{(r)}(\xi_i)}{r!} \Delta_i^{r+1} C_r, \tag{9}$$

where

$$C_r = \int_0^1 (x - \bar{p}_0)^2 \cdot \ldots \cdot (x - \bar{p}_{r-2})\,dx \qquad (C_r < 0), \tag{10}$$

and $\bar{p}_j$ are equidistant nodes in $[0,1]$. For $r=2$, (9) holds with $C_2 = 1/12$.
Let r be odd, $r \geq 3$. We have for $y \in [y_i, y_{i+1}]$

$$\hat{e}_i(y) = \frac{g^{(r)}(\xi_{i,y})}{r!}(y-p_0)\dots(y-p_{r-1}), \qquad \text{for some} \quad \xi_{i,y} \in [y_i, y_{i+1}].$$

The second integral in the splitting

$$\int_{y_i}^{y_{i+1}} \hat{e}_i(y)\,dy = \int_{y_i}^{p_{r-1}} + \int_{p_{r-1}}^{y_{i+1}}$$

can be written as

$$\int_{p_{r-1}}^{y_{i+1}} \hat{e}_i(y)\,dy = \frac{g^{(r)}(\eta_i)}{r!} \int_{p_{r-1}}^{y_{i+1}} (y-p_0)\dots(y-p_{r-1})\,dy, \qquad \text{for some} \quad \eta_i \in [y_i, y_{i+1}].$$

Since

$$\int_{y_i}^{p_{r-1}} (y-p_0)\dots(y-p_{r-1})\,dy = 0,$$

the first intergral is equal to

$$\int_{y_i}^{p_{r-1}} \hat{e}_i(y)\,dy = \int_{y_i}^{p_{r-1}} \frac{g^{(r)}(\xi_{i,y})}{r!}(y-p_0)\dots(y-p_{r-1})\,dy$$

$$= \frac{g^{(r)}(\eta_i)}{r!} \int_{y_i}^{p_{r-1}} \frac{g^{(r)}(\xi_{i,y}) - g^{(r)}(\eta_i)}{g^{(r)}(\eta_i)}(y-p_0)\dots(y-p_{r-1})\,dy.$$

We denote

$$\gamma_i^y = \frac{g^{(r)}(\xi_{i,y}) - g^{(r)}(\eta_i)}{g^{(r)}(\eta_i)}. \tag{11}$$

The quantity γ_i^y continuously depends on y; this follows from the well known interpolation remainder formula written with the use of the divided difference of g expressed in the integral form. Summing up the two integrals we get for odd r, $r \geq 3$

$$\int_{y_i}^{y_{i+1}} \hat{e}_i(y)\,dy = \frac{g^{(r)}(\eta_i)}{r!} \int_{p_{r-1}}^{y_{i+1}} (y-p_0)\dots(y-p_{r-1})\,dy\,(1+\kappa_i), \tag{12}$$

where

$$\kappa_i = \frac{\displaystyle\int_{y_i}^{p_{r-1}} \gamma_i^y (y-p_0)\dots(y-p_{r-1})\,dy}{\displaystyle\int_{p_{r-1}}^{y_{i+1}} (y-p_0)\dots(y-p_{r-1})\,dy}. \tag{13}$$

It is easy to see that

$$|\kappa_i| \le \frac{2r^{r+1}}{(r-1)!} \sup_{y\in[y_i,y_{i+1}]} |\gamma_i^y|. \tag{14}$$

Changing variables as above in the integral in (12), we get

$$\int_{y_i}^{y_{i+1}} \hat{e}_i(y)\,dy = \frac{g^{(r)}(\eta_i)}{r!}\Delta_i^{r+1} C_r(1+\kappa_i), \tag{15}$$

where

$$C_r = \int_{1-1/r}^{1} (x-\bar{p}_0)\ldots(x-\bar{p}_{r-1})\,dx\,, \tag{16}$$

and $\bar{p}_j$, $j = 0,1,\ldots,r$ are equidistant points in $[0,1]$. It is easy to see that the case $r = 1$ is included in (15) with $\kappa_i = 0$. The formulas (9) and (15) will be used in the next section.

For any sequence $a = x_0 < x_1 < \ldots < x_m = b$ under consideration there is a sequence $\eta = y_0 < y_1 < \ldots < y_m$ which satisfies (7), and has the global error bounded (as it can be expected) as in the following proposition. Some bounds obtained in the proof will be used in the sequel. Let $h_i = x_{i+1} - x_i$, $i = 0,1,\ldots,m-1$.

Proposition 1 *Let $f \in F_r$. There is $m_0 \in \mathbf{N}$ such that for any $m \ge m_0$, for any partition $a = x_0 < x_1 < \ldots < x_m = b$ satisfying (2) there exists a sequence $\eta = y_0 < y_1 < \ldots < y_m$ satisfying (7) such that*

$$|y_i - z(x_i)| \le M_i \max_{0\le j\le i-1} h_j^r, \quad i = 0,1,\ldots,m \quad (\textit{with} \max_{0\le j\le -1} = 1). \tag{17}$$

Here $M_0 = 0$, and $M_{i+1} = \exp(Lh_i)M_i + \tilde{M}h_i$, $i = 0,1,\ldots,m-1$, where $\tilde{M}$ is given by (26). Hence, $M_i \le M := \exp(L(b-a))(b-a)\tilde{M}$.

