# A STUDY OF SCHRÖDER'S METHOD FOR THE MATRIX $p$ TH ROOT USING POWER SERIES EXPANSIONS 

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#### Abstract

When $A$ is a matrix with all eigenvalues in the disk $|z-1|<1$, the principal $p$ th root of $A$ can be computed by Schröder's method, among many other methods. In this paper we present a further study of Schröder's method for the matrix $p$ th root, through an examination of power series expansions of some sequences of scalar functions. Specifically, we obtain a new and informative error estimate for the matrix sequence generated by the Schröder's method, a monotonic convergence result when $A$ is a nonsingular $M$-matrix, and a structure preserving result when $A$ is a nonsingular $M$-matrix or a real nonsingular $H$-matrix with positive diagonal entries.


## 1. Introduction

For a given integer $p \geq 2$ and a matrix $A \in \mathbb{C}^{n \times n}$ whose eigenvalues are in the open disk $\{z \in \mathbb{C}:|z-1|<1\}$, the principal $p$ th root of $A$ exists and is denoted by $A^{1 / p}$ [10]. Various methods can be used to compute $A^{1 / p}$; see [2, 7, 8, 2, 10, 11, 12, 13, 14, 15, 18, 20, 23, 24.

In this paper we are concerned with the Schröder family of iterations, also called Schröder's method for short, which is a special case of the dual Padé family of iterations proposed in 24].

In the scalar case of computing $a^{1 / p}$, the dual Padé family of iterations has the form

$$
\begin{equation*}
x_{k+1}=x_{k} \frac{Q_{\ell m}\left(1-a x_{k}^{-p}\right)}{P_{\ell m}\left(1-a x_{k}^{-p}\right)}, \quad x_{0}=1, \tag{1}
\end{equation*}
$$

where $P_{\ell m}(t) / Q_{\ell m}(t)$ is the $[\ell / m]$ Padé approximant to the function $(1-t)^{-1 / p}$, or equivalently $Q_{\ell m}(t) / P_{\ell m}(t)$ is the $[m / \ell]$ Padé approximant to the function $(1-t)^{1 / p}$.

When $\ell=m=1$, we get Halley's method. When $\ell=0$, we get the Schröder family of iterations. Within the Schröder family, we get Newton's method when $m=1$, and get Chebyshev's method when $m=2$.

For $a=1-z$ with $|z|<1$, Each $x_{k}$ from the dual Padé iteration (11) has a power series expansion

$$
\begin{equation*}
x_{k}=\sum_{i=0}^{\infty} c_{k, i} z^{i}, \quad|z|<1 . \tag{2}
\end{equation*}
$$

[^0]It is conjectured in [24] that $c_{k, i}<0$ for $i \geq 1$ (as long as the series in (22) is not reduced to a finite series). The conjecture is an extension of an earlier conjecture in [8 for Newton's method and Halley's method and a similar conjecture in [19] for Chebyshev's method.

For $p=2$, the conjecture for Newton's method is shown to be true in [8], by using a result proved in [22, and a more direct proof is presented in [17] for both Newton's method and Halley's method. For any integer $p \geq 2$, the conjecture has been proved very recently [21] for both Newton's method and Halley's method. The conjecture for Chebyshev's method has remained open even for $p=2$.

In this paper we will prove the conjecture for the whole Schröder family (which include Chebyshev's method) for all $p \geq 2$. However, the conjecture is not true for the whole dual Padé family. Indeed for $\ell=1$ and $m=0$, we have $P_{10}(t)=1+\frac{1}{p} t$ and $Q_{10}(t)=1$. It follows from (1) that

$$
x_{1}=\frac{1}{1+\frac{1}{p}(1-(1-z))}=\frac{1}{1+\frac{1}{p} z}=\sum_{i=0}^{\infty}(-1)^{i} \frac{1}{p^{i}} z^{i} .
$$

From the Schröder's method for computing $a^{1 / p}$, we can get the corresponding Schröder's method for computing $A^{1 / p}$. In particular, we have Chebyshev's method for computing $A^{1 / p}$. Chebyshev's method is called Euler's method in 20] and its efficiency (when properly implemented) has been shown in that paper. This has provided us additional motivation to further study Schröder's method for the matrix $p$ th root.

