# TENSOR EXTRAPOLATION METHODS WITH APPLICATIONS 

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#### Abstract

In this paper, we mainly develop the well-known vector and matrix polynomial extrapolation methods in tensor framework. To this end, some new products between tensors are defined and the concept of positive definitiveness is extended for tensors corresponding to T-product. Furthermore, we discuss on the solution of least-squares problem associated with a tensor equation using Tensor Singular Value Decomposition (TSVD). Motivated by the effectiveness of proposed vector extrapolation method in [Numer. Algorithms, 51 (2009), 195-208], we describe how an extrapolation technique can be also implemented on the sequence of tensors produced by truncated TSVD (TTSVD) for solving possibly ill-posed tensor equations.


Keywords. Extrapolation; Sequence of tensors; Tensor SVD; T-products; Positive definite tensor; Least-squares problem; ill-posed problem.

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1. Introduction. In the last few years, several iterative methods have been proposed for solving large and sparse linear and nonlinear systems of equations. When an iterative process converges slowly, the extrapolation methods are required to obtain rapid convergence. The purpose of vector extrapolation methods is to transform a sequence of vector or matrices generated by some process to a new one that converges faster than the initial sequence. The well known extrapolation methods can be classified into two categories, the polynomial methods that includes the minimal polynomial extrapolation (MPE) method of Cabay and Jackson [6], the modified minimal polynomial extrapolation (MMPE) method of Sidi, Ford ans Smith [38], the reduced rank extrapolation (RRE) method of Eddy [31] and Mesina [34], Brezinski [5] and Pugatchev [36], and the $\epsilon$-type algorithms including the topological $\epsilon$-algorithm of Brezinski [5] and the vector $\epsilon$-algorithm of Wynn [43].
Efficient implementations of some of these extrapolation methods have been proposed by Sidi [39] for the RRE and MPE methods using QR decomposition while Jbilou and sadok [18] gives an efficient implementation of the MMPE based on a LU decomposition with pivoting strategy. It was also shown that when applied to linearly generated vector sequences, RRE and TEA methods are mathematically equivalent to GMRES and Lanczos methods, respectively. Those results were also extended to the block and global cases when dealing with matrix sequences, see $[17,20]$. Our aim in this paper is to define the analogue of these vector and matrix extrapolation methods to the tensor framework.

Basically, in the present paper, we develop some tensor extrapolation methods namely, the Tensor RRE (TRRE), the Tensor MPE (TMPE), the Tensor MMPE (TMMPE) and the Tensor Topological $\epsilon$-Algorithm (TTEA). We give some properties and show how these new tensor extrapolation methods can be applied to sequences obtained by truncation of the Tensor Singular Value Decomposition (TSVD) when applied to linear tensor discrete ill-posed problems.

The remainder of this paper is organized as follows. Before ending this section, we recall some fundamental concepts in tensor framework. In Section 2, we give notations, some basic

[^0]definitions and properties related to tensors. Moreover, we introduce the concept of positive definiteness for tensors with respect to T-product, some new products are also defined between tensors and their properties are analyzed. In Section 3, we introduce the tensor versions of the vector polynomial extrapolation methods namely the Tensor Reduced Rank Extrapolation (TRRE), the Tensor Minimal Polynomial Extrapolation (TMPE) and the Tensor Modified Minimal Polynomial Extrapolation (TMMPE), the Tensor Topological $\epsilon$-Algorithm (TTEA). Section 4 describes the TSVD, the truncated or low rank version of TSVD and shows how to apply the Tensor Reduced Rank Extrapolation method together with the truncated TSVD, to the solution of linear discrete tensor ill-posed problems.
Preliminaries: A tensor is a multidimensional array of data. The number of indices of a tensor is called modes or ways. Notice that a scalar can be regarded as a zero mode tensor, first mode tensors are vectors and matrices are second mode tensor. For a given N-mode tensor $X \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3} \ldots \times n_{N}}$, the notation $x_{i_{1}, \ldots, i_{N}}$ (with $1 \leq i_{j} \leq n_{j}$ and $j=1, \ldots N$ ) stand for the element $\left(i_{1}, \ldots, i_{N}\right)$ of the tensor $\mathcal{X}$. The norm of a tensor $\mathcal{X} \in \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{\ell}}$ is specified by
$$
\|X\|^{2}=\sum_{i_{1}=1}^{n_{1}} \sum_{i_{2}=1}^{n_{2}} \cdots \sum_{i_{\ell}=1}^{n_{\ell}} x_{i_{1} i_{2} \cdots i_{\ell}}^{2}
$$

Corresponding to a given tensor $\mathcal{X} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3} \ldots \times n_{N}}$, the notation

$$
X \underbrace{: \because \ldots:}_{(N-1)-\text { times }} k \text { for } k=1,2, \ldots, n_{N}
$$

denotes a tensor in $\mathbb{R}^{n_{1} \times n_{2} \times n_{3} \ldots \times n_{N-1}}$ which is obtained by fixing the last index and is called frontal slice. Fibers are the higher-order analogue of matrix rows and columns. A fiber is defined by fixing all the indexes except one. A matrix column is a mode- 1 fiber and a matrix row is a mode- 2 fiber. Third-order tensors have column, row and tube fibers. An element $c \in \mathbb{R}^{1 \times 1 \times n}$ is called a tubal-scalar of length $n$ [24]. More details are found in [23, 26].

Fig. 1.1. (a) Frontal, (b) horizontal, and (c) lateral slices of a third order tensor. (d) A mode-3 tube fibers.
2. Definitions and new tensor products. The current section is concerned with two main parts. In the first part, we recall definitions and properties related to T-product. Furthermore, we develop the definition of positive definiteness for tensors and establish some basic results. The second part deals with presenting some new products between tensors which can be used for simplifying the algebraic computations of the main results.
2.1. Definitions and properties. In this part, we briefly review some concepts and notations related to the T-Product, see [3, 13, 24, 25] for more details.

Definition 2.1. The T-product (*) between two tensors $X \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ and $y \in$ $\mathbb{R}^{n_{2} \times m_{2} \times n_{3}}$ is an $n_{1} \times m_{2} \times n_{3}$ tensor given by:

$$
x * y=\operatorname{Fold}(\operatorname{bcirc}(X) \operatorname{MatVec}(y))
$$

where

$$
\begin{aligned}
& \operatorname{bcirc}(X)=\left(\begin{array}{ccccc}
X_{1} & X_{n_{3}} & X_{n_{3-1}} & \ldots & X_{2} \\
X_{2} & X_{1} & X_{n_{3}} & \ldots & X_{3} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
X_{n_{3}} & X_{n_{3-1}} & \ddots & X_{2} & X_{1}
\end{array}\right) \in \mathbb{R}^{n_{1} n_{3} \times m_{2} n_{3}} \\
& \operatorname{MatVec}(y)=\left(\begin{array}{c}
Y_{1} \\
Y_{2} \\
\vdots \\
Y_{n_{3}}
\end{array}\right) \in \mathbb{R}^{n_{2} n_{3} \times m_{2}}, \quad \operatorname{Fold}(\operatorname{MatVec}(y))=y
\end{aligned}
$$