Proof We prove (17) by induction with respect to i. The statement holds for $i = 0$. Suppose that there exist $y_0 < y_1 < \ldots < y_i$ satisfying (7) and (17), and $M_i \le M$. Let

$$F(y) = \int_{y_i}^{y} g(z)\,dz, \quad \hat{F}(y) = \int_{y_i}^{y} \hat{g}_i(z,y)\,dz \quad \text{and} \quad H(y) = F(y) - \hat{F}(y), \;\; y \ge y_i$$

(the notation $\hat{g}_i(\cdot,y)$ reflects the fact that the interpolation polynomial is defined on the interval $[y_i,y]$). Note that F and $\hat{F}$ are continuous functions, $F'(y) = g(y) > 0$, and $F(z_i(x_{i+1})) = x_{i+1} - x_i$. Our aim is to show the existence of a solution $y_{i+1} > y_i$ of the equation $\hat{F}(y) = x_{i+1} - x_i$. Note that $\hat{F}(y_i) = 0 < x_{i+1} - x_i$. We show that there is $\bar{y} > z_i(x_{i+1})$ (which depends on i) such that $\hat{F}(\bar{y}) \geq x_{i+1} - x_i$. This holds iff $H(\bar{y}) \leq F(\bar{y}) - F(z_i(x_{i+1}))$. Using (9) or (15) with the interval $[y_i, y_{i+1}]$ replaced by $[y_i, y]$ we have that

$$H(y) = \frac{g^{(r)}(\bar{\xi}_{i,y})}{r!}(y - y_i)^{r+1} C_r \,(1 + \bar{\kappa}_i^y) \quad \text{(where } \bar{\kappa}_i^y = 0 \text{ for even } r\text{)}, \tag{18}$$

for some $\bar{\xi}_{i,y} \in [y_i, y]$. Since $F(y) - F(z_i(x_{i+1})) = g(\tilde{\xi}_{i,y})(y - z_i(x_{i+1}))$ for some $\tilde{\xi}_{i,y} \in \operatorname{conv}(z_i(x_{i+1}), y)$, the equivalent condition on $\bar{y}$ reads

$$\frac{g^{(r)}(\bar{\xi}_{i,\bar{y}})}{r!}(\bar{y} - y_i)^{r+1} C_r \,(1 + \bar{\kappa}_i^{\bar{y}}) \leq g(\tilde{\xi}_{i,\bar{y}})(\bar{y} - z_i(x_{i+1})). \tag{19}$$

We now show that (19) holds for $\bar{y} = y_i + 2f(y_i)(x_{i+1} - x_i)$. The following auxilliary inequalities hold for sufficiently large m (where the starting value of m only depends on f)

$$y_i \leq z(x_i) + |y_i - z(x_i)| \leq z(x_i) + M_i \max_{0 \leq j \leq i-1} h_j^r \leq z(b) + 1, \tag{20}$$

$$\begin{aligned} z_i(x_{i+1}) \leq z(x_{i+1}) + |z_i(x_{i+1}) - z(x_{i+1})| &\leq z(x_{i+1}) + \exp(Lh_i)|y_i - z(x_i)| \\ &\leq z(b) + 1, \end{aligned} \tag{21}$$

and

$$\begin{aligned} \bar{y} = y_i + 2f(y_i)h_i &\leq z(x_i) + M_i \max_{0 \leq j \leq i-1} h_j^r + 2f(y_i)h_i \\ &\leq z(b) + 1. \end{aligned} \tag{22}$$

Let now

$$C = \sup_{y \in [\eta, z(b)+1]} |g^{(r)}(y)| \quad \text{and} \quad c = \inf_{y \in [\eta, z(b)+1]} g(y).$$

We come back to (19). In order to bound $|\bar{\kappa}_i^{\bar{y}}|$, we use (11) and (14), where the working variable y in these formulas is replaced by z and y_{i+1} replaced by y, with $y = \bar{y}$. Since $g^{(r)}$ is uniformly continuous on $[\eta, z(b) + 1]$, we have for $\bar{y}$ as above that $|\bar{\kappa}_i^{\bar{y}}| \leq 1/2$ for m sufficiently large, where the limit value of m only depends on g. A sufficient condition for (19) can now be written as

$$C|C_r|\, 2^{r+1}/(c^{r+1} r!)(x_{i+1} - x_i)^r \leq 1/3,$$

which holds true for sufficiently large m. Consequently, there exists $y_{i+1} \in (y_i, y_i + 2f(y_i)(x_{i+1} - x_i)]$ such that $\hat{F}(y_{i+1}) = x_{i+1} - x_i$, as claimed.
It remains to show that

$$|y_{i+1} - z(x_{i+1})| \leq M_{i+1} \max_{0 \leq j \leq i} h_j^r \tag{23}$$

and $M_{i+1} \le M$. We remember that

$$H(y_{i+1}) = F(y_{i+1}) - F(z_i(x_{i+1})) = g(\tilde{\xi}_{i,y_{i+1}})(y_{i+1} - z_i(x_{i+1})),$$

and $\tilde{\xi}_{i,y_{i+1}} \in \mathrm{conv}(z_i(x_{i+1}), y_{i+1}) \subset [\eta, z(b)+1]$. From this the local error can be expressed as

$$y_{i+1} - z_i(x_{i+1}) = \frac{g^{(r)}(\bar{\xi}_{i,y_{i+1}})}{r!} \frac{1}{g(\tilde{\xi}_{i,y_{i+1}})} (y_{i+1} - y_i)^{r+1} C_r \,(1 + \bar{\kappa}_i^{y_{i+1}}). \tag{24}$$

Taking into account that $y_{i+1} - y_i \le 2f(y_i)(x_{i+1} - x_i)$ we get for sufficiently large m that

$$|y_{i+1} - z_i(x_{i+1})| \le \frac{C\,|C_r|}{cr!} (y_{i+1} - y_i)^{r+1}(3/2) \le \tilde{M} h_i^{r+1}, \tag{25}$$

where

$$\tilde{M} = (3/2) C\,|C_r|\, 2^{r+1}/(c^{r+2} r!). \tag{26}$$

Finally, the bound (23) on the global error together with the formula for M_{i+1} follow from the inductive assumption and the inequality

$$|y_{i+1} - z(x_{i+1})| \le |y_{i+1} - z_i(x_{i+1})| + |z_i(x_{i+1}) - z(x_{i+1})| \le \tilde{M} h_i^{r+1} + \exp(Lh_i)|y_i - z(x_i)|.$$

This holds for $m \ge m_0$, where m_0 only depends on f (and is indeiendent of i). The proof is completed. ∎

In particular, for $h_j = O(m^{-1})$ the global error of the method is

$$\max_{0 \le i \le m} |y_i - z(x_i)| = O(m^{-r}), \quad m \to \infty,$$

which is known to be optimal, as far as the speed of convergence is concerned. The constant in the $'O'$-notation however depends on a global behavior of f, and it can be large, see the constant $\tilde{M}$ in the statement of Proposition 1. We now take into account a local behavior of f in order to adjust the step sizes h_j to the size of derivatives of f in particular subintervals.