## 2. Preliminaries

Schröder's method for the matrix $p$ th root will be studied through an examination of power series expansions of some sequences of scalar functions.

We start with the rising factorial notation

$$
(x)_{i}=x(x+1) \cdots(x+i-1)
$$

where $x$ is a real number and $i \geq 0$ is an integer. We have $(x)_{0}=1$ by convention.
We have the binomial expansion

$$
\begin{equation*}
(1-t)^{1 / p}=\sum_{i=0}^{\infty} b_{i} t^{i}, \quad|t|<1 \tag{3}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{0}=1, \quad b_{i}=\frac{\left(-\frac{1}{p}\right)_{i}}{i!}<0, \quad i \geq 1 \tag{4}
\end{equation*}
$$

By a limit argument, we can show that the equality in (3) also holds for $t=1$. So we have $\sum_{i=0}^{\infty} b_{i}=0$.

Let $T_{m}(t)$ be the polynomial of degree $m$

$$
T_{m}(t)=\sum_{i=0}^{m} b_{i} t^{i}
$$

which is the sum of the first $m+1$ terms in the power series (3). It is readily seen that we have the Taylor expansion

$$
\begin{equation*}
\left(T_{m}(t)\right)^{-p}=\sum_{i=0}^{\infty} a_{i} t^{i}, \quad|t|<1, \tag{5}
\end{equation*}
$$

where $a_{0}=1$ and $a_{i}>0$ for all $i \geq 0$.
In the scalar case of computing $a^{1 / p}$, we let the residual be $R\left(x_{k}\right)=1-a x_{k}^{-p}$, The Schröder's iteration applied to the function $x^{p}-a$ gives (see [4) the iteration

$$
\begin{equation*}
x_{k+1}=x_{k} T_{m}\left(R\left(x_{k}\right)\right) . \tag{6}
\end{equation*}
$$

So indeed it is a special case of (11) with $\ell=0$. When $m=2$, iteration (6) is the same as Chebyshev's method 3 applied to the function $x^{p}-a$, as noted in 19 .

For iteration (6), we have

$$
\begin{equation*}
x_{k}-x_{k+1}=x_{k}\left(1-T_{m}\left(R\left(x_{k}\right)\right)=x_{k} R\left(x_{k}\right) \sum_{i=1}^{m}\left(-b_{i}\right)\left(R\left(x_{k}\right)\right)^{i-1},\right. \tag{7}
\end{equation*}
$$

and

$$
R\left(x_{k+1}\right)=f\left(R\left(x_{k}\right)\right)
$$

with

$$
\begin{equation*}
f(t)=1-\left(T_{m}(t)\right)^{-p}(1-t) . \tag{8}
\end{equation*}
$$

We have by (5) that $f(t)=\sum_{i=0}^{\infty} c_{i} t^{i}$ with $c_{0}=0$ and $c_{i}=a_{i-1}-a_{i}$ for $i \geq 1$. It is shown in [5] that

$$
\begin{equation*}
f(t)=\sum_{i=0}^{\infty} c_{i} t^{i}=\sum_{i=m+1}^{\infty} c_{i} t^{i} \tag{9}
\end{equation*}
$$

with $c_{i}>0$ for $i \geq m+1$. This means that we actually have $a_{0}=a_{1}=\cdots=a_{m}=1$ in (5).

For $a=1-z$ with $|z|<1$, Schröder's method for finding $(1-z)^{1 / p}$ is

$$
\begin{equation*}
\left.x_{k+1}=x_{k} T_{m}\left(1-(1-z) x_{k}^{-p}\right)\right), \quad x_{0}=1 . \tag{10}
\end{equation*}
$$

To emphasize the dependence of $x_{k}$ on $z$, we will write $x_{k}(z)$ for $x_{k}$. Each $x_{k}(z)$ has a power series expansion

$$
\begin{equation*}
x_{k}(z)=\sum_{i=0}^{\infty} c_{k, i} z^{i}, \quad|z|<1 . \tag{11}
\end{equation*}
$$

The following connection between Schröder's method and the binomial expansion is included in the more general Theorem 6.1 of [24]; it can also be proved in the same way as [8, Theorem 10] is proved for Newton's method and Halley's method.

