Orthogonal sequences constructed from quasi-orthogonal ultraspherical polynomials

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Abstract

Let $\{x_{k,n-1}\}_{k=1}^{n-1}$ and $\{x_{k,n}\}_{k=1}^{n}$, $n \in \mathbb{N}$, be two sets of real, distinct points satisfying the interlacing property $x_{i,n} < x_{i,n-1} < x_{i+1,n}$, $i = 1, 2, \ldots, n-1$. In [10], Wendroff proved that if $p_{n-1}(x) = \prod_{k=1}^{n-1}(x - x_{k,n-1})$ and $p_n(x) = \prod_{k=1}^{n}(x - x_{k,n})$, then p_{n-1} and p_n can be embedded in a non-unique monic orthogonal sequence $\{p_n\}_{n=0}^{\infty}$. We investigate a question raised by Mourad Ismail at OPSFA 2015 as to the nature and properties of orthogonal sequences generated by applying Wendroff's Theorem to the interlacing zeros of $C_{n-1}^{\lambda}(x)$ and $(x^2-1)C_{n-2}^{\lambda}(x)$, where $\{C_k^{\lambda}(x)\}_{k=0}^{\infty}$ is a sequence of monic ultraspherical polynomials and $-3/2 < \lambda < -1/2$, $\lambda \neq -1$. We construct an algorithm for generating infinite monic orthogonal sequences $\{D_k^{\lambda}(x)\}_{k=0}^{\infty}$ from the two polynomials $D_n^{\lambda}(x) := (x^2-1)C_{n-2}^{\lambda}(x)$ and $D_{n-1}^{\lambda}(x) := C_{n-1}^{\lambda}(x)$, which is applicable for each pair of fixed parameters n, λ in the ranges $n \in \mathbb{N}, n \geq 5$ and $\lambda > -3/2$, $\lambda \neq -1, 0, (2k-1)/2, k = 0, 1, \ldots$. We plot and compare the zeros of $D_m^{\lambda}(x)$ and $C_m^{\lambda}(x)$ for several choices of $m \in \mathbb{N}$ and a range of values of the parameters λ and n. For $-3/2 < \lambda < -1$, the curves that the zeros of $D_m^{\lambda}(x)$ and $C_m^{\lambda}(x)$ approach are substantially different for large values of m. When $-1 < \lambda < -1/2$, the two curves have a similar shape while the curves are almost identical for $\lambda > -1/2$.

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1 Introduction

The monic ultraspherical polynomial $C_n^{\lambda}(x)$ is defined by the three term recurrence relation [8, eqn.(8.18)]

$$C_n^{\lambda}(x) = xC_{n-1}^{\lambda}(x) - b_n^{\lambda}C_{n-2}^{\lambda}(x), \;\; \lambda \neq 0, -1, \ldots; \; n = 1, 2, \ldots \quad , \tag{1}$$

where

$$C_{-1}^{\lambda}(x) \equiv 0, \;\; C_0^{\lambda}(x) = 1, \;\; b_n^{\lambda} = \frac{(n-1)(n-2+2\lambda)}{4(n-2+\lambda)(n-1+\lambda)}, \;\; \lambda \neq 0, -1, \ldots; \; n = 1, 2, \ldots \tag{2}$$

For each $\lambda > -\frac{1}{2}$, the sequence $\{C_n^{\lambda}(x)\}_{n=0}^{\infty}$ is orthogonal on $(-1, 1)$ with respect to the weight function $(1-x^2)^{\lambda-\frac{1}{2}}$ and for each $n \in \mathbb{N}$, $n \geq 1$, the zeros of $C_n^{\lambda}(x)$ are real, simple, symmetric, lie in $(-1, 1)$ and the zeros of $C_{n-1}^{\lambda}(x)$ interlace with the zeros of $C_n^{\lambda}(x)$, $n \geq 2$, (see [9, Theorem 3.3.2]) namely,

$$-1 < x_{1,n} < x_{1,n-1} < \cdots < x_{n-1,n} < x_{n-1,n-1} < x_{n,n} < 1. \tag{3}$$

where $\{x_{i,n}\}_{i=1}^{n}$ are the zeros of $C_n^{\lambda}(x)$ in increasing order.

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As λ decreases below $-1/2$, two (symmetric) zeros of $C_n^\lambda(x)$ leave the interval $(-1,1)$ through the endpoints -1 and 1 (see [5, p. 296]) and remain real with absolute value > 1 for each $n \in \mathbb{N}, n \geq 3$, and $-\frac{3}{2} < \lambda < -\frac{1}{2}, \lambda \neq -1$. For $-\frac{3}{2} < \lambda < -\frac{1}{2}, \lambda \neq -1$, the sequence $\{C_n^\lambda(x)\}_{n=0}^\infty$ is quasi-orthogonal of order 2 with respect to the weight function $(1-x^2)^{\lambda+\frac{1}{2}}$, (see [2, Theorem 6] and [3, p.144]) and, for any $n \in \mathbb{N}, n \geq 4$, (see [6, Theorem 3.1]),

$$x_{1,n-1} < x_{1,n} < -1 < x_{2,n} < x_{2,n-1} < \cdots < x_{n-2,n-1} < x_{n-1,n} < 1 < x_{n,n} < x_{n-1,n-1}. \tag{4}$$

It follows from (4) that the zeros of $C_{n-1}^\lambda(x)$ and $C_n^\lambda(x)$ are not interlacing for any $n \in \mathbb{N}, n \geq 4$ and $-\frac{3}{2} < \lambda < -\frac{1}{2}, \lambda \neq -1$, but we see from (3) and (4) that the zeros of $C_n^\lambda(x)$ interlace with the zeros of $(x^2-1)C_{n-1}^\lambda(x)$ for each $n \in \mathbb{N}, n \geq 4$ and each λ with $\lambda > -\frac{3}{2}, \lambda \neq -1$.

In 1961, Wendroff [10, p. 554] proved that, for any fixed positive integer n, $n \geq 2$, if $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^{n-1}$ are two sets of real, distinct points satisfying the interlacing property $x_1 < y_1 < x_2 < y_2 < \cdots < x_{n-1} < y_{n-1} < x_n$, there exist infinitely many sequences $\{p_k(x)\}_{k=0}^\infty$ of monic orthogonal polynomials with $p_n(x) = \prod_{k=1}^{n}(x-x_k)$ and $p_{n-1}(x) = \prod_{k=1}^{n-1}(x-y_k)$. His proof is constructive and for a given, fixed $n \in \mathbb{N}$, $n \geq 2$, each polynomial $p_k(x)$ of degree $k \leq n-2$ is uniquely determined by $p_n(x)$ and $p_{n-1}(x)$. In contrast, the monic polynomials of degree $n+1, n+2, \ldots$ in any orthogonal sequence that includes $p_n(x)$ and $p_{n-1}(x)$ are only constrained by the requirement that any infinite sequence of (monic) orthogonal polynomials satisfies a three term recurrence relation of the form

$$p_{n+k}(x) = (x-a_k)p_{n+k-1}(x) - b_k p_{n+k-2}(x),\ a_k \in \mathbb{R}, b_k > 0,\ k = 1, 2, \ldots. \tag{5}$$

Since there are infinitely many choices of the coefficients a_k and b_k with $a_k \in \mathbb{R}$ and $b_k > 0$ for $k = 1, 2, \ldots$, there are infinitely many distinct monic orthogonal sequences $\{p_k(x)\}_{k=0}^\infty$ that include $p_n(x)$ and $p_{n-1}(x)$.

Here, we fix $n \in \mathbb{N}, n \geq 5$, fix λ, $\lambda > -3/2$, $\lambda \neq -1, 0, (2k-1)/2, k = 0, 1, \ldots$, and define

$$D_{n-1}^\lambda(x) := C_{n-1}^\lambda(x), \quad D_n^\lambda(x) := (x^2-1)C_{n-2}^\lambda(x). \tag{6}$$

We investigate the properties of the zeros of polynomials in monic orthogonal sequences $\{D_m^\lambda\}_{m=0}^\infty$ generated by the Wendroff process. It is important to emphasize the dependence on the "starting value" of $n \in \mathbb{N}$ when generating each monic orthogonal sequence $\{D_m^\lambda\}_{m=0}^\infty$ that includes $D_{n-1}^\lambda(x)$ and $D_n^\lambda(x)$. If $n \in \mathbb{N}$, $n \geq 5$ is large, the number of polynomials (namely, $n-2$) that are uniquely determined in every monic orthogonal sequence that includes $D_{n-1}^\lambda(x)$ and $D_n^\lambda(x)$ is correspondingly large whereas, for example, if $n = 5$ we have two degrees of freedom when generating each of the monic polynomials of degree > 5 and exactly 4 of the polynomials of lower degree are completely determined. The restriction $n \geq 5$ arises from the fact that when $-\frac{3}{2} < \lambda < -1$, the quadratic ultraspherical polynomial $C_2^\lambda(x)$ has two pure imaginary zeros (see [3, p. 144]). If we restrict λ to the range $\lambda > -1$, the results proved here apply for $n \geq 3$.