4 Local error expressions

The local error of the method can be expressed due to (24) as

$$y_{i+1} - z_i(x_{i+1}) = \frac{g^{(r)}(\bar{\xi}_{i,y_{i+1}})}{r!} \frac{1}{g(\tilde{\xi}_{i,y_{i+1}})} (y_{i+1} - y_i)^{r+1} C_r \,(1 + \bar{\kappa}_i^{y_{i+1}}) \tag{27}$$

with $\bar{\xi}_{i,y_{i+1}} \in [y_i, y_{i+1}]$ and $\tilde{\xi}_{i,y_{i+1}} \in \mathrm{conv}(z_i(x_{i+1}), y_{i+1})$.

We shall adopt in what follows a convenient notation for relative errors used in the round off error analysis of numerical algorithms. We have that $y_{i+1} - z_i(x_{i+1}) = y_{i+1} - y_i + y_i - z_i(x_{i+1}) = y_{i+1} - y_i - f(z_i(\alpha_i))h_i$, so that

$$y_{i+1} - y_i = f(z_i(\alpha_i))h_i(1 + \kappa_i),$$

for some $\alpha_i \in [x_i, x_{i+1}]$, where $\kappa_i = (y_{i+1} - z_i(x_{i+1}))/(f(z_i(\alpha_i))\, h_i)$. We can alternatively write the local error as

$$y_{i+1} - z_i(x_{i+1}) = \frac{g^{(r)}(\bar{\xi}_{i,y_{i+1}})}{r! g(\tilde{\xi}_{i,y_{i+1}})\,(g(z_i(\alpha_i)))^{r+1}}\, C_r\,(x_{i+1} - x_i)^{r+1}\,(1 + \bar{\kappa}_i^{y_{i+1}})(1 + \kappa_i)^{r+1}, \quad (28)$$

where $z_i(\alpha_i) \in [y_i, z_i(x_{i+1})]$ and $\max\limits_{0 \le i \le m-1} |\kappa_i|$ tends to zero as $m \to \infty$. The last convergence is uniform with respect to the class of partitions $\{x_i\}$.

The following remarks will be used in what follows.

Remarks

1. Let $\gamma : [\eta, z(b) + 1] \to \mathbf{R}$ be a continuous function of constant sign, and $[\alpha_{i,m}, \beta_{i,m}] \subset [\eta, z(b) + 1]$, $\alpha_{i,m} < \beta_{i,m}$, $i = 0, 1, \ldots, m - 1$. Assume that $\max\limits_{0 \le i \le m-1} (\beta_{i,m} - \alpha_{i,m})$ tends to zero as $m \to \infty$. Then by the uniform continuity, for any $z_1, z_2 \in [\alpha_{i,m}, \beta_{i,m}]$ we have that

$$\gamma(z_1) = \gamma(z_2)(1 + \bar{\kappa}_{i,m}),$$

for some $\bar{\kappa}_{i,m}$, where $\max\limits_{0 \le i \le m-1} |\bar{\kappa}_{i,m}|$ tends to zero as $m \to \infty$.

2. Let $\lim\limits_{m \to \infty} \max\limits_{0 \le i \le m-1} |\bar{\kappa}_{i,m}^j| = 0$ for $j = 1, 2$. Define $\bar{\kappa}_{i,m}^3$ by

$$1+\bar{\kappa}_{i,m}^3 = (1+\bar{\kappa}_{i,m}^1)(1+\bar{\kappa}_{i,m}^2) \;\text{ or }\; 1+\bar{\kappa}_{i,m}^3 = (1+\bar{\kappa}_{i,m}^1)/(1+\bar{\kappa}_{i,m}^2) \;\text{ or }\; 1+\bar{\kappa}_{i,m}^3 = (1+\bar{\kappa}_{i,m}^1)^{r+1}.$$

Then obviously $\lim\limits_{m \to \infty} \max\limits_{0 \le i \le m-1} |\bar{\kappa}_{i,m}^3| = 0$. ■

By these remarks we have the following lemma.

Lemma 1 *The absolute local error of the method (7) is given by*

$$|y_{i+1} - z_i(x_{i+1})| = c_i\, \Delta_i^{r+1}(1 + \kappa_i^1), \quad (29)$$

where $c_i = \sup\limits_{y \in [y_i, y_{i+1}]} \left(|g^{(r)}(y)|/g(y)\right) |C_r|\,/r!$, *and* $\lim\limits_{m \to \infty} \max\limits_{0 \le i \le m-1} |\kappa_i^1| = 0$.

Alternatively,

$$|y_{i+1} - z_i(x_{i+1})| = \bar{c}_i\, h_i^{r+1}(1 + \kappa_i^2), \quad (30)$$

where $\bar{c}_i = \sup_{y \in [z(x_i), z(x_{i+1})]} \left(|g^{(r)}(y)| / \left(g(y)^{r+2}\right) \right) |C_r| / r!$, *and* $\lim_{m\to\infty} \max_{0\le i\le m-1} |\kappa_i^2| = 0$.
The convergence of $\max_{0\le i\le m-1} |\kappa_i^1|$ *and* $\max_{0\le i\le m-1} |\kappa_i^2|$ *in (29) and (30), respectively, is uniform with respect to the partition.*

Proof To show (29) we use (27). Let $s_i \in [y_i, y_{i+1}]$ be a point at which the supremum in the definition of c_i is achieved. We use Remark 1 with $\alpha_{i,m} = \min\{y_i, z(x_i)\}$ and $\beta_{i,m} = \max\{y_{i+1}, z_i(x_{i+1}), z(x_{i+1})\}$, with function $\gamma(y) = |g^{(r)}(y)|$ or $\gamma(y) = g(y)$ and a point z_1 suitably chosen, and with z_2 fixed to be $z_2 = s_i$. The number κ_i^1 absorbes all numbers $\bar{\kappa}_{i,m}$ that appear when applying Remark 1, in accordance with Remark 2.
To show (30), we use (28) and Remarks 1 and 2 in a similar way. ■

The unknown numbers c_i and $\bar{c}_i$ depend on a local behavior of the function g.

