# Exponential moment bounds and strong convergence rates for tamed-truncated numerical approximations of stochastic convolutions

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## Abstract

In this article we establish exponential moment bounds, moment bounds in fractional order smoothness spaces, a uniform Hölder continuity in time, and strong convergence rates for a class of fully discrete exponential Euler-type numerical approximations of infinite dimensional stochastic convolution processes. The considered approximations involve specific taming and truncation terms and are therefore well suited to be used in the context of SPDEs with non-globally Lipschitz continuous nonlinearities.

# Contents

# 1 Introduction

Stochastic partial differential equations (SPDEs) of evolutionary type are important modeling tools in economics and the natural sciences (see, e.g., Harms et al. [12, Theorem 3.5], Filipović et al. [10, Equation (1.2)], Blömker & Romito [5, Equation (1)], Hairer [11, Equation (3)], Mourrat & Weber [21, Equation (1.1)], Birnir [3, Equation (7)], and Birnir [4, Equation (1.5)]). However, exact solutions to SPDEs are usually not known explicitly. Therefore, it has been and still is a very active research area to develop and analyze numerical approximation methods which approximate the exact solutions of SPDEs with a reasonable approximation accuracy in a reasonable computational time. It is known that in order to approximate the exact solutions of SPDEs appropriately, the numerical methods employed should enjoy similar statistical properties, such as finite uniform moment bounds (see, e.g., Hutzenthaler & Jentzen [14] and the references therein). Unfortunately, moments of the easily realizable Euler-Maruyama and exponential Euler approximation methods are known to diverge for some stochastic differential equations (SDEs) and SPDEs with superlinearly growing nonlineaties (see, e.g., Hutzenthaler et al. [15, 17]). This poses the challenge to develop new efficient approximation methods which preserve finite moments (see, e.g., Hutzenthaler et al. [16, Theorem 1.1 and Lemma 3.9], Gan & Wang [26, Theorem 3.2 and Lemma 3.4], Hutzenthaler & Jentzen [14, Corollary 2.21 and Theorem 3.15], Tretyakov & Zhang [25, Theorem 2.1], Sabanis [22, Theorem 2.2, Corollary 2.3, and Lemmas 3.2, 3.3], and Sabanis [23, Theorems 1–3, Lemmas 1,2] for finite dimensional stochastic evolution equations and, e.g., [19, Proposition 7.3, Theorem 7.6], Becker & Jentzen [2, Corollaries 5.1, 6.15, and 6.17], Becker & Jentzen [1, Lemma 5.4, Theorem 1.1] for infinite dimensional SPDEs). In this context, it has been revealed recently in, e.g., Hutzenthaler & Jentzen [13, Theorem 1.3] (cf., e.g., Dörsek [9, Proposition 3.1], Hutzenthaler et al. [18, Corollary 2.10], and [20, Corollary 3.4]) that finite exponential moments of numerical schemes are crucial for deriving strong convergence with rates in the case of SDEs and SPDEs with non-globally Lipschitz continuous and non-globally monotone nonlinearities.

In this article we derive finite uniform exponential moment bounds for a class of fully discrete exponential Euler-type numerical approximations of infinite dimensional stochastic convolution processes (see Corollary 3.4 in Section 3 below). The considered numerical approximations involve specific taming and truncation terms and are therefore well suited to be applied in the context of semi-linear SPDEs with non-globally Lipschitz continuous and non-globally monotone nonlinearities. In addition to deriving exponential moment bounds we also establish finite uniform moment bounds in fractional order smoothness spaces (see Corollary 3.1 in Section 3 below), a uniform Hölder continuity in time (see Corollary 3.2 in Section 3 below) as well as strong convergences rates for the considered numerical approximations (see Corollary 3.3 in Section 3 below). The application of our results to semi-linear SPDEs such as stochastic Burgers equation will be the subject of a future research article. In Theorem 1.1 below we illustrate the results established in Corollary 3.3 and Corollary 3.4. The stochastic convolution process and its numerical approximations are denoted by $O\colon [0,T] \times \Omega \to D((-A)^\gamma)$ and $\mathbf{O}^{M,N}\colon [0,T] \times \Omega \to P_N(H)$, $M,N \in \mathbb{N}$, respectively.

**Theorem 1.1.** *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $(h_n)_{n\in\mathbb{N}} \subseteq H$ be an orthonormal basis of $H$, let $(\mathfrak{v}_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy $\sup_{n\in\mathbb{N}} \mathfrak{v}_n < 0$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies $D(A) = \{v \in H\colon \sum_{n=1}^\infty |\mathfrak{v}_n \langle h_n, v\rangle_H|^2 < \infty\}$ and $\forall\, v \in D(A)\colon Av = \sum_{n=1}^\infty \mathfrak{v}_n \langle h_n, v\rangle_H h_n$, let $p, T \in (0,\infty)$, $\beta \in [0,\infty)$, $\gamma \in [0, 1/2 + \beta)$, $\eta \in [0, 1/2 + \beta - \gamma)$, $\rho \in [0, 1/2 + \beta - \gamma) \cap [0, 1/2)$, $B \in \mathrm{HS}(U, D((-A)^\beta))$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ be*

*a filtered probability space which fulfills the usual conditions, let $(W_t)_{t\in[0,T]}$ be an $\mathrm{Id}_U$-cylindrical $(\mathbb{F}_t)_{t\in[0,T]}$-Wiener process, let $O\colon [0,T]\times\Omega \to D((-A)^\gamma)$ be a stochastic process which satisfies for every $t\in[0,T]$ that $\mathbb{P}(O_t = \int_0^t e^{(t-s)A}B\,dW_s) = 1$, let $(P_N)_{N\in\mathbb{N}} \subseteq L(H)$ satisfy for every $N\in\mathbb{N}$, $x\in H$ that $P_N(x) = \sum_{n=1}^N \langle h_n, x\rangle_H h_n$, let $W^N\colon [0,T]\times\Omega \to P_N(H)$, $N\in\mathbb{N}$, be stochastic processes which satisfy for every $N\in\mathbb{N}$, $t\in[0,T]$ that $\mathbb{P}(W_t^N = \int_0^t P_N B\,dW_s) = 1$, let $\chi^{M,N}\colon [0,T]\times\Omega \to [0,1]$, $M,N\in\mathbb{N}$, be $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes which satisfy $\sup_{M,N\in\mathbb{N}} \sup_{s\in[0,T]} (\mathbb{E}[|\chi_s^{M,N} - 1|^{\max\{p,2\}}]M^{\max\{p,2\}\rho}) < \infty$, and let $\mathbf{O}^{M,N}\colon [0,T]\times\Omega \to P_N(H)$, $M,N\in\mathbb{N}$, be stochastic processes which satisfy for every $M,N\in\mathbb{N}$, $m\in\{0,1,\ldots,M-1\}$, $t\in[{}^{mT}/_M, {}^{(m+1)T}/_M)$ that $\mathbf{O}_0^{M,N} = 0$ and*

$$\mathbf{O}_t^{M,N} = e^{(t-mT/M)A}\left(\mathbf{O}_{mT/M}^{M,N} + \chi_{mT/M}^{M,N}\left[\frac{W_t^N - W_{mT/M}^N}{1+\|W_t^N - W_{mT/M}^N\|_H^2}\right]\right). \tag{1}$$

*Then*

(i) *it holds that $\inf_{\varepsilon\in(0,\infty)} \sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\varepsilon\|\mathbf{O}_t^{M,N}\|_H^2\right)\right] < \infty$ and*

(ii) *there exists a real number $C\in\mathbb{R}$ such that for every $M,N\in\mathbb{N}$ it holds that*

$$\sup_{t\in[0,T]}\left(\mathbb{E}\left[\|(-A)^\gamma(\mathbf{O}_t^{M,N} - O_t)\|_H^p\right]\right)^{1/p} \leq C\left((\inf_{n\in\{N+1,N+2,\ldots\}}|\mathfrak{v}_n|)^{-\eta} + M^{-\rho}\right). \tag{2}$$

Observe that item (i) in Theorem 1.1 is a direct consequence of Corollary 3.4 (with $H = H$, $U = U$, $\mathbb{H} = (h_n)_{n\in\mathbb{N}}$, $\mathfrak{v} = \mathfrak{v}$, $A = A$, $\beta = \beta$, $T = T$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $B = B$, $P_{\{h_1,\ldots,h_N\}} = P_N$, $\hat{P}_\mathbb{U} = \mathrm{Id}_U$, $\chi^{\{0,T/M,\ldots,T\},\{h_1,\ldots,h_N\},\mathbb{U}} = \chi^{M,N}$, $\mathbf{O}^{\{0,T/M,\ldots,T\},\{h_1,\ldots,h_N\},\mathbb{U}} = \mathbf{O}^{M,N}$, $\varepsilon = \varepsilon$ for $M,N\in\mathbb{N}$, $\varepsilon\in[0, {}^1/_{(8[\max\{\|B\|_{\mathrm{HS}(U,H)},1\}]^2\max\{T,1\})^2})$ in the notation of Corollary 3.4) and Hölder's inequality. Moreover, note that item (ii) in Theorem 1.1 follows from Corollary 3.3 (with $H = H$, $U = U$, $\mathbb{H} = (h_n)_{n\in\mathbb{N}}$, $\mathfrak{v} = \mathfrak{v}$, $A = A$, $\beta = \beta$, $T = T$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $(\mathbb{F}_t)_{t\in[0,T]} = (\mathbb{F}_t)_{t\in[0,T]}$, $(W_t)_{t\in[0,T]} = (W_t)_{t\in[0,T]}$, $B = B$, $P_{\{h_1,\ldots,h_N\}} = P_N$, $\hat{P}_\mathbb{U} = \mathrm{Id}_U$, $\chi^{\{0,T/M,\ldots,T\},\{h_1,\ldots,h_N\},\mathbb{U}} = \chi^{M,N}$, $\mathbf{O}^{\{0,T/M,\ldots,T\},\{h_1,\ldots,h_N\},\mathbb{U}} = \mathbf{O}^{M,N}$, $p = \max\{p,1\}$, $C = C$, $\gamma = \gamma$, $\eta = \eta$, $\rho = \rho$, $O = O$ for $M,N\in\mathbb{N}$ in the notation of Corollary 3.3).

We would like to point out that the exponential moment bound in Corollary 3.4 below (see also item (i) above) is not a direct consequence of the one established in [20]. The difference between the numerical method in [20, (1) in Section 1] and (1) above lies, roughly speaking, in the less restrictive choice for the truncation functions $\chi^{M,N}\colon [0,T]\times\Omega \to [0,1]$, $M,N\in\mathbb{N}$, in (1) compared to [20, (1) in Section 1]. This extended class of discrete approximations allows to truncate the numerical method independently of the current value of the approximation process itself. The flexibility in the choice of the truncation function is, in turn, important for applying Corollaries 3.1–3.4 to establish strong convergence rates for fully discrete numerical approximations in the case of SPDEs with non-globally Lipschitz continuous and non-globally monotone nonlinearities.

The remainder of this article is structured as follows. In Section 2 we analyze a temporally semi-discrete version of our approximation scheme (see (4) in Setting 2.1 below). In Subsection 2.1 the considered numerical approximations are rewritten as Itô processes and finite moment bounds in fractional order smoothness spaces are derived. The Itô representation enables us to establish Hölder regularity in time in Subsection 2.2. Furthermore, under additional assumptions on the truncation function we establish temporal strong convergence rates of the proposed numerical methods in Subsection 2.3. In Subsection 2.4 we further improve our moment estimates from Subsection 2.1 in order to derive finite exponential moment bounds in Lemma 2.8. The results

from Section 2 are combined in Section 3 to establish uniform moment bounds in fractional order smoothness spaces, a uniform Hölder regularity in time, strong convergence rates, and uniform exponential moment bounds for fully discrete tamed-truncated numerical approximations in Corollaries 3.1–3.4, respectively.

## 1.1 General setting

Throughout this article the following setting is frequently used.

**Setting 1.1.** *For every set $X$ let $\mathcal{P}(X)$ be the power set of $X$, for every set $X$ let $\mathcal{P}_0(X)$ be the set given by $\mathcal{P}_0(X) = \{\theta \in \mathcal{P}(X) \colon \theta \text{ has finitely many elements}\}$, for every $T \in (0, \infty)$ let $\varpi_T$ be the set given by $\varpi_T = \{\theta \in \mathcal{P}_0([0, T]) \colon \{0, T\} \subseteq \theta\}$, for every $T \in (0, \infty)$ let $|\cdot|_T \colon \varpi_T \to [0, T]$ be the function which satisfies for every $\theta \in \varpi_T$ that*

$$|\theta|_T = \max\left\{ x \in (0, \infty) \colon \left(\exists\, a, b \in \theta \colon \left[x = b - a \text{ and } \theta \cap (a, b) = \emptyset\right]\right) \right\} \in (0, T], \qquad (3)$$

*for every $\theta \in (\cup_{T \in (0, \infty)} \varpi_T)$ let $\llcorner \cdot \lrcorner_\theta \colon [0, \infty) \to [0, \infty)$ be the function which satisfies for every $t \in (0, \infty)$ that $\llcorner t \lrcorner_\theta = \max([0, t) \cap \theta)$ and $\llcorner 0 \lrcorner_\theta = 0$, for every measure space $(\Omega, \mathcal{F}, \mu)$, every measurable space $(S, \mathcal{S})$, every set $R$, and every function $f \colon \Omega \to R$ let $[f]_{\mu, \mathcal{S}}$ be the set given by $[f]_{\mu, \mathcal{S}} = \{g \colon \Omega \to S \colon (\exists\, A \in \mathcal{F} \colon [\mu(A) = 0 \text{ and } \{\omega \in \Omega \colon f(\omega) \neq g(\omega)\} \subseteq A]) \text{ and } (\forall\, A \in \mathcal{S} \colon g^{-1}(A) \in \mathcal{F})\}$, let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of $H$, let $\mathfrak{v} \colon \mathbb{H} \to \mathbb{R}$ be a function which satisfies $\sup_{h \in \mathbb{H}} \mathfrak{v}_h < 0$, let $A \colon D(A) \subseteq H \to H$ be the linear operator which satisfies $D(A) = \{v \in H \colon \sum_{h \in \mathbb{H}} |\mathfrak{v}_h \langle h, v \rangle_H|^2 < \infty\}$ and $\forall\, v \in D(A) \colon Av = \sum_{h \in \mathbb{H}} \mathfrak{v}_h \langle h, v \rangle_H h$, and let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., [24, Section 3.7]).*

# 2 Regularity properties of temporally semi-discrete tamed-truncated approximations of stochastic convolutions

**Setting 2.1.** *Assume Setting 1.1, let $\beta \in [0, \infty)$, $\gamma \in [0, 1/2 + \beta)$, $T \in (0, \infty)$, $\theta \in \varpi_T$, $B \in \mathrm{HS}(U, H_\beta)$, let $\mathbb{B} \in L(H, U)$ be the bounded linear operator which satisfies for every $u \in U$, $h \in H$ that $\langle Bu, h \rangle_H = \langle u, \mathbb{B}h \rangle_U$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which fulfills the usual conditions, let $(W_t)_{t \in [0, T]}$ be an $\mathrm{Id}_U$-cylindrical $(\mathbb{F}_t)_{t \in [0, T]}$-Wiener process, let $\chi \colon [0, T] \times \Omega \to [0, 1]$ be an $(\mathbb{F}_t)_{t \in [0, T]}$-adapted stochastic process, and let $\mathbf{O} \colon [0, T] \times \Omega \to H_\gamma$ be a stochastic process which satisfies for every $t \in [0, T]$ that $\mathbf{O}_0 = 0$ and*

$$[\mathbf{O}_t]_{\mathbb{P}, \mathcal{B}(H_\gamma)} = [e^{(t - \llcorner t \lrcorner_\theta)A} \mathbf{O}_{\llcorner t \lrcorner_\theta}]_{\mathbb{P}, \mathcal{B}(H_\gamma)} + \frac{\int_{\llcorner t \lrcorner_\theta}^t \chi_{\llcorner t \lrcorner_\theta}\, e^{(t - \llcorner t \lrcorner_\theta)A} B\, dW_s}{1 + \|\int_{\llcorner t \lrcorner_\theta}^t B\, dW_s\|_H^2}. \qquad (4)$$

## 2.1 Moment bounds for temporally semi-discrete approximations of stochastic convolutions

In this subsection we provide in Lemma 2.1 a representation of the approximation process $\mathbf{O} \colon [0, T] \times \Omega \to H_\gamma$ from Setting 2.1 as a mild Itô process (cf. Da Prato et al. [7, Definition 1]). This enables us to obtain certain moment bounds for $\mathbf{O} \colon [0, T] \times \Omega \to H_\gamma$ in Lemmas 2.2 and 2.3.