2 Notation

For each $m \in \mathbb{N}$, denote

$$C_m^\lambda(x) = x^m + \sum_{j=1}^m \alpha_{j,m}\, x^{m-j} = \prod_{j=1}^m (x - x_{j,m}), \quad c_m = \sum_{j=1}^m x_{j,m}. \tag{7}$$

$$D_m^\lambda(x) = x^m + \sum_{j=1}^m \beta_{j,m}\, x^{m-j} = \prod_{j=1}^m (x - y_{j,m}), \quad d_m = \sum_{j=1}^m y_{j,m}. \tag{8}$$

The zeros $\{x_{j,m}\}_{j=1}^m$ of $C_m^\lambda(x)$ are distinct, real and symmetric with respect to the origin for $m \geq 3$ so that $c_m = 0$ for all $m \in \mathbb{N}, m \geq 3$ while the zeros $\{y_{j,m}\}_{j=1}^m$ of $D_m^\lambda(x)$ are distinct, real and symmetric when $m = n$ or $m = n-1$ so that $d_n = d_{n-1} = 0$. Note that

$$y_{1,n} = x_{1,n-2}, \quad y_{2,n} = -1,\ y_{3,n} = x_{2,n-2}, \ldots, y_{n-1,n} = 1, \quad y_{n,n} = x_{n-2,n-2} \tag{9}$$

and

$$y_{k,n-1} = x_{k,n-1}, \quad k = 1, 2, \ldots . n-1. \tag{10}$$

3 Orthogonal sequences generated by $C^{\lambda}_{n-1}(x)$ and $(x^2-1)C^{\lambda}_{n-2}(x)$, $n \geq 5$

Our main result is the following theorem.

Theorem 3.1 *Let $\{C^{\lambda}_m(x)\}_{m=0}^{\infty}$ be the sequence of monic ultraspherical polynomials defined by (1) and (2). Fix $n \in \mathbb{N}$, $n \geq 5$, fix $\lambda \in (-\frac{3}{2}, -\frac{1}{2})$, $\lambda \neq -1$, and suppose that $\epsilon > 0$ is abitrary.*
Define

$$a_n := x_{n-2,n-2} + \epsilon \tag{11}$$

where $x_{n-2,n-2} > 1$ is the largest zero of $C^{\lambda}_{n-2}(x)$.

Let the sequence of monic polynomials $\{D^{\lambda}_m(x)\}_{m=0}^{\infty}$ be defined by:

$$D^{\lambda}_n(x) = (x^2-1)C^{\lambda}_{n-2}(x), \quad D^{\lambda}_{n-1}(x) = C^{\lambda}_{n-1}(x), \tag{12}$$

$$D^{\lambda}_{n-j}(x) = -\frac{1}{\ell_{n-j+2}}\left[D^{\lambda}_{n-j+2}(x) - xD^{\lambda}_{n-j+1}(x)\right], \; j = 2, 3, \ldots, n \tag{13}$$

$$D^{\lambda}_{n+j}(x) = xD^{\lambda}_{n+j-1}(x) - \ell_{n+j}D^{\lambda}_{n+j-2}(x), \; j = 1, 2, \ldots \tag{14}$$

where

$$\ell_{n-j} = \beta_{2,n-j-1} - \beta_{2,n-j} > 0, \; j = 0, 1, \ldots, n-2, \tag{15}$$

$$\ell_{n+j} \in \left(0, \frac{a_n\, D^{\lambda}_{n+j-1}(a_n)}{D^{\lambda}_{n+j-2}(a_n)}\right), \; j = 1, 2, \ldots, \tag{16}$$

and $\beta_{2,m}$ is the coefficient of x^{m-2} in $D^{\lambda}_m(x)$, see (8). Then the sequence $\{D^{\lambda}_m(x)\}_{m=0}^{\infty}$ is symmetric and orthogonal with respect to a positive measure supported on the interval $(-a_n, a_n)$.

Proof of Theorem 3.1

Fix $n \in \mathbb{N}, n \geq 5$. The monic polynomial $D^{\lambda}_{n-2}(x)$ is uniquely determined by

$$D^{\lambda}_n(x) - xD^{\lambda}_{n-1}(x) = -\ell_n D^{\lambda}_{n-2}(x) \tag{17}$$

where $D^{\lambda}_n(x)$ and $D^{\lambda}_{n-1}(x)$ are defined by (12) and the coefficient ℓ_n is chosen so that $D^{\lambda}_{n-2}(x)$ is monic. The positivity of ℓ_n follows from the interlacing property of the zeros of $D^{\lambda}_n(x)$ and $D^{\lambda}_{n-1}(x)$. In the same way, for each $j = 3, 4, \ldots, n$, the polynomial $D^{\lambda}_{n-j}(x)$ is constructed from the polynomials $D^{\lambda}_{n-j+1}(x)$ and $D^{\lambda}_{n-j+2}(x.)$ The process is repeated until we obtain $D^{\lambda}_0(x) = 1$. The polynomials $D^{\lambda}_{n+j}(x)$, $j = 1, 2, \ldots$ are constructed recursively using the three-term recurrence relation

$$D_{n+j}(x) = xD^{\lambda}_{n+j-1}(x) - \ell_{n+j}D_{n+j-2}(x), \;\; j = 1, 2, \ldots, \tag{18}$$

choosing positive coefficients ℓ_{n+j} for $j = 1, 2, \ldots$. This ensures (Favard's Theorem) that the infinite sequence $\{D^{\lambda}_m(x)\}_{m=0}^{\infty}$ is orthogonal with respect to a positive measure. Wendroff mentions in his proof [10, p. 554] that the coefficients ℓ_{n+j} can be chosen in such a way that all zeros of $D^{\lambda}_{n+j}(x)$ lie in the interval $(-a_n, a_n)$ for each $j \geq 1$ but does not indicate how to choose the coefficients to achieve this outcome. Here, we prove that the choice of ℓ_{n+1} given in (16) ensures that all the zeros of $D^{\lambda}_{n+1}(x)$ lie in the interval $(-a_n, a_n)$. Applying the same argument iteratively, it can be shown that the choice of ℓ_{n+j} given in (16) ensures that the zeros of $D^{\lambda}_{n+j}(x)$ lie in the interval $(-a_n, a_n)$ for each $j \in \mathbb{N}$.

Suppose $\ell_{n+1} \in \left(0, \frac{a_n\, D^{\lambda}_n(a_n)}{D^{\lambda}_{n-1}(a_n)}\right)$. We show that the zeros of D^{λ}_{n+1} lie in the interval $(-a_n, a_n)$. From (4) with n replaced by $n-1$, it follows immediately that the zeros of $D^{\lambda}_n(x) = (x^2-1)C^{\lambda}_{n-2}(x)$ and $D^{\lambda}_{n-1}(x) = C^{\lambda}_{n-1}(x)$

lie in the open interval $(-a_n, a_n)$. In addition, $D_n^\lambda(x)$ and $D_{n-1}^\lambda(x)$ are monic polynomials with no zeros greater than $x_{n-2,n-2}$ so $D_n^\lambda(a_n) > 0$ and $D_{n-1}^\lambda(a_n) > 0$. Since $a_n > 0$, it follows that $\frac{a_n D_n^\lambda(a_n)}{D_{n-1}^\lambda(a_n)} > 0$. By construction, the zeros of D_{n+1}^λ and D_n^λ interlace and the zeros of D_n^λ lie in the interval $(-a_n, a_n)$ so it follows that $y_{1,n+1} < y_{1,n} < y_{2,n+1} < y_{2,n} < \cdots < y_{n,n+1} < y_{n,n} < y_{n+1,n+1}$ and $-a_n < y_{1,n} < y_{n,n} < a_n$. We need to show that the largest zero $y_{n+1,n+1}$ of D_{n+1}^λ satisfies $y_{n+1,n+1} < a_n$.

