# A STUDY OF SCHRÖDER'S METHOD FOR THE MATRIX *p*TH ROOT USING POWER SERIES EXPANSIONS

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ABSTRACT. When A is a matrix with all eigenvalues in the disk |z-1| < 1, the principal *p*th root of A can be computed by Schröder's method, among many other methods. In this paper we present a further study of Schröder's method for the matrix *p*th root, through an examination of power series expansions of some sequences of scalar functions. Specifically, we obtain a new and informative error estimate for the matrix sequence generated by the Schröder's method, a monotonic convergence result when A is a nonsingular M-matrix, and a structure preserving result when A is a nonsingular M-matrix or a real nonsingular H-matrix with positive diagonal entries.

# 1. INTRODUCTION

For a given integer  $p \ge 2$  and a matrix  $A \in \mathbb{C}^{n \times n}$  whose eigenvalues are in the open disk  $\{z \in \mathbb{C} : |z - 1| < 1\}$ , the principal *p*th root of A exists and is denoted by  $A^{1/p}$  [10]. Various methods can be used to compute  $A^{1/p}$ ; see [2, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 20, 23, 24].

In this paper we are concerned with the Schröder family of iterations, also called Schröder's method for short, which is a special case of the dual Padé family of iterations proposed in [24].

In the scalar case of computing  $a^{1/p}$ , the dual Padé family of iterations has the form

(1) 
$$x_{k+1} = x_k \frac{Q_{\ell m} (1 - a x_k^{-p})}{P_{\ell m} (1 - a x_k^{-p})}, \quad x_0 = 1,$$

where  $P_{\ell m}(t)/Q_{\ell m}(t)$  is the  $[\ell/m]$  Padé approximant to the function  $(1-t)^{-1/p}$ , or equivalently  $Q_{\ell m}(t)/P_{\ell m}(t)$  is the  $[m/\ell]$  Padé approximant to the function  $(1-t)^{1/p}$ .

When  $\ell = m = 1$ , we get Halley's method. When  $\ell = 0$ , we get the Schröder family of iterations. Within the Schröder family, we get Newton's method when m = 1, and get Chebyshev's method when m = 2.

For a = 1 - z with |z| < 1, Each  $x_k$  from the dual Padé iteration (1) has a power series expansion

(2) 
$$x_k = \sum_{i=0}^{\infty} c_{k,i} z^i, \quad |z| < 1.$$

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It is conjectured in [24] that  $c_{k,i} < 0$  for  $i \ge 1$  (as long as the series in (2) is not reduced to a finite series). The conjecture is an extension of an earlier conjecture in [8] for Newton's method and Halley's method and a similar conjecture in [19] for Chebyshev's method.

For p = 2, the conjecture for Newton's method is shown to be true in [8], by using a result proved in [22], and a more direct proof is presented in [17] for both Newton's method and Halley's method. For any integer  $p \ge 2$ , the conjecture has been proved very recently [21] for both Newton's method and Halley's method. The conjecture for Chebyshev's method has remained open even for p = 2.

In this paper we will prove the conjecture for the whole Schröder family (which include Chebyshev's method) for all  $p \ge 2$ . However, the conjecture is not true for the whole dual Padé family. Indeed for  $\ell = 1$  and m = 0, we have  $P_{10}(t) = 1 + \frac{1}{p}t$  and  $Q_{10}(t) = 1$ . It follows from (1) that

$$x_1 = \frac{1}{1 + \frac{1}{p}(1 - (1 - z))} = \frac{1}{1 + \frac{1}{p}z} = \sum_{i=0}^{\infty} (-1)^i \frac{1}{p^i} z^i.$$

From the Schröder's method for computing  $a^{1/p}$ , we can get the corresponding Schröder's method for computing  $A^{1/p}$ . In particular, we have Chebyshev's method for computing  $A^{1/p}$ . Chebyshev's method is called Euler's method in [20] and its efficiency (when properly implemented) has been shown in that paper. This has provided us additional motivation to further study Schröder's method for the matrix *pth* root.

### 2. Preliminaries

Schröder's method for the matrix pth root will be studied through an examination of power series expansions of some sequences of scalar functions.

We start with the rising factorial notation

$$(x)_i = x(x+1)\cdots(x+i-1),$$

where x is a real number and  $i \ge 0$  is an integer. We have  $(x)_0 = 1$  by convention. We have the binomial expansion

(3) 
$$(1-t)^{1/p} = \sum_{i=0}^{\infty} b_i t^i, \quad |t| < 1.$$

where

(4) 
$$b_0 = 1, \quad b_i = \frac{\left(-\frac{1}{p}\right)_i}{i!} < 0, \quad i \ge 1.$$

By a limit argument, we can show that the equality in (3) also holds for t = 1. So we have  $\sum_{i=0}^{\infty} b_i = 0$ .

