# Fast and accurate evaluation of dual Bernstein polynomials 

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#### Abstract

Dual Bernstein polynomials find many applications in approximation theory, computational mathematics, numerical analysis and computer-aided geometric design. In this context, one of the main problems is fast and accurate evaluation both of these polynomials and their linear combinations. New simple recurrence relations of low order satisfied by dual Bernstein polynomials are given. In particular, a first-order non-homogeneous recurrence relation linking dual Bernstein and shifted Jacobi orthogonal polynomials has been obtained. When used properly, it allows to propose fast and numerically efficient algorithms for evaluating all $n+1$ dual Bernstein polynomials of degree $n$ with $O(n)$ computational complexity.


Keywords: Recurrence relations; Bernstein basis polynomials; Dual Bernstein polynomials; Jacobi polynomials.

## 1. Introduction

Let us introduce the inner product $\langle\cdot, \cdot\rangle_{\alpha, \beta}$ by

$$
\begin{equation*}
\langle f, g\rangle_{\alpha, \beta}:=\int_{0}^{1}(1-x)^{\alpha} x^{\beta} f(x) g(x) \mathrm{d} x \tag{1.1}
\end{equation*}
$$

where $\alpha, \beta>-1$.
Let $\Pi_{n}(n \in \mathbb{N})$ denote the set of polynomials of degree at most $n$. Recall that the shifted Jacobi polynomial of degree $n, R_{n}^{(\alpha, \beta)} \in \Pi_{n}$, is defined by

$$
\begin{equation*}
R_{n}^{(\alpha, \beta)}(x):=\frac{(\alpha+1)_{n}}{n!} \sum_{k=0}^{n} \frac{(-n)_{k}(n+\alpha+\beta+1)_{k}}{k!(\alpha+1)_{k}}(1-x)^{k} \quad(n=0,1, \ldots) \tag{1.2}
\end{equation*}
$$

Here $(c)_{l}(c \in \mathbb{C} ; l \in \mathbb{N})$ denotes the Pochhammer symbol,

$$
(c)_{0}:=1, \quad(c)_{l}:=c(c+1) \ldots(c+l-1) \quad(l \geq 1)
$$

These polynomials satisfy the second-order recurrence relation of the form

$$
\begin{equation*}
\xi_{0}(n) R_{n}^{(\alpha, \beta)}(x)+\xi_{1}(n) R_{n+1}^{(\alpha, \beta)}(x)+\xi_{2}(n) R_{n+2}^{(\alpha, \beta)}(x)=0 \quad(n=0,1, \ldots) \tag{1.3}
\end{equation*}
$$

[^0]where
\[

$$
\begin{align*}
& \xi_{0}(n):=-2(n+\alpha+1)(n+\beta+1)(2 n+\sigma+3),  \tag{1.4}\\
& \xi_{1}(n):=(2 n+\sigma+2)\left\{(2 n+\sigma+1)(2 n+\sigma+3)(2 x-1)+\alpha^{2}-\beta^{2}\right\},  \tag{1.5}\\
& \xi_{2}(n):=-2(n+2)(n+\sigma+1)(2 n+\sigma+1), \tag{1.6}
\end{align*}
$$
\]

and $\sigma:=\alpha+\beta+1$ (cf., e.g., $[1, \S 1.8]$ ).
Remark 1.1. Notice that the recurrence relation (1.3) can be used, for example, in fast and accurate methods for evaluating the values $R_{n}^{(\alpha, \beta)}(x)$ for a given $x, \alpha, \beta$ and all $0 \leq n \leq N$, where $N$ is a fixed natural number, with $O(N)$ computational complexity. For more details about performing computations with recurrence relations properly, see [2].

Shifted Jacobi polynomials are orthogonal with respect to the inner product (1.1), i.e.,

$$
\left\langle R_{k}^{(\alpha, \beta)}, R_{l}^{(\alpha, \beta)}\right\rangle_{\alpha, \beta}=\delta_{k l} h_{k} \quad(k, l \in \mathbb{N}),
$$

where $\delta_{k l}$ is the Kronecker delta ( $\delta_{k l}=0$ for $k \neq l$ and $\delta_{k k}=1$ ) and

$$
h_{k}:=K \frac{(\alpha+1)_{k}(\beta+1)_{k}}{k!(2 k / \sigma+1)(\sigma)_{k}} \quad(k=0,1, \ldots)
$$

with $K:=\Gamma(\alpha+1) \Gamma(\beta+1) / \Gamma(\sigma+1)$. For more properties and applications of polynomials $R_{n}^{(\alpha, \beta)}$, see, e.g., [1, 3].

Let $B_{0}^{n}, B_{1}^{n}, \ldots, B_{n}^{n} \in \Pi_{n}$ be Bernstein basis polynomials given by

$$
\begin{equation*}
B_{i}^{n}(x):=\binom{n}{i} x^{i}(1-x)^{n-i} \quad(i=0,1, \ldots, n ; n \in \mathbb{N}) . \tag{1.7}
\end{equation*}
$$

Definition $1.2([4, \S 5])$. Dual Bernstein polynomials of degree $n$,

$$
\begin{equation*}
D_{0}^{n}(x ; \alpha, \beta), D_{1}^{n}(x ; \alpha, \beta), \ldots, D_{n}^{n}(x ; \alpha, \beta) \in \Pi_{n}, \tag{1.8}
\end{equation*}
$$

are defined so that the following conditions hold:

$$
\left\langle B_{i}^{n}, D_{j}^{n}(\cdot ; \alpha, \beta)\right\rangle_{\alpha, \beta}=\delta_{i j} \quad(i, j=0,1, \ldots, n)
$$

(cf. (1.1)). We adopt the convention that $D_{i}^{n}(x ; \alpha, \beta):=0$ for $i<0$ or $i>n$.
Let us mention that in the case $\alpha=\beta=0$ these polynomials were introduced earlier by Ciesielski in [5].