From (14) with $j = 1$, we have $D_{n+1}^\lambda(x) = xD_n^\lambda(x) - \ell_{n+1}D_{n-1}^\lambda(x)$ so that

$$D_{n+1}^\lambda(y_{n,n}) = -\ell_{n+1}D_{n-1}^\lambda(y_{n,n}) < 0. \tag{19}$$

On the other hand, since we are assuming that $\ell_{n+1} \in \left(0, \frac{a_n D_n^\lambda(a_n)}{D_{n-1}^\lambda(a_n)}\right)$, we have

$$D_{n+1}^\lambda(a_n) = a_nD_n^\lambda(a_n) - \ell_{n+1}D_{n-1}^\lambda(a_n) \; > \; a_nD_n^\lambda(a_n) - \frac{a_nD_n^\lambda(a_n)}{D_{n-1}^\lambda(a_n)}D_{n-1}^\lambda(a_n) = 0. \tag{20}$$

Therefore, $D_{n+1}^\lambda(y_{n,n}) < 0$ and $D_{n+1}^\lambda(a_n) > 0$ so that $y_{n+1,n+1} < a_n$ as required.

■

Remark 3.1. The sequences of polynomials $\{D_m^\lambda(x)\}_{m=0}^\infty$ defined in Theorem 3.1 with $D_{n-1}^\lambda(x) = C_{n-1}^\lambda(x)$ and $D_n^\lambda(x) = (x^2-1)C_{n-2}^\lambda(x)$ are (by construction) orthogonal for each $n \geq 5$ and $\lambda > -\frac{1}{2}$ satisfying $\lambda \neq 0, \lambda \neq \frac{2k-1}{2}$, $k = 1, 2 \ldots$. It is therefore natural to compare orthogonal sequences $\{D_m^\lambda(x)\}_{m=0}^\infty$ generated by the Wendroff process with the sequences $\{C_m^\lambda(x)\}_{m=0}^\infty$ of ultraspherical polynomials orthogonal on $(-1, 1)$. In this case, we can choose $a_n = 1$ so the interval $(-1, 1)$ contains the zeros of all the polynomials in the sequence $\{D_m^\lambda(x)\}_{m=0}^\infty$ as well as the sequence $\{C_m^\lambda(x)\}_{m=0}^\infty$.

Remark 3.2. For $-3/2 < \lambda < -1/2$, $\lambda \neq -1$, and $n \geq 5$ fixed, the largest (real) zero $x_{n-2,n-2}$ of $C_{n-2}^\lambda(x)$ is bounded above by $\left(\frac{n-3}{2\lambda+n-2}\right)^{1/2}$, see [7, (4)]. An alternative upper bound for $x_{n-2,n-2}$ is given by $1 - \frac{2\lambda+1}{(n-2)(n+2\lambda-2)}$, see [7, (15)]. These bounds give estimates for the interval of orthogonality $(-a_n, a_n)$ of the sequences $\{D_m^\lambda(x)\}_{m=0}^\infty$, where $D_{n-1}^\lambda(x) = C_{n-1}^\lambda(x)$ and $D_n^\lambda(x) = (x^2 - 1)C_{n-2}^\lambda(x)$.

Remark 3.3. For $-3/2 < \lambda < -1/2$, $\lambda \neq -1$, and $n \geq 5$ fixed, we can choose $\epsilon > 0$ in Theorem 3.1 in such a way that the interval $(-a_n, a_n)$ contains all the zeros of the polynomials $\{C_m^\lambda(x)\}_{m=3}^\infty$. To this end, we use the estimates $x_{m,m} < \left(\frac{m-1}{2\lambda+m}\right)^{1/2}$ and $x_{m,m} < 1 - \frac{2\lambda+1}{m(m+2\lambda)}$, $m \geq 3$, see [7], where $x_{m,m}$ is the largest zero of C_m^λ. Because $\max\{\left(\frac{m-1}{2\lambda+m}\right)^{1/2} : m \geq 3, \lambda > -3/2\} = \left(\frac{2}{2\lambda+3}\right)^{1/2}$ and $\max\{1 - \frac{2\lambda+1}{m(m+2\lambda)} : m \geq 3, \lambda > -3/2\} = \frac{4(2+\lambda)}{3(3+2\lambda)}$, we may choose $a_n := a$ independent of $n \in \mathbb{N}$, namely

$$a = A_1(\lambda) := \left(\frac{2}{2\lambda + 3}\right)^{1/2} \tag{21}$$

or

$$a = A_2(\lambda) := \frac{4(2 + \lambda)}{3(3 + 2\lambda)}. \tag{22}$$

To choose the sharper of the two bounds $A_1(\lambda)$ and $A_2(\lambda)$, we use the following comparisons: $A_1(-5/4) = A_2(-5/4)$, $A_1(\lambda) < A_2(\lambda)$ if $\lambda \in (-3/2, -5/4)$ and $A_1(\lambda) > A_2(\lambda)$ if $\lambda \in (-5/4, -1/2)$. Note that $A_1(-1/2) = A_2(-1/2)$ and $\lim\limits_{\lambda\to(-3/2)+} A_k(\lambda) = +\infty$ for $k = 1, 2$.

Remark 3.4. When developing an algorithm for generating orthogonal sequences, we can choose $\ell_{n+j} = \frac{aD_{n+j-1}^\lambda(a)}{\sigma D_{n+j-2}^\lambda(a)}$, $j = 1, 2, \ldots$, where $\sigma > 1$. Using this expression for ℓ_{n+j} and putting $x = a$ into (14), we obtain $\ell_{n+j+1} = \frac{(\sigma-1)}{\sigma^2}a^2$ for all $j = 1, 2, \ldots$. The advantage of choosing $\ell_{n+j} = \frac{aD_{n+j-1}^\lambda(a)}{\sigma D_{n+j-2}^\lambda(a)}$, $j = 1, 2, \ldots$, with $\sigma > 1$, is that all coefficients ℓ_{n+j+1}, $j = 1, 2, \ldots$ are equal, namely, $\ell_{n+j+1} = \frac{(\sigma-1)}{\sigma^2}a^2$ for $j = 1, 2, \ldots$. This results in a significant reduction in the computational complexity of the algorithm.

4 Algorithm for construction of $\{D_m^\lambda(x)\}_{m=0}^{n+k}$, n fixed, $n \geq 5$, $k \in \mathbb{N}$

Using Theorem 3.1 and Remarks 1,3,4, we present an algorithm for construction of the first $n+k+1$ terms of orthogonal sequences $\{D_m^\lambda(x)\}_{m=0}^\infty$.

1. Choose integers $n \geq 5$ and $k \geq 1$.
2. Choose $\lambda \in (-\frac{3}{2}, +\infty)$ with $\lambda \neq -1, 0, (2k-1)/2$, $k = 0, 1, 2, \ldots$.
3. If $-3/2 < \lambda < -5/4$, define $a = A_1(\lambda) = \left(\frac{2}{2\lambda+3}\right)^{1/2}$. If $-5/4 \leq \lambda < -1/2$, define $a = A_2(\lambda) = \frac{4(2+\lambda)}{3(3+2\lambda)}$. If $\lambda > -\frac{1}{2}$, define $a = 1$.
4. Choose $\sigma > 1$.
5. Let $D_n^\lambda(x) = (x^2-1)C_{n-2}^\lambda(x)$, $\quad D_{n-1}^\lambda(x) = C_{n-1}^\lambda(x)$.
6. For $j = 0, 1$, let $\beta_{2,n-j}$ be the coefficient of x^{n-j-2} in $D_{n-j}^\lambda(x)$.
7. For $j = 2, 3, \ldots, n$,
   let $\ell_{n-j+2} = \beta_{2,n-j+1} - \beta_{2,n-j+2}$;
   let $D_{n-j}^\lambda(x) = -\frac{1}{\ell_{n-j+2}}\left[D_{n-j+2}^\lambda(x) - xD_{n-j+1}^\lambda(x)\right]$;
   let $\beta_{2,n-j}$ be the coefficient of x^{n-j-2} in $D_{n-j}^\lambda(x)$.
8. Let $\ell_{n+1} = \frac{a\,D_n^\lambda(a)}{\sigma D_{n-1}^\lambda(a)}$ and $D_{n+1}^\lambda(x) = xD_n^\lambda(x) - \ell_{n+1}D_{n-1}^\lambda(x)$.
9. For $j = 2, \ldots, k$, let $D_{n+j}^\lambda(x) = xD_{n+j-1}^\lambda(x) - \frac{(\sigma-1)}{\sigma^2}a^2 D_{n+j-2}^\lambda(x)$.