5 Adaptive (nonconstructive) selection of mesh points

We now show how to (approximately) minimize the maximal absolute local error skipping for a moment the question whether the mesh points can be constructed or not. From (30)

$$|y_{i+1} - z_i(x_{i+1})| = \bar{c}_i \, (x_{i+1} - x_i)^{r+1} (1 + \kappa_i^2). \tag{31}$$

with $\bar{c}_i = \sup_{y \in [z(x_i), z(x_{i+1})]} \left(|g^{(r)}(y)| / \left(g(y)^{r+2}\right) \right) |C_r|/r!$.
We note that for any f and $\alpha \in (0, 1/2)$ there exists m_0 such that for any $m \ge m_0$ and any partition $\{x_i\}$ under consideration it holds

$$(1-\alpha) \max_{0\le i\le m-1} \bar{c}_i (x_{i+1}-x_i)^{r+1} \le \max_{0\le i\le m-1} |y_{i+1}-z_i(x_{i+1})| \le (1+\alpha) \max_{0\le i\le m-1} \bar{c}_i (x_{i+1}-x_i)^{r+1}. \tag{32}$$

Consider now the minimization problem

$$\max_{0\le i\le m-1} \bar{c}_i \, (x_{i+1} - x_i)^{r+1} \to \text{MIN with respect to } x_1, x_2, \ldots, x_{m-1}, \tag{33}$$

with $x_0 = a$ and $x_m = b$.
Define the functions of variables $x_1, x_2, \ldots, x_{m-1}$ by

$$p(x_i, x_{i+1}) = \bar{c}_i (x_{i+1} - x_i)^{r+1}, \quad x_{i+1} \ge x_i,$$

and

$$P_m(x_0, x_1, \ldots, x_m) = \max_{0\le i\le m-1} p(x_i, x_{i+1}).$$

Note that p is a continuous function of (x_i, x_{i+1}), it is an increasing function of x_{i+1} for fixed x_i, and a decreasing function of x_i for fixed x_{i+1}. The function P_m is continuous on the compact set $a = x_0 \le x_1 \le \ldots \le x_m = b$, so that it attains its infimum for some $a = x_0^* \le x_1^* \le \ldots \le x_m^* = b$. The corresponding $\bar{c}_i$ are denoted by $\bar{c}_i^*$. We note that the infimum

$$\inf_{x_0, x_1, \ldots, x_m} P_m(x_0, x_1, \ldots, x_m)$$

is a nonincreasing function of m, since $P_m(x_0, x_1, \ldots, x_m)$ is equal to $P_{m+1}(x_0, x_1, \ldots, x_m, x_m)$ for any $x_0 \le x_1 \le \ldots \le x_m$.

Proposition 2 *It holds*

$$p(x_i^*, x_{i+1}^*) = \bar{c}_i^* \, (x_{i+1}^* - x_i^*)^{r+1} = k_m^* = \text{const}, \quad i = 0, 1, \ldots, m-1. \tag{34}$$

The number k_m^* *equals the minimal value in (33),* $k_m^* = \min\limits_{x_0, x_1, \ldots, x_m} \max\limits_{0 \le i \le m-1} \bar{c}_i \, (x_{i+1} - x_i)^{r+1}$.
The points $a = x_0^*, x_1^*, \ldots, x_{m-1}^*, x_m^* = b$ *are unique.*
Furthermore,

$$x_{i+1}^* - x_i^* = (b-a) \frac{(1/\bar{c}_i^*)^{1/(r+1)}}{\sum\limits_{i=0}^{m-1} (1/\bar{c}_i^*)^{1/(r+1)}} \qquad \text{and} \qquad k_m^* = \frac{(b-a)^{r+1}}{\left(\sum\limits_{i=0}^{m-1} (1/\bar{c}_i^*)^{1/(r+1)} \right)^{r+1}}. \tag{35}$$

Proof The proof of (34) follows from the following observation. Suppose that there is i such that

$$p(x_i^*, x_{i+1}^*) < p(x_{i+1}^*, x_{i+2}^*)$$

(the case $' >'$ is analogous). Then we can decrease $\max\{p(x_i^*, x_{i+1}^*), p(x_{i+1}^*, x_{i+2}^*)\}$ by slightly increasing x_{i+1}^*. Applying this observation if necessary a number of times, we can also decrease $P_m(x_0^*, x_1^*, \ldots, x_m^*)$, which is a contradiction.
Given x_i^*, the point x_{i+1}^* is a solution of $p(x_i^*, x_{i+1}) = k_m^*$, where $k_m^* = \inf\limits_{x_0, x_1, \ldots, x_m} P_m(x_0, x_1, \ldots, x_m)$.
The solution is unique, since $p(x_i^*, \cdot)$ is an increasing function.
Relations (35) follow from (34) and the fact that $\sum\limits_{i=0}^{m-1} (x_{i+1}^* - x_i^*) = b - a$.

■

The sequence $\{k_m^*\}$ is nonincreasing. A convenient expression for k_m^* is the following

$$k_m^* = \left(\frac{b-a}{m} \right)^{r+1} S(m), \tag{36}$$

where

$$S(m) = \frac{1}{\left(\frac{1}{m}\sum_{i=0}^{m-1}(1/\bar{c}_i^*)^{1/(r+1)}\right)^{r+1}}. \tag{37}$$

Note that $S(m)$ plays here the role of a constant, since the dependence on m is weak: for any $m \geq 1$ we have

$$0 < c(f) \leq S(m) \leq C(f), \tag{38}$$

where

$$c(f) = \inf_{y\in[\eta,z(b)]} b_g(y) \quad \text{and} \quad C(f) = \sup_{y\in[\eta,z(b)]} b_g(y)$$

with $b_g(y) = \left(|g^{(r)}(y)|/\left(g(y)^{r+2}\right)\right)|C_r|/r!$. We see that the factor $S(m)$ in (36) does not improve the speed of convergence of k_m^*, which remains of order $\Theta\left(m^{-(r+1)}\right)$ as $m \to \infty$. However, the gain in the coefficient can be significant. The number $C(f)$ in the 'a priori' bound (38) can be large; it reflects a global behavior of the function g in the entire interval $[\eta, z(b)]$. This bound is sharp if $\bar{c}_i^*$ are essentially constant. In the opposite case, when $\bar{c}_i^*$ are all small except for a single large one equal to $C(f)$, the sum $\sum_{i=0}^{m-1}(1/\bar{c}_i^*)^{1/(r+1)}$ can be much larger than