Certainly, $\operatorname{lin}\left\{B_{k}^{n}: 0 \leq k \leq n\right\}=\Pi_{n}$. One can also prove that dual Bernstein polynomials (1.8) form a basis of the $\Pi_{n}$ space.

It is well-known that for many years Bernstein basis polynomials have been used in computer-aided geometric design, approximation theory, numerical analysis and computational mathematics. See, e.g., books [6, 7] and article [8], as well as papers cited therein.

For a given function $f$, let us define a polynomial $p_{n}^{*}$ of the following Bernstein-Bézier form:

$$
p_{n}^{*}(x):=\sum_{k=0}^{n} I_{k} B_{k}^{n}(x),
$$

where

$$
\begin{equation*}
I_{k}:=\left\langle f, D_{k}^{n}(\cdot ; \alpha, \beta)\right\rangle_{\alpha, \beta}=\int_{0}^{1}(1-x)^{\alpha} x^{\beta} f(x) D_{k}^{n}(x ; \alpha, \beta) \mathrm{d} x \quad(0 \leq k \leq n) \tag{1.9}
\end{equation*}
$$

Recall that the polynomial $p_{n}^{*}$ minimizes the value of the least-square error

$$
\left\|f-p_{n}\right\|_{2}^{2}:=\left\langle f-p_{n}, f-p_{n}\right\rangle_{\alpha, \beta}=\int_{0}^{1}(1-x)^{\alpha} x^{\beta}\left(f(x)-p_{n}(x)\right)^{2} \mathrm{~d} x \quad\left(p_{n} \in \Pi_{n}\right)
$$

(cf. 19, Lemma 2.2]).
This is one of the main reasons that dual Bernstein polynomials have recently been extensively studied and found many theoretical (see [4, 5, $9-12]$ ) and practical applications. For example, these dual polynomials are very useful in: curve intersection using Bézier clipping ( $13-15]$ ); degree reduction and merging of Bézier curves ( $16-19]$ ); polynomial approximation of rational Bézier curves ([20]); numerical solving of boundary value problems ( 21$]$ ) or even fractional partial differential equations ([22, 23]). Skillful use of these polynomials often results in less costly algorithms of solving many computational problems.

In some of the mentioned tasks, as well as in finding the solution of the least-square problem in the Bernstein-Bézier form, it is necessary to compute the numerical approximations of the collection of integrals (1.9) for all $k=0,1, \ldots, n$ and a given function $f$. In general, to do so, one has to use quadrature rules (see, e.g., $24, \S 5]$ ), but it requires fast evaluation of all $n+1$ dual Bernstein polynomials of degree $n$ in many nodes. Thus, the authors consider the following problem.

Problem 1.3 (cf. [25]). Let us fix numbers: $n \in \mathbb{N}, x \in[0,1]$ and $\alpha, \beta>-1$. Compute the values

$$
D_{i}^{n}(x ; \alpha, \beta)
$$

for all $i=0,1, \ldots, n$.
Using new differential-recurrence properties of polynomials $D_{i}^{n}(x ; \alpha, \beta)$ obtained in [25], the authors have recently constructed the fourth-order recurrence relation of the following form:

$$
\begin{equation*}
\sum_{j=-2}^{2} v_{j}(i) D_{i+j}^{n}(x ; \alpha, \beta)=0 \tag{1.10}
\end{equation*}
$$

satisfied by dual Bernstein polynomials, where coefficients $v_{j}(-2 \leq j \leq 2)$ are quintic polynomials in $i$. See [25, Corollary 4.2 and Eq. (4.2)]. Notice that this result follows from relations between dual Bernstein and shifted Jacobi polynomials (cf. (1.2)), as well as so-called Hahn orthogonal polynomials (see, e.g., [1, §1.5]).

The recurrence relation (1.10) has been used to propose an algorithm which solves Problem 1.3 with $O(n)$ computational complexity. Notice that previously known methods have $O\left(n^{2}\right)$ or even $O\left(n^{3}\right)$ computational complexity. Experiments have shown that the new method is much faster and gives good numerical results for low $n(n \approx 20,30)$. See 25, §6].

The first goal of this paper is to derive new recurrence relations of lower order for dual Bernstein polynomials. More specifically, in Section [2.1, a simple first-order non-homogeneous recurrence relation for dual Bernstein polynomials is given. Next, in Section 2.2, we find homogeneous recurrence relations of the second and third order for polynomials $D_{i}^{n}(x ; \alpha, \beta)$.

The second goal is to propose fast and accurate algorithms which solve Problem 1.3 and have $O(n)$ computational complexity, as well as work even for large values of $n(n \approx$ 1000,2000 ) (cf. [26], where evaluation of high-degree polynomials in Bernstein-Bézier form was examined, or signal processing, as well as numerical integration of highly-oscillating functions, where high-degree polynomials may appear). Such efficient methods-based on the relation obtained in 2.1 are presented in Section 3. Results of numerical experiments are given in $\{4$,

## 2. New recurrence relations

Now, let us recall some properties of dual Bernstein polynomials (1.8) and shifted Jacobi polynomials (1.2) which allow us to derive new recurrence relations for $D_{i}^{n}(x ; \alpha, \beta)$.