The above algorithm generates the first $n+k+1$ terms of a sequence of symmetric polynomials $\{D_m^\lambda(x)\}_{m=0}^\infty$ orthogonal with respect to some positive measure supported on the interval $(-a, a)$, which contains all the zeros of the symmetric polynomials $\{C_m^\lambda(x)\}_{m=0}^\infty$. If $\lambda \in (-3/2, -1/2)$ and $\lambda \neq -1$, the sequence $\{C_m^\lambda(x)\}_{m=0}^\infty$ is quasi-orthogonal of order 2 on $(-1, 1)$ with respect to the weight function $(1-x^2)^{\lambda+1/2}$; if $\lambda \in (-1/2, +\infty)$ and $\lambda \neq 0$, $\lambda \neq \frac{2k-1}{2}$ for $k = 1, 2, \ldots$, the sequence $\{C_m^\lambda(x)\}_{m=0}^\infty$ is orthogonal with respect to the weight function $(1-x^2)^{\lambda-1/2}$ on the interval $(-1, 1)$.

Example 4.1. In this example we present the first 11 terms of the sequence $\{D_m^\lambda\}_{m=0}^\infty$ using our algorithm with $n = 5$, $k = 5$, $\sigma = 2$, and $a = \frac{4(2+\lambda)}{3(3+2\lambda)}$, where $\lambda \in (-\frac{3}{2}, +\infty)$, $\lambda \neq -1, 0$ and $\lambda \neq (2k-1)/2$, $k = 0, 1, 2, \ldots$:

$$D_0^\lambda(x) = 1,$$
$$D_1^\lambda(x) = x,$$
$$D_2^\lambda(x) = x^2 - \frac{2\lambda^2+7\lambda+9}{2(2\lambda^3+7\lambda^2+9\lambda+6)},$$
$$D_3^\lambda(x) = x^3 - \frac{3(2\lambda+5)}{2(2\lambda^2+7\lambda+9)}x,$$
$$D_4^\lambda(x) = x^4 - \frac{3}{\lambda+3}x^2 + \frac{3}{4(\lambda^2+5\lambda+6)},$$
$$D_5^\lambda(x) = x^5 - \frac{(2\lambda+7)}{2\lambda+4}x^3 + \frac{3}{2\lambda+4}x,$$
$$D_6^\lambda(x) = x^6$$
$$+\frac{-26624\lambda^8 - 315136\lambda^7 - 1452096\lambda^6 - 3030464\lambda^5 - 1350544\lambda^4 + 6634848\lambda^3 + 14325052\lambda^2 + 11993936\lambda + 3814971)}{18(\lambda+2)(2\lambda+3)^2(512\lambda^5 + 2944\lambda^4 + 5208\lambda^3 + 4\lambda^2 - 8638\lambda - 6429)}x^4$$
$$-\frac{(-8192\lambda^7 - 55552\lambda^6 - 93408\lambda^5 + 238480\lambda^4 + 1249616\lambda^3 + 2167224\lambda^2 + 1773274\lambda + 577325)}{6(\lambda+2)(2\lambda+3)^2(512\lambda^5 + 2944\lambda^4 + 5208\lambda^3 + 4\lambda^2 - 8638\lambda - 6429)}x^2$$
$$+\frac{4(2\lambda+1)^2(80\lambda^3 + 426\lambda^2 + 753\lambda + 442)}{3(2\lambda+3)^2(512\lambda^5 + 2944\lambda^4 + 5208\lambda^3 + 4\lambda^2 - 8638\lambda - 6429)},$$
$$D_7^\lambda(x) = x^7$$

$$+\frac{(-10240\lambda^8-121088\lambda^7-561408\lambda^6-1198624\lambda^5-656672\lambda^4+2255736\lambda^3+5154212\lambda^2+4387984\lambda+1408809)}{6(\lambda+2)(2\lambda+3)^2(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x^5$$
$$+\frac{4096\lambda^8+78848\lambda^7+458688\lambda^6+1074016\lambda^5+295376\lambda^4-3734688\lambda^3-8130836\lambda^2-7213054\lambda-2452023}{18(\lambda+2)(2\lambda+3)^2(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x^3$$
$$-\frac{2(512\lambda^6+3328\lambda^5+7048\lambda^4+828\lambda^3-19042\lambda^2-28747\lambda-13742)}{3(2\lambda+3)^2(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x,$$
$$D_8^\lambda(x)=x^8$$
$$+\frac{-34816\lambda^8-411392\lambda^7-1916352\lambda^6-4161280\lambda^5-2589488\lambda^4+6899568\lambda^3+16600220\lambda^2+14333968\lambda+4637883}{18(\lambda+2)(2\lambda+3)^2(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x^6$$
$$+\Big[\frac{253952\lambda^{10}+4967424\lambda^9+36636672\lambda^8+134987136\lambda^7+240904128\lambda^6+28003872\lambda^5-813980016\lambda^4}{162(\lambda+2)(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}$$
$$-\frac{1823644104\lambda^3+1962244068\lambda^2+1102018370\lambda+259653399}{162(\lambda+2)(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}\Big]x^4$$
$$-\frac{8(6656\lambda^8+61760\lambda^7+214784\lambda^6+252224\lambda^5-437944\lambda^4-1922704\lambda^3-2812378\lambda^2-1984129\lambda-566938)}{27(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x^2$$
$$-\frac{16(\lambda+2)^3(2\lambda+1)^2(80\lambda^2+266\lambda+221)}{27(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)},$$
$$D_9^\lambda(x)=x^9$$
$$+\frac{-38912\lambda^8-459520\lambda^7-2148480\lambda^6-4726688\lambda^5-3208960\lambda^4+7031928\lambda^3+17737804\lambda^2+15503984\lambda+5049339}{18(\lambda+2)(2\lambda+3)^2(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x^7$$
$$+\Big[\frac{376832\lambda^{10}+6912000\lambda^9+49677312\lambda^8+182130432\lambda^7+333265728\lambda^6+89989248\lambda^5-952585632\lambda^4}{162(\lambda+2)(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}$$
$$-\frac{2231977416\lambda^3+2437175184\lambda^2+1380264434\lambda+327276231}{162(\lambda+2)(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}\Big]x^5$$
$$-\frac{2(4096\lambda^9+166912\lambda^8+1357504\lambda^7+4568800\lambda^6+5470096\lambda^5-8399264\lambda^4-38672660\lambda^3-57223262\lambda^2-40687679\lambda-11707302)}{81(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x^3$$
$$+\frac{8(\lambda+2)^3(512\lambda^5+1664\lambda^4-328\lambda^3-8108\lambda^2-13238\lambda-7313)}{27(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x,$$
$$D_{10}^\lambda(x)=x^{10}$$
$$+\frac{(-14336\lambda^8-169216\lambda^7-793536\lambda^6-1764032\lambda^5-1276144\lambda^4+2388096\lambda^3+6291796\lambda^2+5558000\lambda+1820265)}{6(\lambda+2)(2\lambda+3)^2(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x^8$$
$$+\Big[\frac{57344\lambda^10+1012736\lambda^9+7164672\lambda^8+26224384\lambda^7+48985088\lambda^6+18933696\lambda^5-120883088\lambda^4}{18(\lambda+2)(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}$$
$$-\frac{296145544\lambda^3+327852636\lambda^2+187090450\lambda+44609151}{18(\lambda+2)(2\lambda+3)^4(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}\Big]x^6$$
$$-\Big[\frac{4(\lambda+2)(200704\lambda^10+5561856\lambda^9+45776256\lambda^8+174881280\lambda^7+305832480\lambda^6-27524064\lambda^5)}{729(2\lambda+3)^6(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}$$
$$-\frac{1252067256\lambda^4+2680180104\lambda^3+2833083030\lambda^2+1572495406\lambda+366899565}{729(2\lambda+3)^6(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}\Big]x^4$$
$$+\frac{8(\lambda+2)^3(45056\lambda^7+308992\lambda^6+680928\lambda^5-126736\lambda^4-3262160\lambda^3-6273816\lambda^2-5263402\lambda-1726229)}{243(2\lambda+3)^6(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}x^2$$
$$+\frac{64(\lambda+2)^5(2\lambda+1)^2(80\lambda^2+266\lambda+221)}{243(2\lambda+3)^6(512\lambda^5+2944\lambda^4+5208\lambda^3+4\lambda^2-8638\lambda-6429)}.$$

5 The zeros of $D_m^\lambda(x)$ and $C_m^\lambda(x)$

In this section, we plot and compare the zeros of $D_m^\lambda(x)$ constructed using the algorithm in Section 4 with the zeros of $C_m^\lambda(x)$, where $m=3,4,\dots,n+k$ and $n\geq 5$, $k\geq 1$ are fixed integers.