**Lemma 2.1.** *Assume Setting 2.1 and for every $s \in [0,T]$ let $X_{s,(\cdot)}(\cdot) = (X_{s,t}(\omega))_{(t,\omega) \in [s,T] \times \Omega} \colon [s,T] \times \Omega \to H_\beta$ be an $(\mathbb{F}_t)_{t \in [s,T]}$-adapted stochastic process with continuous sample paths which satisfies for every $t \in [s,T]$ that $[X_{s,t}]_{\mathbb{P},\mathcal{B}(H_\beta)} = \int_s^t B\,dW_u$. Then it holds for every $t \in [0,T]$ that*

$$
\begin{aligned}
[\mathbf{O}_t]_{\mathbb{P},\mathcal{B}(H_\gamma)} &= \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \left[ \frac{B}{1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2} - \frac{2 X_{\llcorner u \lrcorner_\theta, u} \langle X_{\llcorner u \lrcorner_\theta, u}, B(\cdot) \rangle_H}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right] dW_u \\
&\quad + \left[ \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \left( \frac{4\|\mathbb{B} X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2 B \mathbb{B} X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right) du \right]_{\mathbb{P},\mathcal{B}(H_\gamma)}.
\end{aligned}
\tag{5}
$$

*Proof of Lemma 2.1.* Throughout this proof let $\psi \colon H \to H$ be the function which satisfies for every $v \in H$ that $\psi(v) = \frac{v}{1 + \|v\|_H^2}$ and let $\mathbb{U} \subseteq U$ be an orthonormal basis of $U$. Note that for every $u, v, z \in H$ it holds that

$$
\psi'(z)(u) = \frac{u}{1 + \|z\|_H^2} - \frac{2z \langle z, u \rangle_H}{(1 + \|z\|_H^2)^2}
\tag{6}
$$

and

$$
\psi''(z)(u,v) = -\frac{2[u \langle z, v \rangle_H + v \langle z, u \rangle_H + z \langle u, v \rangle_H]}{(1 + \|z\|_H^2)^2} + \frac{8z \langle z, u \rangle_H \langle z, v \rangle_H}{(1 + \|z\|_H^2)^3}.
\tag{7}
$$

Itô's formula (see, e.g., Brzeźniak et al. [6, Theorem 2.4] with $H = U$, $E = H$, $F = H$, $f = ([0,T] \times H \ni (t,x) \mapsto \psi(x) \in H)$, $\Phi = ([0,T] \times \Omega \ni (t,\omega) \mapsto B \in \mathrm{HS}(U,H))$ in the notation of Brzeźniak et al. [6, Theorem 2.4]) hence proves that for every $s \in [0,T]$, $t \in [s,T]$ it holds that

$$
\begin{aligned}
[\psi(X_{s,t})]_{\mathbb{P},\mathcal{B}(H_\beta)} &= \int_s^t \left( \frac{B}{1 + \|X_{s,u}\|_H^2} - \frac{2 X_{s,u} \langle X_{s,u}, B(\cdot) \rangle_H}{(1 + \|X_{s,u}\|_H^2)^2} \right) dW_u \\
&\quad + \left[ \int_s^t \left( \sum_{\mathbf{u} \in \mathbb{U}} \frac{4 X_{s,u} |\langle X_{s,u}, B\mathbf{u} \rangle_H|^2}{(1 + \|X_{s,u}\|_H^2)^3} - \frac{2 B\mathbf{u} \langle X_{s,u}, B\mathbf{u} \rangle_H + X_{s,u} \|B\mathbf{u}\|_H^2}{(1 + \|X_{s,u}\|_H^2)^2} \right) du \right]_{\mathbb{P},\mathcal{B}(H_\beta)} \\
&= \int_s^t \left( \frac{B}{1 + \|X_{s,u}\|_H^2} - \frac{2 X_{s,u} \langle X_{s,u}, B(\cdot) \rangle_H}{(1 + \|X_{s,u}\|_H^2)^2} \right) dW_u \\
&\quad + \left[ \int_s^t \left( \frac{4 X_{s,u} \|\mathbb{B} X_{s,u}\|_U^2}{(1 + \|X_{s,u}\|_H^2)^3} - \frac{2 B \mathbb{B} X_{s,u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{s,u}}{(1 + \|X_{s,u}\|_H^2)^2} \right) du \right]_{\mathbb{P},\mathcal{B}(H_\beta)}.
\end{aligned}
\tag{8}
$$

Therefore, we obtain for every $t \in [0,T]$ that

$$
\begin{aligned}
[\mathbf{O}_t]_{\mathbb{P},\mathcal{B}(H_\gamma)} &= [e^{(t - \llcorner t \lrcorner_\theta)A}(\mathbf{O}_{\llcorner t \lrcorner_\theta} + \chi_{\llcorner t \lrcorner_\theta} \psi(X_{\llcorner t \lrcorner_\theta, t}))]_{\mathbb{P},\mathcal{B}(H_\gamma)} \\
&= [e^{(t - \llcorner t \lrcorner_\theta)A} \mathbf{O}_{\llcorner t \lrcorner_\theta}]_{\mathbb{P},\mathcal{B}(H_\gamma)} + \int_{\llcorner t \lrcorner_\theta}^t \chi_{\llcorner t \lrcorner_\theta} e^{(t - \llcorner t \lrcorner_\theta)A} \left( \frac{B}{1 + \|X_{\llcorner t \lrcorner_\theta, u}\|_H^2} - \frac{2 X_{\llcorner t \lrcorner_\theta, u} \langle X_{\llcorner t \lrcorner_\theta, u}, B(\cdot) \rangle_H}{(1 + \|X_{\llcorner t \lrcorner_\theta, u}\|_H^2)^2} \right) dW_u \\
&\quad + \left[ \int_{\llcorner t \lrcorner_\theta}^t \chi_{\llcorner t \lrcorner_\theta} e^{(t - \llcorner t \lrcorner_\theta)A} \left( \frac{4 X_{\llcorner t \lrcorner_\theta, u} \|\mathbb{B} X_{\llcorner t \lrcorner_\theta, u}\|_U^2}{(1 + \|X_{\llcorner t \lrcorner_\theta, u}\|_H^2)^3} - \frac{2 B \mathbb{B} X_{\llcorner t \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner t \lrcorner_\theta, u}}{(1 + \|X_{\llcorner t \lrcorner_\theta, u}\|_H^2)^2} \right) du \right]_{\mathbb{P},\mathcal{B}(H_\gamma)}.
\end{aligned}
\tag{9}
$$

This completes the proof of Lemma 2.1. $\qquad\square$

**Lemma 2.2.** *Assume Setting 2.1, let $p \in [2,\infty)$, $\rho \in [\beta, 1/2 + \beta)$, $\eta \in [\beta, 1 + \beta)$, and for every $s \in [0,T]$ let $X_{s,(\cdot)}(\cdot) = (X_{s,t}(\omega))_{(t,\omega) \in [s,T] \times \Omega} \colon [s,T] \times \Omega \to H_\beta$ be an $(\mathbb{F}_t)_{t \in [s,T]}$-adapted stochastic process with continuous sample paths which satisfies for every $t \in [s,T]$ that $[X_{s,t}]_{\mathbb{P},\mathcal{B}(H_\beta)} = \int_s^t B\,dW_u$. Then it holds for every $s \in [0,T]$, $t \in [s,T]$ that*

$$
\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \frac{B}{1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2}\,dW_u \right\|_{L^p(\mathbb{P};H_\rho)} \leq \|B\|_{\mathrm{HS}(U,H_\beta)} \frac{p(t-s)^{1/2 + \beta - \rho}}{\sqrt{2(1 + 2\beta - 2\rho)}},
\tag{10}
$$

$$\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta) A} \frac{2 X_{\llcorner u \lrcorner_\theta, u} \langle X_{\llcorner u \lrcorner_\theta, u}, B\, dW_u \rangle_H}{(1 + \| X_{\llcorner u \lrcorner_\theta, u} \|_H^2)^2} \right\|_{L^p(\mathbb{P}; H_\rho)} \tag{11}$$
$$\leq 2\sqrt{2} p^3 \| B \|_{\mathrm{HS}(U, H_\beta)}^3 | \sup_{h \in \mathbb{H}} \mathfrak{v}_h |^{-2\beta} \frac{(t-s)^{1/2 + \beta - \rho}}{\sqrt{1 + 2\beta - 2\rho}} | \theta |_T,$$

*and*

$$\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta) A} \left( \frac{4 \| \mathbb{B} X_{\llcorner u \lrcorner_\theta, u} \|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \| X_{\llcorner u \lrcorner_\theta, u} \|_H^2)^3} - \frac{2 B \mathbb{B} X_{\llcorner u \lrcorner_\theta, u} + \| B \|_{\mathrm{HS}(U, H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \| X_{\llcorner u \lrcorner_\theta, u} \|_H^2)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P}; H_\eta)} \tag{12}$$
$$\leq 2\sqrt{2} p \| B \|_{\mathrm{HS}(U, H_\beta)}^3 | \sup_{h \in \mathbb{H}} \mathfrak{v}_h |^{-2\beta} \frac{(t-s)^{1 + \beta - \eta}}{1 + \beta - \eta} [ | \theta |_T ]^{1/2}.$$

*Proof of Lemma 2.2.* Note that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] shows for every $s \in [0, T]$, $t \in [s, T]$ that

$$\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta) A} \frac{B}{1 + \| X_{\llcorner u \lrcorner_\theta, u} \|_H^2} dW_u \right\|_{L^p(\mathbb{P}; H_\rho)}^2$$
$$\leq \frac{p(p-1)}{2} \int_s^t \left\| e^{(t - \llcorner u \lrcorner_\theta) A} \frac{B}{1 + \| X_{\llcorner u \lrcorner_\theta, u} \|_H^2} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathrm{HS}(U, H_\rho))}^2 du \tag{13}$$
$$\leq \frac{p^2}{2} \int_s^t \| (-A)^{\rho - \beta} e^{(t - \llcorner u \lrcorner_\theta) A} \|_{L(H)}^2 \left\| \frac{B}{1 + \| X_{\llcorner u \lrcorner_\theta, u} \|_H^2} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathrm{HS}(U, H_\beta))}^2 du$$
$$\leq \frac{p^2}{2} \| B \|_{\mathrm{HS}(U, H_\beta)}^2 \int_s^t (t - \llcorner u \lrcorner_\theta)^{2\beta - 2\rho} du.$$

In addition, observe for every $s \in [0, T]$, $t \in [s, T]$ that

$$\int_s^t (t - \llcorner u \lrcorner_\theta)^{2\beta - 2\rho} du \leq \int_s^t (t - u)^{2\beta - 2\rho} du \leq \frac{(t-s)^{1 + 2\beta - 2\rho}}{1 + 2\beta - 2\rho}. \tag{14}$$

Combining this and (13) establishes (10). Furthermore, note that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] proves that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta) A} \frac{2 X_{\llcorner u \lrcorner_\theta, u} \langle X_{\llcorner u \lrcorner_\theta, u}, B(\cdot) \rangle_H}{(1 + \| X_{\llcorner u \lrcorner_\theta, u} \|_H^2)^2} dW_u \right\|_{L^p(\mathbb{P}; H_\rho)}^2$$
$$\leq \frac{p(p-1)}{2} \int_s^t \left\| e^{(t - \llcorner u \lrcorner_\theta) A} \frac{2 X_{\llcorner u \lrcorner_\theta, u} \langle X_{\llcorner u \lrcorner_\theta, u}, B(\cdot) \rangle_H}{(1 + \| X_{\llcorner u \lrcorner_\theta, u} \|_H^2)^2} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathrm{HS}(U, H_\rho))}^2 du \tag{15}$$
$$\leq 2 p^2 \int_s^t \left\| e^{(t - \llcorner u \lrcorner_\theta) A} X_{\llcorner u \lrcorner_\theta, u} \langle X_{\llcorner u \lrcorner_\theta, u}, B(\cdot) \rangle_H \right\|_{\mathcal{L}^p(\mathbb{P}; \mathrm{HS}(U, H_\rho))}^2 du$$
$$\leq 2 p^2 \int_s^t \left\| \| (-A)^{\rho - \beta} e^{(t - \llcorner u \lrcorner_\theta) A} \|_{L(H)} \| (-A)^\beta X_{\llcorner u \lrcorner_\theta, u} \|_H \| \langle X_{\llcorner u \lrcorner_\theta, u}, B(\cdot) \rangle_H \|_{\mathrm{HS}(U, \mathbb{R})} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})}^2 du.$$

The Hölder inequality, the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8], and the fact that $\| B \|_{\mathrm{HS}(U, H)} \leq | \sup_{h \in \mathbb{H}} \mathfrak{v}_h |^{-\beta} \| B \|_{\mathrm{HS}(U, H_\beta)}$ therefore ensure that for

every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \frac{2 X_{\llcorner u \lrcorner_\theta, u} \langle X_{\llcorner u \lrcorner_\theta, u}, B(\cdot) \rangle_H}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \, dW_u \right\|_{L^p(\mathbb{P}; H_\rho)}^2
$$

$$
\leq 2p^2 \|B\|_{\mathrm{HS}(U,H)}^2 \int_s^t \|(-A)^{\rho - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}^2 \|X_{\llcorner u \lrcorner_\theta, u}\|_{\mathcal{L}^{2p}(\mathbb{P}; H_\beta)}^2 \|X_{\llcorner u \lrcorner_\theta, u}\|_{\mathcal{L}^{2p}(\mathbb{P}; H)}^2 \, du
$$

$$
\leq 8p^6 \|B\|_{\mathrm{HS}(U,H)}^2 \int_s^t \|(-A)^{\rho - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}^2 \left[ \int_{\llcorner u \lrcorner_\theta}^u \|B\|_{\mathrm{HS}(U,H_\beta)}^2 \, dw \right] \left[ \int_{\llcorner u \lrcorner_\theta}^u \|B\|_{\mathrm{HS}(U,H)}^2 \, dw \right] du \quad (16)
$$

$$
\leq 8p^6 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-4\beta} \|B\|_{\mathrm{HS}(U,H_\beta)}^6 \int_s^t \|(-A)^{\rho - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}^2 (u - \llcorner u \lrcorner_\theta)^2 \, du
$$

$$
\leq 8p^6 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-4\beta} \|B\|_{\mathrm{HS}(U,H_\beta)}^6 \int_s^t (t - \llcorner u \lrcorner_\theta)^{2\beta - 2\rho} \, du [|\theta|_T]^2.
$$