Let  $T_m(t)$  be the polynomial of degree m

$$T_m(t) = \sum_{i=0}^m b_i t^i,$$

which is the sum of the first m + 1 terms in the power series (3). It is readily seen that we have the Taylor expansion

(5) 
$$(T_m(t))^{-p} = \sum_{i=0}^{\infty} a_i t^i, \quad |t| < 1,$$

where  $a_0 = 1$  and  $a_i > 0$  for all  $i \ge 0$ .

In the scalar case of computing  $a^{1/p}$ , we let the residual be  $R(x_k) = 1 - ax_k^{-p}$ , The Schröder's iteration applied to the function  $x^p - a$  gives (see [4]) the iteration

So indeed it is a special case of (1) with  $\ell = 0$ . When m = 2, iteration (6) is the same as Chebyshev's method [3] applied to the function  $x^p - a$ , as noted in [19].

For iteration (6), we have

(7) 
$$x_k - x_{k+1} = x_k (1 - T_m(R(x_k))) = x_k R(x_k) \sum_{i=1}^m (-b_i) (R(x_k))^{i-1},$$

and

$$R(x_{k+1}) = f(R(x_k))$$

with

(8) 
$$f(t) = 1 - (T_m(t))^{-p}(1-t).$$

We have by (5) that  $f(t) = \sum_{i=0}^{\infty} c_i t^i$  with  $c_0 = 0$  and  $c_i = a_{i-1} - a_i$  for  $i \ge 1$ . It is shown in [5] that

(9) 
$$f(t) = \sum_{i=0}^{\infty} c_i t^i = \sum_{i=m+1}^{\infty} c_i t^i$$

with  $c_i > 0$  for  $i \ge m+1$ . This means that we actually have  $a_0 = a_1 = \cdots = a_m = 1$  in (5).

For a = 1 - z with |z| < 1, Schröder's method for finding  $(1 - z)^{1/p}$  is

(10) 
$$x_{k+1} = x_k T_m (1 - (1 - z) x_k^{-p})), \quad x_0 = 1.$$

To emphasize the dependence of  $x_k$  on z, we will write  $x_k(z)$  for  $x_k$ . Each  $x_k(z)$  has a power series expansion

(11) 
$$x_k(z) = \sum_{i=0}^{\infty} c_{k,i} z^i, \quad |z| < 1.$$

The following connection between Schröder's method and the binomial expansion is included in the more general Theorem 6.1 of [24]; it can also be proved in the same way as [8, Theorem 10] is proved for Newton's method and Halley's method.

**Theorem 1.** For Schröder's method,  $c_{k,i} = b_i$  for  $k \ge 0$  and  $0 \le i \le (m+1)^k - 1$ .

From the theorem, we know that  $c_{k,0} = 1$  for all  $k \ge 0$  and that  $c_{k,i} < 0$  for  $k \ge 1$ and  $1 \le i \le (m+1)^k - 1$ . To obtain some new results for Schröder's method for the matrix *p*th root, we need to prove that  $c_{k,i} \le 0$  for  $k \ge 1$  and all  $i \ge (m+1)^k$ . When k = 1, we have  $x_1(z) = T_m(1 - (1 - z)) = T_m(z)$ , so  $c_{1,i} = 0$  for  $i \ge m + 1$ . 3. SIGN PATTERN OF COEFFICIENTS  $c_{k,i}$  in power series expansions

To determine the sign pattern of  $c_{k,i}$  for Schröder's method, we will show that  $c_{k,i}$  decreases when k increases (and  $i \ge 1$  is fixed), as in [21] for Newton's method and Halley's method. In this process, we will need a useful recursion for the coefficients  $a_i$  in (5), and some good luck as well!

For the scalar case of computing  $(1-z)^{1/p}$ , where |z| < 1, the residual is  $R(x_k) = 1 - (1-z)x_k^{-p}$ . We have the following result.

**Lemma 2.** The coefficients in the power series expansion (in the variable z) of  $R(x_k)$  are all nonnegative.

*Proof.* We have  $R(x_0) = z$ . The result is proved by induction since

$$R(x_{k+1}) = f(R(x_k)) = \sum_{i=m+1}^{\infty} c_i (R(x_k))^i$$

with  $c_i > 0$  for  $i \ge m + 1$ .

We will show that the coefficients in the power series expansion of  $x_k R(x_k)$  are also all nonnegative.