Example 5.1. Let $n=5$ and $\sigma=2$. Choose $k=5$, $\lambda=-5/4$ and $a=\frac{4(2+\lambda)}{3(3+2\lambda)}=2$. The polynomials $D_m^\lambda, 0\leq m\leq 10$, are listed below with the approximate values of their zeros in curly brackets $\{\}$:

$$D_0^\lambda(x)=1,\{\}$$
$$D_1^\lambda(x)=x,\{0\},$$

$$D_2^\lambda(x) = x^2 - \frac{18}{19}, \{-0.973329, 0.973329\},$$
$$D_3^\lambda(x) = x^3 - \frac{10}{9}x, \{-1.05409, 0, 1.05409\},$$
$$D_4^\lambda(x) = x^4 - \frac{12}{7}x^2 + \frac{4}{7}, \{-1.12303, -0.673114, 0.673114, 1.12303\},$$
$$D_5^\lambda(x) = x^5 - 3x^3 + 2x, \{-1.41421, -1, 0, 1, 1.41421\},$$
$$D_6^\lambda(x) = x^6 - \frac{72}{17}x^4 + \frac{70}{17}x^2 - \frac{12}{17}, \{-1.7026, -1.05773, -0.466529, 0.466529, 1.05773, 1.7026\},$$
$$D_7^\lambda(x) = x^7 - \frac{89}{17}x^5 + \frac{121}{17}x^3 - \frac{46}{17}x, \{-1.83123, -1.10502, -0.812906, 0, 0.812906, 1.10502, 1.83123\},$$
$$D_8^\lambda(x) = x^8 - \frac{106}{17}x^6 + \frac{193}{17}x^4 - \frac{116}{17}x^2 + \frac{12}{17}, \{-1.89282, -1.23417, -1, -0.359651, 0.359651, 1, 1.23417, 1.89282\},$$
$$D_9^\lambda(x) = x^9 - \frac{123}{17}x^7 + \frac{282}{17}x^5 - \frac{237}{17}x^3 + \frac{58}{17}x,$$
$$\{-1.92625, -1.41421, -1.05407, -0.643268, 0, 0.643268, 1.05407, 1.41421, 1.92625\},$$
$$D_{10}^\lambda(x) = x^{10} - \frac{140}{17}x^8 + \frac{388}{17}x^6 - \frac{430}{17}x^4 + \frac{174}{17}x^2 - \frac{12}{17},$$
$$\{-1.94625, -1.55305, -1.09439, -0.867151, -0.292897, 0.292897, 0.867151, 1.09439, 1.55305, 1.94625\}.$$

Note that the zeros of $D_5^{-5/4}(x)$ are $-\sqrt{2}; -1, 0, 1, \sqrt{2}$. The largest and smallest zeros of $D_{10}^{-5/4}(x)$ are close to the limits -2 and 2.

In Figures 1 through 4, the y-coordinates of the plotted points are the zeros of $D_m^{-5/4}(x)$ (diamond, brown) and $C_m^{-5/4}(x)$ (round, blue) for $m = 3, 4, 5, 10$. The figures suggest that the greatest difference between the zeros of $D_m^{-5/4}(x)$ and $C_m^{-5/4}(x)$ are at the extreme zeros.

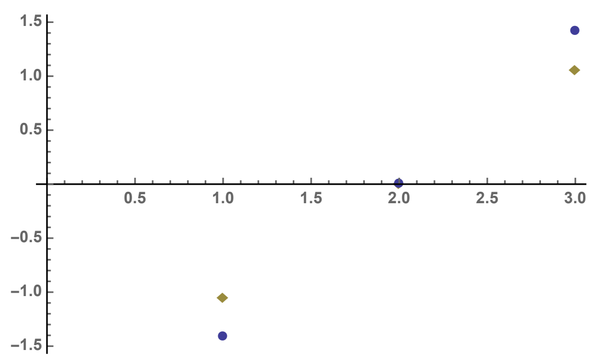

Figure 1: $n = 5, \lambda = -5/4, m = 3$. The polynomials $C_3^{-5/4}$ and $D_3^{-5/4}$ have a common zero at the origin.

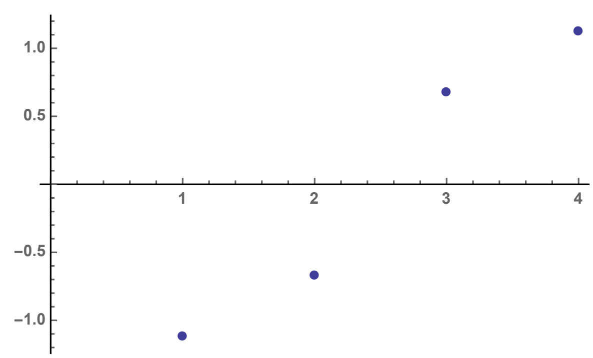

Figure 2: $n = 5, \lambda = -5/4, m = 4$. Since $D_4^{-5/4}(x) = C_4^{-5/4}(x)$, their zeros are equal.

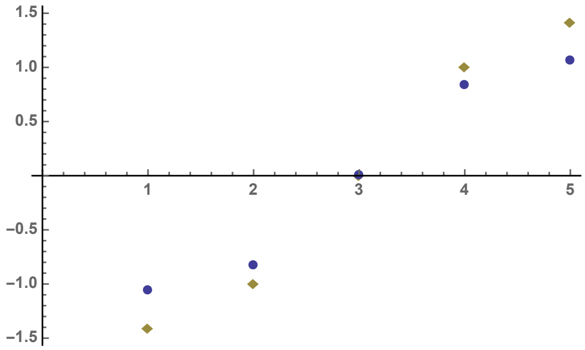

Figure 3: $n = 5, \lambda = -5/4, m = 5$. By construction, the zeros of $D_5^{-5/4}(x)$ are the zeros of $C_3^{-5/4}(x)$ together with the points -1 and 1.

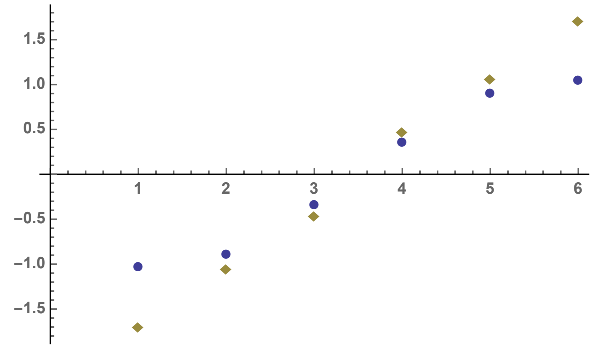

Figure 4: $n = 5, \lambda = -5/4, m = 10$.

Example 5.1 provides numerical confirmation that the relative ordering of the zeros of $D_{n+1}^{-5/4}, D_n^{-5/4}$, and $D_{n-1}^{-5/4}$, is consistent with [1, Theorem 4]. Replacing n by $n-1$ and putting $b_n = 0$ in (7) and (8) in [1, Theorem 4], the negative zeros of $D_m^{-5/4}$, $m \in \{n-1, n, n+1\}$, should satisfy

$$y_{1,n+1} < y_{1,n} < y_{1,n-1} < y_{2,n+1} < y_{2,n} < y_{2,n-1} \dots$$

while the positive zeros of $D_m^{-5/4}$, $m \in \{n-1, n, n+1\}$, should satisfy

$$y_{n+1,n+1} > y_{n,n} > y_{n-1,n-1} > y_{n,n+1} > y_{n-1,n} \dots$$

From Example 5.1, we see that the zeros of $D_4^{-5/4}$, $D_5^{-5/4}$, and $D_6^{-5/4}$ satisfy

$$y_{1,6} < y_{1,5} < y_{1,4} < y_{2,6} < y_{2,5} < y_{2,4} < y_{3,6} < y_{3,5} = 0$$

and

$$y_{6,6} > y_{5,5} > y_{4,4} > y_{5,6} > y_{4,5} > y_{3,4} > y_{4,6} > y_{3,5} = 0.$$

as expected.

In the examples and figures that follow, we plot the zeros of $D_m^\lambda(x)$ and $C_m^\lambda(x)$ for selected values of n (the "starting value"), m and λ.

Example 5.2. Let $n = 5$, $k = 5$, and $\sigma = 2$, as in the previous example. In Figures 5 and 6, the y-coordinates of the plotted points are the zeros of D_{10}^λ (diamond, brown) and C_{10}^λ (round, blue) respectively for $\lambda = -3/4$, $a = \frac{4(2+\lambda)}{3(3+2\lambda)} = \frac{10}{9}$ and $\lambda = -1/4$, $a = \frac{4(2+\lambda)}{3(3+2\lambda)} = \frac{14}{15}$.

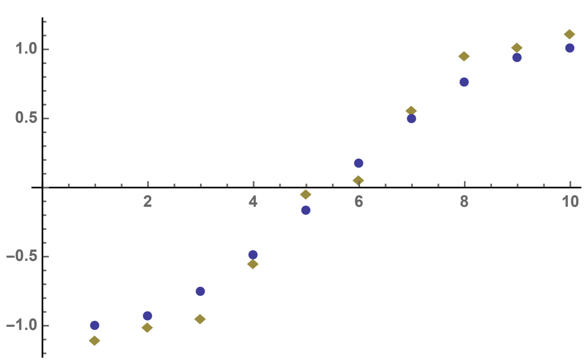

Figure 5: $n = 5, m = 10, \lambda = -3/4$.