here for $i=1, \ldots, n_{3}, X_{i}$ and $Y_{i}$ are frontal slices of the tensors $X$ and $y$, respectively.
The $n_{1} \times n_{1} \times n_{3}$ identity tensor $\mathcal{J}_{n_{1} n_{1} n_{3}}$ is the tensor whose first frontal slice is the $n_{1} \times n_{1}$ identity matrix, and whose other frontal slices are all zeros, that is

$$
\operatorname{MatVec}\left(\mathcal{J}_{n_{1} n_{1} n_{3}}\right)=\left(\begin{array}{c}
I_{n_{1} n_{1}} \\
0_{n_{1} n_{1}} \\
\vdots \\
0_{n_{1} n_{1}}
\end{array}\right)
$$

where $I_{n_{1} n_{1}}$ is the identity matrix.
In the special case in which $n_{1}=1$, the identity tensor is a tubal-scalar and denoted by $\mathbf{e}$, in other words $\operatorname{MatVec}(\mathbf{e})=(1,0,0 \ldots, 0)^{T}$.

Definition 2.2. We have the following definitions

1. An $n_{1} \times n_{1} \times n_{3}$ tensor $\mathcal{A}$ is invertible, if there exists a tensor $\mathcal{B}$ of order $n_{1} \times n_{1} \times n_{3}$ such that

$$
\mathcal{A} * \mathcal{B}=\mathcal{J}_{n_{1} n_{1} n_{3}} \quad \text { and } \quad \mathcal{B} * \mathcal{A}=\mathcal{J}_{n_{1} n_{1} n_{3}}
$$

It is clear that $\mathcal{A}$ is invertible if and only if $\operatorname{bcirc}(\mathcal{A})$ is invertible (see [35]).
2. If $\mathcal{A}$ is an $n_{1} \times n_{2} \times n_{3}$ tensor, then $\mathcal{A}^{T}$ is the $n_{2} \times n_{1} \times n_{3}$ tensor obtained by transposing each of the front-back frontal slices and then reversing the order of transposed frontal slices 2 through $n_{3}$.
Exemple 1. If $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times 5}$ and its frontal slices are given by the $n_{1} \times n_{2}$ matrices $A_{1}, A_{2}, A_{3}, A_{4}, A_{5}$, then

$$
\mathcal{A}^{T}=\text { Fold }\left(\begin{array}{l}
A_{1}^{T} \\
A_{5}^{T} \\
A_{4}^{T} \\
A_{3}^{T} \\
A_{2}^{T}
\end{array}\right)
$$

Remark 2.1. As pointed out earlier the tensor $\mathcal{A}$ of order $m \times m \times n$ is invertible iff $\operatorname{bcirc}(\mathcal{A})$ is invertible. It is equivalent to say that $\mathcal{A}$ is invertible iff $\mathcal{A} * \mathcal{X}=\mathcal{O}$ implies $X=\mathcal{O}$ where $X \in \mathbb{R}^{m \times 1 \times n}$ and $\mathcal{O}$ is zero tensor.

Here we define the notion of positive definiteness for tensors in term of T-product which can be seen as a natural extension of the same concept for matrices.

Definition 2.3. The tensor $\mathcal{A} \in \mathbb{R}^{m \times m \times n}$ is said to be positive (semi) definite if

$$
\left(X^{T} * \mathcal{A} * X\right)_{:: 1}>(\geq) 0
$$

for all nonzero tensors $X \in \mathbb{R}^{m \times 1 \times n}$.
Remark 2.2. In view of Remark 2.1, it is immediate to see that every positive definite tensor is invertible. For any tensor $B \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ and arbitrary nonzero tensor $\mathcal{X} \in \mathbb{R}^{n_{2} \times 1 \times n_{3}}$, it can be seen that

$$
\left(\mathcal{X}^{T} * \mathcal{B}^{T} * \mathcal{B} * \mathcal{X}\right)_{:: 1}=\left((\mathcal{B} * \mathcal{X})^{T} * \mathcal{B} * \mathcal{X}\right)_{:: 1}=\|\mathcal{B} * \mathcal{X}\|^{2} \geq 0
$$

in which the last equality follows from [25]. As a result, the tensor $\mathcal{B}^{T} * \mathcal{B}$ is positive semi definite. Evidently, for any scalar $\epsilon>0$,

$$
\left(X^{T} *\left(\mathcal{B}^{T} * \mathcal{B}+\epsilon \mathcal{J}_{n_{2} n_{2} n_{3}}\right) * X\right)_{:: 1}=\|\mathcal{B} * X\|^{2}+\epsilon\|\mathcal{X}\|^{2}>0
$$

This shows that the tensor $\mathcal{B}^{T} * \mathcal{B}+\epsilon \mathcal{J}_{n_{2} n_{2} n_{3}}$ is positive definite.
Definition 2.4. Let $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$. Then the tensor $\mathcal{X} \in \mathbb{R}^{n_{2} \times n_{1} \times n_{3}}$ satisfying the following four conditions
(a) $\mathcal{A} * \mathcal{X} * \mathcal{A}=\mathcal{A}$
(b) $\quad X * \mathcal{A} * X=X$,
(c) $(\mathcal{A} * \mathcal{X})^{T}=\mathcal{A} * \mathcal{X}$,
(d) $(X * \mathcal{A})^{T}=X \neq \mathcal{A}$.
is called the Moore-Penrose inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^{\dagger}$. In view of [25, Lemmas 3.3 and 3.16] and by using straightforward computations, one can easily observe that conditions (a)-(d) determine $\mathcal{A}^{\dagger}$ uniquely. Here we further comment that if $\mathcal{A}$ is invertible, then $\mathcal{A}^{\dagger}=\mathcal{A}^{-1}$. For $\mathcal{X}, \mathcal{y}$ two tensors in $\mathbb{R}^{n_{1} \times 1 \times n_{3}}$, theT-scalar product $\langle.,$.$\rangle is a bilinear form defined by:$

$$
\left\{\begin{array}{cl}
\mathbb{R}^{n_{1} \times 1 \times n_{3}} \times \mathbb{R}^{n_{1} \times 1 \times n_{3}} & \longrightarrow \mathbb{R}^{1 \times 1 \times n_{3}}  \tag{2.1}\\
(X, y) & \longrightarrow\langle X, y\rangle=X^{T} * y
\end{array}\right.
$$

Let $X_{1}, \ldots, X_{\ell}$ a collection of $\ell$ third tensors in $\mathbb{R}^{n_{1} \times 1 \times n_{3}}$, if

$$
\left\langle X_{i}, X_{j}\right\rangle= \begin{cases}\alpha_{i} \mathbf{e} & i=j \\ 0 & i \neq j\end{cases}
$$

where $\alpha_{i}$ is a non-zero scalar, then the set $X_{1}, \ldots, X_{\ell}$ is said to be an orthogonal collection of tensors. The collection is called orthonormal if $\alpha_{i}=1, i=1, \ldots, l$.