Combining this and (14) assures that (11) holds. In the next step we observe that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \left( \frac{4 \|\mathbb{B} X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B} X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P}; H_\eta)}
$$

$$
\leq \int_s^t \left\| e^{(t - \llcorner u \lrcorner_\theta)A} \left( \frac{4 \|\mathbb{B} X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B} X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right) \right\|_{\mathcal{L}^p(\mathbb{P}; H_\eta)} du
$$

$$
\leq \int_s^t \|(-A)^{\eta - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)} \left\| \frac{4 \|\mathbb{B} X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B} X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right\|_{\mathcal{L}^p(\mathbb{P}; H_\beta)} du
$$

$$
\leq \int_s^t \|(-A)^{\eta - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}
$$

$$
\cdot \left\| \left( \frac{4 \|\mathbb{B}\|_{L(H,U)}^2 \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} + \frac{2 \|B\mathbb{B}\|_{L(H_\beta, H_\beta)} + \|B\|_{\mathrm{HS}(U,H)}^2}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right) \|X_{\llcorner u \lrcorner_\theta, u}\|_{H_\beta} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})} du \quad (17)
$$

$$
\leq \int_s^t \|(-A)^{\eta - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}
$$

$$
\cdot \left\| \left( \|B\|_{\mathrm{HS}(U,H)}^2 + 2 \|B\|_{L(U,H_\beta)} \|\mathbb{B}\|_{L(H_\beta, U)} + \|B\|_{\mathrm{HS}(U,H)}^2 \right) \|X_{\llcorner u \lrcorner_\theta, u}\|_{H_\beta} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})} du.
$$

This and the fact that $\|\mathbb{B}\|_{L(H_\beta, U)} \leq |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|\mathbb{B}\|_{L(H,U)}$ demonstrate that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \left( \frac{4 \|\mathbb{B} X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B} X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P}; H_\eta)}
$$

$$
\leq 2 \int_s^t \|(-A)^{\eta - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}
$$

$$
\cdot \left( \|B\|_{\mathrm{HS}(U,H)}^2 + |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|B\|_{\mathrm{HS}(U,H_\beta)} \|\mathbb{B}\|_{L(H,U)} \right) \|X_{\llcorner u \lrcorner_\theta, u}\|_{\mathcal{L}^p(\mathbb{P}; H_\beta)} \, du \quad (18)
$$

$$
\leq 2 \int_s^t \|B\|_{\mathrm{HS}(U,H)} \|(-A)^{\eta - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}
$$

$$
\cdot \left( \|B\|_{\mathrm{HS}(U,H)} + |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|B\|_{\mathrm{HS}(U,H_\beta)} \right) \|X_{\llcorner u \lrcorner_\theta, u}\|_{\mathcal{L}^p(\mathbb{P}; H_\beta)} \, du.
$$

The Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] and the fact that $\|B\|_{\mathrm{HS}(U,H)} \leq |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|B\|_{\mathrm{HS}(U,H_\beta)}$ hence establish for every $s \in [0, T]$, $t \in [s, T]$

that

$$\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left( \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}\|_U^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}+\|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P};H_\eta)}$$

$$\leq 2\|B\|_{\mathrm{HS}(U,H)}\Big( \|B\|_{\mathrm{HS}(U,H)} + |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-\beta}\|B\|_{\mathrm{HS}(U,H_\beta)} \Big)$$

$$\cdot \int_s^t \|(-A)^{\eta-\beta}e^{(t-\llcorner u \lrcorner_\theta)A}\|_{L(H)} \left( \frac{p(p-1)}{2} \int_{\llcorner u \lrcorner_\theta}^u \|B\|_{\mathrm{HS}(U,H_\beta)}^2 \, ds \right)^{1/2} du \tag{19}$$

$$\leq 2\sqrt{2}p\|B\|_{\mathrm{HS}(U,H_\beta)}^3 |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} \int_s^t \|(-A)^{\eta-\beta}e^{(t-\llcorner u \lrcorner_\theta)A}\|_{L(H)} (u-\llcorner u \lrcorner_\theta)^{1/2} \, du$$

$$\leq 2\sqrt{2}p\|B\|_{\mathrm{HS}(U,H_\beta)}^3 |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} [|\theta|_T]^{1/2} \int_s^t (t-\llcorner u \lrcorner_\theta)^{\beta-\eta} \, du.$$

Moreover, note that for every $s \in [0,T]$, $t \in [s,T]$ it holds that

$$\int_s^t (t-\llcorner u \lrcorner_\theta)^{\beta-\eta} \, du \leq \int_s^t (t-u)^{\beta-\eta} \, du \leq \frac{(t-s)^{1+\beta-\eta}}{1+\beta-\eta}. \tag{20}$$

Combining this and (19) establishes (12). The proof of Lemma 2.2 is thus completed. $\qquad\square$

**Lemma 2.3.** *Assume Setting 2.1, assume that $\beta \leq \gamma$, and let $p \in [2,\infty)$, $t \in [0,T]$. Then it holds that*

$$\|\mathbf{O}_t\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \leq 3p\|B\|_{\mathrm{HS}(U,H_\beta)} \frac{[\max\{T,1\}]^{3/2}}{1+2\beta-2\gamma} \Big( 1 + 4p^2\|B\|_{\mathrm{HS}(U,H_\beta)}^2 |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} \Big). \tag{21}$$

*Proof of Lemma 2.3.* Throughout this proof for every $s \in [0,T]$ let $X_{s,(\cdot)}(\cdot) = (X_{s,u}(\omega))_{(u,\omega)\in[s,T]\times\Omega}\colon [s,T] \times \Omega \to H_\beta$ be an $(\mathbb{F}_u)_{u\in[s,T]}$-adapted stochastic process with continuous sample paths which satisfies for every $u \in [s,T]$ that $[X_{s,u}]_{\mathbb{P},\mathcal{B}(H_\beta)} = \int_s^u B \, dW_\tau$. Lemma 2.1 (with $X_{s,u} = X_{s,u}$ for $u \in [s,T]$, $s \in [0,T]$ in the notation of Lemma 2.1) implies that

$$\|\mathbf{O}_t\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \leq \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} \left( \frac{e^{(t-\llcorner u \lrcorner_\theta)A}B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2e^{(t-\llcorner u \lrcorner_\theta)A}X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u},B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right) dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}$$

$$+ \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left[ \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}\|_U^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}+\|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] du \right\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)}. \tag{22}$$

Moreover, observe that the triangle inequality proves that

$$\left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} \left( \frac{e^{(t-\llcorner u \lrcorner_\theta)A}B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2e^{(t-\llcorner u \lrcorner_\theta)A}X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u},B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right) dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}$$

$$\leq \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} \, dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \tag{23}$$

$$+ \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} \frac{2e^{(t-\llcorner u \lrcorner_\theta)A}X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u},B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \, dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}.$$

Next note that Lemma 2.2 (with $p = p$, $\rho = \gamma$, $\eta = \gamma$, $X_{s,\tau} = X_{s,\tau}$ for $\tau \in [s,T]$, $s \in [0,T]$ in the notation of Lemma 2.2) shows that

$$\left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} \, dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \leq \|B\|_{\mathrm{HS}(U,H_\beta)} \frac{pt^{1/2+\beta-\gamma}}{\sqrt{2(1+2\beta-2\gamma)}} \tag{24}$$

$$\left\| \int_0^t \chi_{\llcorner u \lrcorner \theta} \frac{2e^{(t-\llcorner u \lrcorner \theta)A} X_{\llcorner u \lrcorner \theta, u} \langle X_{\llcorner u \lrcorner \theta, u}, B(\cdot) \rangle_H}{(1+\|X_{\llcorner u \lrcorner \theta, u}\|_H^2)^2} \, dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)}$$

$$\leq 2\sqrt{2} p^3 \|B\|_{\mathrm{HS}(U, H_\beta)}^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \frac{t^{1/2+\beta-\gamma}}{\sqrt{1+2\beta-2\gamma}} |\theta|_T, \tag{25}$$

and

$$\left\| \int_0^t \chi_{\llcorner u \lrcorner \theta} e^{(t-\llcorner u \lrcorner \theta)A} \left( \frac{4\|\mathbb{B} X_{\llcorner u \lrcorner \theta, u}\|_U^2 X_{\llcorner u \lrcorner \theta, u}}{(1+\|X_{\llcorner u \lrcorner \theta, u}\|_H^2)^3} - \frac{2B \mathbb{B} X_{\llcorner u \lrcorner \theta, u} + \|B\|_{\mathrm{HS}(U, H)}^2 X_{\llcorner u \lrcorner \theta, u}}{(1+\|X_{\llcorner u \lrcorner \theta, u}\|_H^2)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P}; H_\gamma)}$$

$$\leq 2\sqrt{2} p \|B\|_{\mathrm{HS}(U, H_\beta)}^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \frac{t^{1+\beta-\gamma}}{1+\beta-\gamma} [|\theta|_T]^{1/2}. \tag{26}$$

Next we combine (23)–(25) to obtain that

$$\left\| \int_0^t \chi_{\llcorner u \lrcorner \theta} \left( \frac{e^{(t-\llcorner u \lrcorner \theta)A} B}{1+\|X_{\llcorner u \lrcorner \theta, u}\|_H^2} - \frac{2e^{(t-\llcorner u \lrcorner \theta)A} X_{\llcorner u \lrcorner \theta, u} \langle X_{\llcorner u \lrcorner \theta, u}, B(\cdot) \rangle_H}{(1+\|X_{\llcorner u \lrcorner \theta, u}\|_H^2)^2} \right) dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)}$$

$$\leq \|B\|_{\mathrm{HS}(U, H_\beta)} \frac{p t^{1/2+\beta-\gamma}}{\sqrt{2(1+2\beta-2\gamma)}} \left( 1 + 4p^2 \|B\|_{\mathrm{HS}(U, H_\beta)}^2 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} |\theta|_T \right). \tag{27}$$

This, (22), (26), the fact that $\sqrt{2} + \frac{1}{\sqrt{2}} \leq 3$, the fact that $1 + 2\beta - 2\gamma \leq 2(1 + \beta - \gamma)$, and the fact that $\forall\, x \in (0, 1]: {}^1\!/\!\sqrt{x} \leq {}^1\!/x$ demonstrate that

$$\|\mathbf{O}_t\|_{\mathcal{L}^p(\mathbb{P}; H_\gamma)} \leq 4\sqrt{2} p \|B\|_{\mathrm{HS}(U, H_\beta)}^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \frac{[\max\{T, 1\}]^{3/2}}{1+2\beta-2\gamma}$$

$$+ \frac{p}{\sqrt{2}} \|B\|_{\mathrm{HS}(U, H_\beta)} \frac{[\max\{T, 1\}]^{3/2}}{1+2\beta-2\gamma} \left( 1 + 4p^2 \|B\|_{\mathrm{HS}(U, H_\beta)}^2 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \right) \tag{28}$$

$$\leq 3p \|B\|_{\mathrm{HS}(U, H_\beta)} \frac{[\max\{T, 1\}]^{3/2}}{1+2\beta-2\gamma} \left( 1 + 4p^2 \|B\|_{\mathrm{HS}(U, H_\beta)}^2 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \right).$$

The proof of Lemma 2.3 is thus completed. $\qquad\square$

## 2.2 Hölder continuity of temporally semi-discrete approximations of stochastic convolutions

In this subsection we combine Lemma 2.1 and Lemma 2.2 to establish in Lemma 2.4 a temporal regularity property for the approximation process $\mathbf{O}\colon [0, T] \times \Omega \to H_\gamma$ from Setting 2.1.

**Lemma 2.4.** *Assume Setting 2.1, assume that $\beta \leq \gamma$, and let $p \in [2, \infty)$, $\rho \in [0, {}^1\!/2 + \beta - \gamma)$. Then it holds for every $s \in [0, T]$, $t \in [s, T]$ that*

$$\|\mathbf{O}_t - \mathbf{O}_s\|_{\mathcal{L}^p(\mathbb{P}; H_\gamma)} \leq \frac{3p^3 \|B\|_{\mathrm{HS}(U, H_\beta)} [\max\{T, 1\}]^2 \max\{|\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}, 1\} (1 + 8\|B\|_{\mathrm{HS}(U, H_\beta)}^2)}{\sqrt{1+2(\beta-\gamma-\rho)}} (t-s)^\rho. \tag{29}$$

*Proof of Lemma 2.4.* Throughout this proof for every $s \in [0, T]$ let $X_{s, (\cdot)}(\cdot) = (X_{s, t}(\omega))_{(t, \omega) \in [s, T] \times \Omega}\colon [s, T] \times \Omega \to H_\beta$ be an $(\mathbb{F}_t)_{t \in [s, T]}$-adapted stochastic process with continuous sample paths which satisfies for every $t \in [s, T]$ that $[X_{s, t}]_{\mathbb{P}, \mathcal{B}(H_\beta)} = \int_s^t B \, dW_u$. Lemma 2.1 (with $X_{s, t} = X_{s, t}$ for $t \in [s, T]$, $s \in [0, T]$ in the notation of Lemma 2.1) and the triangle inequality prove for every

9

$s \in [0, T]$, $t \in [s, T]$ that

$$
\begin{aligned}
&\|\mathbf{O}_t - \mathbf{O}_s\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \\
&\leq \left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \\
&+ \left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left[ \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}\|_U^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}+\|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] du \right\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \quad (30) \\
&+ \left\| \int_0^s \chi_{\llcorner u \lrcorner_\theta} \left[ e^{(t-\llcorner u \lrcorner_\theta)A} - e^{(s-\llcorner u \lrcorner_\theta)A} \right] \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \\
&+ \int_0^s \left\| \left[ e^{(t-\llcorner u \lrcorner_\theta)A} - e^{(s-\llcorner u \lrcorner_\theta)A} \right] \left[ \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}\|_U^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}+\|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] \right\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} du.
\end{aligned}
$$

Furthermore, observe that the triangle inequality implies for every $s \in [0, T]$, $t \in [s, T]$ that

$$
\begin{aligned}
&\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \\
&\leq \left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \quad (31) \\
&+ \left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}.
\end{aligned}
$$

Next note that Lemma 2.2 (with $p = p$, $\rho = \gamma$, $\eta = \gamma$, $X_{s,t} = X_{s,t}$ for $t \in [s, T]$, $s \in [0, T]$ in the notation of Lemma 2.2) shows for every $s \in [0, T]$, $t \in [s, T]$ that

$$
\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \leq \frac{p\|B\|_{\mathrm{HS}(U,H_\beta)}}{\sqrt{2(1+2\beta-2\gamma)}}(t-s)^{1/2+\beta-\gamma}, \quad (32)
$$

$$
\begin{aligned}
&\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B\, dW_u\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right\|_{L^p(\mathbb{P};H_\gamma)} \\
&\leq \frac{2\sqrt{2}p^3\|B\|_{\mathrm{HS}(U,H_\beta)}^3 |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta}}{\sqrt{1+2\beta-2\gamma}} |\theta|_T (t-s)^{1/2+\beta-\gamma},
\end{aligned} \quad (33)
$$

and

$$
\begin{aligned}
&\left\| \int_s^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left[ \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}\|_U^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}+\|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] du \right\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \\
&\leq \frac{2\sqrt{2}p}{1+\beta-\gamma}\|B\|_{\mathrm{HS}(U,H_\beta)}^3 |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} [|\theta|_T]^{1/2}(t-s)^{1+\beta-\gamma}.
\end{aligned} \quad (34)
$$