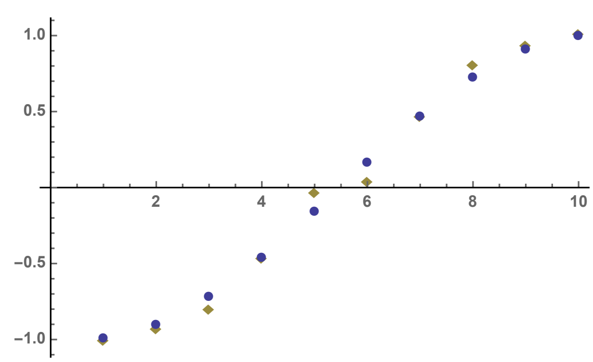

Figure 6: $n = 5, m = 10, \lambda = -1/4$.

Example 5.3. Let $n = 10$, $k = 58$, $\sigma = 2$. Choose $\lambda = -5/4$ and $a = \frac{4(2+\lambda)}{3(3+2\lambda)} = 2$. In Figures 7 through 14 the y-coordinates of the plotted points are the zeros of D_m^λ (diamond, brown) and C_m^λ (round, blue) for selected integer values of m between 3 and 68. The figures suggest that, as m increases, the curves that fit the zeros of $D_m^{-5/4}$ and $C_m^{-5/4}$ are significantly different.

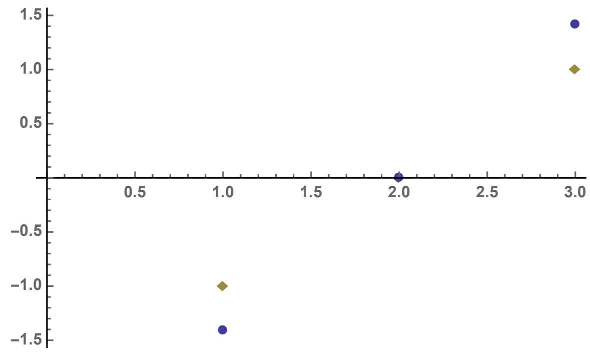

Figure 7: $n = 10, \lambda = -5/4, m = 3$.

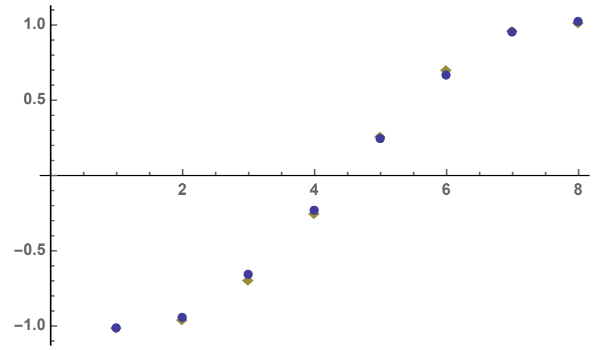

Figure 8: $n = 10, \lambda = -5/4, m = 8$.

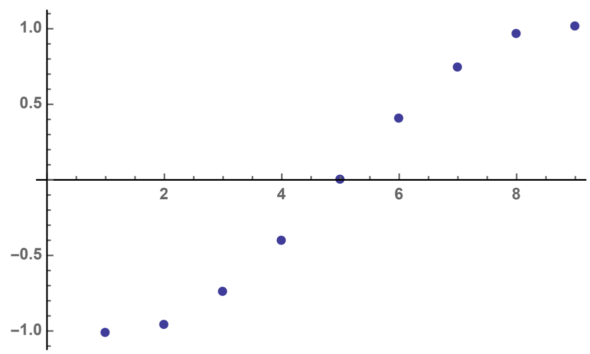

Figure 9: $n = 10, \lambda = -5/4, m = 9$.

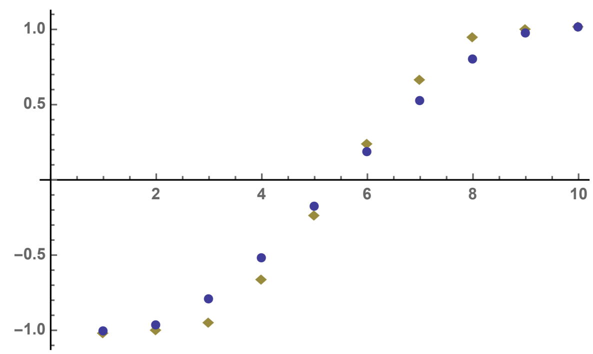

Figure 10: $n = 10, \lambda = -5/4, m = 10$.

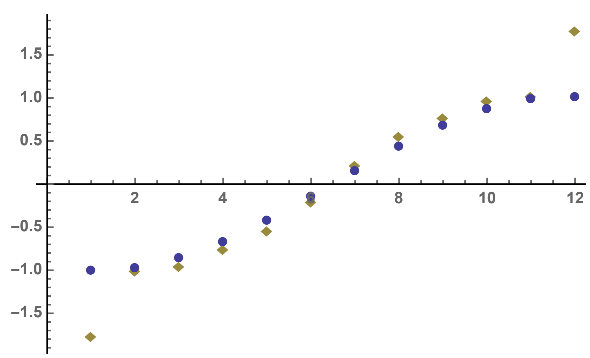

Figure 11: $n = 10, \lambda = -5/4, m = 12$.

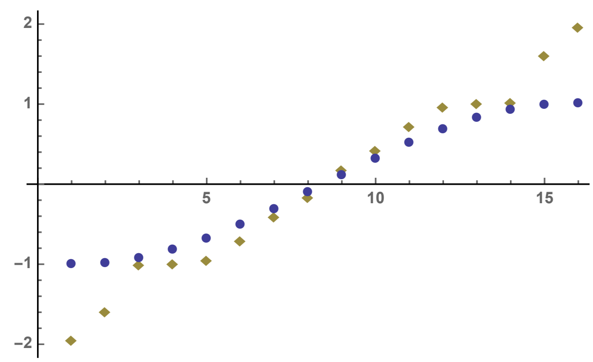

Figure 12: $n = 10, \lambda = -5/4, m = 16$.

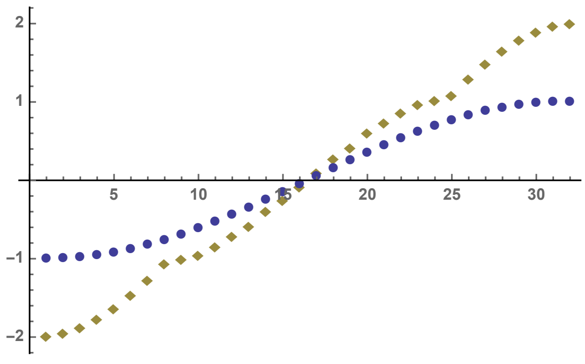

Figure 13: $n = 10, \lambda = -5/4, m = 32$.

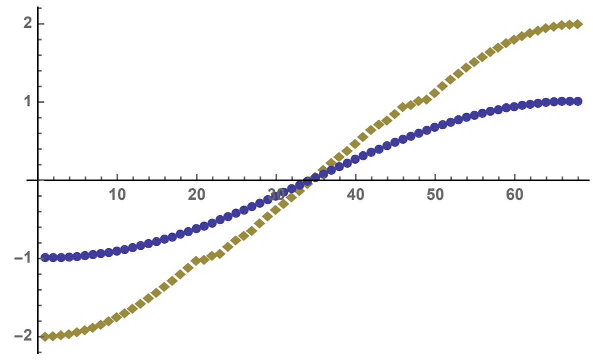

Figure 14: $n = 10, \lambda = -5/4, m = 68$.

Example 5.4. Let $n = 10$, $k = 58$, $\sigma = 2$. Choose $\lambda = -3/4$ and $a = \frac{4(2+\lambda)}{3(3+2\lambda)} = \frac{10}{9}$. In Figures 15 through 20 the y-coordinates of the plotted points are the zeros of D_m^λ and C_m^λ for selected integer values of m between 8 and 68.