Definition 2.5. An $n \times n \times \ell$ real-values tensor $\mathcal{Q}$ is said to be is orthogonal if $\mathbb{Q}^{T} * \mathcal{Q}=$ $Q * \mathbb{Q}^{T}=\mathcal{J}_{\text {nn } \ell}$.
We end this part with the following proposition.
Proposition 2.6. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two tensors of order $n_{1} \times n_{2} \times n_{3}$. Then

$$
\left(\mathcal{A}^{T} * \mathcal{B}\right)_{i j:}=(\mathcal{A}(:, i,:))^{T} * \mathcal{B}(:, j,:) .
$$

Proof. Let $\widetilde{\mathcal{J}}_{\tau}$ be an $n_{2} \times 1 \times n_{3}$ tensor whose all frontal slices are zero except its first frontal being the $\tau$-th column of the $n_{2} \times n_{2}$ identity matrix. Then we have

$$
\mathcal{B} * \widetilde{\mathcal{J}}_{j}=\mathcal{B}(:, j,:) \quad \text { and } \quad \widetilde{\mathfrak{J}}_{i}^{T} * \mathcal{A}^{T}=\mathcal{A}^{T}(i,:,:)=(\mathcal{A}(:, i,:))^{T}
$$

Notice also that

$$
\left(\mathcal{A}^{T} * \mathcal{B}\right)_{i j:}=\widetilde{\mathfrak{J}}_{i}^{T} *\left(\mathcal{A}^{T} * \mathcal{B}\right) * \widetilde{\mathcal{J}}_{j} .
$$

From [25, Lemma 3.3], it is known that

$$
\tilde{\mathfrak{J}}_{i}^{T} *\left(\mathcal{A}^{T} * \mathcal{B}\right) * \tilde{\mathcal{J}}_{j}=\left(\tilde{\mathfrak{J}}_{i}^{T} * \mathcal{A}^{T}\right) *\left(\mathcal{B} * \tilde{\mathcal{J}}_{j}\right)
$$

which completes the proof.
2.2. New tensor products . In order to simplify derivation of generalized extrapolation methods in tensor format, we need to define new tensor products.

Definition 2.7. Let $\mathcal{A}$ and $\mathcal{B}$ be 4-mode tensors with frontal slices $\mathcal{A}_{i} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ and $\mathcal{B}_{j} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ for $i=1,2, \ldots, \ell$ and $j=1,2, \ldots, k$, respectively. The product $\mathcal{A} \diamond \mathcal{B}$ is defined as a 5-mode tensor of order $n_{2} \times n_{2} \times n_{3} \times k \times \ell$ defined as follows:

$$
(\mathcal{A} \diamond \mathcal{B})_{::: j i}=\mathcal{A}_{i}^{T} * \mathcal{B}_{j}
$$

for $i=1,2, \ldots, \ell$ and $j=1,2, \ldots, k$. In the case where $k=1$, i.e. $\mathcal{B} \in \mathbb{R}^{n_{1} \times s \times n_{3}}, \mathcal{A} \diamond \mathcal{B}$ is a 4-mode tenor whose $i$-th frontal slice is given by $\mathcal{A}_{i}^{T} * \mathcal{B}$ for $i=1,2, \ldots, \ell$.

Here we comment that the $\diamond$ product can be seen as a generalization of $\diamond$-product between two matrices given in [2]. In the sequel, we further present an alternative product called $\star$-product which can be seen as an extension of the $*$-product between a set of matrices and vectors; see [19] for more details.

Definition 2.8. Let $\mathcal{A}$ be a 5-mode tensor of order $n_{2} \times n_{1} \times n_{3} \times k \times \ell$ and $\mathcal{B}$ be a 4-mode tensor with frontal slices $\mathcal{B}_{1}, \ldots, \mathcal{B}_{k} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$. The product $\mathcal{A} \star \mathcal{B}$ is defined as a 4-mode tensor of order $n_{2} \times n_{2} \times n_{3} \times \ell$ such that

$$
(\mathcal{A} \star \mathcal{B})_{::: i}=\sum_{j=1}^{k} \mathcal{A}_{::: j i} * \mathcal{B}_{j}, \quad i=1,2, \ldots, \ell
$$

As a natural way, for the case that $\mathcal{A}$ is a 4-mode tensor of order $n_{2} \times n_{1} \times n_{3} \times \ell$ then $\mathcal{A} \star \mathcal{B} \in \mathbb{R}^{n_{2} \times n_{2} \times n_{3}}$ defined by $\mathcal{A} \star \mathcal{B}=\sum_{\eta=1}^{k} \mathcal{A}_{\eta} * \mathcal{B}_{\eta}$ where $\mathcal{A}_{i}$ stands for the $i$-th frontal slice of $\mathcal{A}$ for $i=1,2, \ldots, \ell$.

In the main results, it is worth to define a notion of the left inverse of a 5 -mode tensors dealing with $*, \star$ and $\diamond$ products. To this end, we define the $\bar{\star}$-product between two 5 -mode tensors which can be seen as an extension of $\star$ product.

DEFINITION 2.9. Let $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3} \times k \times \ell}$ and $\mathcal{B} \in \mathbb{R}^{n_{2} \times n_{1} \times n_{3} \times k \times \ell}$, the product $\mathcal{A} \mp \mathcal{B}$ is a 5-mode tensor of order $n_{1} \times n_{1} \times n_{3} \times k \times k$ such that for $\tau, \eta=1,2, \ldots, k$,

$$
(\mathcal{A} \mp \mathcal{B})_{::: \tau \eta}=\sum_{j=1}^{\ell} \mathcal{A}_{::: \eta j} * \mathcal{B}_{::: \tau j} .
$$

Definition 2．10．The tensor $\mathcal{B}^{+} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3} \times k \times \ell}$ is called a left inverse of $\mathcal{B} \in$ $\mathbb{R}^{n_{2} \times n_{1} \times n_{3} \times k \times \ell}$ ，if

$$
\left(\mathcal{B}^{+} \text {天 } \mathcal{B}\right)_{::: \tau \eta}= \begin{cases}\mathcal{J}_{n_{1} n_{1} n_{3}} & \tau=\eta \\ \mathcal{O} & \tau \neq \eta\end{cases}
$$

for $\tau, \eta=1,2, \ldots, k$ ．Here $\mathcal{O}$ is the 3－mode zero tensor of order $n_{1} \times n_{1} \times n_{3}$ ．
Now we establish a proposition which reveals the relation between the two proposed products $\star$ and 天．

Proposition 2．11．Let $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3} \times k \times k}, \mathcal{B} \in \mathbb{R}^{n_{2} \times n_{1} \times n_{3} \times k \times k}$ and $y \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3} \times k}$ ． Then，the following relation holds

$$
(\mathcal{A} 天 \mathcal{B}) \star y=\mathcal{A}^{*} \star(\mathcal{B} \star y) .
$$

Here $\mathcal{A}^{*}$ stands for an 5－mode tensor of order $n_{1} \times n_{2} \times n_{3} \times k \times k$ associated with $\mathcal{A}$ where $\mathcal{A}_{::: i j}^{*}=\mathcal{A}_{::: j i}$ for $1 \leq i, j \leq k$ ．