Moreover, observe that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato

10

& Zabczyk [8] assures for every $s \in [0, T]$, $t \in [s, T]$ that

$$
\left\| \int_0^s \chi_{\llcorner u \lrcorner_\theta} \left[ e^{(t-\llcorner u \lrcorner_\theta)A} - e^{(s-\llcorner u \lrcorner_\theta)A} \right] \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}^2
$$

$$
\leq \frac{p^2}{2} \int_0^s \left\| e^{(s-\llcorner u \lrcorner_\theta)A}(e^{(t-s)A} - \mathrm{Id}_H) \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] \right\|_{\mathcal{L}^p(\mathbb{P};\mathrm{HS}(U,H_\gamma))}^2 du
$$

$$
\leq \frac{p^2}{2} \int_0^s \left\| (-A)^{\gamma+\rho-\beta} e^{(s-\llcorner u \lrcorner_\theta)A} \right\|_{L(H)}^2 \left\| (-A)^{-\rho}(e^{(t-s)A} - \mathrm{Id}_H) \right\|_{L(H)}^2
$$

$$
\cdot \left\| \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right\|_{\mathcal{L}^p(\mathbb{P};\mathrm{HS}(U,H_\beta))}^2 du \tag{35}
$$

$$
\leq \frac{p^2}{2} \int_0^s \left\| (-A)^{\gamma+\rho-\beta} e^{(s-\llcorner u \lrcorner_\theta)A} \right\|_{L(H)}^2 (t-s)^{2\rho}
$$

$$
\cdot \left\| \frac{\|B\|_{\mathrm{HS}(U,H_\beta)}}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} + \frac{2\|X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H\|_{\mathrm{HS}(U,H_\beta)}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}^2 du
$$

$$
\leq \frac{p^2}{2} \int_0^s \left\| (-A)^{\gamma+\rho-\beta} e^{(s-\llcorner u \lrcorner_\theta)A} \right\|_{L(H)}^2 (t-s)^{2\rho}
$$

$$
\cdot \left\| \|B\|_{\mathrm{HS}(U,H_\beta)} + \frac{2\|X_{\llcorner u \lrcorner_\theta,u}\|_{H_\beta}\|X_{\llcorner u \lrcorner_\theta,u}\|_H\|B\|_{\mathrm{HS}(U,H)}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}^2 du.
$$

Furthermore, note that the fact that $\gamma + \rho - \beta \in [0, 1/2)$ and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] hence imply for every $s \in [0, T]$, $t \in [s, T]$ that

$$
\left\| \int_0^s \chi_{\llcorner u \lrcorner_\theta} \left[ e^{(t-\llcorner u \lrcorner_\theta)A} - e^{(s-\llcorner u \lrcorner_\theta)A} \right] \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}^2
$$

$$
\leq \frac{p^2}{2} \int_0^s \left\| (-A)^{\gamma+\rho-\beta} e^{(s-\llcorner u \lrcorner_\theta)A} \right\|_{L(H)}^2 (t-s)^{2\rho} \left[ \|B\|_{\mathrm{HS}(U,H_\beta)} + \|X_{\llcorner u \lrcorner_\theta,u}\|_{\mathcal{L}^p(\mathbb{P};H_\beta)} \|B\|_{\mathrm{HS}(U,H)} \right]^2 du
$$

$$
\leq \frac{p^2}{2} \int_0^s \left\| (-A)^{\gamma+\rho-\beta} e^{(s-\llcorner u \lrcorner_\theta)A} \right\|_{L(H)}^2 (t-s)^{2\rho}
$$

$$
\cdot \left[ \|B\|_{\mathrm{HS}(U,H_\beta)} + \left[ \frac{p(p-1)}{2} \int_{\llcorner u \lrcorner_\theta}^u \|B\|_{\mathrm{HS}(U,H_\beta)}^2 dw \right]^{1/2} \|B\|_{\mathrm{HS}(U,H)} \right]^2 du \tag{36}
$$

$$
\leq \frac{p^2}{2} \int_0^s \left\| (-A)^{\gamma+\rho-\beta} e^{(s-\llcorner u \lrcorner_\theta)A} \right\|_{L(H)}^2 (t-s)^{2\rho}
$$

$$
\cdot \left[ \|B\|_{\mathrm{HS}(U,H_\beta)} + p\|B\|_{\mathrm{HS}(U,H_\beta)}\|B\|_{\mathrm{HS}(U,H)}(u - \llcorner u \lrcorner_\theta)^{1/2} \right]^2 du
$$

$$
\leq \frac{p^2}{2}(t-s)^{2\rho} \|B\|_{\mathrm{HS}(U,H_\beta)}^2 \left( 1 + p\sqrt{T}\|B\|_{\mathrm{HS}(U,H)} \right)^2 \int_0^s \frac{1}{(s-\llcorner u \lrcorner_\theta)^{2(\gamma+\rho-\beta)}} du.
$$

In addition, observe that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\int_0^s \frac{du}{(s-\llcorner u \lrcorner_\theta)^{2(\gamma+\rho-\beta)}} \leq \int_0^s \frac{du}{(s-u)^{2(\gamma+\rho-\beta)}} = \frac{s^{1+2(\beta-\gamma-\rho)}}{1+2(\beta-\gamma-\rho)} \leq \frac{T^{1+2(\beta-\gamma-\rho)}}{1+2(\beta-\gamma-\rho)}. \tag{37}
$$

Combining this and (36) demonstrates that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\left\| \int_0^s \chi_{\llcorner u \lrcorner_\theta} \left[ e^{(t-\llcorner u \lrcorner_\theta)A} - e^{(s-\llcorner u \lrcorner_\theta)A} \right] \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}
$$

$$
\leq \frac{p}{\sqrt{2}}(t-s)^\rho \|B\|_{\mathrm{HS}(U,H_\beta)} \left( 1 + p\sqrt{T}\|B\|_{\mathrm{HS}(U,H)} \right) \frac{T^{1/2+\beta-\gamma-\rho}}{\sqrt{1+2(\beta-\gamma-\rho)}}. \tag{38}
$$

11

In the next step we observe that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\int_0^s \left\| \left[ e^{(t - \llcorner u \lrcorner_\theta)A} - e^{(s - \llcorner u \lrcorner_\theta)A} \right] \left[ \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U, H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right] \right\|_{\mathcal{L}^p(\mathbb{P}; H_\gamma)} du
$$

$$
\leq \int_0^s \left\| \|(-A)^{\gamma + \rho - \beta} e^{(s - \llcorner u \lrcorner_\theta)A}\|_{L(H)} \|(-A)^{-\rho}(e^{(t-s)A} - \mathrm{Id}_H)\|_{L(H)} \right.
$$
$$
\left. \cdot \left\| \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U, H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right\|_{H_\beta} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})} du
$$

$$
\leq \int_0^s \|(-A)^{\gamma + \rho - \beta} e^{(s - \llcorner u \lrcorner_\theta)A}\|_{L(H)} (t - s)^\rho \tag{39}
$$
$$
\cdot \left\| \left[ \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta, u}\|_U^2}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} + \frac{2\|B\mathbb{B}\|_{L(H_\beta, H_\beta)} + \|B\|_{\mathrm{HS}(U, H)}^2}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right] \|X_{\llcorner u \lrcorner_\theta, u}\|_{H_\beta} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})} du
$$

$$
\leq \int_0^s \|(-A)^{\gamma + \rho - \beta} e^{(s - \llcorner u \lrcorner_\theta)A}\|_{L(H)} (t - s)^\rho
$$
$$
\cdot \left\| \left[ \frac{4\|\mathbb{B}\|_{L(H, U)}^2 \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} + \frac{2\|B\|_{L(U, H_\beta)} \|\mathbb{B}\|_{L(H_\beta, U)} + \|B\|_{\mathrm{HS}(U, H)}^2}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right] \|X_{\llcorner u \lrcorner_\theta, u}\|_{H_\beta} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})} du.
$$

This, the fact that $\|\mathbb{B}\|_{L(H_\beta, U)} \leq |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|\mathbb{B}\|_{L(H, U)}$, and the fact that $\|B\|_{\mathrm{HS}(U, H)} \leq |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|B\|_{\mathrm{HS}(U, H_\beta)}$ show for every $s \in [0, T]$, $t \in [s, T]$ that

$$
\int_0^s \left\| \left[ e^{(t - \llcorner u \lrcorner_\theta)A} - e^{(s - \llcorner u \lrcorner_\theta)A} \right] \left[ \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U, H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right] \right\|_{\mathcal{L}^p(\mathbb{P}; H_\gamma)} du
$$

$$
\leq (t - s)^\rho \int_0^s \|(-A)^{\gamma + \rho - \beta} e^{(s - \llcorner u \lrcorner_\theta)A}\|_{L(H)}
$$
$$
\cdot \left\| \left( \|\mathbb{B}\|_{L(H, U)}^2 + 2\|B\|_{L(U, H_\beta)} \|\mathbb{B}\|_{L(H_\beta, U)} + \|B\|_{\mathrm{HS}(U, H)}^2 \right) \|X_{\llcorner u \lrcorner_\theta, u}\|_{H_\beta} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})} du
$$

$$
\leq (t - s)^\rho \int_0^s \|(-A)^{\gamma + \rho - \beta} e^{(s - \llcorner u \lrcorner_\theta)A}\|_{L(H)} \tag{40}
$$
$$
\cdot \left( \|\mathbb{B}\|_{L(H, U)}^2 + 2|\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|B\|_{\mathrm{HS}(U, H_\beta)} \|\mathbb{B}\|_{L(H, U)} + \|B\|_{\mathrm{HS}(U, H)}^2 \right) \|X_{\llcorner u \lrcorner_\theta, u}\|_{\mathcal{L}^p(\mathbb{P}; H_\beta)} du
$$

$$
\leq 4(t - s)^\rho \|B\|_{\mathrm{HS}(U, H_\beta)}^2 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}
$$
$$
\cdot \int_0^s \|X_{\llcorner u \lrcorner_\theta, u}\|_{\mathcal{L}^p(\mathbb{P}; H_\beta)} \|(-A)^{\gamma + \rho - \beta} e^{(s - \llcorner u \lrcorner_\theta)A}\|_{L(H)} du.
$$

The Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] therefore proves for every $s \in [0, T]$, $t \in [s, T]$ that

$$
\int_0^s \left\| \left[ e^{(t - \llcorner u \lrcorner_\theta)A} - e^{(s - \llcorner u \lrcorner_\theta)A} \right] \left[ \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U, H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right] \right\|_{\mathcal{L}^p(\mathbb{P}; H_\gamma)} du
$$

$$
\leq 4(t - s)^\rho \|B\|_{\mathrm{HS}(U, H_\beta)}^2 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}
$$
$$
\cdot \int_0^s \left[ \frac{p(p-1)}{2} \int_{\llcorner u \lrcorner_\theta}^u \|B\|_{\mathrm{HS}(U, H_\beta)}^2 dw \right]^{1/2} \|(-A)^{\gamma + \rho - \beta} e^{(s - \llcorner u \lrcorner_\theta)A}\|_{L(H)} du \tag{41}
$$

$$
\leq 2\sqrt{2} p (t - s)^\rho \|B\|_{\mathrm{HS}(U, H_\beta)}^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}
$$
$$
\cdot \int_0^s (u - \llcorner u \lrcorner_\theta)^{1/2} \|(-A)^{\gamma + \rho - \beta} e^{(s - \llcorner u \lrcorner_\theta)A}\|_{L(H)} du
$$

$$
\leq 2\sqrt{2} p [|\theta|_T]^{1/2} (t - s)^\rho \|B\|_{\mathrm{HS}(U, H_\beta)}^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \int_0^s \frac{du}{(s - \llcorner u \lrcorner_\theta)^{\gamma + \rho - \beta}}.
$$

Moreover, note that for every $s \in [0, T]$ it holds that

$$
\int_0^s (s - \llcorner u \lrcorner_\theta)^{\beta - \gamma - \rho} du \leq \int_0^s (s - u)^{\beta - \gamma - \rho} du = \frac{s^{1 + \beta - \gamma - \rho}}{1 + \beta - \gamma - \rho} \leq \frac{T^{1 + \beta - \gamma - \rho}}{1 + \beta - \gamma - \rho}. \tag{42}
$$

Estimate (41) hence establishes for every $s \in [0, T]$, $t \in [s, T]$ that

$$
\begin{aligned}
&\int_0^s \left\| \left[ e^{(t - \llcorner u \lrcorner_\theta)A} - e^{(s - \llcorner u \lrcorner_\theta)A} \right] \left[ \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta, u}\|_U^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right] \right\|_{\mathcal{L}^p(\mathbb{P}; H_\gamma)} du \\
&\leq 2\sqrt{2} p [|\theta|_T]^{1/2} (t - s)^\rho \|B\|_{\mathrm{HS}(U, H_\beta)}^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \frac{T^{1+\beta-\gamma-\rho}}{1+\beta-\gamma-\rho}.
\end{aligned} \tag{43}
$$

Combining (30)–(34), the fact that $\|B\|_{\mathrm{HS}(U,H)} \leq |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|B\|_{\mathrm{HS}(U,H_\beta)}$, (38), (43), the fact that $\sqrt{1 + 2\beta - 2\gamma} \leq 1 + \beta - \gamma$, the fact that $\sqrt{1 + 2(\beta - \gamma - \rho)} \leq 1 + \beta - \gamma - \rho$, and the fact that $\frac{1}{\sqrt{2}} + \frac{p}{\sqrt{2}} \|B\|_{\mathrm{HS}(U,H_\beta)} + 2\sqrt{2} \|B\|_{\mathrm{HS}(U,H_\beta)}^2 \leq 2(1 + \|B\|_{\mathrm{HS}(U,H_\beta)}^2) p$ implies that for every $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\begin{aligned}
(\mathbb{E}[\|\mathbf{O}_t - \mathbf{O}_s\|_{H_\gamma}^p])^{1/p} &\leq \frac{p\|B\|_{\mathrm{HS}(U,H_\beta)} \max\{|\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}, 1\}}{\sqrt{2(1+2\beta-2\gamma)}} \\
&\quad \cdot \left( (t-s)^{1/2+\beta-\gamma} + 4p^2 \|B\|_{\mathrm{HS}(U,H_\beta)}^2 |\theta|_T (t-s)^{1/2+\beta-\gamma} + 4\|B\|_{\mathrm{HS}(U,H_\beta)}^2 [|\theta|_T]^{1/2} (t-s)^{1+\beta-\gamma} \right) \\
&\quad + \frac{p}{\sqrt{2}} (t-s)^\rho \|B\|_{\mathrm{HS}(U,H_\beta)} \left( 1 + p\sqrt{T} |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|B\|_{\mathrm{HS}(U,H_\beta)} \right) \frac{T^{1/2+\beta-\gamma-\rho}}{\sqrt{1+2(\beta-\gamma-\rho)}} \\
&\quad + 2\sqrt{2} p [|\theta|_T]^{1/2} (t-s)^\rho \|B\|_{\mathrm{HS}(U,H_\beta)}^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \frac{T^{1+\beta-\gamma-\rho}}{1+\beta-\gamma-\rho} \\
&\leq \frac{p\|B\|_{\mathrm{HS}(U,H_\beta)} \max\{|\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}, 1\} \max\{T, 1\}}{\sqrt{2(1+2\beta-2\gamma)}} \left( 1 + 4p^2 \|B\|_{\mathrm{HS}(U,H_\beta)}^2 + 4\|B\|_{\mathrm{HS}(U,H_\beta)}^2 \right) \\
&\quad \cdot (t-s)^{1/2+\beta-\gamma} + p(t-s)^\rho \|B\|_{\mathrm{HS}(U,H_\beta)} \max\left\{ |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}, 1 \right\} \\
&\quad \cdot \left( \frac{1}{\sqrt{2}} + \frac{p}{\sqrt{2}} \|B\|_{\mathrm{HS}(U,H_\beta)} + 2\sqrt{2} \|B\|_{\mathrm{HS}(U,H_\beta)}^2 \right) \frac{[\max\{T,1\}]^{3/2+\beta}}{\sqrt{1+2(\beta-\gamma-\rho)}} \\
&\leq \frac{p^3 \|B\|_{\mathrm{HS}(U,H_\beta)} \max\{|\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}, 1\} \max\{T, 1\}}{\sqrt{2(1+2(\beta-\gamma-\rho))}} \left( 1 + 8\|B\|_{\mathrm{HS}(U,H_\beta)}^2 \right) (t-s)^{1/2+\beta-\gamma} \\
&\quad + 2p^2 (t-s)^\rho \|B\|_{\mathrm{HS}(U,H_\beta)} \max\left\{ |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}, 1 \right\} \left( 1 + \|B\|_{\mathrm{HS}(U,H_\beta)}^2 \right) \frac{[\max\{T,1\}]^{3/2+\beta}}{\sqrt{1+2(\beta-\gamma-\rho)}} \\
&\leq \frac{3p^3 \|B\|_{\mathrm{HS}(U,H_\beta)} [\max\{T,1\}]^{3/2+\beta} \max\{|\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}, 1\} (1 + 8\|B\|_{\mathrm{HS}(U,H_\beta)}^2)}{\sqrt{1+2(\beta-\gamma-\rho)}} (t-s)^\rho.
\end{aligned} \tag{44}
$$