$$m\left(\frac{1}{\max_{0\leq i\leq m-1}\bar{c}_i^*}\right)^{1/(r+1)}.$$

In this case $S(m)$ is much smaller than $C(f)$ and k_m^* is much smaller than the 'a priori' bound,

$$k_m^* << \left(\frac{b-a}{m}\right)^{r+1} C(f). \tag{39}$$

The gain from adjusting the mesh points to $\bar{c}_i^*$ is then significant.
For comparison, consider the equidistant mesh points, $x_i = a + i(b-a)/m$. Then

$$\max_{0\leq i\leq m-1}\bar{c}_i\left(x_{i+1}-x_i\right)^{r+1} = \max_{0\leq i\leq m-1}\bar{c}_i\left(\frac{b-a}{m}\right)^{r+1} = C(f)\left(\frac{b-a}{m}\right)^{r+1}.$$

Hence, for the equidistant mesh the 'a priori' upper bound in (39) is attained. The points x_i^* defined in Proposition 2 can do much better.
We stress again that the points x_i^* for which the gain is achieved depend on unknown quantities; we do not show at this point how to construct them.
Note also that

$$\max_{0\leq i\leq m-1}(x_{i+1}^* - x_i^*) \leq \frac{b-a}{m}\cdot\left(\frac{C(f)}{c(f)}\right)^{1/(r+1)},$$

so that $\{x_i^*\}$ is an admissible partition (for any $l(m) \geq 1/m$ and $K \geq (b-a)\left(C(f)/c(f)\right)^{1/(r+1)}$ in (2)).

In many cases we are interested in computing approximations with the absolute local error not exceeding a prescribed level $\varepsilon \in (0,1)$. That is, we wish to find the minimal number $m = m(\varepsilon)$ such that $k_m^* \leq \varepsilon$. Hence,

$$k_m^* = \left(\frac{b-a}{m}\right)^{r+1} S(m) \leq \varepsilon,$$

which gives us that $m(\varepsilon)$ is the minimal m such that

$$m \geq (b-a) S(m)^{1/(r+1)} \left(\frac{1}{\varepsilon}\right)^{1/(r+1)}. \tag{40}$$

The 'a priori' bounds on $S(m)$ lead to 'a priori' bounds on $m(\varepsilon)$

$$(b-a)c(f)^{1/(r+1)} \left(\frac{1}{\varepsilon}\right)^{1/(r+1)} \leq m(\varepsilon) < (b-a)C(f)^{1/(r+1)} \left(\frac{1}{\varepsilon}\right)^{1/(r+1)} + 1, \tag{41}$$

so that $m(\varepsilon) = \Theta\left((1/\varepsilon)^{1/(r+1)}\right)$ as $\varepsilon \to 0$. The actual value of $m(\varepsilon)$ can be however much smaller than the upper bound, since $S(m)$ for all m can be much smaller than $C(f)$.

It is clear that the number of subdivision intervals $m(\varepsilon)$ will be crucial for establishing the minimal cost of computing a constructive approximation with the absolute local error at most ε.

Proposition 2 leads to the following result about minimization of the maximal absolute local error. Let

$$L^m = \min_{x_0, x_1, \ldots, x_m} \max_{0 \leq i \leq m-1} |y_{i+1} - z_i(x_{i+1})|. \tag{42}$$

The value of L^m is asymptotically equal to k_m^*, up to an arbitrarily small positive constant α.

Proposition 3 *For any f and $\alpha \in (0, 1/2)$ there exists m_0 such that for any $m \geq m_0$ the minimal error satisfies*

$$(1-\alpha) k_m^* \leq L^m \leq (1+\alpha) k_m^*, \tag{43}$$

and

$$(1-2\alpha) \max_{0 \leq i \leq m-1} |y_{i+1}^* - z_i(x_{i+1}^*)| \leq L^m \leq \max_{0 \leq i \leq m-1} |y_{i+1}^* - z_i(x_{i+1}^*)|. \tag{44}$$

Hence, up to a (possibly small) constant α, the mesh points $\{x_i^*\}$ are optimal.

(Here $\{y_i^*\}$ are given for $\{x_i^*\}$ by (7), and z_i denotes the solution of the local problem with the initial condition $z_i(x_i^*) = y_i^*$.)

Proof The proof follows from (32). ■

Hence, the quantity k_m^* is equal to the minimal maximum local error, and the points x_i^* define the best partition (up to the constant α). The method (7) needs at each step to compute the interpolation polynomial $\hat{g}_i$. In order to have local errors at level ε, the cost is thus at least $r\,m(\varepsilon)$ evaluations of the function g. In the next section we effectively construct the mesh points and modify (7) to compute approximations with the local errors proportional to ε, with cost proportional to $m(\varepsilon)$.

6 Adaptive constructive selection of mesh points

Let $\varepsilon \in (0,1)$. We shall slightly modify (7) by replacing the interpolation polynomial $\hat{g}_i$ (defined on $[y_i, y_{i+1}]$) by an interpolation polynomial $\hat{g}_i^1$ defined on the interval dependent only on y_i. The approximation to y_{i+1} will be obtained by a number of steps of the bisection method. The replacement will allow us to use the same polynomial in all iterations. This makes it possible to avoid the $\log 1/\varepsilon$ factor in the cost bound, at expence of an additional factor dependent only on r in the error bound. The dependence of the cost on g and ε, and possible gain discussed in the previous section, will be hidden in the quantity $m(\varepsilon)$.