Proof．It is clear that both side of the above relation are 4－mode tensors of order $n_{1} \times n_{2} \times$ $n_{3} \times k$ ．To prove the assertion，we show that the frontal slices of both sides are equal．Let $1 \leq z \leq k$ ，we can observe that

$$
\begin{aligned}
((\mathcal{A} 天 \mathcal{B}) \star \mathcal{y})_{::: z} & =\sum_{\ell=1}^{k}(\mathcal{A} 天 \mathcal{B})_{::: \ell z} * y_{\ell} \\
& =\sum_{\ell=1}^{k} \sum_{\mu=1}^{k} \mathcal{A}_{::: z \mu} * \mathcal{B}_{::: \ell \mu} * y_{\ell} \\
& =\sum_{\mu=1}^{k} \mathcal{A}_{::: z \mu} *\left(\sum_{\ell=1}^{k} \mathcal{B}_{::: \ell \mu} * y_{\ell}\right) \\
& =\sum_{\mu=1}^{k} \mathcal{A}_{::: z \mu} *(\mathcal{B} \star y)_{::: \mu} \\
& =\sum_{\mu=1}^{k} \mathcal{A}_{::: \mu z}^{*} *(\mathcal{B} \star y)_{::: \mu}=\left(\mathcal{A}^{*} \star(\mathcal{B} \star y)\right)_{::: z}
\end{aligned}
$$

The result follows immediately from the above computations．
We end this section by a proposition which can be established by using straightforward algebraic computations．

Proposition 2．12．Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{n_{1} \times s \times n_{3} \times \ell}$ and $\mathcal{D} \in \mathbb{R}^{s \times n_{2} \times n_{3}}$ ．The following state－ ments hold：

1．$(\mathcal{A}+\mathcal{B}) \diamond \mathfrak{C}=\mathcal{A} \diamond \mathfrak{C}+\mathcal{B} \diamond \mathfrak{C}$
2． $\mathcal{A} \diamond(\mathcal{B}+\mathcal{C})=\mathcal{A} \diamond \mathcal{B}+\mathcal{A} \diamond \mathcal{C}$
3．$(\mathcal{A} \diamond \mathcal{B}) \star \mathcal{D}=\mathcal{A} \diamond(\mathcal{B} \star \mathcal{D})$ ．
3．Extrapolation methods based on tensor formats．In this section，we define new tensor extrapolation methods．In the first part，we present in the tensor polynomial－type extrapolation methods by using the new tensor products introduced in the preceding section． In the second part，a tensor topological $\epsilon$－algorithm is developed．We notice that when we are dealing with vectors，all these new methods become just the classical vector extrapolation methods．
3.1. Tensor polynomial-type extrapolation methods. Corresponding to a given sequence of tensors $\left(\bar{S}_{n}\right)$ in $\mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$, we consider the transformation $\left(\bar{T}_{k}\right)$ defined by

$$
T_{k}\left(S_{n}\right)=T_{k}^{(n)}:=\left\{\begin{array}{cl}
\mathbb{R}^{n_{1} \times n_{2} \times n_{3}} & \rightarrow \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}  \tag{3.1}\\
S_{n} & \mapsto T_{k}^{(n)}=S_{n}+\mathcal{G}_{k, n} \star \alpha_{k}
\end{array}\right.
$$

where the 4 -mode $\mathcal{G}_{k, n}$ with frontal slices $G_{i}(n) \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ is given for $i=1, \ldots, k$. The 4-mode tensor $\alpha_{k}$ is unknown whose frontal slices are denoted by $\alpha_{i}^{(n)} \in \mathbb{R}^{n_{2} \times n_{2} \times n_{3}}$ for $i=$ $1, \ldots, k$. As the vector and matrix case, in extrapolation methods, we aim to determine the unknown tensors. To this end, we use the transformation $\widetilde{T}_{k}^{(n)}$ obtained from $T_{k}^{(n)}$ as follows:

$$
\widetilde{T}_{k}^{(n)}=\widetilde{T}_{k}\left(S_{n}\right)=S_{n+1}+\mathcal{G}_{k, n+1} \star \alpha_{k}
$$

here, the 4-mode $\mathcal{G}_{k, n+1}$ has frontal slices $G_{i}(n+1) \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ for $i=1, \ldots, k$. Let $\Delta$ denote the forward difference operator on the index $n$ such that $\Delta S_{n}=S_{n+1}-S_{n}$ and $\Delta \mathcal{G}_{k, n}$ stand for the 4 -mode tensor whose frontal slices are given by $\Delta G_{i}(n)=G_{i}(n+1)-G_{i}(n)$ for $i=1, \ldots, k$. The generalized residual of $T_{k}^{(n)}$ is represented by $R\left(T_{k}^{(n)}\right)$ defined as follows:

$$
\begin{align*}
R\left(T_{k}^{(n)}\right) & =\widetilde{T}_{k}\left(S_{n}\right)-T_{k}\left(S_{n}\right) \\
& =\Delta S_{n}+\Delta \mathcal{G}_{k, n} \star \alpha_{k} \tag{3.2}
\end{align*}
$$

For an arbitrary given set of tensors $Y_{1}^{(n)}, \ldots, Y_{k}^{(n)} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$, let $\widetilde{\mathbf{H}}_{k, n}$ and $\widetilde{\mathbf{L}}_{k, n}$ denote the subspaces generated by $\Delta G_{1}(n), \ldots, \Delta G_{k}(n)$ and $Y_{1}^{(n)}, \ldots, Y_{k}^{(n)}$ respectively. Evidently, we have

$$
\begin{equation*}
R\left(T_{k}^{(n)}\right)-\Delta S_{n} \in \widetilde{\mathbf{H}}_{k, n} \tag{3.3}
\end{equation*}
$$

and the unknown tensors $\alpha_{i}^{(n)}$ are determined by imposing the following condition,

$$
R\left(T_{k}^{(n)}\right) \in \widetilde{\mathbf{L}}_{k, n}^{\perp}
$$

Let $\mathcal{L}_{k, n}$ be a 4 -mode tensor with frontal slices $Y_{1}^{(n)}, \ldots, Y_{k}^{(n)}$, the above relation can be equivalently expressed by

$$
\begin{equation*}
\mathcal{L}_{k, n} \diamond R\left(T_{k}^{(n)}\right)=\mathcal{O} \tag{3.4}
\end{equation*}
$$

where $\mathcal{O}$ is a zero tensor of order $1 \times 1 \times n_{3} \times k$. Therefore, from Proposition 2.12, we have

$$
\begin{aligned}
\mathcal{O} & =\mathcal{L}_{k, n} \diamond R\left(T_{k}^{(n)}\right) \\
& =\mathcal{L}_{k, n} \diamond\left(\Delta S_{n}+\Delta \mathcal{G}_{k, n} \star \alpha_{k}\right) \\
& =\mathcal{L}_{k, n} \diamond \Delta S_{n}+\mathcal{L}_{k, n} \diamond\left(\Delta \mathcal{G}_{k, n} \star \alpha_{k}\right)=\mathcal{L}_{k, n} \diamond \Delta S_{n}+\left(\mathcal{L}_{k, n} \diamond \Delta \mathcal{G}_{k, n}\right) \star \alpha_{k} .
\end{aligned}
$$