In [4, Theorem 5.1], the following relation between dual Bernstein polynomials of degrees $n, n+1$, as well as the shifted Jacobi polynomial of degree $n+1$ has been proven:

$$
\begin{equation*}
D_{i}^{n+1}(x ; \alpha, \beta)=\left(1-\frac{i}{n+1}\right) D_{i}^{n}(x ; \alpha, \beta)+\frac{i}{n+1} D_{i-1}^{n}(x ; \alpha, \beta)+C_{n i}^{(\alpha, \beta)} R_{n+1}^{(\alpha, \beta)}(x), \tag{2.1}
\end{equation*}
$$

where $0 \leq i \leq n+1$, and

$$
\begin{equation*}
C_{n i}^{(\alpha, \beta)}:=(-1)^{n-i+1} \frac{(2 n+\sigma+2)(\sigma+1)_{n}}{K(\alpha+1)_{n-i+1}(\beta+1)_{i}} . \tag{2.2}
\end{equation*}
$$

Note that for $\alpha=\beta=0$, this identity was found earlier by Ciesielski in (5].
It is also known that dual Bernstein polynomials satisfy a symmetry relation of the type

$$
\begin{equation*}
D_{i}^{n}(x ; \alpha, \beta)=D_{n-i}^{n}(1-x ; \beta, \alpha) \quad(i=0,1, \ldots, n) \tag{2.3}
\end{equation*}
$$

See [4, Corollary 5.3].
The polynomial $D_{i}^{n}(x ; \alpha, \beta)$ can be expressed as a short linear combination of $\min (i, n-$ $i)+1$ shifted Jacobi polynomials with shifted parameters:

$$
\begin{aligned}
D_{i}^{n}(x ; \alpha, \beta) & =\frac{(-1)^{n-i}(\sigma+1)_{n}}{K(\alpha+1)_{n-i}(\beta+1)_{i}} \sum_{k=0}^{i} \frac{(-i)_{k}}{(-n)_{k}} R_{n-k}^{(\alpha, \beta+k+1)}(x), \\
D_{n-i}^{n}(x ; \alpha, \beta) & =\frac{(-1)^{i}(\sigma+1)_{n}}{K(\alpha+1)_{i}(\beta+1)_{n-i}} \sum_{k=0}^{i}(-1)^{k} \frac{(-i)_{k}}{(-n)_{k}} R_{n-k}^{(\alpha+k+1, \beta)}(x),
\end{aligned}
$$

where $i=0,1, \ldots, n$. See [4, Corollary 5.4]. In particular, we have

$$
\begin{align*}
D_{0}^{n}(x ; \alpha, \beta) & =\frac{(-1)^{n}(\sigma+1)_{n}}{K(\alpha+1)_{n}} R_{n}^{(\alpha, \beta+1)}(x),  \tag{2.4}\\
D_{n}^{n}(x ; \alpha, \beta) & =\frac{(\sigma+1)_{n}}{K(\beta+1)_{n}} R_{n}^{(\alpha+1, \beta)}(x) . \tag{2.5}
\end{align*}
$$

Using [25, Eq. (3.1)], so-called Chu-Vandermonde identity (see, e.g., [3, Corollary 2.3.]) and symmetry (2.3), one can check that

$$
\begin{align*}
D_{i}^{n}(1 ; \alpha, \beta) & =(-1)^{n-i} \frac{(\sigma+1)_{n}(n-i+\alpha+2)_{i}}{K n!(\beta+1)_{i}},  \tag{2.6}\\
D_{i}^{n}(0 ; \alpha, \beta) & =(-1)^{i} \frac{(\sigma+1)_{n}(i+\beta+2)_{n-i}}{K n!(\alpha+1)_{n-i}} \quad(0 \leq i \leq n) . \tag{2.7}
\end{align*}
$$

From [3, Eq. (6.4.20) and (6.4.23)], it follows that

$$
\begin{align*}
(1-x) R_{n}^{(\alpha+1, \beta)}(x) & =-\frac{n+1}{2 n+\sigma+1} R_{n+1}^{(\alpha, \beta)}(x)+\frac{n+\alpha+1}{2 n+\sigma+1} R_{n}^{(\alpha, \beta)}(x),  \tag{2.8}\\
x R_{n}^{(\alpha, \beta+1)}(x) & =\frac{n+1}{2 n+\sigma+1} R_{n+1}^{(\alpha, \beta)}(x)+\frac{n+\beta+1}{2 n+\sigma+1} R_{n}^{(\alpha, \beta)}(x) \tag{2.9}
\end{align*}
$$

### 2.1. First-order non-homogeneous recurrence relation

Using the results mentioned above and relation (1.3), one can justify a simple first-order non-homogeneous recurrence relation for dual Bernstein polynomials.

Theorem 2.1. For $i=0,1, \ldots, n$, the following relation holds:

$$
\begin{equation*}
(x-1)(i+1) D_{i}^{n}(x ; \alpha, \beta)+x(n-i) D_{i+1}^{n}(x ; \alpha, \beta)=\frac{-C_{n, i+1}^{(\alpha, \beta)}}{2 n+\sigma+2} T_{n i}^{(\alpha, \beta)}(x), \tag{2.10}
\end{equation*}
$$

where the notation used is that of (2.2), and

$$
\begin{equation*}
T_{n i}^{(\alpha, \beta)}(x):=(n-i)(n+\alpha+1) x R_{n}^{(\alpha, \beta+1)}(x)+(i+1)(n+\beta+1)(1-x) R_{n}^{(\alpha+1, \beta)}(x) . \tag{2.11}
\end{equation*}
$$

Proof. In the sequel, we need the following identity

$$
\begin{align*}
\frac{2 n+\sigma+1}{n+1} T_{n i}^{(\alpha, \beta)}(x)=(n+\alpha & +1)(n+\beta+1) R_{n}^{(\alpha, \beta)}(x) \\
& +((n-i)(n+\alpha+1)-(i+1)(n+\beta+1)) R_{n+1}^{(\alpha, \beta)}(x) \tag{2.12}
\end{align*}
$$

which can be verified using (2.8) and (2.9).
Let us use induction on $n$. First, observe that for any $i=n$, the relation (2.10) immediately follows from (2.5). So, in particular, it also holds for $n=0$.