Theorem 1. For Schröder's method, $c_{k, i}=b_{i}$ for $k \geq 0$ and $0 \leq i \leq(m+1)^{k}-1$.
From the theorem, we know that $c_{k, 0}=1$ for all $k \geq 0$ and that $c_{k, i}<0$ for $k \geq 1$ and $1 \leq i \leq(m+1)^{k}-1$. To obtain some new results for Schröder's method for the matrix $p$ th root, we need to prove that $c_{k, i} \leq 0$ for $k \geq 1$ and all $i \geq(m+1)^{k}$. When $k=1$, we have $x_{1}(z)=T_{m}(1-(1-z))=T_{m}(z)$, so $c_{1, i}=0$ for $i \geq m+1$.

## 3. Sign pattern of coefficients $c_{k, i}$ IN POWER SERIES EXPANSIONS

To determine the sign pattern of $c_{k, i}$ for Schröder's method, we will show that $c_{k, i}$ decreases when $k$ increases (and $i \geq 1$ is fixed), as in 21] for Newton's method and Halley's method. In this process, we will need a useful recursion for the coefficients $a_{i}$ in (5), and some good luck as well!

For the scalar case of computing $(1-z)^{1 / p}$, where $|z|<1$, the residual is $R\left(x_{k}\right)=$ $1-(1-z) x_{k}^{-p}$. We have the following result.

Lemma 2. The coefficients in the power series expansion (in the variable $z$ ) of $R\left(x_{k}\right)$ are all nonnegative.

Proof. We have $R\left(x_{0}\right)=z$. The result is proved by induction since

$$
R\left(x_{k+1}\right)=f\left(R\left(x_{k}\right)\right)=\sum_{i=m+1}^{\infty} c_{i}\left(R\left(x_{k}\right)\right)^{i}
$$

with $c_{i}>0$ for $i \geq m+1$.

We will show that the coefficients in the power series expansion of $x_{k} R\left(x_{k}\right)$ are also all nonnegative.

Note that

$$
\begin{equation*}
x_{k+1} R\left(x_{k+1}\right)=x_{k} g\left(R\left(x_{k}\right)\right) \tag{12}
\end{equation*}
$$

where

$$
g(t)=T_{m}(t) f(t)
$$

It follows from (9) that $g(t)$ has a Taylor expansion $g(t)=\sum_{i=m+1}^{\infty} d_{i} t^{i}$, where

$$
d_{i}=b_{0} c_{i}+b_{1} c_{i-1}+\cdots+b_{m} c_{i-m}
$$

for each $i \geq m+1$. We are going to prove that $d_{i}>0$ for all $i \geq m+1$. Since $b_{0}=1$ and $c_{k}=a_{k-1}-a_{k}$ for $k \geq 1$, we need to show for $i \geq m+1$ that

$$
\begin{equation*}
-a_{i}+\left(b_{0}-b_{1}\right) a_{i-1}+\left(b_{1}-b_{2}\right) a_{i-2}+\cdots+\left(b_{m-1}-b_{m}\right) a_{i-m}+b_{m} a_{i-m-1}>0 \tag{13}
\end{equation*}
$$

The following recursion about the coefficients $a_{i}$ in (5) will play an important role. It is equation (2.11) in [5], with some notation changes.

Lemma 3. For each $k \geq m-1$,

$$
a_{k+1}=\frac{1}{k+1} \sum_{s=0}^{m-1}(k-s+p(s+1))\left(-b_{s+1}\right) a_{k-s} .
$$

By Lemma 3 with $k=i-1$, (13) becomes

$$
\begin{equation*}
\sum_{s=0}^{m-1} \alpha_{s} a_{i-1-s}+b_{m} a_{i-m-1}>0 \tag{14}
\end{equation*}
$$

where for $s=0,1, \ldots, m-1$

$$
\begin{aligned}
\alpha_{s} & =b_{s}-b_{s+1}+\frac{1}{i}(i-1-s+p(s+1)) b_{s+1} \\
& =b_{s}+\frac{1}{i}(p-1)(s+1) b_{s+1} \\
& =b_{s}+\frac{1}{i}(p-1)(s+1) b_{s} \frac{-\frac{1}{p}+s}{s+1} \\
& =b_{s}\left(1+\frac{1}{i}(p-1)\left(-\frac{1}{p}+s\right)\right)
\end{aligned}
$$

Thus (14) is equivalent to
(15) $\frac{p(i-1)+1}{p i} a_{i-1}+\sum_{s=1}^{m-1} b_{s}\left(1+\frac{1}{i}(p-1)\left(-\frac{1}{p}+s\right)\right) a_{i-1-s}+b_{m} a_{i-m-1}>0$.