To be specific, we define points $\hat{x}_i$ as follows. We set $\hat{x}_0 = a$, $\hat{y}_0 = \eta$. For a given $\hat{x}_i$ and $\hat{y}_i$, we compute the divided difference

$$g[\bar{z}_0^i, \bar{z}_1^i, \ldots, \bar{z}_r^i],$$

where $\bar{z}_j^i$ are equidistant points from $[\hat{y}_i, \hat{y}_i + \varepsilon^{1/(r+1)}]$ (including the end points).
Then we set

$$\hat{c}_i = \hat{c}_i(\hat{y}_i) = \frac{2^{r+1}|g[\bar{z}_0^i, \bar{z}_1^i, \ldots, \bar{z}_r^i]|}{g(\hat{y}_i)^{r+2}}. \tag{45}$$

The point $\hat{x}_{i+1}$ is defined as the solution of

$$\hat{c}_i(\hat{y}_i)(\hat{x}_{i+1} - \hat{x}_i)^{r+1} = \frac{2^{r+1}}{|C_r|}\,\frac{1}{1-\alpha}\,\varepsilon, \quad i = 0, 1, \ldots. \tag{46}$$

Let $\bar{y}_i = \hat{y}_i + 2f(\hat{y}_i)(\hat{x}_{i+1} - \hat{x}_i)$. Let $\hat{g}_i^1$ be the Lagrange interpolation polynomial for g of degree at most $r-1$ based on r equidistant points from $[\hat{y}_i, \bar{y}_i]$ for $r \geq 2$ (including the end points), and $\hat{g}_i^1(y) \equiv g(\hat{y}_i)$ for $r = 1$. We define $\hat{y}_{i+1}$ as the solution of

$$\hat{F}^1(y) := \int_{\hat{y}_i}^{y} \hat{g}_i^1(z)\,dz = \hat{x}_{i+1} - \hat{x}_i$$

in the interval $[\hat{y}_i, \bar{y}_i]$. The existence of $\hat{y}_{i+1}$ follows (for sufficiently small ε) from the arguments used in the proof of Proposition 1 with $\{x_i\}$ and $\{y_i\}$ replaced by $\{\hat{x}_i\}$ and $\{\hat{y}_i\}$. We use the function $H^1(y) = F(y) - \hat{F}^1(y)$, where

$$F(y) = \int_{\hat{y}_i}^{y} g(z)\, dz.$$

By the standard interpolation error formula we have

$$|H^1(y)| \leq \sup_{\xi \in [\hat{y}_i, \bar{y}_i]} |g^{(r)}(\xi)| (\bar{y}_i - \hat{y}_i)^r (y - \hat{y}_i)/r!$$

$$= \sup_{\xi \in [\hat{y}_i, \bar{y}_i]} |g^{(r)}(\xi)| 2^r (\hat{x}_{i+1} - \hat{x}_i)^r f(\hat{y}_i)^r (y - \hat{y}_i)/r!, \quad y \in [\hat{y}_i, \bar{y}_i].$$

We define $\hat{m}$ to be the minimal i for which $\hat{x}_i \geq b$.

Consider now the local errors $\hat{y}_{i+1} - z_i(\hat{x}_{i+1})$ of the pairs $(\hat{x}_i, \hat{y}_i)$ (z_i is the solution of (3) such that $z_i(\hat{x}_i) = \hat{y}_i$). Similarly as in the proof of Proposition 1, we have that

$$\hat{y}_{i+1} - z_i(\hat{x}_{i+1}) = H^1(\hat{y}_{i+1})/g(\eta_i),$$

for some $\eta_i \in \mathrm{conv}(\hat{y}_{i+1}, z_i(\hat{x}_{i+1}))$. Hence

$$|\hat{y}_{i+1} - z_i(\hat{x}_{i+1})| \leq \gamma_i (\hat{x}_{i+1} - \hat{x}_i)^{r+1},$$

where

$$\gamma_i = \frac{\sup\limits_{\xi \in [\hat{y}_i, \bar{y}_i]} |g^{(r)}(\xi)|\, f(\hat{y}_i)^{r+1}\, 2^{r+1}}{r! \inf\limits_{\xi \in \mathrm{conv}(\hat{y}_{i+1}, z_i(\hat{x}_{i+1}))} g(\xi)}.$$

We now use Remarks 1 and 2 to replace γ_i by $\hat{c}_i$ to get

$$|\hat{y}_{i+1} - z_i(\hat{x}_{i+1})| \leq \hat{c}_i (\hat{x}_{i+1} - \hat{x}_i)^{r+1} (1 + \kappa_i^2), \tag{47}$$

with some κ_i^2 such that $\max\limits_{0 \leq i \leq \hat{m}-1} |\kappa_i^2|$ tends to 0 as $\varepsilon \to 0$. This together with the definition of $\hat{x}_{i+1}$ yields that for sufficiently small ε we have

$$|\hat{y}_{i+1} - z_i(\hat{x}_{i+1})| \leq \frac{1+\alpha}{1-\alpha} \frac{2^{r+1}}{|C_r|} \varepsilon. \tag{48}$$

We now show that $\hat{m} \leq m(\varepsilon)$, where $m(\varepsilon)$, which is our reference quantity, has been defined to be the minimal value of m such that $k_m^* \leq \varepsilon$. This will follow from the fact that for the optimal points we have $x_i^* \leq \hat{x}_i$, $i = 0, 1, \ldots$. Indeed, this holds for $i = 0$. Let

$x_i^* \le \hat{x}_i$ for some i. If $x_{i+1}^* \le \hat{x}_i$, then obviously $x_{i+1}^* \le \hat{x}_{i+1}$, so that it suffices to consider the case $\hat{x}_i < x_{i+1}^*$. Then

$$\varepsilon \ge k_m^* = \bar{c}_i(x_i^*, x_{i+1}^*)(x_{i+1}^* - x_i^*)^{r+1} \ge \bar{c}_i(\hat{x}_i, x_{i+1}^*)(x_{i+1}^* - \hat{x}_i)^{r+1}.$$

For convenience, we have explicitly written here the arguments that $\bar{c}_i$ depends on. It follows from Remarks 1 and 2 that for sufficiently small ε

$$\varepsilon \ge \bar{c}_i(\hat{x}_i, x_{i+1}^*)(x_{i+1}^* - \hat{x}_i)^{r+1} = \hat{c}_i(\hat{y}_i)(x_{i+1}^* - \hat{x}_i)^{r+1}|C_r|(1+\kappa_i^3)/2^{r+1}$$
$$\ge \hat{c}_i(\hat{y}_i)(x_{i+1}^* - \hat{x}_i)^{r+1}|C_r|(1-\alpha)/2^{r+1}.$$

Thus,

$$\hat{c}_i(\hat{y}_i)(x_{i+1}^* - \hat{x}_i)^{r+1} \le 2^{r+1}/(|C_r|(1-\alpha))\,\varepsilon,$$

which yields that $\hat{x}_{i+1} \ge x_{i+1}^*$, as claimed.