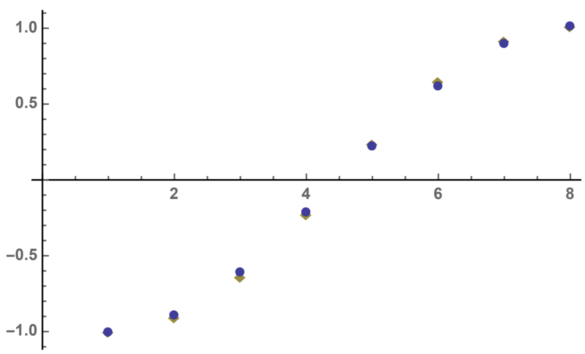

Figure 15: $n = 10, \lambda = -3/4, m = 8$.

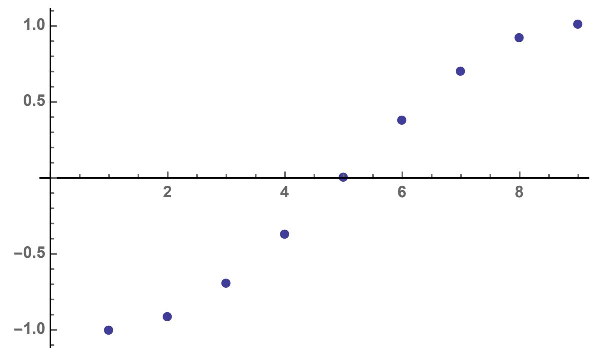

Figure 16: $n = 10, \lambda = -3/4, m = 9$. Since $C_9^{-3/4}(x) = D_9^{-3/4}(x)$, their zeros are equal.

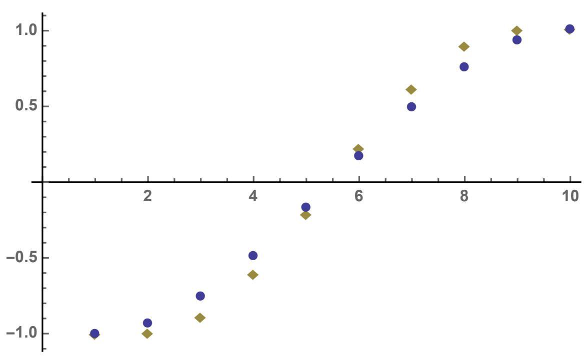

Figure 17: $n = 10, \lambda = -3/4, m = 10$.

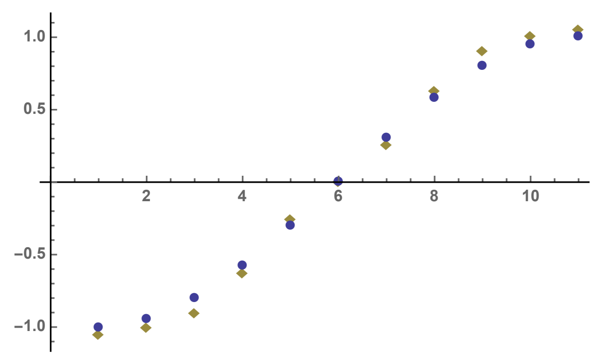

Figure 18: $n = 10, \lambda = -3/4, m = 11$.

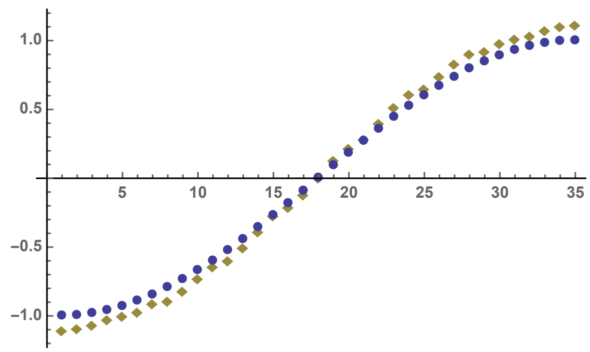

Figure 19: $n = 10, \lambda = -3/4, m = 35$.

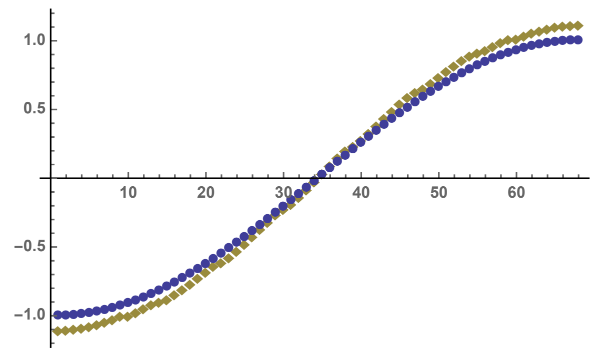

Figure 20: $n = 10, \lambda = -3/4, m = 68$.

Example 5.5. Let $n = 10$, $k = 58$, $\sigma = 2$. Choose $\lambda = -1/4$ and $a = \frac{4(2+\lambda)}{3(3+2\lambda)} = \frac{14}{15}$. In Figures 21 through 26 the y-coordinates of the plotted points are the zeros of D_m^λ and C_m^λ for several integer values of m between 8 and 68.

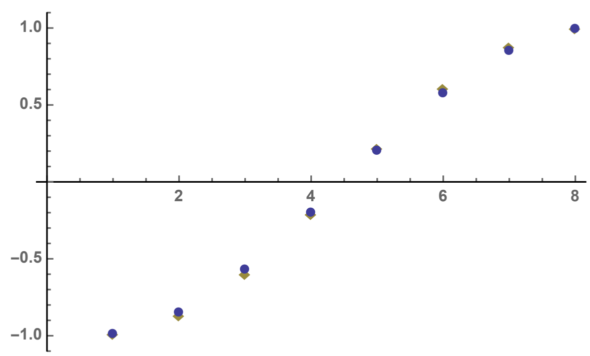

Figure 21: $n = 10, \lambda = -1/4, m = 8$.

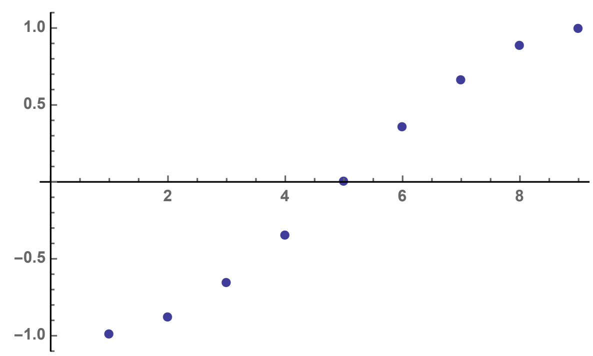

Figure 22: $n = 10, \lambda = -1/4, m = 9$.

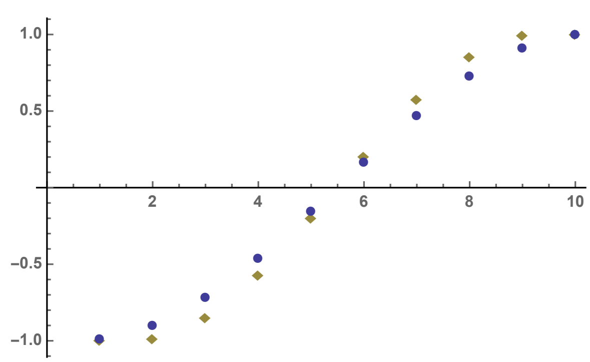

Figure 23: $n = 10, \lambda = -1/4, m = 10$.

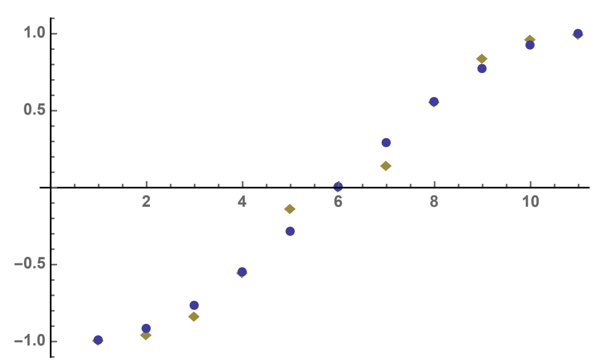

Figure 24: $n = 10, \lambda = -1/4, m = 11$.

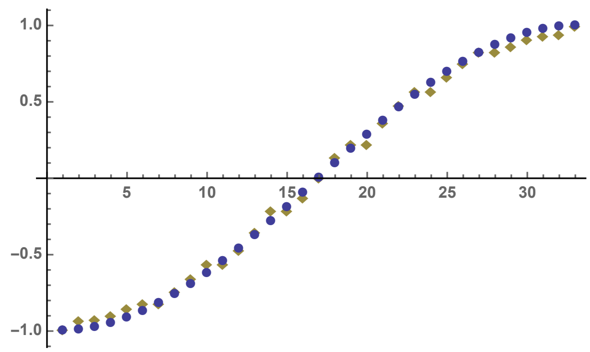

Figure 25: $n = 10, \lambda = -1/4, m = 34$.