By Lemma 3 with $k=i-2$, we have

$$
\begin{aligned}
a_{i-1} & =\frac{1}{i-1} \sum_{s=0}^{m-1}(i-2-s+p(s+1))\left(-b_{s+1}\right) a_{i-2-s} \\
& =\frac{1}{i-1} \sum_{s=1}^{m}(i-1-s+p s)\left(-b_{s}\right) a_{i-1-s}
\end{aligned}
$$

Thus (15) becomes

$$
\begin{equation*}
\sum_{s=1}^{m-1} \beta_{s} b_{s} a_{i-1-s}+\beta_{m} b_{m} a_{i-m-1}>0 \tag{16}
\end{equation*}
$$

where

$$
\beta_{m}=\frac{p(i-1)+1}{p i} \frac{-1}{i-1}(i-1-m+p m)+1
$$

and for $s=1, \ldots, m-1$

$$
\beta_{s}=\frac{p(i-1)+1}{p i} \frac{-1}{i-1}(i-1-s+p s)+1+\frac{1}{i}(p-1)\left(-\frac{1}{p}+s\right) .
$$

Since $a_{i}>0$ for $i \geq 0$ and $b_{i}<0$ for $i \geq 1$, a sufficient condition for (16) to hold is that, for $s=1, \ldots, m, \beta_{s}<0$, or equivalently $-p i(i-1) \beta_{s}>0$. Luckily, the sufficient condition does hold. Indeed,

$$
-p i(i-1) \beta_{m}=(p(i-1)+1)(i-1-m+p m)-p i(i-1)>p(i-1) i-p i(i-1)=0
$$

and for $s=1, \ldots, m-1$

$$
\begin{aligned}
-p i(i-1) \beta_{s}= & (p(i-1)+1)(i-1-s+p s)-p i(i-1)-(i-1)(p-1)(-1+p s) \\
= & p(i-1) i+p(i-1)(-1+p s)-p(i-1) s+i-1-s+p s \\
& -p i(i-1)-(i-1)(p-1)(-1+p s) \\
= & (-1+p s)(p(i-1)-(i-1)(p-1))-p(i-1) s+i-1-s+p s \\
= & (-1+p s)(i-1)-p(i-1) s+i-1-s+p s \\
= & (p-1) s>0 .
\end{aligned}
$$

Therefore, (13) holds for all $i \geq m+1$. We have thus proved the following result.

Lemma 4. The function $g(t)$ has a Taylor expansion $g(t)=\sum_{i=m+1}^{\infty} d_{i} t^{i}$ with $d_{i}>0$ for $i \geq m+1$.

We are now ready to prove the following result.
Lemma 5. For each $k \geq 0$, the coefficients in the power series expansion of $x_{k} R\left(x_{k}\right)$ are all nonnegative.
Proof. We have $x_{0} R\left(x_{0}\right)=z$ and

$$
x_{k+1} R\left(x_{k+1}\right)=x_{k} g\left(R\left(x_{k}\right)\right)=x_{k} R\left(x_{k}\right) \sum_{i=m+1}^{\infty} d_{i}\left(R\left(x_{k}\right)\right)^{i-1}
$$

The result is then proved by induction, using Lemmas 2 and 4 .
For Schröder's method, we have $x_{k}(z)=\sum_{i=0}^{\infty} c_{k, i} z^{i}$ with $c_{k, 0}=1$ for all $k \geq 0$. We already know that $c_{0, i}=0$ for all $i \geq 1, c_{1, i}<0$ for $1 \leq i \leq m$ and $c_{1, i}=0$ for $i \geq m+1$. The next result determines the sign pattern of $c_{k, i}$ for $k \geq 2$ and $i \geq 1$.