The proof of Lemma 2.4 is thus completed. $\qquad\square$

## 2.3 Error estimates for temporally semi-discrete approximations of stochastic convolutions

In this subsection we combine Lemma 2.1 and Lemma 2.2 above to establish in Lemma 2.5 below for every $p \in [2, \infty)$ an upper bound for the strong $L^p$-distance between the approximation process $\mathbf{O} \colon [0, T] \times \Omega \to H_\gamma$ from Setting 2.1 and a suitable stochastic convolution process related to $\mathbf{O} \colon [0, T] \times \Omega \to H_\gamma$.

**Lemma 2.5.** *Assume Setting 2.1, let $C \in [1, \infty)$, $\tilde{B} \in \mathrm{HS}(U, H_\beta)$, $p \in [2, \infty)$, $\eta \in [0, 1/2 + \beta - \gamma)$, $\rho \in [0, 1/2 + \beta - \gamma) \cap [0, 1/2)$, assume for every $s \in [0, T]$ that $\|\chi_{\llcorner s \lrcorner_\theta} - 1\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})} \leq C[|\theta|_T]^\rho$, and let $O \colon [0, T] \times \Omega \to H_\gamma$ be a stochastic process which satisfies for every $t \in [0, T]$ that $[O_t]_{\mathbb{P}, \mathcal{B}(H_\gamma)} =$*

$\int_0^t e^{(t-s)A} \tilde{B} \, dW_s$. *Then*

$$\sup_{t \in [0,T]} \|\mathbf{O}_t - O_t\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \leq \frac{p[\max\{T,1\}]^{1/2+\beta}}{\sqrt{2(1-2\max\{0,\gamma+\eta-\beta\})}} \|(-A)^{\min\{0,\gamma+\eta-\beta\}}\|_{L(H)} \|\tilde{B}-B\|_{\mathrm{HS}(U,H_{\beta-\eta})}$$
$$+ \frac{8p^3 C[\max\{T,1\}]^{3/2+\beta}}{\sqrt{1-2\rho-2\max\{0,\gamma-\beta\}}} \|B\|_{\mathrm{HS}(U,H_\beta)} \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \quad (45)$$
$$\cdot \left(1 + |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|^2_{\mathrm{HS}(U,H_\beta)}\right)[|\theta|_T]^\rho.$$

*Proof of Lemma 2.5.* Throughout this proof for every $s \in [0,T]$ let $X_{s,(\cdot)}(\cdot) = (X_{s,t}(\omega))_{(t,\omega) \in [s,T] \times \Omega}$: $[s,T] \times \Omega \to H_\beta$ be an $(\mathbb{F}_t)_{t \in [s,T]}$-adapted stochastic process with continuous sample paths which satisfies for every $t \in [s,T]$ that $[X_{s,t}]_{\mathbb{P},\mathcal{B}(H_\beta)} = \int_s^t B \, dW_u$. Observe that Lemma 2.1 (with $X_{s,t} = X_{s,t}$ for $t \in [s,T]$, $s \in [0,T]$ in the notation of Lemma 2.1) and the triangle inequality prove for every $t \in [0,T]$ that

$$\|\mathbf{O}_t - O_t\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \leq \left\| \int_0^t e^{(t-u)A}(B-\tilde{B}) \, dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}$$
$$+ \left\| \int_0^t \left( \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left( \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u},B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right) - e^{(t-u)A}B \right) dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \quad (46)$$
$$+ \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left( \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}\|_U^2 X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta,u}+\|B\|^2_{\mathrm{HS}(U,H)}X_{\llcorner u \lrcorner_\theta,u}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)}.$$

In the next step we note that the fact that $\eta < 1/2 + \beta - \gamma$ ensures that $\max\{\gamma+\eta-\beta,0\} < 1/2$. The Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] hence shows that for every $t \in [0,T]$ it holds that

$$\left\| \int_0^t e^{(t-u)A}(B-\tilde{B}) \, dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \left( \int_0^t \|e^{(t-u)A}(\tilde{B}-B)\|^2_{\mathrm{HS}(U,H_\gamma)} \, du \right)^{1/2}$$
$$\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \|\tilde{B}-B\|_{\mathrm{HS}(U,H_{\beta-\eta})} \left( \int_0^t \|(-A)^{\gamma+\eta-\beta}e^{(t-u)A}\|^2_{L(H)} \, du \right)^{1/2}$$
$$\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \|\tilde{B}-B\|_{\mathrm{HS}(U,H_{\beta-\eta})} \left( \int_0^t \|(-A)^{\min\{0,\gamma+\eta-\beta\}}\|^2_{L(H)} \|(-A)^{\max\{0,\gamma+\eta-\beta\}}e^{(t-u)A}\|^2_{L(H)} \, du \right)^{1/2}$$
$$\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \|\tilde{B}-B\|_{\mathrm{HS}(U,H_{\beta-\eta})} \|(-A)^{\min\{0,\gamma+\eta-\beta\}}\|_{L(H)} \left( \int_0^t (t-u)^{-2\max\{0,\gamma+\eta-\beta\}} \, du \right)^{1/2}$$
$$= \frac{\sqrt{p(p-1)}}{\sqrt{2}} \|\tilde{B}-B\|_{\mathrm{HS}(U,H_{\beta-\eta})} \|(-A)^{\min\{0,\gamma+\eta-\beta\}}\|_{L(H)} \frac{t^{1/2-\max\{0,\gamma+\eta-\beta\}}}{\sqrt{1-2\max\{0,\gamma+\eta-\beta\}}}.$$
$$(47)$$

Furthermore, observe that the triangle inequality implies for every $t \in [0,T]$ that

$$\left\| \int_0^t \left( \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u},B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] - e^{(t-u)A}B \right) dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}$$
$$\leq \left\| \int_0^t \left( \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - e^{(t-u)A}B \right) dW_u \right\|_{L^p(\mathbb{P};H_\gamma)} \quad (48)$$
$$+ \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u},B \, dW_u\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right\|_{L^p(\mathbb{P};H_\gamma)}.$$

In addition, note that the triangle inequality assures for every $t \in [0, T]$ that

$$
\begin{aligned}
&\left\| \int_0^t \left( \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \frac{B}{1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2} - e^{(t-u)A} B \right) dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)} \\
&\leq \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \left[ \frac{B}{1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2} - B \right] dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)} \\
&\quad + \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} \left( e^{(t - \llcorner u \lrcorner_\theta)A} - e^{(t-u)A} \right) B \, dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)} \\
&\quad + \left\| \int_0^t (\chi_{\llcorner u \lrcorner_\theta} - 1) e^{(t-u)A} B \, dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)}.
\end{aligned}
\tag{49}
$$

Moreover, observe that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] demonstrates that for every $t \in [0, T]$ it holds that

$$
\begin{aligned}
&\left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \left[ \frac{B}{1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2} - B \right] dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)} \\
&\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \left( \int_0^t \left\| e^{(t - \llcorner u \lrcorner_\theta)A} \frac{\|X_{\llcorner u \lrcorner_\theta, u}\|_H^2 B}{1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathrm{HS}(U, H_\gamma))}^2 du \right)^{1/2} \\
&\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \left( \int_0^t \|(-A)^{\gamma - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}^2 \left\| \frac{\|X_{\llcorner u \lrcorner_\theta, u}\|_H^2 B}{1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2} \right\|_{\mathcal{L}^p(\mathbb{P}; \mathrm{HS}(U, H_\beta))}^2 du \right)^{1/2} \\
&\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \left( \int_0^t \|(-A)^{\gamma - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}^2 \|X_{\llcorner u \lrcorner_\theta, u}\|_{\mathcal{L}^{2p}(\mathbb{P}; H)}^4 \|B\|_{\mathrm{HS}(U, H_\beta)}^2 du \right)^{1/2}.
\end{aligned}
\tag{50}
$$

The Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] and the fact that $\|B\|_{\mathrm{HS}(U,H)} \leq |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-\beta} \|B\|_{\mathrm{HS}(U, H_\beta)}$ hence ensure that for every $t \in [0, T]$ it holds that

$$
\begin{aligned}
&\left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t - \llcorner u \lrcorner_\theta)A} \left[ \frac{B}{1 + \|X_{\llcorner u \lrcorner_\theta, u}\|_H^2} - B \right] dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)} \\
&\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \left( \int_0^t \|(-A)^{\gamma - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}^2 \|B\|_{\mathrm{HS}(U, H_\beta)}^2 \left( p(2p-1) \int_{\llcorner u \lrcorner_\theta}^u \|B\|_{\mathrm{HS}(U, H)}^2 ds \right)^2 du \right)^{1/2} \\
&\leq \sqrt{2} p^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|_{\mathrm{HS}(U, H_\beta)}^3 \left( \int_0^t \|(-A)^{\gamma - \beta} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}^2 (u - \llcorner u \lrcorner_\theta)^2 du \right)^{1/2} \\
&\leq \sqrt{2} p^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|_{\mathrm{HS}(U, H_\beta)}^3 \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)} \\
&\quad \cdot \left( \int_0^t \|(-A)^{\max\{0, \gamma - \beta\}} e^{(t - \llcorner u \lrcorner_\theta)A}\|_{L(H)}^2 (u - \llcorner u \lrcorner_\theta)^2 du \right)^{1/2} \\
&\leq \sqrt{2} p^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|_{\mathrm{HS}(U, H_\beta)}^3 \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)} \\
&\quad \cdot |\theta|_T \left( \int_0^t (t - \llcorner u \lrcorner_\theta)^{-2 \max\{0, \gamma - \beta\}} du \right)^{1/2}.
\end{aligned}
\tag{51}
$$

Furthermore, observe that the fact that $1 - 2\max\{0, \gamma - \beta\} = \max\{1, 1 - 2\gamma + 2\beta\} > 0$ ensures that for every $t \in [0, T]$ it holds that

$$
\int_0^t (t - \llcorner u \lrcorner_\theta)^{-2 \max\{0, \gamma - \beta\}} du \leq \int_0^t (t - u)^{-2 \max\{0, \gamma - \beta\}} du = \frac{t^{1 - 2 \max\{0, \gamma - \beta\}}}{1 - 2 \max\{0, \gamma - \beta\}}.
\tag{52}
$$

15

Combining this and (51) implies that for every $t \in [0, T]$ it holds that

$$
\begin{aligned}
&\left\| \int_0^t \chi_{\llcorner u \lrcorner \theta} e^{(t - \llcorner u \lrcorner \theta)A} \left[ \frac{B}{1 + \|X_{\llcorner u \lrcorner \theta, u}\|_H^2} - B \right] dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)} \\
&\leq \sqrt{2} p^3 |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|_{\mathrm{HS}(U, H_\beta)}^3 \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)} \frac{t^{1/2 - \max\{0, \gamma - \beta\}}}{\sqrt{1 - 2\max\{0, \gamma - \beta\}}} |\theta|_T.
\end{aligned}
\tag{53}
$$

In the next step we note that the fact that $1 - 2\rho - 2\max\{0, \gamma - \beta\} = 1 - 2\rho + 2\min\{0, \beta - \gamma\} > 0$ and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] show that for every $t \in [0, T]$ it holds that

$$
\begin{aligned}
&\left\| \int_0^t \chi_{\llcorner u \lrcorner \theta} (e^{(t - \llcorner u \lrcorner \theta)A} - e^{(t - u)A}) B \, dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)} \\
&\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \left( \int_0^t \|(e^{(t - \llcorner u \lrcorner \theta)A} - e^{(t - u)A})B\|_{\mathrm{HS}(U, H_\gamma)}^2 \, du \right)^{1/2} \\
&\leq \frac{p}{\sqrt{2}} \left( \int_0^t \|(-A)^{\gamma + \rho - \beta} e^{(t - u)A}\|_{L(H)}^2 \|(-A)^{-\rho}(e^{(u - \llcorner u \lrcorner \theta)A} - \mathrm{Id}_H)\|_{L(H)}^2 \|B\|_{\mathrm{HS}(U, H_\beta)}^2 \, du \right)^{1/2} \\
&\leq \frac{p}{\sqrt{2}} \|B\|_{\mathrm{HS}(U, H_\beta)} \left( \int_0^t \|(-A)^{\gamma + \rho - \beta} e^{(t - u)A}\|_{L(H)}^2 (u - \llcorner u \lrcorner \theta)^{2\rho} \, du \right)^{1/2} \\
&\leq \frac{p}{\sqrt{2}} \|B\|_{\mathrm{HS}(U, H_\beta)} \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)} \\
&\quad \cdot \left( \int_0^t \|(-A)^{\rho + \max\{0, \gamma - \beta\}} e^{(t - u)A}\|_{L(H)}^2 (u - \llcorner u \lrcorner \theta)^{2\rho} \, du \right)^{1/2} \\
&\leq \frac{p}{\sqrt{2}} \|B\|_{\mathrm{HS}(U, H_\beta)} \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)} \left( \int_0^t \frac{|\theta|_T^{2\rho}}{(t - u)^{2\rho + 2\max\{0, \gamma - \beta\}}} \, du \right)^{1/2} \\
&\leq \frac{p}{\sqrt{2}} \|B\|_{\mathrm{HS}(U, H_\beta)} \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)} \frac{t^{1/2 - \rho - \max\{0, \gamma - \beta\}}}{\sqrt{1 - 2\rho - 2\max\{0, \gamma - \beta\}}} [|\theta|_T]^\rho.
\end{aligned}
\tag{54}
$$