In fact the unknown tensor $\alpha_{k}$ can be seen as the solution of thee following tensor equation,

$$
\left(\mathcal{L}_{k, n} \diamond \Delta \mathcal{G}_{k, n}\right) \star \alpha_{k}=-\mathcal{L}_{k, n} \diamond \Delta S_{n} .
$$

The choices of sequence of tensors $G_{1}(n), \ldots, G_{k}(n)$ and $Y_{1}^{(n)}, \ldots, Y_{k}^{(n)}$ determine the type of the tensor polynomial extrapolation method. In fact, for all these polynomial-type methods,
the auxiliary sequence of tensors is given by $G_{i}(n)=\Delta S_{n+i-1}$ for $i=1, \ldots, k(n \geq 0)$. The following choices for $Y_{1}^{(n)}, \ldots, Y_{k}^{(n)}$ can be used,

$$
\begin{array}{lc}
Y_{i}^{(n)}=\Delta S_{n+i-1} & \text { for TMPE } \\
Y_{i}^{(n)}=\Delta^{2} S_{n+i-1} & \text { for TRRE } \\
Y_{i}^{(n)}=Y_{i} & \text { for TMMPE }
\end{array}
$$

where the operator $\Delta^{2}$ refers to the second forward difference with respect to the index $n$ such that

$$
\Delta^{2} S_{n}=\Delta S_{n+1}-\Delta S_{n} \quad \text { and } \quad \Delta^{2} G_{i}(n)=\Delta G_{i}(n+1)-\Delta G_{i}(n)
$$

for $i=1, \ldots, k$. The approximation $T_{k}^{(n)}$ produced by TMPE, TRRE and TMMPE can be also expressed as follows:

$$
T_{k}^{(n)}=\sum_{j=0}^{k} S_{n+j} * \gamma_{j}^{(k)}
$$

and the unknown tensors $\gamma_{0}^{(k)}, \gamma_{1}^{(k)}, \ldots, \gamma_{k}^{(k)}$ are determined by imposing the following condition

$$
\begin{equation*}
\sum_{j=0}^{k} \gamma_{j}^{(k)}=\mathcal{J}_{n_{2} n_{2} n_{3}} \quad \text { and } \quad \sum_{j=0}^{k} \eta_{i, j}^{(n)} * \gamma_{j}^{(k)}=\mathcal{O}_{n_{2} n_{2} n_{3}} \quad 0 \leq i<k \tag{3.5}
\end{equation*}
$$

where $\mathcal{O}_{n_{2} n_{2} n_{3}} \in \mathbb{R}^{n_{2} \times n_{2} \times n_{3}}$ is the tensor which all entries equal to zero, $\eta_{i, j}^{(n)}=\left(Y_{i+1}^{(n)}\right)^{T} * \Delta S_{n+j}$.
From now on and for simplification, we set $n=0$. The system of equations (3.5) is given in the following form

$$
\left\{\begin{array}{l}
\gamma_{0}^{(k)}+\gamma_{1}^{(k)}+\cdots+\gamma_{k}^{(k)}=\mathcal{J}_{n_{2} n_{2} n_{3}}  \tag{3.6}\\
\left(Y_{1}^{(0)}\right)^{T} * \Delta S_{0} * \gamma_{0}^{(k)}+\left(Y_{1}^{(0)}\right)^{T} * \Delta S_{1} * \gamma_{1}^{(k)}+\cdots+\left(Y_{1}^{(0)}\right)^{T} * \Delta S_{k} * \gamma_{k}^{(k)}=\mathcal{O}_{n_{2} n_{2} n_{3}} \\
\left(Y_{2}^{(0)}\right)^{T} * \Delta S_{0} * \gamma_{0}^{(k)}+\left(Y_{2}^{(0)}\right)^{T} * \Delta S_{1} * \gamma_{1}^{(k)}+\cdots+\left(Y_{2}^{(0)}\right)^{T} * \Delta S_{k} * \gamma_{k}^{(k)}=\mathcal{O}_{n_{2} n_{2} n_{3}} \\
\cdots \cdots \cdots \cdots \cdots \cdots \\
\left(Y_{k}^{(0)}\right)^{T} * \Delta S_{0} * \gamma_{0}^{(k)}+\left(Y_{k}^{(0)}\right)^{T} * \Delta S_{1} * \gamma_{1}^{(k)}+\cdots+\left(Y_{k}^{(0)}\right)^{T} * \Delta S_{k} * \gamma_{k}^{k}=\mathcal{O}_{n_{2} n_{2} n_{3}}
\end{array} .\right.
$$

Let $\beta_{i}^{(k)}=\gamma_{i}^{(k)} *\left(\gamma_{k}^{(k)}\right)^{-1}$ where $\left(\gamma_{k}^{(k)}\right)^{-1}$ is the inverse of $\gamma_{k}^{(k)}$, i.e., $\gamma_{k}^{(k)} *\left(\gamma_{k}^{(k)}\right)^{-1}=\mathcal{J}_{n_{2} n_{2} n_{3}}$, for $0 \leq l \leq k$. Then, it is not difficult to verify that

$$
\begin{equation*}
\gamma_{i}^{(k)}=\beta_{i}^{(k)} *\left(\sum_{i=0}^{k} \beta_{i}^{(k)}\right)^{-1} \quad \text { for } \quad 0 \leq l<k \quad \text { and } \quad \beta_{k}^{(k)}=\mathcal{J}_{n_{2} n_{2} n_{3}} \tag{3.7}
\end{equation*}
$$

The system of equations (??sysofequa2) can be rewritten as follows

$$
\left\{\begin{array}{l}
\left(Y_{1}^{(0)}\right)^{T} * \Delta S_{0} * \beta_{0}^{(k)}+\cdots+\left(Y_{1}^{(0)}\right)^{T} * \Delta S_{k-1} * \beta_{k-1}^{(k)}=-\left(Y_{1}^{(0)}\right)^{T} * \Delta S_{k}  \tag{3.8}\\
\cdots \cdots \cdots \cdots \cdots \cdots \\
\left(Y_{k}^{(0)}\right)^{T} * \Delta S_{0} * \beta_{0}^{(k)}+\cdots+\left(Y_{k}^{(0)}\right)^{T} * \Delta S_{k-1} * \beta_{k-1}^{(k)}=-\left(Y_{k}^{(0)}\right)^{T} * \Delta S_{k}
\end{array}\right.
$$