Now, suppose that (2.10) is true for some natural number $n$ and all $0 \leq i \leq n$. One has to prove that

$$
(x-1)(i+1) D_{i}^{n+1}(x ; \alpha, \beta)+x(n-i+1) D_{i+1}^{n+1}(x ; \alpha, \beta)+\frac{C_{n+1, i+1}^{(\alpha, \beta)}}{2 n+\sigma+4} T_{n+1, i}^{(\alpha, \beta)}(x) \equiv 0,
$$

where $0 \leq i \leq n+1$. We already know that it holds for $i=n+1$. Assume that $0 \leq i \leq n$. Applying twice (2.1) to the left-hand side, using (2.12) and doing simple algebra, one can obtain its equivalent form

$$
\begin{aligned}
& \frac{n-i+1}{n+1}\left[(x-1)(i+1) D_{i}^{n}(x ; \alpha, \beta)+x(n-i) D_{i+1}^{n}(x ; \alpha, \beta)\right] \\
& \quad+\frac{i+1}{n+1}\left[(x-1) i D_{i-1}^{n}(x ; \alpha, \beta)+x(n-i+1) D_{i}^{n}(x ; \alpha, \beta)\right] \\
& +((x-1)(i+1)
\end{aligned} \begin{aligned}
& \left.C_{n i}^{(\alpha, \beta)}+x(n-i+1) C_{n, i+1}^{(\alpha, \beta)}+C_{n+1, i+1}^{(\alpha, \beta)} \frac{(n+\alpha+2)(n+\beta+2)}{(n+2)^{-1}(2 n+\sigma+3)_{2}}\right) R_{n+1}^{(\alpha, \beta)}(x) \\
& \quad+\frac{C_{n+1, i+1}^{(\alpha, \beta)}(n+2)}{(2 n+\sigma+3)_{2}}((n-i+1)(n+\alpha+2)-(i+1)(n+\beta+2)) R_{n+2}^{(\alpha, \beta)}(x) .
\end{aligned}
$$

Applying twice the induction assumption to terms in square brackets and after some algebra, we have

$$
\begin{equation*}
G_{n i}^{(\alpha, \beta)}\left(\xi_{0}(n) R_{n}^{(\alpha, \beta)}(x)+\xi_{1}(n) R_{n+1}^{(\alpha, \beta)}(x)+\xi_{2}(n) R_{n+2}^{(\alpha, \beta)}(x)\right), \tag{2.13}
\end{equation*}
$$

where the notation used is that of (1.4)-(1.6), and

$$
G_{n i}^{(\alpha, \beta)}:=-C_{n i}^{(\alpha, \beta)} \frac{(n-i+1)(n-i+\alpha+1)-(i+1)(i+\beta+1)}{2(n+1)(2 n+\sigma+2)_{2}(n-i+\alpha+1)} .
$$

Indeed, it follows from (1.3) that (2.13) is equal to zero. At the end, note the special case: in (2.13), if $i=n-i$ and $\alpha=\beta$, both the expression in brackets and $G_{n i}^{(\alpha, \beta)}$ are equal to zero.

Let us stress that shifted Jacobi polynomials appearing in (2.11) (cf. (2.12)) do not depend on $i$. Now, solving this first-order non-homogeneous recurrence relation and using (2.4) allows us to obtain-after some algebra-a more explicit formula for dual Bernstein polynomials.

Corollary 2.2. For $i=0,1, \ldots, n$, we have

$$
\begin{aligned}
D_{i}^{n}(x ; \alpha, \beta)=\binom{n}{i}^{-1} & \frac{(-1)^{n-i}(\sigma+1)_{n}}{K(\alpha+1)_{n}(\beta+1)_{n}}\left(\frac{x-1}{x}\right)^{i} \\
& \times\left[R_{n}^{(\alpha, \beta+1)}(x) S_{n i}^{(\alpha+1, \beta)}\left(\frac{x}{x-1}\right)-R_{n}^{(\alpha+1, \beta)}(x) S_{n, i-1}^{(\alpha, \beta+1)}\left(\frac{x}{x-1}\right)\right],
\end{aligned}
$$

where

$$
S_{m k}^{(a, b)}(z):=(b+1)_{m} \sum_{j=0}^{k} \frac{(-m)_{j}(-m-a)_{j}}{j!(b+1)_{j}} z^{j} \quad(0 \leq k \leq m)
$$

(cf. (1.2)).

### 2.2. Homogeneous recurrence relations of order 2 and 3

Using Theorem [2.1, one can obtain a homogeneous relations of order 2 and 3 by eliminating the non-homogeneity. The proofs are quite technical therefore we omit them. Let us only mention that in the construction of third-order recurrence relation for dual Bernstein polynomials we used the same idea as in the proof of [25, Collorary 5.2], where recurrence of the form (1.10) has been derived.

Corollary 2.3. Dual Bernstein polynomials satisfy the second-order recurrence relation of the form

$$
u_{0}(i) D_{i}^{n}(x ; \alpha, \beta)+u_{1}(i) D_{i+1}^{n}(x ; \alpha, \beta)+u_{2}(i) D_{i+2}^{n}(x ; \alpha, \beta)=0 \quad(0 \leq i \leq n-2),
$$

where

$$
\begin{aligned}
& u_{0}(i):=(x-1)(i+1)(n-i+\alpha) T_{n, i+1}^{(\alpha, \beta)}(x), \\
& u_{1}(i):=x(n-i)(n-i+\alpha) T_{n, i+1}^{(\alpha, \beta)}(x)+(x-1)(i+2)(i+\beta+2) T_{n i}^{(\alpha, \beta)}(x), \\
& u_{2}(i):=x(n-i-1)(i+\beta+2) T_{n i}^{(\alpha, \beta)}(x),
\end{aligned}
$$

where the notation used is that of (2.11).

The coefficients $u_{j}(j=0,1,2)$ are not simple because they depend on two shifted Jacobi polynomials of degree $n$ in $x$. However, these polynomials are independent of $i$ and can be efficiently computed with the recurrence (1.3) (cf. Remark 1.1) and re-used for all remaining $i$. Thus, Corollary 2.3 may be useful in numerical practice.