Note that

(12) 
$$x_{k+1}R(x_{k+1}) = x_k g(R(x_k)),$$

where

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$$g(t) = T_m(t)f(t).$$

It follows from (9) that g(t) has a Taylor expansion  $g(t) = \sum_{i=m+1}^{\infty} d_i t^i$ , where

$$d_i = b_0 c_i + b_1 c_{i-1} + \dots + b_m c_{i-m}$$

for each  $i \ge m + 1$ . We are going to prove that  $d_i > 0$  for all  $i \ge m + 1$ . Since  $b_0 = 1$  and  $c_k = a_{k-1} - a_k$  for  $k \ge 1$ , we need to show for  $i \ge m + 1$  that

$$(13) - a_i + (b_0 - b_1)a_{i-1} + (b_1 - b_2)a_{i-2} + \dots + (b_{m-1} - b_m)a_{i-m} + b_m a_{i-m-1} > 0.$$

The following recursion about the coefficients  $a_i$  in (5) will play an important role. It is equation (2.11) in [5], with some notation changes.

**Lemma 3.** For each  $k \ge m - 1$ ,

$$a_{k+1} = \frac{1}{k+1} \sum_{s=0}^{m-1} (k-s+p(s+1))(-b_{s+1})a_{k-s}.$$

By Lemma 3 with k = i - 1, (13) becomes

(14) 
$$\sum_{s=0}^{m-1} \alpha_s a_{i-1-s} + b_m a_{i-m-1} > 0,$$

where for s = 0, 1, ..., m - 1

$$\alpha_s = b_s - b_{s+1} + \frac{1}{i}(i - 1 - s + p(s+1))b_{s+1}$$
  
=  $b_s + \frac{1}{i}(p-1)(s+1)b_{s+1}$   
=  $b_s + \frac{1}{i}(p-1)(s+1)b_s \frac{-\frac{1}{p} + s}{s+1}$   
=  $b_s \left(1 + \frac{1}{i}(p-1)\left(-\frac{1}{p} + s\right)\right).$ 

Thus (14) is equivalent to

(15) 
$$\frac{p(i-1)+1}{pi}a_{i-1} + \sum_{s=1}^{m-1} b_s \left(1 + \frac{1}{i}(p-1)\left(-\frac{1}{p}+s\right)\right) a_{i-1-s} + b_m a_{i-m-1} > 0.$$

By Lemma 3 with k = i - 2, we have

$$a_{i-1} = \frac{1}{i-1} \sum_{s=0}^{m-1} (i-2-s+p(s+1))(-b_{s+1})a_{i-2-s}$$
$$= \frac{1}{i-1} \sum_{s=1}^{m} (i-1-s+p_s)(-b_s)a_{i-1-s}.$$

Thus (15) becomes

(16) 
$$\sum_{s=1}^{m-1} \beta_s b_s a_{i-1-s} + \beta_m b_m a_{i-m-1} > 0,$$

where

$$\beta_m = \frac{p(i-1)+1}{pi} \frac{-1}{i-1}(i-1-m+pm) + 1,$$

and for s = 1, ..., m - 1

$$\beta_s = \frac{p(i-1)+1}{pi} \frac{-1}{i-1}(i-1-s+ps) + 1 + \frac{1}{i}(p-1)\left(-\frac{1}{p}+s\right).$$

Since  $a_i > 0$  for  $i \ge 0$  and  $b_i < 0$  for  $i \ge 1$ , a sufficient condition for (16) to hold is that, for s = 1, ..., m,  $\beta_s < 0$ , or equivalently  $-pi(i-1)\beta_s > 0$ . Luckily, the sufficient condition does hold. Indeed,

$$-pi(i-1)\beta_m = (p(i-1)+1)(i-1-m+pm) - pi(i-1) > p(i-1)i - pi(i-1) = 0,$$

and for 
$$s = 1, ..., m - 1$$

$$\begin{aligned} -pi(i-1)\beta_s &= (p(i-1)+1)(i-1-s+ps) - pi(i-1) - (i-1)(p-1)(-1+ps) \\ &= p(i-1)i + p(i-1)(-1+ps) - p(i-1)s + i - 1 - s + ps \\ -pi(i-1) - (i-1)(p-1)(-1+ps) \\ &= (-1+ps)(p(i-1) - (i-1)(p-1)) - p(i-1)s + i - 1 - s + ps \\ &= (-1+ps)(i-1) - p(i-1)s + i - 1 - s + ps \\ &= (p-1)s > 0. \end{aligned}$$

Therefore, (13) holds for all  $i \ge m + 1$ . We have thus proved the following result.