The above system can be mentioned in the following form

$$
\begin{equation*}
\left(\mathcal{L}_{k, n} \diamond \mathcal{V}_{k}\right) \star \beta_{k}=-\left(\mathcal{L}_{k, n} \diamond \Delta S_{k}\right) \tag{3.9}
\end{equation*}
$$

where $\beta_{k}$ is the 4 -mode tensor with the $k$ frontal slices $\beta_{0}^{(k)}, \ldots, \beta_{k-1}^{(k)}$ and $\mathcal{V}_{k}$ is a 4-mode tensor whose $i$-th frontal slice is given by $\Delta S_{i-1}$ for $i=1,2, \ldots, k$.
Having $\gamma_{0}, \gamma_{1}, \ldots, \gamma_{k}$ computed, we set

$$
\begin{equation*}
\alpha_{0}^{(k)}=\mathcal{J}_{n_{2} n_{2} n_{3}}-\gamma_{0}^{(k)}, \quad \alpha_{j}^{(k)}=\alpha_{j-1}^{(k)}-\gamma_{j}^{(k)}, \quad 1 \leq j<k \quad \text { and } \quad \alpha_{k-1}^{(k)}=\gamma_{k}^{(k)} . \tag{3.10}
\end{equation*}
$$

Setting $T_{k}=T_{k}^{(0)}$, we get

$$
\begin{equation*}
T_{k}=S_{0}+\sum_{j=0}^{k-1} V_{j} * \alpha_{j}^{(k)}=S_{0}+\mathcal{V}_{k} \star \alpha_{k} \tag{3.11}
\end{equation*}
$$

where $V_{j}=\Delta S_{j}$ the $(j+1)$-th frontal slice of $\mathcal{V}_{k}$ for $j=0 \ldots, k-1$ and $\alpha_{k}$ is a 4-mode tensor with frontal slices $\alpha_{0}^{(k)}, \ldots, \alpha_{k-1}^{(k)}$. To determine $\gamma_{i}^{(k)}$ for $i=0,1, \ldots, k$, we first we need to compute $\beta^{(k)}$ by solving system of equations (3.9). Using (3.2), (3.10) and (3.11), the generalized residual $R\left(T_{k}\right)$ can be also seen as follows:

$$
\begin{equation*}
R\left(T_{k}\right)=\sum_{i=0}^{k} V_{i} * \gamma_{i}^{(k)}=\mathcal{V}_{k} \star \gamma_{k} \tag{3.12}
\end{equation*}
$$

in which $\gamma_{k}$ and $\mathcal{V}_{k}$ are 4 -mode tensors with whose $i$-th frontal slices are respectively given by $\gamma_{i-1}^{(k)}$ and $\bar{V}_{i-1}$ for $i=1,2, \ldots, k$.
3.2. The tensor toplogical $\epsilon$ - transformation. For vector sequences, Brezinski [5] proposed the well known topological $\epsilon$-algorithm (TEA) which is a generalization of the scalar $\epsilon$-algorithm [43] known as a technique for transforming slowly convergent or divergent sequences. In this section, we briefly see how to extend this idea in tensor framework and define the Tensor Toplogical $\epsilon$-Transformation (TTET).
Let $\left(S_{n}\right)$ a given sequence of tensors in $\mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$, and consider approximations of tensors $E_{k}\left(S_{n}\right)=E_{k}^{(n)}$ of the limit of sequence $\left(S_{n}\right)_{n \in \mathbb{N}}$ defined as

$$
\begin{equation*}
E_{k}^{(n)}=S_{n}+\sum_{i=1}^{k} \Delta S_{n+i-1} * \beta_{i}^{(n)}, \quad n \geq 0 \tag{3.13}
\end{equation*}
$$

where $\beta_{i}^{(n)} \in \mathbb{R}^{n_{2} \times n_{2} \times n_{3}}$ are the unknown tensors to be determined for $i=1 \ldots, k$. We set

$$
\widetilde{E}_{k, j}^{(n)}=S_{n+j}+\sum_{i=1}^{k} \Delta S_{n+i+j-1} * \beta_{i}^{(n)} \quad j=1, \ldots, k,
$$

where $\widetilde{E}_{k, 0}^{(n)}=E_{k}^{(n)}$. Let $\widetilde{R}_{j}\left(\bar{E}_{k}^{(n)}\right)$ denote the $j$-th generalized residual tensor, i.e.,

$$
R_{j}\left(E_{k}^{(n)}\right)=\widetilde{E}_{k, j}^{(n)}-\widetilde{E}_{k, j-1}^{(n)}
$$

For given third order tensor $Y \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ be given. The coefficients $\beta_{i}^{(n)}$ in (3.13) are computed such that

$$
\sum_{i=1}^{k}\left(Y^{T} * \Delta^{2} S_{n+i+j-1}\right) * \beta_{i}^{(n)}=\mathcal{O}, \quad j=0,1 \ldots, k-1
$$

where $\mathcal{O} \in \mathbb{R}^{n_{2} \times n_{2} \times n_{3}}$. The above conditions result the following system of equations:

$$
\left\{\begin{array}{l}
\left(Y^{T} * \Delta^{2} S_{n}\right) * \beta_{1}^{(n)}+\cdots+\left(Y^{T} * \Delta^{2} S_{n+k-1}\right) * \beta_{k}^{(n)}=-Y^{T} * \Delta S_{n} \\
\quad \vdots \\
\left(Y^{T} * \Delta^{2} S_{n+k-1}\right) * \beta_{1}^{(n)}+\cdots+\left(Y^{T} * \Delta^{2} S_{n+2 k-2}\right) * \beta_{k}^{(n)}=-Y^{T} * \Delta S_{n+k-1}
\end{array}\right.
$$

In the case that the above system is uniquely solvable we can obtain $E_{k}^{(n)}$.
4. Application of tensor extrapolation methods to solve ill-posed tensor problems. The purpose of this section is to adopt the idea used by Jbilou et al. [22] for solving a class of ill-posed tensor equations. To this end, we start by recalling the truncated tensor SVD for the third order-tensors (TTSVD), then we prove a theorem which gives the minimum norm (least-squares) solution of our mentioned tensor equation. In addition, we present an algorithm for approximating the Moore-Penros inverse of tensor. In the second subsection, we combine TTSVD with the TSRRE method to resolve tensor ill-posed problems.
4.1. Truncated tensor singular value decomposition. The truncated SVD (T-SVD) of matrices is efficient for approximating Moore-Penrose inverse and solving least-squares problem as the truncated version consumes less space of storage in comparison with SVD when the rank of matrix is not very large. This inspired Miao et al. [35] to extend the theory of tensor SVD (TSVD) [25] to truncated tensor SVD (TTSVD); here the decompositions are based on the T-product of tensors. In this part, first, the TSVD and TTSVD are recalled. Then we derive the explicit form for minimum norm solution of least-square problem in tensor framework.

In the following, an F-diagonal tensor refers to a third order tensor whose all frontal slices are diagonal.