It remains to show how we compute an approximation to $\hat{y}_{i+1}$. We apply the bisection method to the equation $\hat{F}^1(y) = \hat{x}_{i+1} - \hat{x}_i$, starting from the interval $[\hat{y}_i, \hat{y}_i + 2f(\hat{y}_i)(\hat{x}_{i+1} - \hat{x}_i)]$ (see the proof of Proposition 1). After $l_i \ge 1$ steps the length of the interval is reduced to $f(\hat{y}_i)(\hat{x}_{i+1} - \hat{x}_i)/2^{l_i-1}$. We choose l_i to the minimal number such that

$$f(\hat{y}_i)(\hat{x}_{i+1} - \hat{x}_i)/2^{l_i-1} \le \varepsilon/2. \tag{49}$$

Equivalently, inserting $\hat{x}_{i+1} - \hat{x}_i$, we get that l_i is the minimal number such that

$$\frac{8f(\hat{y}_i)}{\hat{c}_i^{1/(r+1)}\,|C_r|^{1/(r+1)}} \left(\frac{1}{1-\alpha}\right)^{1/(r+1)} \varepsilon^{1/(r+1)-1} \le 2^{l_i}. \tag{50}$$

We note that the bisection process does not require any new evaluations of g.
Any point of the last bisection interval can be taken as an approximation to $\hat{y}_{i+1}$. We denote the selected approximation by the same symbol $\hat{y}_{i+1}$, and get

$$|\hat{y}_{i+1} - z_i(\hat{x}_{i+1})| \le \left(\frac{1+\alpha}{1-\alpha}\,\frac{2^{r+1}}{|C_r|} + \frac{1}{2}\right)\varepsilon. \tag{51}$$

We will refer to the above bisection procedure as BISEC. These considerations are summarized in the following algorithm for solving (1).

Algorithm **ADMESH**

1 Set $\varepsilon \in (0,1)$, $\alpha \in (0,1/2)$, $\hat{x}_0 = a$, $\hat{y}_0 = \eta$, $i = -1$

2 $i := i+1$

3 Compute $d_r^i = g[\bar{z}_0^i, \bar{z}_1^i, \dots, \bar{z}_r^i]$, where $\bar{z}_j^i$ are equidistant points in $[\hat{y}_i, \hat{y}_i + \varepsilon^{1/(r+1)}]$ (including the end points)

4 Compute $\hat{c}_i = 2^{r+1} |d_r^i| f(\hat{y}_i)^{r+2}$ and $\hat{x}_{i+1} = \hat{x}_i + 2/\left(|C_r|\, \hat{c}_i\, (1-\alpha)\right)^{1/(r+1)}\ \varepsilon^{1/(r+1)}$. If $\hat{x}_{i+1} \geq b$ then $\hat{x}_{i+1} := b$

5 Compute the interpolation polynomial $\hat{g}_i^1$

6 Compute $\hat{y}_{i+1}$ by the algorithm BISEC applied to the equation $\hat{F}^1(y) = \hat{x}_{i+1} - \hat{x}_i$ with l_i steps starting from $[\hat{y}_i, \hat{y}_i + 2f(\hat{y}_i)(\hat{x}_{i+1} - \hat{x}_i)]$, with l_i given by (50). If $\hat{x}_{i+1} = b$ then go to STOP

7 Go to 2

STOP

The following theorem summarizes the error and cost properties of the algorithm ADMESH.

Theorem 1 *Let $f \in F_r$ and $\alpha \in (0,1/2)$. There exists $\varepsilon_0 = \varepsilon_0(f,\alpha)$ such that for any $\varepsilon \leq \varepsilon_0$ the algorithm ADMESH computes pairs $(\hat{x}_i, \hat{y}_i)$, $i = 0,1,\dots,\hat{m}$, with the following error/cost properties. The maximum local error is bounded by*

$$\max_{0 \leq i \leq \hat{m}-1} |\hat{y}_{i+1} - z_i(\hat{x}_{i+1})| \leq \left(\frac{1+\alpha}{1-\alpha} \frac{2^{r+1}}{|C_r|} + \frac{1}{2} \right) \varepsilon, \tag{52}$$

where z_i is the solution of the local problem (3) with the initial condition $z_i(\hat{x}_i) = \hat{y}_i$. The cost of the algorithm $\mathrm{cost}(f,\alpha,\varepsilon)$ measured by the number of evaluations of f is bounded by

$$\mathrm{cost}(f,\alpha,\varepsilon) \leq 2r\, m(\varepsilon), \tag{53}$$

where $m(\varepsilon)$ is the (almost optimal) number of subintervals given in (40).

Proof The bound (52) follows from (51). The cost related to the ith interval consists of $r+1$ function evaluations to compute d_r^i and additional $r-1$ function evaluations to compute the interpolation polynomial $\hat{g}_i^1$. This and the bound $\hat{m} \leq m(\varepsilon)$ yield the cost bound (53).

■

We comment on this result. Note first that α can be an arbitrary small positive number which does not play any crucial role. The accuracy achieved by the algorithm ADMESH differs from the accuracy achieved by the almost optimal points x_i^* only by the explicitly known factor dependent on r (and independent of f), see Proposition 3. The cost of ADMESH is proportional, with coefficient $2r$, to the reference value $m(\varepsilon)$. It follows from the discussion after Proposition 2, see (41), that the 'a priori' upper bound on the cost is

$$\mathrm{cost}(f,\alpha,\varepsilon) \le 2r\left((b-a)C(f)^{1/(r+1)}\left(\frac{1}{\varepsilon}\right)^{1/(r+1)}+1\right). \tag{54}$$

This upper bound is essentially achieved by the equidistant mesh. The advantage of the mesh points constructed in the algorithm ADMESH lies in the fact that $m(\varepsilon)$, where the dependence on f is hidden, can be much smaller than the upper bound given in (41), see the discussion after Proposition 2. Consequently, the actual cost of getting the accuracy proportional to ε can be much smaller than the upper bound in (54).
Note also that we can have the error bound in (52) equal to a given number ε_1, by running the algorithm with $\varepsilon := \varepsilon_1\left(\frac{1+\alpha}{1-\alpha}\frac{2^{r+1}}{|C_r|}+\frac{1}{2}\right)^{-1}$.