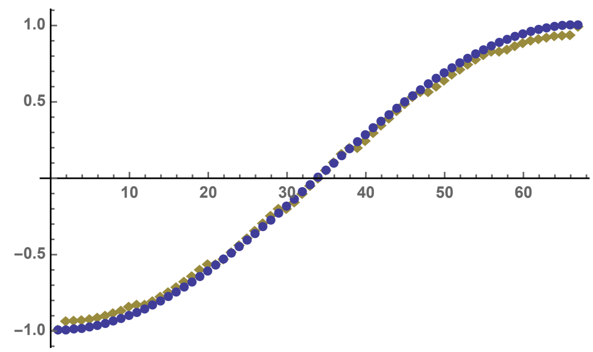

Figure 26: $n = 10, \lambda = -1/4, m = 67$.

Remark 5.2. As m increases, the zeros of $D_m^\lambda(x)$ and $C_m^\lambda(x)$ appear to be asymptotically equal for $\lambda = -1/4$. This is not unexpected since $\lambda = -1/4$ lies in the orthogonal range $\lambda > -1/2$ for ultraspherical polynomials. Note that the interval of orthogonality is $(-a, a)$, where $a < 1$.

Example 5.6. Let $n = 5$, $k = 18$, $\sigma = 2$. In Figures 27 through 34, the y-coordinates of the plotted points are the zeros of D_m^λ and C_m^λ for a selection of values of $\lambda \in (-\frac{3}{2}, +\infty)$ with $\lambda \neq -1, 0, (2k-1)/2$, $k = 0, 1, 2, \ldots$ where $m = 23$ is fixed. We choose $a = \frac{4(2+\lambda)}{3(3+2\lambda)}$ if $\lambda < -1/2$ and $a = 1$ if $\lambda > -1/2$; the zeros of D_m^λ and C_m^λ are contained in $(-a, a)$.
For $-3/2 < \lambda < -1$, the curves to which the zeros of $D_m^\lambda(x)$ and $C_m^\lambda(x)$ can be fitted are substantially different for some values of m. As λ approaches $-1/2$ from the left, the two curves are very similar, and, as $\lambda >$ increases further, the curves are almost identical.

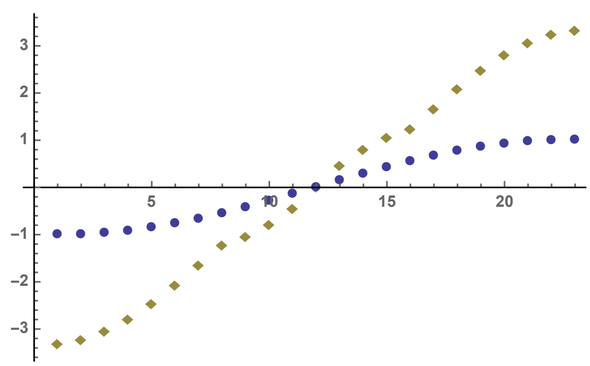

Figure 27: $n = 5, m = 23, \lambda = -11/8, a = 10/3$.

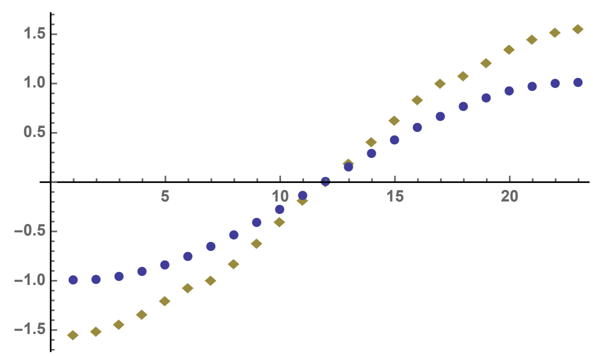

Figure 28: $n = 5, m = 23, \lambda = -9/8, a = 14/9$.

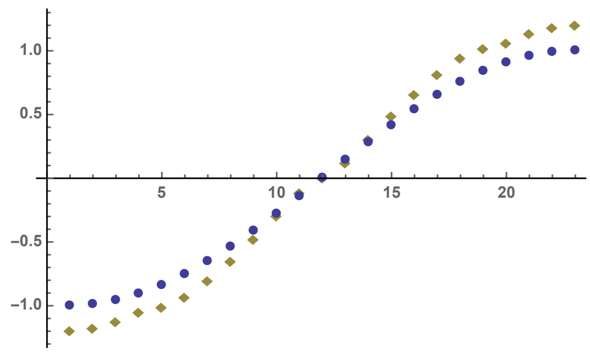

Figure 29: $n = 5, m = 23, \lambda = -7/8, a = 6/5$.

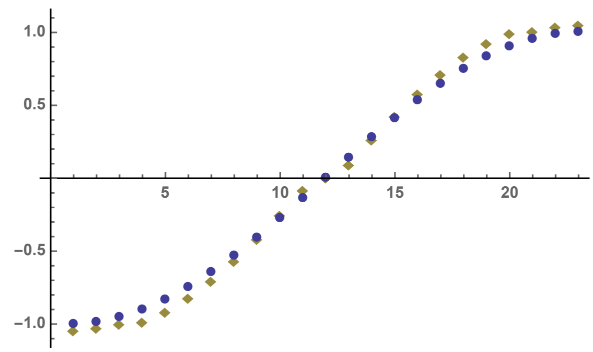

Figure 30: $n = 5, m = 23, \lambda = -5/8, a = 22/21$.

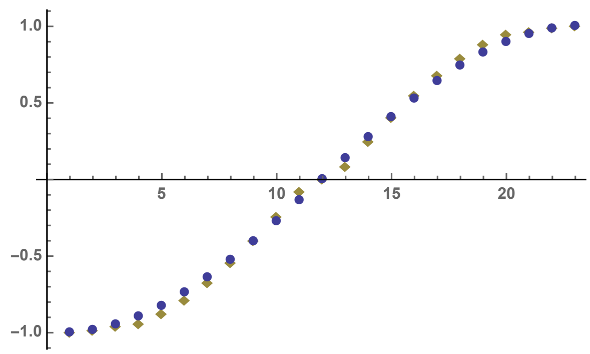

Figure 31: $n = 5, m = 23, \lambda = -3/8, a = 1$.

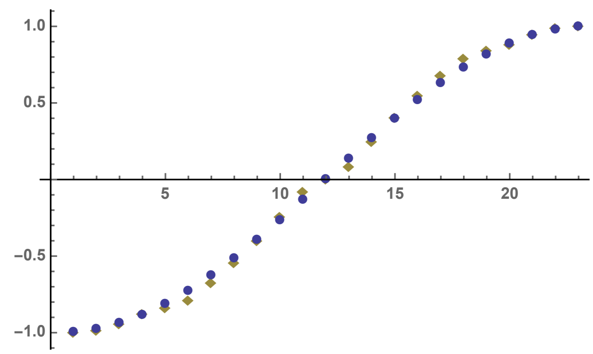

Figure 32: $n = 5, m = 23, \lambda = 1/8, a = 1$.

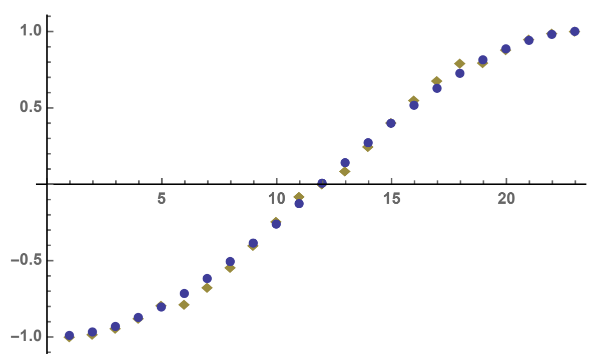

Figure 33: $n = 5, m = 23, \lambda = 3/8, a = 1$.

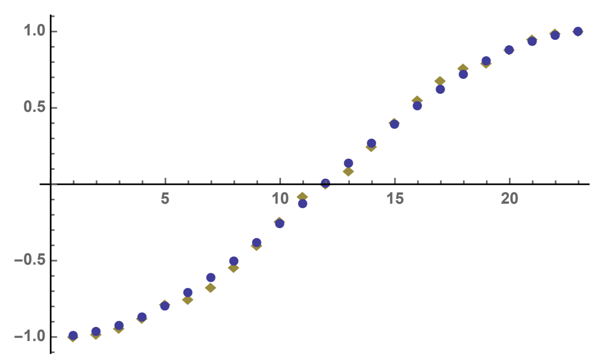

Figure 34: $n = 5, m = 23, \lambda = 5/8, a = 1$.

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7 Bibliography

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