Corollary 2.4. For $0 \leq i \leq n-3$, the polynomials $D_{i}^{n}(x ; \alpha, \beta)$ satisfy the following thirdorder recurence relation:

$$
\begin{equation*}
w_{0}(i) D_{i}^{n}(x ; \alpha, \beta)+w_{1}(i) D_{i+1}^{n}(x ; \alpha, \beta)+w_{2}(i) D_{i+2}^{n}(x ; \alpha, \beta)+w_{3}(i) D_{i+3}^{n}(x ; \alpha, \beta)=0 . \tag{2.14}
\end{equation*}
$$

Here

$$
\begin{aligned}
& w_{0}(i):=(x-1)(i+1)(n-i+\alpha-1)_{2}, \\
& w_{1}(i):=(n-i+\alpha-1)[x(n-i)(n-i+\alpha)+2(x-1)(i+2)(i+\beta+2)], \\
& w_{2}(i):=(i+\beta+2)[(x-1)(i+3)(i+\beta+3)+2 x(n-i-1)(n-i+\alpha-1)], \\
& w_{3}(i):=x(n-i-2)(i+\beta+2)_{2} .
\end{aligned}
$$

Notice that, compared to (1.10), the recurrence (2.14) is simpler: i) it has lower order (third instead of fourth); ii) its coefficients $w_{j}(0 \leq j \leq 3)$ are cubic polynomials in $i$ (not quintic; cf. [25, Eq. (4.2)]).

Remark 2.5. Using three new recurrence relations given in $\$ 2.1$ and 2.2 one can solve Problem 1.3 with $O(n)$ computational complexity (cf. Remark 1.1).

## 3. Algorithms for evaluating dual Bernstein polynomials

Let us come back to Problem 1.3 of computing all $n+1$ dual Bernstein polynomials of degree $n$ for fixed $n \in \mathbb{N}, \alpha, \beta>-1$ and $x \in[0,1]$. Recall that these polynomials are dual to Bernstein basis polynomials (1.7) in the interval $[0,1]$ (see (1.1) and Definition [1.2). So, in the context of applications of polynomials $D_{i}^{n}(x ; \alpha, \beta)$ presented in Section 11, the issue of their evaluation for $0 \leq x \leq 1$ is the most important.

If $x \in\{0,1\}$ then the value of the dual Bernstein polynomial can be easily obtained (cf. (2.6) or (2.7)). Now, suppose that $x \in(0,1)$. In this section we propose algorithms for evaluating polynomials $D_{i}^{n}(x ; \alpha, \beta)(0 \leq i \leq n)$ using the first-order non-homogeneous recurrence relation (cf. Theorem [2.1).

To obtain accurate methods, it is necessary to be mindful of numerical difficulties arising when recurrent computations are performed. See [2]. This is the reason that, in the sequel, we consider two ways of using relation (2.10), i.e., with a forward and a backward direction of computations.

For a fixed $0 \leq i \leq n$, let us define a forward computation of $D_{i}^{n}(x ; \alpha, \beta)$ as a computation that, starting from $D_{0}^{n}(x ; \alpha, \beta)$, computes $D_{i}^{n}(x ; \alpha, \beta)$ using Theorem [2.1] Analogously, a backward computation of $D_{i}^{n}(x ; \alpha, \beta)$ starts with $D_{n}^{n}(x ; \alpha, \beta)$ and uses Theorem 2.1] as well. As mentioned before, some numerical difficulties may arise when performing these computations - especially for sufficiently large $n$ and $i$. One can mitigate this issue by performing, for certain parameter $J \in \mathbb{N}(0 \leq J \leq n)$, a forward computation of $D_{i}^{n}(x ; \alpha, \beta)$ for $i=0,1, \ldots, J$ and a backward computation of $D_{i}^{n}(x ; \alpha, \beta)$ for $i=J+1, J+2, \ldots, n$. Note that, using (2.3), a backward computation can be expressed as a forward computation with changed parameters.

We have found that, in order to determine the value of $J$, one can use the function

$$
\begin{equation*}
J \equiv J(n, x)=\operatorname{round}(n \cdot p(x)), \tag{3.1}
\end{equation*}
$$

where $p$ is a cubic polynomial in $x$ which satisfies the following interpolation conditions:

$$
\begin{array}{c|c|c|c|c}
x & 0.01 & 0.3 & 0.7 & 0.99 \\
\hline p(x) & 0.1 & 0.4 & 0.6 & 0.9
\end{array}
$$

and $\operatorname{round}(z)$ denotes the nearest integer to the real number $z$. It can be checked that

$$
\begin{aligned}
p(x)= & 1.58084223194525186 \ldots \cdot x^{3}-2.37126334791787779 \ldots \cdot x^{2} \\
& +1.62239798468112882 \ldots \cdot x+0.08401156564574855 \ldots
\end{aligned}
$$

Let us stress that such choice of $J$ has been established experimentally and is used in all algorithms, as well as numerical tests, presented in this paper.

### 3.1. Algorithms

For given $n \in \mathbb{N}$ and $\alpha, \beta>-1$, an implementation of a forward computation of $D_{i}^{n}(x ; \alpha, \beta)$ for $i=0,1, \ldots, j$ and a fixed $0 \leq j \leq n$ at one point $x \in(0,1)$ is presented in Algorithm $\mathbb{1}$.

```
\(\overline{\text { Algorithm } 1 \text { Computation of } j+1 \text { first dual Bernstein polynomials of degree } n \text { at point }}\)
\(x \in(0,1)\)
    procedure DualBer \((n, \alpha, \beta, x, j, K)\)
        \(\alpha 1 \leftarrow \alpha+1, \beta 1 \leftarrow \beta+1\)
        \(n 1 \leftarrow n+\alpha 1, x 1 x \leftarrow(x-1) / x\)
        \(C \leftarrow(-1)^{n+1} \cdot K / n 1 \cdot \prod_{j=0}^{n-1}(1+\beta 1 /(j+\alpha 1))\)
        \(R 1 \leftarrow n 1 \cdot R_{n}^{(\alpha, \beta 1)}(x)\)
        \(R 2 \leftarrow x 1 x \cdot(n+\beta 1) \cdot R_{n}^{(\alpha 1, \beta)}(x)\)
        \(D[0] \leftarrow-C \cdot R 1\)
        for \(i \leftarrow 1, j\) do
            \(p \leftarrow i-n-1\)
            \(q \leftarrow i / p\)
        \(C \leftarrow C \cdot(p-\alpha 1) /(i+\beta)\)
        \(D[i] \leftarrow q \cdot x 1 x \cdot D[i-1]-C \cdot(R 1+q \cdot R 2)\)
    return \(D\)
```