Theorem 6. For each $i \geq 1, c_{k, i}$ decreases as $k$ increases and becomes equal to $b_{i}$ for all $k$ sufficiently large. In particular, $c_{k, i} \leq 0$ for $k \geq 0$ and $i \geq 1$. Moreover, $c_{k, i}<0$ for $k \geq 2$ and $i \geq 1$.

Proof. That $c_{k, i}$ decreases as $k$ increases follows directly from (7), Lemma[2, Lemma 55, and the fact that $b_{i}<0$ for $i \geq 1$. We know from Theorem 1 that $c_{k, i}=b_{i}$ when $i \leq(m+1)^{k}-1$, i.e., when $k \geq \ln (i+1) / \ln (m+1)$. Since $c_{0, i}=0$ for all $i \geq 1$, the monotonicity of $c_{k, i}$ implies that $c_{k, i} \leq 0$ for $k \geq 0$ and $i \geq 1$. To show $c_{k, i}<0$ for $k \geq 2$ and $i \geq 1$, we only need to show that $c_{2, i}<0$ for $i \geq 1$. By (7) and (12)

$$
\begin{aligned}
x_{1}-x_{2} & =x_{1} R\left(x_{1}\right) \sum_{i=1}^{m}\left(-b_{i}\right)\left(R\left(x_{1}\right)\right)^{i-1} \\
& =x_{0} g\left(R\left(x_{0}\right)\right) \sum_{i=1}^{m}\left(-b_{i}\right)\left(R\left(x_{1}\right)\right)^{i-1} \\
& =g(z) \sum_{i=1}^{m}\left(-b_{i}\right)\left(R\left(x_{1}\right)\right)^{i-1}
\end{aligned}
$$

It follows from Lemmas 2 and 4 that

$$
x_{1}-x_{2}=\sum_{i=m+1}^{\infty} e_{i} z^{i}
$$

with $e_{i} \geq d_{i}\left(-b_{1}\right)>0$ for $i \geq m+1$. Therefore, for $i \geq m+1$ we have $c_{2, i}<c_{1, i}=0$ and for $1 \leq i \leq m$ we have $c_{2, i}=c_{1, i}<0$. We have thus proved that $c_{2, i}<0$ for $i \geq 1$.

## 4. SChröder's method for the matrix $p$ Th root

In the matrix case, Schröder's method for finding $A^{1 / p}$ is given by

$$
\begin{equation*}
X_{k+1}=X_{k} T_{m}\left(R\left(X_{k}\right)\right), \quad X_{0}=I \tag{17}
\end{equation*}
$$

where

$$
R\left(X_{k}\right)=I-A X_{k}^{-p}
$$

Note that we have $X_{k} A=A X_{k}$ for Schröder's method whenever $X_{k}$ is defined. Using this commutativity and its consequences, we immediately get the following result from Theorem 1.

Theorem 7. Suppose that all eigenvalues of $A$ are in $\{z:|z-1|<1\}$ and write $A=I-B($ so $\rho(B)<1)$. Let $(I-B)^{1 / p}=\sum_{i=0}^{\infty} b_{i} B^{i}$ be the binomial expansion (where the coefficients $b_{i}$ are given by (41). Then the sequence $\left\{X_{k}\right\}$ generated by Schröder's method has the power series expansion $X_{k}=\sum_{i=0}^{\infty} c_{k, i} B^{i}$, with $c_{k, i}=b_{i}$ for $i=0,1, \ldots,(m+1)^{k}-1$.

Recall that $\sum_{i=0}^{\infty} b_{i}=0$ for the coefficients $b_{i}$. Let $s_{k}=\sum_{i=0}^{k-1} b_{i}$. Then $s_{k}=$ $\sum_{i=k}^{\infty}\left(-b_{i}\right), s_{k} \leq 1$ for all $k \geq 1$, and $\lim _{k \rightarrow \infty} s_{k}=0$.