Theorem 4.1. [25] Let $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ be a real valued-tensor, then there exists orthogonal tensors $\mathcal{U} \in \mathbb{R}^{n_{1} \times n_{1} \times n_{3}}$ and $\mathcal{V} \in \mathbb{R}^{n_{2} \times n_{2} \times n_{3}}$ such that

$$
\begin{equation*}
\mathcal{A}=\mathcal{U} * \mathcal{S} * \mathcal{V}^{T} \tag{4.1}
\end{equation*}
$$

in which $\mathcal{S} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ is an $F$-diagonal tensor .
ThEOREM 4.2. [35] Let $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ be a real valued-tensor, then there exist unitary tensors $\mathcal{U}_{(k)} \in \mathbb{R}^{n_{1} \times k \times n_{3}}$ and $\mathcal{V}_{(k)} \in \mathbb{R}^{n_{2} \times k \times n_{3}}$ such that

$$
\begin{equation*}
\mathcal{A} \approx \mathcal{A}_{k}=\mathcal{U}_{(k)} * \mathcal{S}_{(k)} * \mathcal{V}_{(k)}^{T} \tag{4.2}
\end{equation*}
$$

and the Moore-Penrose inverse of tensor $\mathcal{A}_{k}$ is given by

$$
\mathcal{A}_{k}^{\dagger}=\mathcal{V}_{(k)} * \mathcal{S}_{(k)}^{\dagger} * \mathcal{U}_{(k)}^{T}
$$

where $\mathcal{S}_{k} \in \mathbb{R}^{k \times k \times n_{3}}$ is an $F$-diagonal tensor and $k<\min (n, m)$ is called the tubal-rank of $\mathcal{A}$ based on the T-product.

In [35], the decomposition (4.2) is called the tensor compact SVD (T-CSVD) of $\mathcal{A}$ and the tensor $\mathcal{A}_{k}$ can be regarded the rank-k approximation of the tensor $\mathcal{A}$. Here we call it TTSVD. We comment here that a similar compression strategy to (4.2) is also given by Kilmer and Martin[25].
The TTSVD of $\mathcal{A}_{k}$ can be expressed as follows

$$
\begin{equation*}
\mathcal{A}_{k}=\sum_{j=1}^{k} \bar{U}_{j} * d_{j} * \bar{V}_{j}^{T} \tag{4.3}
\end{equation*}
$$

where $\bar{U}_{j}=\mathcal{U}_{(k)}(:, j,:) \in \mathbb{R}^{n_{1} \times 1 \times n_{3}}, \bar{V}_{j}=\mathcal{V}_{(k)}(:, j,:) \in \mathbb{R}^{n_{2} \times 1 \times n_{3}}$ and $d_{j}=\mathcal{S}(j, j,:) \in \mathbb{R}^{1 \times 1 \times n_{3}}$ for $j=1,2, \ldots, k$.

Remark 4.1. Let $\mathcal{A}=\mathcal{U} * \mathcal{S} * \mathcal{V}^{T}$ be the TSVD of $\mathcal{A}$. In view of Proposition 2.6, we can see that

$$
\bar{U}_{i} * \bar{U}_{j}=\left(\mathcal{U}^{T} * \mathcal{U}\right)_{i j:} \quad \text { and } \quad \overline{\mathrm{V}}_{\mathrm{i}} * \overline{\mathrm{~V}}_{\mathrm{j}}=\left(\mathcal{V}^{\mathrm{T}} * \mathcal{V}\right)_{\mathrm{ij}:}
$$

Therefore, we have $\bar{U}_{i} * \bar{U}_{j}$ and $\bar{V}_{i} * \bar{V}_{j}$ are zero tubal-scalar of length $n_{3}$ for $i \neq j ; \bar{U}_{i} * \bar{U}_{i}=\mathbf{e}$ and $\bar{V}_{i} * \bar{V}_{i}=\mathbf{e}$.

The following theorem reveals that $\mathcal{A}_{k}$ is an optimal approximation of a tensor, see [25] for the proof.

THEOREM 4.3. Let the TSVD of $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ be given by $\mathcal{A}=\mathcal{U} * \mathcal{S} * \mathcal{V}^{T}$ and for $k<\min \left(n_{1}, n_{2}\right)$, the tensor $\mathcal{A}_{k}$ given by (4.3) satisfies

$$
\mathcal{A}_{k}=\operatorname{argmin}_{\tilde{\mathcal{A}} \in M}\|\mathcal{A}-\tilde{\mathcal{A}}\|
$$

where $M=\left\{\mathcal{C}=X * y \mid X \in \mathbb{R}^{n_{1} \times k \times n_{3}}, y \in \mathbb{R}^{k \times n_{2} \times n_{3}}\right\}$.
In view of Theorem 4.3, the Moore-Penrose inverse of tensor $\mathcal{A}$ is efficiently estimated on $\widetilde{M}=\left\{\mathcal{C}=X * y \mid X \in \mathbb{R}^{n_{2} \times k \times n_{3}}, y \in \mathbb{R}^{k \times n_{1} \times n_{3}}\right\}$ by

$$
\begin{equation*}
\mathcal{A}^{\dagger} \approx \sum_{j=1}^{k} \bar{V}_{j} * d_{j}^{\dagger} * \bar{U}_{j}^{T} \tag{4.4}
\end{equation*}
$$

in which tubal-scalar $d_{j}^{\dagger} \in \mathbb{R}^{1 \times 1 \times n_{3}}$ stands for the $(j, j,:)$ entry of $\mathcal{S}_{(k)}^{\dagger}$.
The following theorem has a key role in deriving the results of the next section.
Theorem 4.4. Assume that $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ and $\mathcal{B} \in \mathbb{R}^{n_{1} \times s \times n_{3}}$. If $\hat{X}=\mathcal{A}^{\dagger} * \mathcal{B}$, then

$$
\begin{equation*}
\|\mathcal{A} * \hat{X}-\mathcal{B}\|=\min _{x \in \mathbb{R}^{n_{2} \times s \times n_{3}}}\|\mathcal{A} * \mathcal{X}-\mathcal{B}\| \tag{4.5}
\end{equation*}
$$

For any $\tilde{X} \in \mathbb{R}^{n_{1} \times s \times n_{3}}$ such that $\tilde{X} \neq \hat{X}$ and $\|\mathcal{A} * \hat{X}-\mathcal{B}\|=\|\mathcal{A} * \tilde{X}-\mathcal{B}\|$, then $\|\hat{X}\|<\|\tilde{X}\|$.
Proof. Let $A=U^{T} * \mathcal{S} * \mathcal{V}$ be the TSVD of $\mathcal{A}$. From [25, Lemma 3.19], it can be seen that

$$
\|\mathcal{A} * \mathcal{X}-\mathcal{B}\|=\left\|\mathcal{U}^{T} *(\mathcal{A} * X-\mathcal{B})\right\| .
$$

By some straightforward computations, we have

$$
\left\|\mathcal{U}^{T} *(\mathcal{A} * \mathcal{X}-\mathcal{B})\right\|=\left\|\mathcal{S} * \mathcal{V}^{T} * \mathcal{X}-\mathcal{U}^{T} * \mathcal{B}\right\| .
$$