For fixed $n \in \mathbb{N}$ and $\alpha, \beta>-1$, Algorithm 2 computes the values of all $n+1$ dual Bernstein polynomials of degree $n$ at one point $x \in(0,1)$. It computes the value $J$ (cf. (3.1)) and then performs two forward computations, utilizing Algorithm 1. This algorithm returns an array $D \equiv D[0 . . n]$, where

$$
D[i]=D_{i}^{n}(x ; \alpha, \beta) \quad(0 \leq i \leq n) .
$$

Remark 3.1. Shifted Jacobi polynomials $R_{n}^{(\alpha, \beta+1)}$ and $R_{n}^{(\alpha+1, \beta)}$ (cf. lines 5 , 6 in Algorithm (1) can be evaluated using recurrence relation (1.3) (cf. Remark (1.1) or even explicit formula (1.2). Thus, the computational complexity of Algorithm 2 is $O(n)$.

```
\(\overline{\text { Algorithm } 2}\) Computation of all \(n+1\) dual Bernstein polynomials of degree \(n\) at point
\(x \in(0,1)\)
    procedure \(\operatorname{AllDualBer}(n, \alpha, \beta, x)\)
        \(\alpha 1 \leftarrow \alpha+1, \beta 1 \leftarrow \beta+1\)
        \(K \leftarrow \Gamma(\alpha 1+\beta 1) /(\Gamma(\alpha 1) \cdot \Gamma(\beta 1))\)
        \(J \leftarrow J(n, x)\)
        \(D[0 . . J] \leftarrow \operatorname{DualBer}(n, \alpha, \beta, x, J, K)\)
        \(D[J+1 . . n] \leftarrow \operatorname{ReverseArray}(\operatorname{DualBer}(n, \beta, \alpha, 1-x, n-J-1, K))\)
        return \(D\)
```

Note that the quantities $q$ and $C$ in Algorithm [1 as well as the quantity $K$ in Algorithm [2, are independent of $x$. They can be, therefore, computed once for given $n \in \mathbb{N}, \alpha, \beta>-1$ and used across multiple instances of Problem 1.3 for different values of $x \in(0,1)$. This approach is realized in Algorithms 3 and 4 . Note that they require $O(n)$ additional memory to store $C$ and $q$.

```
\(\overline{\text { Algorithm } 3 \text { Computation of } j+1 \text { first dual Bernstein polynomials of degree } n \text { at point }}\)
\(x \in(0,1)\) - with preprocessing
    procedure DualBer-2( \(n, \alpha, \beta, x, j, q, C)\)
        \(\alpha 1 \leftarrow \alpha+1, \beta 1 \leftarrow \beta+1\)
        \(n 1 \leftarrow n+\alpha 1, x 1 x \leftarrow(x-1) / x\)
        \(R 1 \leftarrow n 1 \cdot R_{n}^{(\alpha, \beta 1)}(x)\)
        \(R 2 \leftarrow x 1 x \cdot(n+\beta 1) \cdot R_{n}^{(\alpha 1, \beta)}(x)\)
        \(D[0] \leftarrow C[0] \cdot R 1 / n 1\)
        for \(i \leftarrow 1, j\) do
        \(D[i] \leftarrow q[i-1] \cdot x 1 x \cdot D[i-1]-C[i] \cdot(R 1+q[i-1] \cdot R 2)\)
    return \(D\)
```

After executing Algorithm 4 we obtain a two-dimensional array $D \equiv D[0 . . M, 0 . . n]$, where

$$
D[m, i]=D_{i}^{n}\left(x_{m} ; \alpha, \beta\right) \quad(0 \leq m \leq M ; 0 \leq i \leq n) .
$$

The computational complexity of this algorithm is $O(n M)$ (cf. Remark 3.1).

## 4. Numerical experiments

The algorithms presented in the previous section have been tested for numerical stability. The computations have been performed in the computer algebra system Maple ${ }^{T M} 14$-using single (Digits:=8), double (Digits:=18) and quadruple (Digits:=32) precision-on a computer with Intel(R) Core(TM) i5-2540M CPU @ 2.60 GHz processor and 4 GB of RAM.

We measure the accuracy of approximation $\widetilde{v}$ of a nonzero number $v$ by computing the quantity

$$
\begin{equation*}
\operatorname{acc}(\widetilde{v}, v):=-\log _{10}\left|1-\frac{\widetilde{v}}{v}\right| . \tag{4.1}
\end{equation*}
$$

Hence, $\operatorname{acc}(\widetilde{v}, v)$ is the number of exact significant decimal digits (acc in short) in the approximation $\widetilde{v}$ of the number $v$.

```
\(\overline{\text { Algorithm } 4 \text { Computation of all } n+1 \text { dual Bernstein polynomials of degree } n \text { at multiple }}\)
points \(x_{0}, x_{1}, \ldots, x_{M} \in(0,1)\)
    procedure AllDualBer-2 \(\left(n, \alpha, \beta,\left[x_{0}, x_{1}, \ldots, x_{M}\right]\right)\)
        \(\alpha 1 \leftarrow \alpha+1, \beta 1 \leftarrow \beta+1\)
        \(K \leftarrow \Gamma(\alpha 1+\beta 1) /(\Gamma(\alpha 1) \cdot \Gamma(\beta 1))\)
        \(q[0] \leftarrow-1 / n\)
        \(C[0] \leftarrow(-1)^{n} \cdot K \cdot \prod_{j=0}^{n-1}(1+\beta 1 /(j+\alpha 1))\)
        \(C[1] \leftarrow C[0] / \beta 1\)
        for \(i \leftarrow 1, n-1\) do
            \(p \leftarrow i-n\)
            \(q[i] \leftarrow(i+1) / p\)
            \(C[i+1] \leftarrow C[i] \cdot(p-\alpha 1) /(i+\beta 1)\)
        for \(m \leftarrow 0, M\) do
            \(J \leftarrow J\left(n, x_{m}\right)\)
            \(D[m, 0 . . J] \leftarrow\) DuALBER-2 \(\left(n, \alpha, \beta, x_{m}, J, q, C\right)\)
            \(D[m, J+1 . . n] \leftarrow \operatorname{ReverseArray}\left(\operatorname{DuALBER}-2\left(n, \beta, \alpha, 1-x_{m}, n-J-1, q, C\right)\right)\)
        return \(D\)
```