Moreover, observe that the assumption that $\forall\, u \in [0, T], \theta \in \varpi_T \colon \|\chi_{\llcorner u \lrcorner \theta} - 1\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})} \leq C[|\theta|_T]^\rho$ and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] establish for every $t \in [0, T]$ that

$$
\begin{aligned}
&\left\| \int_0^t (\chi_{\llcorner u \lrcorner \theta} - 1) e^{(t - u)A} B \, dW_u \right\|_{L^p(\mathbb{P}; H_\gamma)} \\
&\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \left( \int_0^t \left\| (\chi_{\llcorner u \lrcorner \theta} - 1) e^{(t - u)A} B \right\|_{\mathcal{L}^p(\mathbb{P}; \mathrm{HS}(U, H_\gamma))}^2 \, du \right)^{1/2} \\
&\leq \frac{p}{\sqrt{2}} \left( \int_0^t \|\chi_{\llcorner u \lrcorner \theta} - 1\|_{\mathcal{L}^p(\mathbb{P}; \mathbb{R})}^2 \|(-A)^{\gamma - \beta} e^{(t - u)A}\|_{L(H)}^2 \|B\|_{\mathrm{HS}(U, H_\beta)}^2 \, du \right)^{1/2} \\
&\leq \frac{pC}{\sqrt{2}} \|B\|_{\mathrm{HS}(U, H_\beta)} [|\theta|_T]^\rho \left( \int_0^t \|(-A)^{\gamma - \beta} e^{(t - u)A}\|_{L(H)}^2 \, du \right)^{1/2} \\
&\leq \frac{pC}{\sqrt{2}} \|B\|_{\mathrm{HS}(U, H_\beta)} [|\theta|_T]^\rho \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)} \left( \int_0^t \|(-A)^{\max\{0, \gamma - \beta\}} e^{(t - u)A}\|_{L(H)}^2 \, du \right)^{1/2} \\
&\leq \frac{pC}{\sqrt{2}} \|B\|_{\mathrm{HS}(U, H_\beta)} [|\theta|_T]^\rho \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)} \left( \int_0^t (t - u)^{-2\max\{0, \gamma - \beta\}} \, du \right)^{1/2} \\
&= \frac{pC}{\sqrt{2}} \|B\|_{\mathrm{HS}(U, H_\beta)} [|\theta|_T]^\rho \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)} \frac{t^{1/2 - \max\{0, \gamma - \beta\}}}{\sqrt{1 - 2\max\{0, \gamma - \beta\}}}.
\end{aligned}
\tag{55}
$$

Combining this, (49), (53), (54), and Hölder's inequality demonstrates that for every $t \in [0, T]$ it

holds that

$$\left\| \int_0^t \left( \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta, u}\|_H^2} - e^{(t-u)A}B \right) dW_u \right\|_{L^p(\mathbb{P};H_\gamma)}$$

$$\leq \sqrt{2}p^3 |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|_{\mathrm{HS}(U,H_\beta)}^3 \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \frac{t^{1/2-\max\{0,\gamma-\beta\}}}{\sqrt{1-2\max\{0,\gamma-\beta\}}} |\theta|_T$$

$$+ \frac{p}{\sqrt{2}} \|B\|_{\mathrm{HS}(U,H_\beta)} \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \frac{t^{1/2-\rho-\max\{0,\gamma-\beta\}}}{\sqrt{1-2\rho-2\max\{0,\gamma-\beta\}}} [|\theta|_T]^\rho$$

$$+ \frac{pC}{\sqrt{2}} \|B\|_{\mathrm{HS}(U,H_\beta)} [|\theta|_T]^\rho \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \frac{t^{1/2-\max\{0,\gamma-\beta\}}}{\sqrt{1-2\max\{0,\gamma-\beta\}}}$$

$$\leq \frac{pC}{\sqrt{2(1-2\rho-2\max\{0,\gamma-\beta\})}} \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \|B\|_{\mathrm{HS}(U,H_\beta)} \left( 1 + |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|_{\mathrm{HS}(U,H_\beta)}^2 \right) \tag{56}$$

$$\cdot \max\left\{ t^{1/2+\min\{0,\beta-\gamma\}}, t^{1/2-\rho+\min\{0,\beta-\gamma\}} \right\} \left[ 2p^2 |\theta|_T + [|\theta|_T]^\rho + [|\theta|_T]^\rho \right]$$

$$\leq \frac{pC[\max\{T,1\}]^{1/2+\beta}}{\sqrt{2(1-2\rho-2\max\{0,\gamma-\beta\})}} \|B\|_{\mathrm{HS}(U,H_\beta)} \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)}$$

$$\cdot \left( 1 + |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|_{\mathrm{HS}(U,H_\beta)}^2 \right) \left[ 2p^2 |\theta|_T + 2[|\theta|_T]^\rho \right]$$

$$\leq \frac{3p^3 C[\max\{T,1\}]^{3/2+\beta}}{\sqrt{2(1-2\rho-2\max\{0,\gamma-\beta\})}} \|B\|_{\mathrm{HS}(U,H_\beta)} \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)}$$

$$\cdot \left( 1 + |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|_{\mathrm{HS}(U,H_\beta)}^2 \right) [|\theta|_T]^\rho.$$

Furthermore, note that the fact that $\gamma = \max\{\beta,\gamma\} + \min\{0,\gamma-\beta\}$ and Lemma 2.2 (with $p=p$, $\rho = \max\{\beta,\gamma\}$, $\eta = \max\{\beta,\gamma\}$, $X_{s,t} = X_{s,t}$ for $t\in[s,T]$, $s\in[0,T]$, $h\in(0,T]$ in the notation of Lemma 2.2) prove that for every $t\in[0,T]$ it holds that

$$\left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{2X_{\llcorner u \lrcorner_\theta, u}\langle X_{\llcorner u \lrcorner_\theta, u}, B\, dW_u\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right\|_{L^p(\mathbb{P};H_\gamma)}$$

$$\leq \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \frac{2X_{\llcorner u \lrcorner_\theta, u}\langle X_{\llcorner u \lrcorner_\theta, u}, B\, dW_u\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right\|_{L^p(\mathbb{P};H_{\max\{\beta,\gamma\}})} \tag{57}$$

$$\leq 2\sqrt{2}p^3 \|B\|_{\mathrm{HS}(U,H_\beta)}^3 |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \frac{t^{1/2+\beta-\max\{\beta,\gamma\}}}{\sqrt{1+2\beta-2\max\{\beta,\gamma\}}} |\theta|_T$$

and

$$\left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left( \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta, u}\|_H^2 X_{\llcorner u \lrcorner_\theta, u}}{(1+\|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1+\|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)}$$

$$\leq \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)}$$

$$\cdot \left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left( \frac{4\|\mathbb{B}X_{\llcorner u \lrcorner_\theta, u}\|_H^2 X_{\llcorner u \lrcorner_\theta, u}}{(1+\|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u \lrcorner_\theta, u} + \|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u \lrcorner_\theta, u}}{(1+\|X_{\llcorner u \lrcorner_\theta, u}\|_H^2)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P};H_{\max\{\beta,\gamma\}})} \tag{58}$$

$$\leq 2\sqrt{2}p \|B\|_{\mathrm{HS}(U,H_\beta)}^3 |\sup_{h\in\mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \frac{t^{1+\beta-\max\{\beta,\gamma\}}}{1+\beta-\max\{\beta,\gamma\}} [|\theta|_T]^{1/2}.$$

Moreover, observe that

$$\frac{1}{\sqrt{1+2\beta-2\max\{\beta,\gamma\}}} = \frac{1}{\sqrt{1-2\max\{0,\gamma-\beta\}}} \leq \frac{1}{\sqrt{1-2\rho-2\max\{0,\gamma-\beta\}}}. \tag{59}$$

Combining (46), (47), (48), (56), (57), (58), the fact that $\sqrt{1+2\beta-2\max\{\beta,\gamma\}} \leq 1+\beta-\max\{\beta,\gamma\}$, and the fact that $\frac{3}{\sqrt{2}} + 2\sqrt{2} + 2\sqrt{2} \leq 8$ hence ensures that for every $t\in[0,T]$ it holds

17

that

$$
\begin{aligned}
\|\mathbf{O}_t - O_t\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \leq\ & \|\tilde{B} - B\|_{\mathrm{HS}(U,H_{\beta-\eta})}\|(-A)^{\min\{0,\gamma+\eta-\beta\}}\|_{L(H)}\frac{p[\max\{T,1\}]^{1/2+\beta}}{\sqrt{2(1-2\max\{0,\gamma+\eta-\beta\})}} \\
& + \frac{3p^3 C[\max\{T,1\}]^{3/2+\beta}}{\sqrt{2(1-2\rho-2\max\{0,\gamma-\beta\})}}\|B\|_{\mathrm{HS}(U,H_\beta)}\|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \\
& \cdot \Big(1 + |\sup_{h\in\mathbb{H}}\mathfrak{v}_h|^{-2\beta}\|B\|^2_{\mathrm{HS}(U,H_\beta)}\Big)[|\theta|_T]^\rho \\
& + 2\sqrt{2}p^3\|B\|^3_{\mathrm{HS}(U,H_\beta)}|\sup_{h\in\mathbb{H}}\mathfrak{v}_h|^{-2\beta}\|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)}\frac{[\max\{T,1\}]^{1/2+\beta}}{\sqrt{1+2\beta-2\max\{\beta,\gamma\}}}|\theta|_T \\
& + 2\sqrt{2}p\|B\|^3_{\mathrm{HS}(U,H_\beta)}|\sup_{h\in\mathbb{H}}\mathfrak{v}_h|^{-2\beta}\|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)}\frac{[\max\{T,1\}]^{1+\beta}}{\sqrt{1+2\beta-2\max\{\beta,\gamma\}}}[|\theta|_T]^{1/2} \\
\leq\ & \frac{p[\max\{T,1\}]^{1/2+\beta}}{\sqrt{2(1-2\max\{0,\gamma+\eta-\beta\})}}\|(-A)^{\min\{0,\gamma+\eta-\beta\}}\|_{L(H)}\|\tilde{B}-B\|_{\mathrm{HS}(U,H_{\beta-\eta})} \\
& + \frac{8p^3 C[\max\{T,1\}]^{3/2+\beta}}{\sqrt{1-2\rho-2\max\{0,\gamma-\beta\}}}\|B\|_{\mathrm{HS}(U,H_\beta)}\|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \\
& \cdot \Big(1 + |\sup_{h\in\mathbb{H}}\mathfrak{v}_h|^{-2\beta}\|B\|^2_{\mathrm{HS}(U,H_\beta)}\Big)[|\theta|_T]^\rho.
\end{aligned}
\tag{60}
$$

The proof of Lemma 2.5 is thus completed. $\qquad\square$

## 2.4 Exponential moments of temporally semi-discrete approximations of stochastic convolutions

In this subsection we first derive two auxiliary lemmas (see Lemma 2.6 and Lemma 2.7 below) which we then combine to establish in Lemma 2.8 below appropriate exponential moment bounds for the approximation process $\mathbf{O}\colon [0,T]\times\Omega\to H_\gamma$ from Setting 2.1.

**Lemma 2.6.** *Assume Setting 2.1, let $p \in [1,\infty)$, and for every $s \in [0,T]$ let $X_{s,(\cdot)}(\cdot) = (X_{s,t}(\omega))_{(t,\omega)\in[s,T]\times\Omega}\colon [s,T] \times \Omega \to H_\beta$ be an $(\mathbb{F}_t)_{t\in[s,T]}$-adapted stochastic process with continuous sample paths which satisfies for every $t \in [s,T]$ that $[X_{s,t}]_{\mathbb{P},\mathcal{B}(H_\beta)} = \int_s^t B\,dW_u$. Then it holds for every $t \in [0,T]$ that*

$$
\begin{aligned}
&\left\| \int_0^t \chi_{\llcorner u\lrcorner_\theta} e^{(t-\llcorner u\lrcorner_\theta)A}\left( \frac{4\|\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}\|^2_U X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^3} - \frac{2B\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}+\|B\|^2_{\mathrm{HS}(U,H)}X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq 2\|B\|^2_{\mathrm{HS}(U,H)}t.
\end{aligned}
\tag{61}
$$

*Proof of Lemma 2.6.* Note that for every $t \in [0,T]$ it holds that

$$
\begin{aligned}
&\left\| \int_0^t \chi_{\llcorner u\lrcorner_\theta} e^{(t-\llcorner u\lrcorner_\theta)A}\left( \frac{4\|\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}\|^2_U X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^3} - \frac{2B\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}+\|B\|^2_{\mathrm{HS}(U,H)}X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^2} \right) du \right\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \int_0^t \left\| e^{(t-\llcorner u\lrcorner_\theta)A}\left( \frac{4\|\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}\|^2_U X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^3} - \frac{2B\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}+\|B\|^2_{\mathrm{HS}(U,H)}X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^2} \right) \right\|_{\mathcal{L}^p(\mathbb{P};H)} du \\
&\leq \int_0^t \left\| \frac{4\|\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}\|^2_U X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^3} - \frac{2B\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}+\|B\|^2_{\mathrm{HS}(U,H)}X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^2} \right\|_{\mathcal{L}^p(\mathbb{P};H)} du \\
&\leq \int_0^t \left\| \frac{4\|\mathbb{B}\|^2_{L(H,U)}\|X_{\llcorner u\lrcorner_\theta,u}\|^3_H}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^3} + \frac{(2\|B\mathbb{B}\|_{L(H,H)}+\|B\|^2_{\mathrm{HS}(U,H)})\|X_{\llcorner u\lrcorner_\theta,u}\|_H}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|^2_H)^2} \right\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} du \\
&\leq \int_0^t \left( \frac{\|B\|^2_{\mathrm{HS}(U,H)}}{2} + \|B\|_{L(U,H)}\|\mathbb{B}\|_{L(H,U)} + \frac{\|B\|^2_{\mathrm{HS}(U,H)}}{2} \right) du.
\end{aligned}
\tag{62}
$$

This completes the proof of Lemma 2.6. $\qquad\square$

18

**Lemma 2.7.** *Assume Setting 2.1 and for every $s \in [0,T]$ let $X_{s,(\cdot)}(\cdot) = (X_{s,t}(\omega))_{(t,\omega)\in[s,T]\times\Omega}\colon [s,T]\times \Omega \to H_\beta$ be an $(\mathbb{F}_t)_{t\in[s,T]}$-adapted stochastic process with continuous sample paths which satisfies for every $t \in [s,T]$ that $[X_{s,t}]_{\mathbb{P},\mathcal{B}(H_\beta)} = \int_s^t B\, dW_u$. Then it holds for every $n \in \mathbb{N}$, $t \in [0,T]$ that*

$$\left\| \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u \right\|_{L^{2n}(\mathbb{P};H)}^{2n} \tag{63}$$
$$\leq \frac{9^n (2n)!}{8^n n!} \|B\|_{\mathrm{HS}(U,H)}^{2n} t^n.$$

*Proof of Lemma 2.7.* Throughout this proof let $\mathbb{U} \subseteq U$ be an orthonormal basis of $U$, let $Z\colon [0,T]\times \Omega \to H$ be an $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process which satisfies for every $t \in [0,T]$ that

$$[Z_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t \chi_{\llcorner u \lrcorner_\theta} e^{(t-\llcorner u \lrcorner_\theta)A} \left[ \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u, \tag{64}$$

and let $\mathcal{Z}\colon [0,T]\times \Omega \to H$ be an $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process with left-continuous sample paths and finite right limits which satisfies for every $t \in [0,T]$ that

$$[\mathcal{Z}_t]_{\mathbb{P},\mathcal{B}(H)} = [Z_{\llcorner t \lrcorner_\theta}]_{\mathbb{P},\mathcal{B}(H)} + \int_{\llcorner t \lrcorner_\theta}^t \chi_{\llcorner t \lrcorner_\theta} \left[ \frac{B}{1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner t \lrcorner_\theta,u}\langle X_{\llcorner t \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u. \tag{65}$$