**Lemma 4.** The function g(t) has a Taylor expansion  $g(t) = \sum_{i=m+1}^{\infty} d_i t^i$  with  $d_i > 0$  for  $i \ge m+1$ .

We are now ready to prove the following result.

**Lemma 5.** For each  $k \ge 0$ , the coefficients in the power series expansion of  $x_k R(x_k)$  are all nonnegative.

*Proof.* We have  $x_0 R(x_0) = z$  and

$$x_{k+1}R(x_{k+1}) = x_k g(R(x_k)) = x_k R(x_k) \sum_{i=m+1}^{\infty} d_i (R(x_k))^{i-1}$$

The result is then proved by induction, using Lemmas 2 and 4.

For Schröder's method, we have  $x_k(z) = \sum_{i=0}^{\infty} c_{k,i} z^i$  with  $c_{k,0} = 1$  for all  $k \ge 0$ . We already know that  $c_{0,i} = 0$  for all  $i \ge 1$ ,  $c_{1,i} < 0$  for  $1 \le i \le m$  and  $c_{1,i} = 0$  for  $i \ge m + 1$ . The next result determines the sign pattern of  $c_{k,i}$  for  $k \ge 2$  and  $i \ge 1$ .

**Theorem 6.** For each  $i \ge 1$ ,  $c_{k,i}$  decreases as k increases and becomes equal to  $b_i$  for all k sufficiently large. In particular,  $c_{k,i} \le 0$  for  $k \ge 0$  and  $i \ge 1$ . Moreover,  $c_{k,i} < 0$  for  $k \ge 2$  and  $i \ge 1$ .

*Proof.* That  $c_{k,i}$  decreases as k increases follows directly from (7), Lemma 2, Lemma 5, and the fact that  $b_i < 0$  for  $i \ge 1$ . We know from Theorem 1 that  $c_{k,i} = b_i$  when  $i \le (m+1)^k - 1$ , i.e., when  $k \ge \ln(i+1)/\ln(m+1)$ . Since  $c_{0,i} = 0$  for all  $i \ge 1$ , the monotonicity of  $c_{k,i}$  implies that  $c_{k,i} \le 0$  for  $k \ge 0$  and  $i \ge 1$ . To show  $c_{k,i} < 0$  for  $k \ge 2$  and  $i \ge 1$ , we only need to show that  $c_{2,i} < 0$  for  $i \ge 1$ . By (7) and (12)

$$x_1 - x_2 = x_1 R(x_1) \sum_{i=1}^m (-b_i) (R(x_1))^{i-1}$$
  
=  $x_0 g(R(x_0)) \sum_{i=1}^m (-b_i) (R(x_1))^{i-1}$   
=  $g(z) \sum_{i=1}^m (-b_i) (R(x_1))^{i-1}.$ 

It follows from Lemmas 2 and 4 that

$$x_1 - x_2 = \sum_{i=m+1}^{\infty} e_i z^i$$

with  $e_i \ge d_i(-b_1) > 0$  for  $i \ge m+1$ . Therefore, for  $i \ge m+1$  we have  $c_{2,i} < c_{1,i} = 0$ and for  $1 \le i \le m$  we have  $c_{2,i} = c_{1,i} < 0$ . We have thus proved that  $c_{2,i} < 0$  for  $i \ge 1$ .

# 4. Schröder's method for the matrix pth root

In the matrix case, Schröder's method for finding  $A^{1/p}$  is given by

(17) 
$$X_{k+1} = X_k T_m(R(X_k)), \quad X_0 = I$$

where

$$R(X_k) = I - AX_k^{-p}.$$

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Note that we have  $X_k A = A X_k$  for Schröder's method whenever  $X_k$  is defined. Using this commutativity and its consequences, we immediately get the following result from Theorem 1.

**Theorem 7.** Suppose that all eigenvalues of A are in  $\{z : |z - 1| < 1\}$  and write A = I - B (so  $\rho(B) < 1$ ). Let  $(I - B)^{1/p} = \sum_{i=0}^{\infty} b_i B^i$  be the binomial expansion (where the coefficients  $b_i$  are given by (4)). Then the sequence  $\{X_k\}$  generated by Schröder's method has the power series expansion  $X_k = \sum_{i=0}^{\infty} c_{k,i} B^i$ , with  $c_{k,i} = b_i$  for  $i = 0, 1, \ldots, (m + 1)^k - 1$ .

Recall that  $\sum_{i=0}^{\infty} b_i = 0$  for the coefficients  $b_i$ . Let  $s_k = \sum_{i=0}^{k-1} b_i$ . Then  $s_k = \sum_{i=k}^{\infty} (-b_i)$ ,  $s_k \leq 1$  for all  $k \geq 1$ , and  $\lim_{k \to \infty} s_k = 0$ .