For fixed $n \in \mathbb{N}$ and $\alpha, \beta>-1$, the experiments involved computing values of all $n+1$ dual Bernstein polynomials of degree $n$ for $x \in\{0.01,0.02, \ldots, 0.99\}$ using Algorithm (4) where Maple ${ }^{\text {TM }} 14$ GAMMA and JacobiP procedures have been used to compute values of $\Gamma$ function and shifted Jacobi polynomials, respectively. For each of $(n+1) \cdot 99$ obtained values, we have computed the number of exact significant decimal digits (cf. (4.1)), where results computed by the same algorithm but in a 512 -digit arithmetic (Digits:=512) have been assumed to be accurate while comparing to these done for Digits: $=8,18,32$.

The experiments have been performed for dual Bernstein polynomials of degrees $n \in$ $\{10,20,50,100,200,500,1000,2000,5000\}$ and three $\alpha, \beta$ choices - Legendre's $(\alpha=\beta=0)$, Chebyshev's $(\alpha=\beta=-0.5)$, and a non-standard choice $(\alpha=-0.33, \beta=5.6)$. A mean (Table 11), first percentile (Table 2) and minimal (Table 3) number of exact significant decimal digits have been computed.

The numerical results show that the proposed method for evaluating dual Bernstein polynomials works very well even for large degrees. Note that results given in Tables 11 and 2 are almost the same. It indicates that at least $99 \%$ of obtained values have greater or similar number of exact significant decimal digits than these given in Table $\mathbb{1}$ thus making the presented algorithms useful (for example, in numerical evaluation of integrals (1.9), even for large $n)$. Even though in pessimistic cases the algorithms lose a significant amount of precision (especially for Digits:=8; see Table 3), they happen rarely (compare with Table 2) and do not significantly affect the average number of correct decimal digits (cf. Table [1).

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|  |  | Digits:=8 | Digits:=18 | Digits:=32 |
| :---: | :---: | :---: | :---: | :---: |
| $n=10$ | $\alpha=\beta=0$ | 7.64 | 17.67 | 31.80 |
|  | $\alpha=\beta=-0.5$ | 6.97 | 17.03 | 31.04 |
|  | $\alpha=-0.33, \beta=5.6$ | 7.14 | 17.39 | 31.27 |
| $n=20$ | $\alpha=\beta=0$ | 7.24 | 17.15 | 31.14 |
|  | $\alpha=\beta=-0.5$ | 6.72 | 16.82 | 30.95 |
|  | $\alpha=-0.33, \beta=5.6$ | 6.65 | 17.47 | 31.17 |
| $n=50$ | $\alpha=\beta=0$ | 6.77 | 17.58 | 30.46 |
|  | $\alpha=\beta=-0.5$ | 6.58 | 17.47 | 30.55 |
|  | $\alpha=-0.33, \beta=5.6$ | 7.00 | 17.43 | 30.94 |
| $n=100$ | $\alpha=\beta=0$ | 6.98 | 16.80 | 30.32 |
|  | $\alpha=\beta=-0.5$ | 6.46 | 17.06 | 31.00 |
|  | $\alpha=-0.33, \beta=5.6$ | 6.79 | 17.30 | 31.16 |
| $n=200$ | $\alpha=\beta=0$ | 7.28 | 16.56 | 31.18 |
|  | $\alpha=\beta=-0.5$ | 6.41 | 16.12 | 30.65 |
|  | $\alpha=-0.33, \beta=5.6$ | 6.21 | 17.02 | 31.00 |
| $n=500$ | $\alpha=\beta=0$ | 6.65 | 17.01 | 30.95 |
|  | $\alpha=\beta=-0.5$ | 6.08 | 16.36 | 30.70 |
|  | $\alpha=-0.33, \beta=5.6$ | 6.13 | 16.80 | 30.91 |
| $n=1000$ | $\alpha=\beta=0$ | 6.51 | 16.31 | 30.23 |
|  | $\alpha=\beta=-0.5$ | 6.23 | 16.56 | 29.99 |
|  | $\alpha=-0.33, \beta=5.6$ | 5.85 | 15.73 | 29.73 |
| $n=2000$ | $\alpha=\beta=0$ | 6.09 | 16.88 | 29.64 |
|  | $\alpha=\beta=-0.5$ | 6.87 | 16.43 | 29.56 |
|  | $\alpha=-0.33, \beta=5.6$ | 6.21 | 15.81 | 30.43 |
| $n=5000$ | $\alpha=\beta=0$ | 5.57 | 15.36 | 30.12 |
|  | $\alpha=\beta=-0.5$ | 5.62 | 15.45 | 29.57 |
|  | $\alpha=-0.33, \beta=5.6$ | 5.29 | 15.42 | 30.51 |