Note that Itô's formula proves for every $p \in [2,\infty)$, $t \in [0,T]$ that

$$[\|\mathcal{Z}_t\|_H^p]_{\mathbb{P},\mathcal{B}(\mathbb{R})} = [\|Z_{\llcorner t \lrcorner_\theta}\|_H^p]_{\mathbb{P},\mathcal{B}(\mathbb{R})}$$
$$+ \int_{\llcorner t \lrcorner_\theta}^t \chi_{\llcorner u \lrcorner_\theta} p\|\mathcal{Z}_u\|_H^{p-2} \left\langle \mathcal{Z}_u, \frac{B}{1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner t \lrcorner_\theta,u}\langle X_{\llcorner t \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2)^2} \right\rangle_H dW_u$$
$$+ \left[ \frac{1}{2} \int_{\llcorner t \lrcorner_\theta}^t \left[ \chi_{\llcorner u \lrcorner_\theta} \sum_{\mathbf{u}\in\mathbb{U}} \left\{ p\|\mathcal{Z}_u\|_H^{p-2} \left\| \frac{B\mathbf{u}}{1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner t \lrcorner_\theta,u}\langle X_{\llcorner t \lrcorner_\theta,u}, B\mathbf{u}\rangle_H}{(1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2)^2} \right\|_H^2 \right. \right. \tag{66}$$
$$\left. \left. + p(p-2)\chi_{\llcorner t \lrcorner_\theta} \mathbb{1}_{\{\mathcal{Z}_u\neq 0\}} \|\mathcal{Z}_u\|_H^{p-4} \left| \left\langle \mathcal{Z}_u, \frac{B\mathbf{u}}{1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner t \lrcorner_\theta,u}\langle X_{\llcorner t \lrcorner_\theta,u}, B\mathbf{u}\rangle_H}{(1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2)^2} \right\rangle_H \right|^2 \right\} \right] du \right]_{\mathbb{P},\mathcal{B}(\mathbb{R})}.$$

In addition, observe that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] ensures for every $p \in [2,\infty)$, $t \in [0,T]$ that

$$\|\mathcal{Z}_t\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \|Z_{\llcorner t \lrcorner_\theta}\|_{\mathcal{L}^p(\mathbb{P};H)} + \left\| \int_{\llcorner t \lrcorner_\theta}^t \chi_{\llcorner t \lrcorner_\theta} \left[ \frac{B}{1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner t \lrcorner_\theta,u}\langle X_{\llcorner t \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2)^2} \right] dW_u \right\|_{L^p(\mathbb{P};H)}$$
$$\leq \left[ \frac{p^2}{2} \int_0^{\llcorner t \lrcorner_\theta} \left\| \frac{B}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner u \lrcorner_\theta,u}\langle X_{\llcorner u \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right\|_{\mathcal{L}^p(\mathbb{P};\mathrm{HS}(U,H))}^2 du \right]^{1/2}$$
$$+ \left[ \frac{p^2}{2} \int_{\llcorner t \lrcorner_\theta}^t \left\| \frac{B}{1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2} - \frac{2X_{\llcorner t \lrcorner_\theta,u}\langle X_{\llcorner t \lrcorner_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\llcorner t \lrcorner_\theta,u}\|_H^2)^2} \right\|_{\mathcal{L}^p(\mathbb{P};\mathrm{HS}(U,H))}^2 du \right]^{1/2} \tag{67}$$
$$\leq 2\left[ \frac{p^2}{2} \int_0^t \left\| \frac{\|B\|_{\mathrm{HS}(U,H)}}{1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2} + \frac{2\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2 \|B\|_{\mathrm{HS}(U,H)}}{(1+\|X_{\llcorner u \lrcorner_\theta,u}\|_H^2)^2} \right\|_{\mathcal{L}^p(\mathbb{P};\mathrm{HS}(U,H))}^2 du \right]^{1/2}$$
$$\leq 2\left[ \frac{p^2}{2} \int_0^t 4\|B\|_{\mathrm{HS}(U,H)}^2 du \right]^{1/2} \leq 2\sqrt{2} p \max\{T,1\} \|B\|_{\mathrm{HS}(U,H)}.$$

This implies for every $p \in [2, \infty)$, $t \in [0, T]$ that

$$
\begin{aligned}
\int_{\lfloor t\rfloor_\theta}^{t} & \left\| \chi_{\lfloor u\rfloor_\theta} \|\mathcal{Z}_u\|_H^{p-2} \left\langle \mathcal{Z}_u, \tfrac{B}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} - \tfrac{2X_{\lfloor t\rfloor_\theta,u}\langle X_{\lfloor t\rfloor_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right\rangle_H \right\|_{\mathcal{L}^2(\mathbb{P};\mathrm{HS}(U,H))}^2 du \\
& \leq \int_{\lfloor t\rfloor_\theta}^{t} \left\| \|\mathcal{Z}_u\|_H^{p-2} \left\langle \mathcal{Z}_u, \tfrac{B}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} - \tfrac{2X_{\lfloor t\rfloor_\theta,u}\langle X_{\lfloor t\rfloor_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right\rangle_H \right\|_{\mathcal{L}^2(\mathbb{P};\mathrm{HS}(U,H))}^2 du \\
& \leq \int_{\lfloor t\rfloor_\theta}^{t} \left\| \|\mathcal{Z}_u\|_H^{p-1} \left\| \tfrac{B}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} - \tfrac{2X_{\lfloor t\rfloor_\theta,u}\langle X_{\lfloor t\rfloor_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right\|_{\mathrm{HS}(U,H)} \right\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})}^2 du \quad (68) \\
& \leq \int_{\lfloor t\rfloor_\theta}^{t} \left\| \|\mathcal{Z}_u\|_H^{p-1} \left( \tfrac{\|B\|_{\mathrm{HS}(U,H)}}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} + \tfrac{2\|X_{\lfloor t\rfloor_\theta,u}\|_H^2\|B\|_{\mathrm{HS}(U,H)}}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right) \right\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})}^2 du \\
& \leq 4\|B\|_{\mathrm{HS}(U,H)}^2 \int_{\lfloor t\rfloor_\theta}^{t} \|\mathcal{Z}_u\|_{\mathcal{L}^{2(p-1)}(\mathbb{P};H)}^{2(p-1)} du < \infty.
\end{aligned}
$$

Moreover, note that for every $t \in [0, T]$ it holds that

$$
[Z_t]_{\mathbb{P},\mathcal{B}(H)} = e^{(t-\lfloor t\rfloor_\theta)A}\left[ [Z_{\lfloor t\rfloor_\theta}]_{\mathbb{P},\mathcal{B}(H)} + \int_{\lfloor t\rfloor_\theta}^{t} \chi_{\lfloor t\rfloor_\theta}\left[ \tfrac{B}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} - \tfrac{2X_{\lfloor t\rfloor_\theta,u}\langle X_{\lfloor t\rfloor_\theta,u}, B(\cdot)\rangle_H}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right] dW_u \right]. \quad (69)
$$

Combining this, (66), (68), and Tonelli's theorem establishes that for every $p \in [2, \infty)$, $t \in [0, T]$ it holds that

$$
\begin{aligned}
\mathbb{E}[\|Z_t\|_H^p] &= \mathbb{E}[\|e^{(t-\lfloor t\rfloor_\theta)A}\mathcal{Z}_t\|_H^p] \leq \mathbb{E}[\|\mathcal{Z}_t\|_H^p] = \mathbb{E}[\|Z_{\lfloor t\rfloor_\theta}\|_H^p] \\
& + \tfrac{1}{2}\int_{\lfloor t\rfloor_\theta}^{t} \mathbb{E}\Bigg[ \chi_{\lfloor u\rfloor_\theta} \sum_{\mathbf{u}\in\mathbb{U}} \Bigg\{ p\|\mathcal{Z}_u\|_H^{p-2} \left\| \tfrac{B\mathbf{u}}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} - \tfrac{2X_{\lfloor t\rfloor_\theta,u}\langle X_{\lfloor t\rfloor_\theta,u}, B\mathbf{u}\rangle_H}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right\|_H^2 \\
& + p(p-2)\chi_{\lfloor t\rfloor_\theta}\mathbb{1}_{\{\mathcal{Z}_u\neq 0\}}\|\mathcal{Z}_u\|_H^{p-4}\left| \left\langle \mathcal{Z}_u, \tfrac{B\mathbf{u}}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} - \tfrac{2X_{\lfloor t\rfloor_\theta,u}\langle X_{\lfloor t\rfloor_\theta,u}, B\mathbf{u}\rangle_H}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right\rangle_H \right|^2 \Bigg\} \Bigg] du.
\end{aligned} \quad (70)
$$

The Cauchy-Schwarz inequality hence assures for every $p \in [2, \infty)$, $t \in [0, T]$ that

$$
\begin{aligned}
\mathbb{E}[\|Z_t\|_H^p] &\leq \mathbb{E}[\|\mathcal{Z}_t\|_H^p] \leq \mathbb{E}[\|Z_{\lfloor t\rfloor_\theta}\|_H^p] \\
& + \tfrac{1}{2}\int_{\lfloor t\rfloor_\theta}^{t} \mathbb{E}\Bigg[ \sum_{\mathbf{u}\in\mathbb{U}} \Bigg\{ p\|\mathcal{Z}_u\|_H^{p-2}\left( \tfrac{\|B\mathbf{u}\|_H}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} + \tfrac{2\|X_{\lfloor t\rfloor_\theta,u}\|_H\|X_{\lfloor t\rfloor_\theta,u}\|_H\|B\mathbf{u}\|_H}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right)^2 \\
& + p(p-2)\|\mathcal{Z}_u\|_H^{p-2}\left( \tfrac{\|B\mathbf{u}\|_H}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} + \tfrac{2\|X_{\lfloor t\rfloor_\theta,u}\|_H\|X_{\lfloor t\rfloor_\theta,u}\|_H\|B\mathbf{u}\|_H}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right)^2 \Bigg\} \Bigg] du \\
& \leq \mathbb{E}[\|Z_{\lfloor t\rfloor_\theta}\|_H^p] + p(p-1)\left[\tfrac{1}{2}\right]\|B\|_{\mathrm{HS}(U,H)}^2 \int_{\lfloor t\rfloor_\theta}^{t} \mathbb{E}\left[ \|\mathcal{Z}_u\|_H^{p-2}\left( \tfrac{1}{1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2} + \tfrac{2\|X_{\lfloor t\rfloor_\theta,u}\|_H^2}{(1+\|X_{\lfloor t\rfloor_\theta,u}\|_H^2)^2} \right)^2 \right] du \\
& \leq \mathbb{E}[\|Z_{\lfloor t\rfloor_\theta}\|_H^p] + p(p-1)\left[\tfrac{9}{8}\right]\|B\|_{\mathrm{HS}(U,H)}^2 \int_{\lfloor t\rfloor_\theta}^{t} \mathbb{E}[\|\mathcal{Z}_u\|_H^{p-2}] du.
\end{aligned} \quad (71)
$$

This demonstrates that for every $n \in \mathbb{N}$, $t \in [0, T]$ it holds that

$$
\begin{aligned}
\mathbb{E}[\|Z_t\|_H^{2n}] \leq \mathbb{E}[\|\mathcal{Z}_t\|_H^{2n}] &\leq \mathbb{E}[\|Z_{\lfloor t\rfloor_\theta}\|_H^{2n}] + 2n(2n-1)\left[\tfrac{9}{8}\right]\|B\|_{\mathrm{HS}(U,H)}^2 \int_{\lfloor t\rfloor_\theta}^{t} \mathbb{E}\left[\|\mathcal{Z}_u\|_H^{2n-2}\right] du \\
&\leq \ldots \leq \mathbb{E}[\|Z_0\|_H^{2n}] + 2n(2n-1)\left[\tfrac{9}{8}\right]\|B\|_{\mathrm{HS}(U,H)}^2 \int_{0}^{t} \mathbb{E}\left[\|\mathcal{Z}_u\|_H^{2n-2}\right] du \quad (72) \\
&= 2n(2n-1)\left[\tfrac{9}{8}\right]\|B\|_{\mathrm{HS}(U,H)}^2 \int_{0}^{t} \mathbb{E}\left[\|\mathcal{Z}_u\|_H^{2n-2}\right] du.
\end{aligned}
$$

Therefore, we obtain that for every $n \in \mathbb{N}$, $t_0 \in [0, T]$ it holds that

$$
\mathbb{E}[\|Z_{t_0}\|_H^{2n}] \leq \mathbb{E}[\|\mathcal{Z}_{t_0}\|_H^{2n}] \leq 2n(2n-1)\left[\tfrac{9}{8}\right]\|B\|_{\mathrm{HS}(U,H)}^2 \int_0^{t_0} \mathbb{E}\left[\|\mathcal{Z}_u\|_H^{2n-2}\right] du
$$

$$
\leq \ldots \leq (2n)!\left[\tfrac{9}{8}\right]^n \|B\|_{\mathrm{HS}(U,H)}^{2n} \int_0^{t_0} \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \cdots dt_2\, dt_1 = \tfrac{9^n(2n)!}{8^n n!}\|B\|_{\mathrm{HS}(U,H)}^{2n}(t_0)^n. \tag{73}
$$

The proof of Lemma 2.7 is thus completed. $\qquad\square$

**Lemma 2.8.** *Assume Setting 2.1 and let $\varepsilon \in [0, 1/(8[\max\{\|B\|_{\mathrm{HS}(U,H)},1\}]^2 \max\{T,1\})^2)$. Then it holds for every $t \in [0,T]$ that*

$$
\mathbb{E}[e^{\varepsilon\|\mathbf{O}_t\|_H^2}] \leq \frac{2}{1-\varepsilon^2[8\|B\|_{\mathrm{HS}(U,H)}\max\{\|B\|_{\mathrm{HS}(U,H)},1\}\max\{T,1\}]^4}. \tag{74}
$$

*Proof of Lemma 2.8.* Throughout this proof for every $s \in [0,T]$ let $X_{s,(\cdot)}(\cdot) = (X_{s,t}(\omega))_{(t,\omega)\in[s,T]\times\Omega}$: $[s,T] \times \Omega \to H_\beta$ be an $(\mathbb{F}_u)_{u\in[s,T]}$-adapted stochastic process with continuous sample paths which satisfies for every $u \in [s,T]$ that $[X_{s,u}]_{\mathbb{P},\mathcal{B}(H_\beta)} = \int_s^u B\, dW_\tau$. Lemma 2.1 (with $X_{s,t} = X_{s,t}$ for $t \in [s,T]$, $s \in [0,T]$ in the notation of Lemma 2.1), Lemma 2.6 (with $p = 2n$, $X_{s,t} = X_{s,t}$ for $t \in [s,T]$, $s \in [0,T]$, $n \in \mathbb{N}$ in the notation of Lemma 2.6), Lemma 2.7 (with $X_{s,t} = X_{s,t}$ for $t \in [s,T]$, $s \in [0,T]$, $n \in \mathbb{N}$ in the notation of Lemma 2.7), and the triangle inequality ensure that for every $n \in \mathbb{N}$, $t \in [0,T]$ it holds that

$$
\|\mathbf{O}_t\|_{\mathcal{L}^{2n}(\mathbb{P};H)} \leq \left\| \int_0^t \chi_{\llcorner u\lrcorner_\theta} \left( \frac{e^{(t-\llcorner u\lrcorner_\theta)A}B}{1+\|X_{\llcorner u\lrcorner_\theta,u}\|_H^2} - \frac{2e^{(t-\llcorner u\lrcorner_\theta)A}X_{\llcorner u\lrcorner_\theta,u}\langle X_{\llcorner u\lrcorner_\theta,u},B(\cdot)\rangle_H}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|_H^2)^2} \right) dW_u \right\|_{L^{2n}(\mathbb{P};H)}
$$

$$
+ \left\| \int_0^t \chi_{\llcorner u\lrcorner_\theta} e^{(t-\llcorner u\lrcorner_\theta)A} \left[ \frac{4\|\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}\|_U^2 X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|_H^2)^3} - \frac{2B\mathbb{B}X_{\llcorner u\lrcorner_\theta,u}+\|B\|_{\mathrm{HS}(U,H)}^2 X_{\llcorner u\lrcorner_\theta,u}}{(1+\|X_{\llcorner u\lrcorner_\theta,u}\|_H^2)^2} \right] du \right\|_{\mathcal{L}^{2n}(\mathbb{P};H)} \tag{75}
$$

$$
\leq \tfrac{3}{2\sqrt{2}}\left(\tfrac{(2n)!}{n!}\right)^{1/2n}\|B\|_{\mathrm{HS}(U,H)}t^{1/2} + 2\|B\|_{\mathrm{HS}(U,H)}^2 t
$$

$$
\leq 4\|B\|_{\mathrm{HS}(U,H)}\max\{\|B\|_{\mathrm{HS}(U,H)},1\}\max\{T,1\}\left(\tfrac{(2n)!}{n!}\right)^{1/2n}.
$$