By using Theorems 7 and 6, we can get the following nice error estimate.

**Theorem 8.** Suppose that all eigenvalues of A are in  $\{z : |z - 1| < 1\}$  and write A = I - B. Then, for any matrix norm such that ||B|| < 1, the sequence  $\{X_k\}$  generated by Schröder's method satisfies

$$||X_k - A^{1/p}|| \le ||B||^{(m+1)^k}.$$

*Proof.* We have by Theorem 7 that

$$X_k - A^{1/p} = \sum_{i=(m+1)^k}^{\infty} (c_{k,i} - b_i) B^i.$$

By Theorem 6, we have  $0 \le c_{k,i} - b_i \le -b_i$   $(k \ge 0, i \ge 1)$ . It follows that

$$\begin{aligned} \|X_k - A^{1/p}\| &\leq \sum_{i=(m+1)^k}^{\infty} (-b_i) \|B\|^i \leq \sum_{i=(m+1)^k}^{\infty} (-b_i) \|B\|^{(m+1)^k} \\ &= s_{(m+1)^k} \|B\|^{(m+1)^k} \leq \|B\|^{(m+1)^k}. \end{aligned}$$

Note that we have actually given a sharper upper bound in the proof.

We now consider the computation of  $A^{1/p}$ , where A is a nonsingular *M*-matrix or a real nonsingular *H*-matrix with positive diagonal entries. It is known [1, 6, 16] that  $A^{1/p}$  is a nonsingular *M*-matrix for every nonsingular *M*-matrix *A*. It has been proved in [8] that when A is a real nonsingular *H*-matrix with positive diagonal entries, so is  $A^{1/p}$ .

As in [8], we let  $\mathcal{M}_1$  be the set of all nonsingular *M*-matrices whose diagonal entries are in (0, 1], and  $\mathcal{H}_1$  be the set of all real nonsingular *H*-matrices whose diagonal entries are in (0, 1].

We assume A = I - B is in  $\mathcal{M}_1$  (so  $B \ge 0$ ) or  $\mathcal{H}_1$ . We can see from the binomial expansion that  $A^{1/p} \in \mathcal{M}_1$  when  $A \in \mathcal{M}_1$  and that  $A^{1/p} \in \mathcal{H}_1$  when  $A \in \mathcal{H}_1$ . To find  $(I - B)^{1/p}$  we generate a sequence  $\{X_k\}$  by Schröder's method, with  $X_0 = I$ . When  $A \in \mathcal{M}_1$ , we have the following monotonic convergence result.

**Theorem 9.** Suppose  $A \in \mathcal{M}_1$ . Then the sequence  $\{X_k\}$  generated by Schröder's method is monotonically decreasing and converges to  $A^{1/p}$ .

*Proof.* When  $A \in \mathcal{M}_1$ , we can write A = I - B with  $B \ge 0$  and  $\rho(B) < 1$ . So all eigenvalues of A are in the open disk  $\{z : |z - 1| < 1\}$ . The convergence of  $X_k$  to  $A^{1/p}$  is known from Theorem 8 for example. By Theorem 7, the sequence  $\{X_k\}$  generated by Schröder's method has the power series expansion  $X_k = \sum_{i=0}^{\infty} c_{k,i} B^i = I + \sum_{i=1}^{\infty} c_{k,i} B^i$ . It follows from Theorem 6 that  $X_k \ge X_{k+1}$  for all  $k \ge 0$ .

The following structure-preserving property of Schröder's method follows readily from the above theorem.

**Corollary 10.** Let A be in  $\mathcal{M}_1$  and  $\{X_k\}$  be generated by Schröder's method. Then for all  $k \geq 0$ ,  $X_k$  are in  $\mathcal{M}_1$ .

*Proof.* We know that  $A^{1/p}$  is in  $\mathcal{M}_1$ . For each  $k \geq 0$ ,  $X_k$  is a Z-matrix and  $X_k \geq A^{1/p}$  by Theorem 9. So  $X_k$  is an *M*-matrix. The diagonal entries of  $X_k$  are in (0,1] since  $X_k \leq X_0 = I$ .

We also have the following structure-preserving property.

**Theorem 11.** Let A be in  $\mathcal{H}_1$  and  $\{X_k\}$  be generated by Schröder's method. Then for all  $k \geq 0$ ,  $X_k$  are in  $\mathcal{H}_1$ .

*Proof.* The result can be shown as in [8], using Theorem 6.

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