Remark 3 It would be of interest to generalize the above results to systems of IVPs. One can see that a straightforward generalization is not possible, since there is no counterpart of (6) for systems of IVPs. Preliminary analysis however indicates that a progress in that direction is possible using a different technique. This will be a topic of our future work.

7 Numerical example

To illustrate the behavior of ADMESH, we consider a problem with $r = 2$, dependent on a parameter $\delta > 0$

$$z'(t) = \frac{3}{4}(z(t)-1)^{-3/2}, \quad t \in [0,1], \quad z(0) = 1+\delta. \tag{55}$$

The right hand side function f has the form $f = 1/g$, where $g(z) = (4/3)(z-1)^{3/2}$. The second derivative $g''(z) = (z-1)^{-1/2}$ taken at the initial condition grows to infinity as $1/\sqrt{\delta}$ with $\delta \to 0^+$. For such a function and small δ we should observe a significant advantage of adaptive mesh points over the equidistant points. The solution satisfying the initial condition $z(x) = y \quad (x \ge 0,\ y > 1)$ is given by

$$z(t) = \left(\frac{15}{8}(t-x)+(y-1)^{5/2}\right)^{2/5}+1.$$

The testing program was translated to the C^{++} code by P. Morkisz and B. Bożek.
The following table shows results computed by ADMESH for number of values of ε and δ. In the successive columns we show the values of IADAPT (the number of adaptive mesh points), MAXERR (the maximum local error), MAXERR/BOUND (BOUND is the upper bound given in Theorem 1), MAXERRG (the maximum global error), EQUIDIST/MAXERR (EQUIDIST is the maximal local error obtained with 2∗IADAPT equidistant mesh points), EQUIDISTG/MAXERRG (the same ratio for the maximal global errors). Since ADMESH requires 4 evaluations of f in each subinterval, and the equidistant mesh algorithm 2 evaluations, the results in the latter case are computed for twice as much points. The computer precision is 10^{-16}. We took $\alpha = 0.25$.

ε	δ	IADAPT	MAXERR	$\frac{\text{MAXERR}}{\text{BOUND}}$	MAXERRG	$\frac{\text{EQUIDIST}}{\text{MAXERR}}$	$\frac{\text{EQUIDISTG}}{\text{MAXERRG}}$
0.01	0.1	5	0.23	0.014	0.035	7.39	4.9
0.01	10^{-4}	11	0.02	0.011	0.032	19.57	11.96
0.01	10^{-8}	11	0.02	0.012	0.039	17.79	11.00
10^{-4}	0.1	15	$7.3{*}10^{-4}$	0.046	$3.1{*}10^{-3}$	90.56	21.06
10^{-4}	10^{-4}	27	$6.7{*}10^{-4}$	0.042	$3.0{*}10^{-3}$	369.89	84
10^{-4}	10^{-8}	30	$6.7{*}10^{-4}$	0.042	$3.0{*}10^{-3}$	371.69	101
10^{-8}	0.1	252	$1.09{*}10^{-7}$	0.068	$8.24{*}10^{-6}$	8291	109
10^{-8}	10^{-4}	418	$1.85{*}10^{-7}$	0.115	$8.28{*}10^{-6}$	436463	9732
10^{-8}	10^{-8}	435	$2.30{*}10^{-7}$	0.143	$8.32{*}10^{-6}$	373152	12562
10^{-16}	0.1	115332	$1.59{*}10^{-15}$	0.099	$3.87{*}10^{-11}$	17051	95
10^{-16}	10^{-4}	192546	$1.59{*}10^{-15}$	0.099	$3.90{*}10^{-11}$	$3.7{*}10^{12}$	$1.5{*}10^{8}$
10^{-16}	10^{-8}	200023	$1.39{*}10^{-14}$	0.866	$3.90{*}10^{-11}$	$5.3{*}10^{11}$	$2.2{*}10^{8}$

The 5th column verifies the statement of Theorem 1; all its entries should be at most 1. The 7th column shows how much the local error for equidistant mesh points exceeds that for the adaptive points used by ADMESH. We see that a significant advantage of using adaption is observed for all values of ε and δ. The gain grows when ε or δ go to 0.

Acknowledgments I thank L. Plaskota and P. Przybyłowicz for their comments on the manuscript.

References

[1] Choi, S.T., Ding, Y., Hickernell, F.J, Tong, X. (2017), Lo-

cal adaption for approximation and minimization of univariate functions, http://dx.doi.org/10.1016/j.jco.2016.11.005, to appear in *J. Complexity*.

[2] Jackiewicz, Z. (2002), Implementation of DIMSIMs for stiff differential systems, *Appl. Numer. Math.*, 42, No. 1-3, 251–267.

[3] Kacewicz, B., (1988), Minimum asymptotic error of algorithms for solving ODE, *J. Complexity*, 4, 373–389.

[4] Kacewicz, B., (1990), On sequential and parallel solution of initial value problems, *J. Complexity*, 6, 136–148.

[5] Kacewicz, B., Przybyłowicz, P., (2015), Complexity of the derivative-free solution of systems of IVPs with unknown singularity hypersurface, *J. Complexity*, 31, 75–97.

[6] Lyness, J.N., (1983), When not to use an automatic quadrature routine?, *SIAM Review* , 25, 63–87.

[7] Mazzia, F., Nagy, A.M, (2015), A new mesh selection strategy with stiffness detection for explicit Runge Kutta methods, *Applied Math. and Comp.*, 255, 125–134.

[8] Piessens, R., De Doncker-Kapenga,E., Überhuber,C. W., (1983), *QUADPACK: a subroutine package for automatic integration*, Springer, ISBN: 3-540-12553-1.

[9] Plaskota, L., (2015), Automatic integration using asymptotically optimal adaptive Simpson quadrature, *Numer. Math.*, 131, 173–198.

[10] Plaskota, L., Wasilkowski, G.W., (2009), The power of adaptive algorithms for functions with singularities, *J. Fixed Point Theory Appl.*, 6, 227–248.