This, the fact that for every $x \in [0,\infty)$ it holds that $e^x \leq 2[\sum_{m=0}^\infty \frac{x^{2m}}{(2m)!}]$ (see, e.g., Lemma 2.4 in Hutzenthaler et al. [18]), the dominated convergence theorem, and the fact that $\forall\, m \in \mathbb{N}: (4m)! \leq 2^{4m}[(2m)!]^2$ imply that for every $t \in [0,T]$ it holds that

$$
\mathbb{E}[e^{\varepsilon\|\mathbf{O}_t\|_H^2}] \leq 2\,\mathbb{E}\left[\sum_{n=0}^\infty \frac{\varepsilon^{2n}\|\mathbf{O}_t\|_H^{4n}}{(2n)!}\right] = 2\left[\sum_{n=0}^\infty \mathbb{E}\left[\frac{\varepsilon^{2n}\|\mathbf{O}_t\|_H^{4n}}{(2n)!}\right]\right]
$$

$$
\leq 2\left[\sum_{n=0}^\infty \frac{(4n)!}{[(2n)!]^2}\varepsilon^{2n}[4\|B\|_{\mathrm{HS}(U,H)}\max\{\|B\|_{\mathrm{HS}(U,H)},1\}\max\{T,1\}]^{4n}\right] \tag{76}
$$

$$
\leq 2\left[\sum_{n=0}^\infty 2^{4n}\varepsilon^{2n}[4\|B\|_{\mathrm{HS}(U,H)}\max\{\|B\|_{\mathrm{HS}(U,H)},1\}\max\{T,1\}]^{4n}\right]
$$

$$
= \frac{2}{1-\varepsilon^2[8\|B\|_{\mathrm{HS}(U,H)}\max\{\|B\|_{\mathrm{HS}(U,H)},1\}\max\{T,1\}]^4}.
$$

The proof of Lemma 2.8 is thus completed. $\qquad\square$

# 3 Regularity properties of tamed-truncated space-time approximations of stochastic convolutions

**Setting 3.1.** *Assume Setting 1.1, let $\beta \in [0,\infty)$, $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ be a filtered probability space which fulfills the usual conditions, let $(W_t)_{t\in[0,T]}$ be an $\mathrm{Id}_U$-cylindrical $(\mathbb{F}_t)_{t\in[0,T]}$-*

*Wiener process, let* $B \in \text{HS}(U, H_\beta)$, *let* $\mathbb{U} \subseteq U$ *be an orthonormal basis of* $U$, *let* $P_I : H \rightarrow H$, $I \in \mathcal{P}(\mathbb{H})$, *and* $\hat{P}_J : U \rightarrow U$, $J \in \mathcal{P}(\mathbb{U})$, *be the linear operators which satisfy for every* $x \in H$, $y \in U$, $I \in \mathcal{P}(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$ *that* $P_I(x) = \sum_{h \in I} \langle h, x \rangle_H h$ *and* $\hat{P}_J(y) = \sum_{u \in J} \langle u, y \rangle_U u$, *let* $\chi^{\theta, I, J} : [0, T] \times \Omega \rightarrow [0, 1]$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$, *be* $(\mathbb{F}_t)_{t \in [0,T]}$-*adapted stochastic processes, and let* $\mathbf{O}^{\theta, I, J} : [0, T] \times \Omega \rightarrow P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$, *be stochastic processes which satisfy for every* $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$, $t \in [0, T]$ *that* $\mathbf{O}_0^{\theta, I, J} = 0$ *and*

$$[\mathbf{O}_t^{\theta, I, J}]_{\mathbb{P}, \mathcal{B}(P_I(H))} = [e^{(t - \llcorner t \lrcorner_\theta)A} \mathbf{O}_{\llcorner t \lrcorner_\theta}^{\theta, I, J}]_{\mathbb{P}, \mathcal{B}(P_I(H))} + \frac{\int_{\llcorner t \lrcorner_\theta}^t \chi_{\llcorner t \lrcorner_\theta}^{\theta, I, J} e^{(t - \llcorner t \lrcorner_\theta)A} P_I B \hat{P}_J \, dW_s}{1 + \| \int_{\llcorner t \lrcorner_\theta}^t P_I B \hat{P}_J \, dW_s \|_H^2}. \tag{77}$$

## 3.1 Main results

Here we apply the results from Section 2 in order to obtain our main results concerning tamed-truncated space-time approximations (see (77) above) of stochastic convolutions. A uniform boundedness of moments in fractional order smoothness spaces is presented in Corollary 3.1, a uniform Hölder continuity in time is shown in Corollary 3.2, strong convergence rates are established in Corollary 3.3, and Corollary 3.4 concerns a uniform boundedness of exponential moments.

**Corollary 3.1.** *Assume Setting 3.1 and let* $p \in [1, \infty)$, $\gamma \in [0, 1/2 + \beta)$. *Then it holds that*

$$\sup_{\theta \in \varpi_T} \sup_{J \in \mathcal{P}(\mathbb{U})} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \| \mathbf{O}_t^{\theta, I, J} \|_{\mathcal{L}^p(\mathbb{P}; H_\gamma)} < \infty. \tag{78}$$

*Proof of Corollary 3.1.* Observe that Lemma 2.3 (with $\beta = \beta$, $\gamma = \max\{\gamma, \beta\}$, $T = T$, $\theta = \theta$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_u)_{u \in [0,T]} = (\mathbb{F}_u)_{u \in [0,T]}$, $(W_u)_{u \in [0,T]} = (W_u)_{u \in [0,T]}$, $B = (U \ni u \mapsto P_I B \hat{P}_J(u) \in H_\beta)$, $\chi = \chi^{\theta, I, J}$, $\mathbf{O} = ([0, T] \times \Omega \ni (t, \omega) \mapsto \mathbf{O}_t^{\theta, I, J}(\omega) \in H_{\max\{\gamma, \beta\}})$, $p = \max\{p, 2\}$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$ in the notation of Lemma 2.3) and Hölder's inequality show that (78) holds. The proof of Corollary 3.1 is thus completed. $\square$

**Corollary 3.2.** *Assume Setting 3.1 and let* $p \in [1, \infty)$, $\gamma \in [0, 1/2 + \beta)$, $\rho \in [0, 1/2 + \beta - \gamma) \cap [0, 1/2)$. *Then it holds for every* $s \in [0, T]$, $t \in [s, T]$ *that*

$$\sup_{\theta \in \varpi_T} \sup_{J \in \mathcal{P}(\mathbb{U})} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \| \mathbf{O}_t^{\theta, I, J} - \mathbf{O}_s^{\theta, I, J} \|_{\mathcal{L}^p(\mathbb{P}; H_\gamma)}$$
$$\leq \frac{3 \max[\{p, 2\}]^3 \|B\|_{\text{HS}(U, H_\beta)} [\max\{T, 1\}]^2 \max\{|\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta}, 1\}(1 + 8\|B\|_{\text{HS}(U, H_\beta)}^2) \|(-A)^{\min\{0, \gamma - \beta\}}\|_{L(H)}}{\sqrt{1 + 2(\beta - \max\{\gamma, \beta\} - \rho)}} (t - s)^\rho. \tag{79}$$

*Proof of Corollary 3.2.* Note that Lemma 2.4 (with $H = H$, $\beta = \beta$, $\gamma = \max\{\gamma, \beta\}$, $T = T$, $\theta = \theta$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_t)_{t \in [0,T]} = (\mathbb{F}_t)_{t \in [0,T]}$, $(W_t)_{t \in [0,T]} = (W_t)_{t \in [0,T]}$, $B = (U \ni u \mapsto P_I B \hat{P}_J(u) \in H_\beta)$, $\chi = \chi^{\theta, I, J}$, $\mathbf{O} = ([0, T] \times \Omega \ni (t, \omega) \mapsto \mathbf{O}_t^{\theta, I, J}(\omega) \in H_{\max\{\gamma, \beta\}})$, $p = \max\{p, 2\}$, $\rho = \rho$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$ in the notation of Lemma 2.4) establishes (79). The proof of Corollary 3.2 is thus completed. $\square$

**Corollary 3.3.** *Assume Setting 3.1, let* $p, C \in [1, \infty)$, $\gamma \in [0, 1/2 + \beta)$, $\eta \in [0, 1/2 + \beta - \gamma)$, $\rho \in [0, 1/2 + \beta - \gamma) \cap [0, 1/2)$, *assume for every* $s \in [0, T]$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$ *that* $\|\chi_{\llcorner s \lrcorner_\theta}^{\theta, I, J} - 1\|_{\mathcal{L}^{\max\{p, 2\}}(\mathbb{P}; \mathbb{R})} \leq C[|\theta|_T]^\rho$, *and let* $O : [0, T] \times \Omega \rightarrow H_\gamma$ *be a stochastic process which satisfies for every* $t \in [0, T]$ *that* $[O_t]_{\mathbb{P}, \mathcal{B}(H_\gamma)} = \int_0^t e^{(t-s)A} B \, dW_s$. *Then it holds for every* $I \in \mathcal{P}_0(\mathbb{H})$,

$J \in \mathcal{P}(\mathbb{U})$, $K \in \mathcal{P}(\mathbb{H})$, $\theta \in \varpi_T$ with $I \subseteq K$ that

$$
\begin{aligned}
&\sup_{t \in [0,T]} \|\mathbf{O}_t^{\theta,I,J} - P_K O_t\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \\
&\leq \frac{\max\{p,2\}[\max\{T,1\}]^{1/2+\beta}}{\sqrt{2(1-2\max\{0,\gamma+\eta-\beta\})}} \|(-A)^{\min\{0,\gamma+\eta-\beta\}}\|_{L(H)} \|B - P_I B \hat{P}_J\|_{\mathrm{HS}(U,H_{\beta-\eta})} \\
&\quad + \frac{8[\max\{p,2\}]^3 C[\max\{T,1\}]^{3/2+\beta}}{\sqrt{1-2\rho-2\max\{0,\gamma-\beta\}}} \|B\|_{\mathrm{HS}(U,H_\beta)} \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \\
&\quad \cdot \left(1 + |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|B\|_{\mathrm{HS}(U,H_\beta)}^2\right) [|\theta|_T]^\rho.
\end{aligned}
\tag{80}
$$

*Proof of Corollary 3.3.* Note that Lemma 2.5 (with $H = H$, $\beta = \beta$, $\gamma = \gamma$, $T = T$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_t)_{t \in [0,T]} = (\mathbb{F}_t)_{t \in [0,T]}$, $(W_t)_{t \in [0,T]} = (W_t)_{t \in [0,T]}$, $B = (U \ni u \mapsto P_I B \hat{P}_J(u) \in H_\beta)$, $\chi = \chi^{\theta,I,J}$, $\mathbf{O} = ([0,T] \times \Omega \ni (t,\omega) \mapsto \mathbf{O}_t^{\theta,I,J}(\omega) \in H_\gamma)$, $C = C$, $\tilde{B} = (U \ni u \mapsto P_K B(u) \in H_\beta)$, $p = \max\{p,2\}$, $\eta = \eta$, $\rho = \rho$, $O = ([0,T] \times \Omega \ni (t,\omega) \mapsto P_K O_t(\omega) \in H_\gamma)$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$, $K \in \mathcal{P}(\mathbb{H})$ with $I \subseteq K$ in the notation of Lemma 2.5) ensures that for every $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$, $K \in \mathcal{P}(\mathbb{H})$ with $I \subseteq K$ it holds that

$$
\begin{aligned}
&\sup_{t \in [0,T]} \|\mathbf{O}_t^{\theta,I,J} - P_K O_t\|_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \\
&\leq \frac{\max\{p,2\}[\max\{T,1\}]^{1/2+\beta}}{\sqrt{2(1-2\max\{0,\gamma+\eta-\beta\})}} \|(-A)^{\min\{0,\gamma+\eta-\beta\}}\|_{L(H)} \|P_K B - P_I B \hat{P}_J\|_{\mathrm{HS}(U,H_{\beta-\eta})} \\
&\quad + \frac{8[\max\{p,2\}]^3 C[\max\{T,1\}]^{3/2+\beta}}{\sqrt{1-2\rho-2\max\{0,\gamma-\beta\}}} \|P_I B \hat{P}_J\|_{\mathrm{HS}(U,H_\beta)} \|(-A)^{\min\{0,\gamma-\beta\}}\|_{L(H)} \\
&\quad \cdot \left(1 + |\sup_{h \in \mathbb{H}} \mathfrak{v}_h|^{-2\beta} \|P_I B \hat{P}_J\|_{\mathrm{HS}(U,H_\beta)}^2\right) [|\theta|_T]^\rho.
\end{aligned}
\tag{81}
$$

This completes the proof of Corollary 3.3. $\qquad\square$

**Corollary 3.4.** *Assume Setting 3.1 and let $\varepsilon \in [0, 1/(8[\max\{\|B\|_{\mathrm{HS}(U,H)},1\}]^2 \max\{T,1\})^2)$. Then*

$$
\sup_{\theta \in \varpi_T} \sup_{J \in \mathcal{P}(\mathbb{U})} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{s \in [0,T]} \mathbb{E}[\exp(\varepsilon \|\mathbf{O}_s^{\theta,I,J}\|_H^2)] < \infty.
\tag{82}
$$

*Proof of Corollary 3.4.* Note that Lemma 2.8 (with $H = H$, $\beta = \beta$, $\gamma = 0$, $T = T$, $\theta = \theta$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(\mathbb{F}_t)_{t \in [0,T]} = (\mathbb{F}_t)_{t \in [0,T]}$, $B = (U \ni x \mapsto P_I B \hat{P}_J(x) \in H_\beta)$, $\chi = \chi^{\theta,I,J}$, $\mathbf{O} = ([0,T] \times \Omega \ni (t,\omega) \mapsto \mathbf{O}_t^{\theta,I,J}(\omega) \in H)$, $\varepsilon = \varepsilon$ for $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$, $\theta \in \varpi_T$ in the notation of Lemma 2.8) assures that (82) holds. The proof of Corollary 3.4 is thus completed. $\qquad\square$

### Acknowledgements

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