By using Theorems 7 and 6, we can get the following nice error estimate.
Theorem 8. Suppose that all eigenvalues of $A$ are in $\{z:|z-1|<1\}$ and write $A=I-B$. Then, for any matrix norm such that $\|B\|<1$, the sequence $\left\{X_{k}\right\}$ generated by Schröder's method satisfies

$$
\left\|X_{k}-A^{1 / p}\right\| \leq\|B\|^{(m+1)^{k}}
$$

Proof. We have by Theorem 7 that

$$
X_{k}-A^{1 / p}=\sum_{i=(m+1)^{k}}^{\infty}\left(c_{k, i}-b_{i}\right) B^{i}
$$

By Theorem 6 we have $0 \leq c_{k, i}-b_{i} \leq-b_{i}(k \geq 0, i \geq 1)$. It follows that

$$
\begin{aligned}
\left\|X_{k}-A^{1 / p}\right\| & \leq \sum_{i=(m+1)^{k}}^{\infty}\left(-b_{i}\right)\|B\|^{i} \leq \sum_{i=(m+1)^{k}}^{\infty}\left(-b_{i}\right)\|B\|^{(m+1)^{k}} \\
& =s_{(m+1)^{k}}\|B\|^{(m+1)^{k}} \leq\|B\|^{(m+1)^{k}}
\end{aligned}
$$

Note that we have actually given a sharper upper bound in the proof.
We now consider the computation of $A^{1 / p}$, where $A$ is a nonsingular $M$-matrix or a real nonsingular $H$-matrix with positive diagonal entries. It is known [1, 6, 16 that $A^{1 / p}$ is a nonsingular $M$-matrix for every nonsingular $M$-matrix $A$. It has been proved in [8] that when $A$ is a real nonsingular $H$-matrix with positive diagonal entries, so is $A^{1 / p}$.

As in [8], we let $\mathcal{M}_{1}$ be the set of all nonsingular $M$-matrices whose diagonal entries are in $(0,1]$, and $\mathcal{H}_{1}$ be the set of all real nonsingular $H$-matrices whose diagonal entries are in $(0,1]$.

We assume $A=I-B$ is in $\mathcal{M}_{1}$ (so $B \geq 0$ ) or $\mathcal{H}_{1}$. We can see from the binomial expansion that $A^{1 / p} \in \mathcal{M}_{1}$ when $A \in \mathcal{M}_{1}$ and that $A^{1 / p} \in \mathcal{H}_{1}$ when $A \in \mathcal{H}_{1}$. To find $(I-B)^{1 / p}$ we generate a sequence $\left\{X_{k}\right\}$ by Schröder's method, with $X_{0}=I$.

When $A \in \mathcal{M}_{1}$, we have the following monotonic convergence result.
Theorem 9. Suppose $A \in \mathcal{M}_{1}$. Then the sequence $\left\{X_{k}\right\}$ generated by Schröder's method is monotonically decreasing and converges to $A^{1 / p}$.
Proof. When $A \in \mathcal{M}_{1}$, we can write $A=I-B$ with $B \geq 0$ and $\rho(B)<1$. So all eigenvalues of $A$ are in the open disk $\{z:|z-1|<1\}$. The convergence of $X_{k}$ to $A^{1 / p}$ is known from Theorem 8 for example. By Theorem 7 the
sequence $\left\{X_{k}\right\}$ generated by Schröder's method has the power series expansion $X_{k}=\sum_{i=0}^{\infty} c_{k, i} B^{i}=I+\sum_{i=1}^{\infty} c_{k, i} B^{i}$. It follows from Theorem 6 that $X_{k} \geq X_{k+1}$ for all $k \geq 0$.

The following structure-preserving property of Schröder's method follows readily from the above theorem.

Corollary 10. Let $A$ be in $\mathcal{M}_{1}$ and $\left\{X_{k}\right\}$ be generated by Schröder's method. Then for all $k \geq 0, X_{k}$ are in $\mathcal{M}_{1}$.

Proof. We know that $A^{1 / p}$ is in $\mathcal{M}_{1}$. For each $k \geq 0, X_{k}$ is a $Z$-matrix and $X_{k} \geq A^{1 / p}$ by Theorem 9 So $X_{k}$ is an $M$-matrix. The diagonal entries of $X_{k}$ are in $(0,1]$ since $X_{k} \leq X_{0}=I$.

We also have the following structure-preserving property.
Theorem 11. Let $A$ be in $\mathcal{H}_{1}$ and $\left\{X_{k}\right\}$ be generated by Schröder's method. Then for all $k \geq 0, X_{k}$ are in $\mathcal{H}_{1}$.

Proof. The result can be shown as in [8], using Theorem 6.

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