Table 1: Mean number of acc (cf. (4.1)) obtained by using Algorithm 4 for $x \in\{0.01,0.02, \ldots, 0.99\}$.
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|  |  | Digits:=8 | Digits:=18 | Digits:=32 |
| :---: | :---: | :---: | :---: | :---: |
| $n=10$ | $\alpha=\beta=0$ | 6.34 | 16.36 | 30.47 |
|  | $\alpha=\beta=-0.5$ | 6.01 | 16.18 | 30.41 |
|  | $\alpha=-0.33, \beta=5.6$ | 6.31 | 16.26 | 30.35 |
| $n=20$ | $\alpha=\beta=0$ | 6.12 | 16.23 | 30.25 |
|  | $\alpha=\beta=-0.5$ | 5.99 | 16.16 | 30.32 |
|  | $\alpha=-0.33, \beta=5.6$ | 6.13 | 16.30 | 30.18 |
| $n=50$ | $\alpha=\beta=0$ | 6.31 | 16.44 | 30.16 |
|  | $\alpha=\beta=-0.5$ | 6.22 | 16.28 | 30.26 |
|  | $\alpha=-0.33, \beta=5.6$ | 6.31 | 16.31 | 30.34 |
| $n=100$ | $\alpha=\beta=0$ | 6.32 | 16.38 | 30.15 |
|  | $\alpha=\beta=-0.5$ | 6.17 | 16.36 | 30.36 |
|  | $\alpha=-0.33, \beta=5.6$ | 6.28 | 16.27 | 30.23 |
| $n=200$ | $\alpha=\beta=0$ | 6.11 | 16.13 | 30.21 |
|  | $\alpha=\beta=-0.5$ | 6.05 | 15.89 | 30.11 |
|  | $\alpha=-0.33, \beta=5.6$ | 5.96 | 16.17 | 30.15 |
| $n=500$ | $\alpha=\beta=0$ | 6.17 | 16.18 | 30.15 |
|  | $\alpha=\beta=-0.5$ | 5.90 | 16.02 | 30.17 |
|  | $\alpha=-0.33, \beta=5.6$ | 5.94 | 16.03 | 30.16 |
| $n=1000$ | $\alpha=\beta=0$ | 6.05 | 16.02 | 29.93 |
|  | $\alpha=\beta=-0.5$ | 5.87 | 16.11 | 29.82 |
|  | $\alpha=-0.33, \beta=5.6$ | 5.74 | 15.61 | 29.63 |
| $n=2000$ | $\alpha=\beta=0$ | 5.84 | 16.05 | 29.54 |
|  | $\alpha=\beta=-0.5$ | 6.02 | 16.01 | 29.42 |
|  | $\alpha=-0.33, \beta=5.6$ | 5.86 | 15.61 | 29.96 |
| $n=5000$ | $\alpha=\beta=0$ | 5.46 | 15.30 | 29.65 |
|  | $\alpha=\beta=-0.5$ | 5.49 | 15.36 | 29.46 |
|  | $\alpha=-0.33, \beta=5.6$ | 5.24 | 15.35 | 29.82 |

Table 2: First percentile number of acc (cf. (4.1)) obtained by using Algorithm 4 for $x \in\{0.01,0.02, \ldots, 0.99\}$.
[5] Z. Ciesielski, The basis of B-splines in the space of algebraic polynomials, Ukrainian Mathematical Journal 38 (1987) 311-315.
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|  |  | Digits:=8 | Digits:=18 | Digits:=32 |
| :---: | :---: | :---: | :---: | :---: |
| $n=10$ | $\alpha=\beta=0$ | 4.09 | 14.69 | 28.96 |
|  | $\alpha=\beta=-0.5$ | 4.97 | 15.36 | 29.06 |
|  | $\alpha=-0.33, \beta=5.6$ | 4.93 | 14.96 | 29.09 |
| $n=20$ | $\alpha=\beta=0$ | 5.33 | 15.35 | 29.41 |
|  | $\alpha=\beta=-0.5$ | 4.86 | 14.87 | 29.56 |
|  | $\alpha=-0.33, \beta=5.6$ | 5.39 | 15.12 | 28.91 |
| $n=50$ | $\alpha=\beta=0$ | 4.83 | 14.65 | 28.75 |
|  | $\alpha=\beta=-0.5$ | 4.47 | 14.32 | 28.48 |
|  | $\alpha=-0.33, \beta=5.6$ | 4.65 | 14.13 | 29.16 |
| $n=100$ | $\alpha=\beta=0$ | 4.47 | 14.84 | 28.73 |
|  | $\alpha=\beta=-0.5$ | 4.36 | 14.37 | 28.48 |
|  | $\alpha=-0.33, \beta=5.6$ | 2.98 | 12.84 | 27.35 |
| $n=200$ | $\alpha=\beta=0$ | 3.41 | 13.54 | 27.19 |
|  | $\alpha=\beta=-0.5$ | 3.62 | 13.42 | 27.73 |
|  | $\alpha=-0.33, \beta=5.6$ | 4.17 | 14.31 | 28.04 |
| $n=500$ | $\alpha=\beta=0$ | 3.15 | 13.65 | 27.06 |
|  | $\alpha=\beta=-0.5$ | 2.01 | 12.28 | 26.47 |
|  | $\alpha=-0.33, \beta=5.6$ | 3.37 | 13.37 | 27.26 |
| $n=1000$ | $\alpha=\beta=0$ | 2.92 | 12.97 | 27.30 |
|  | $\alpha=\beta=-0.5$ | 3.21 | 13.41 | 27.56 |
|  | $\alpha=-0.33, \beta=5.6$ | 3.18 | 12.99 | 27.44 |
| $n=2000$ | $\alpha=\beta=0$ | 2.16 | 12.24 | 26.03 |
|  | $\alpha=\beta=-0.5$ | 1.46 | 11.85 | 25.40 |
|  | $\alpha=-0.33, \beta=5.6$ | 2.11 | 11.93 | 25.97 |
| $n=5000$ | $\alpha=\beta=0$ | 0 | 5.38 | 18.85 |
|  | $\alpha=\beta=-0.5$ | 0 | 4.25 | 18.41 |
|  | $\alpha=-0.33, \beta=5.6$ | 0 | 5.64 | 19.33 |

Table 3: Minimal number of acc (cf. (4.1)) obtained by using Algorithm 4 for $x \in\{0.01,0.02, \ldots, 0.99\}$.
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