Setting $\mathcal{Z}=\mathcal{V}^{T} * X$ and $\mathcal{W}=U^{T} * \mathcal{B}$, we get

$$
\begin{align*}
\left\|\mathcal{U}^{T} *(\mathcal{A} * \mathcal{X}-\mathcal{B})\right\| & =\|\mathcal{S} * \mathcal{Z}-\mathcal{W}\| \\
& =\left\|\left(F_{n_{3}} \otimes I_{n_{1}}\right) \operatorname{MatVec}(\mathcal{S} * \mathcal{Z})-\left(F_{n_{3}} \otimes I_{n_{1}}\right) \operatorname{MatVec}(\mathcal{W})\right\|_{F} \\
& =\left\|\left(F_{n_{3}} \otimes I_{n_{1}}\right) \operatorname{bcirc}(\mathcal{S}) \operatorname{MatVec}(\mathcal{Z})-\left(F_{n_{3}} \otimes I_{n_{1}}\right) \operatorname{MatVec}(\mathcal{W})\right\|_{F} \tag{4.6}
\end{align*}
$$

where $\|\cdot\|_{F}$ is the well-known Frobenius matrix norm, the notation $\otimes$ stands for the Kronecker product and the matrix $F_{n_{3}}$ is the discrete Fourier matrix of size $n_{3} \times n_{3}$ defined by (see [8])

$$
F_{n_{3}}=\frac{1}{\sqrt{n_{3}}}\left(\begin{array}{cccccc}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^{2} & \omega^{3} & \cdots & \omega^{n_{3}-1} \\
1 & \omega^{2} & \omega^{4} & \omega^{6} & \cdots & \omega^{2\left(n_{3}-1\right)} \\
1 & \omega^{3} & \omega^{6} & \omega^{9} & \cdots & \omega^{3\left(n_{3}-1\right)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n_{3}-1} & \omega^{2\left(n_{3}-1\right)} & \omega^{3\left(n_{3}-1\right)} & \cdots & \omega^{\left(n_{3}-1\right)\left(n_{3}-1\right)}
\end{array}\right) \text {, }
$$

in which $\omega=e^{-2 \pi \mathbf{i} / n_{3}}$ is the primitive $n_{3}$-th root of unity in which $\mathbf{i}=\sqrt{-1}$ and $F^{*}$ denotes the conjugate transpose of $F$.

For simplicity we set $Z=\left(F_{n_{3}} \otimes I_{n_{2}}\right) \operatorname{MatVec}(\mathcal{Z})$ and $W=\left(F_{n_{3}} \otimes I_{n_{1}}\right) \operatorname{MatVec}(\mathcal{W})$, hence Eq.4.6 can be rewritten as follows:

$$
\left\|U^{T} *(\mathcal{A} * X-\mathcal{B})\right\|=\left\|\left(F_{n_{3}} \otimes I_{n_{1}}\right) \operatorname{bcirc}(\mathcal{S})\left(F_{n_{3}}^{*} \otimes I_{n_{2}}\right) Z-W\right\|_{F} .
$$

Notice that

$$
\begin{equation*}
\|Z\|_{F}=\|\operatorname{MatVec}(\mathcal{Z})\|_{F}=\|\mathcal{Z}\|=\left\|\mathcal{V}^{T} * X\right\|=\|X\| . \tag{4.7}
\end{equation*}
$$

From [25, 35], it is known that there exists diagonal (rectangular) matrices $\Sigma_{1}, \Sigma_{2}, \ldots, \Sigma_{n_{3}}$ such that

$$
\left(F_{n_{3}} \otimes I_{n_{1}}\right) \operatorname{bcirc}(\mathcal{S})\left(F_{n_{3}}^{*} \otimes I_{n_{2}}\right)=\Sigma=\left(\begin{array}{cccc}
\Sigma_{1} & & & \\
& \Sigma_{2} & & \\
& & \ddots & \\
& & & \Sigma_{n_{3}}
\end{array}\right)
$$

and

$$
\left(F_{n_{3}}^{*} \otimes I_{n_{1}}\right) \operatorname{bcirc}\left(\delta^{\dagger}\right)\left(F_{n_{3}} \otimes I_{n_{2}}\right)=\Sigma^{\dagger}=\left(\begin{array}{cccc}
\Sigma_{1}^{\dagger} & & & \\
& \Sigma_{2}^{\dagger} & & \\
& & \ddots & \\
& & & \Sigma_{n_{3}}^{\dagger}
\end{array}\right)
$$

From the above computations, we have

$$
\|\mathcal{A} * \mathcal{X}-\mathcal{B}\|=\|\Sigma Z-W\|_{F}
$$

It is well-known from the literature that the minimum Frobenius norm solution of $\|\Sigma Z-W\|_{F}$ over $\mathbb{R}^{n_{2} n_{3} \times s}$ is given by $\hat{Z}=\Sigma^{\dagger} W$, i.e.,

$$
\hat{Z}=\operatorname{argmin}_{Z \in \mathbb{R}^{n_{2} n_{3} \times s}}\|\Sigma Z-W\|_{F} .
$$

Let $\hat{X}$ be the tensor such that

$$
\begin{aligned}
\operatorname{MatVec}\left(\mathcal{V}^{T} * \hat{X}\right) & =\left(F_{n_{3}}^{*} \otimes I_{n_{2}}\right) \Sigma^{\dagger} W \\
& =\left(F_{n_{3}}^{*} \otimes I_{n_{2}}\right) \Sigma^{\dagger}\left(F_{n_{3}} \otimes I_{n_{1}}\right) \operatorname{MatVec}(\mathcal{W}) \\
& =\operatorname{bcirc}\left(\mathcal{S}^{\dagger}\right) \operatorname{MatVec}\left(\mathcal{U}^{T} * \mathcal{B}\right)
\end{aligned}
$$

It is immediate to deduce the following equality

$$
\operatorname{MatVec}\left(\mathcal{V}^{T} * \hat{X}\right)=\operatorname{MatVec}\left(\mathcal{S}^{\dagger} * \mathcal{U}^{T} * \mathcal{B}\right)
$$

which is equivalent to say that $\mathcal{V}^{T} * \hat{X}=\mathcal{S}^{\dagger} * U^{T} * \mathcal{B}$. Finally, the result follows from (4.7).
Similar to the TSVD [25], the TTSVD can be obtained using the fast Fourier transform. Notice that circulant matrices are diagonalizable via the normalized DFT. This fact was used for proving Theorem 4.1 and deriving the Matlab pseudocode for TSVD; see Algorithm 1 for more details. Here, we further use this fact and present a Matlab pseudocode for computing TTSVD and the corresponding approximation of Moore-Penrose inverse in Algorithm 2. Note that the Matlab function pinv(.) reduces to $\operatorname{inv}($.$) when the diagonal matrix S$ in Step 2 of the algorithm is nonsingular.

```
Algorithm 1 Matlab pseudocode for TSVD [24]
Input. \(\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}\);
Step 1. Set \(\mathcal{D}=\operatorname{fft}(\mathcal{A},[], 3)\);
Step 2. for \(i=1,2, \ldots, n_{3}\)
    \([U, S, V]=\operatorname{svd}(\mathcal{D}(:,:, i))\);
    \(\mathcal{U}(:,:, i)=U ; \mathcal{V}(:,:, i)=V ; \mathcal{S}(:,:, i)=S ;\)
    end
```

Step 3. $\mathcal{U}=\operatorname{ifft}(\mathcal{U},[], 3) ; \mathcal{V}=\operatorname{ifft}(\mathcal{V},[], 3) ; \mathcal{S}=\operatorname{ifft}(\mathcal{S},[], 3)$;
Output. $\mathcal{A}=\mathcal{U} * \mathcal{S} * \mathcal{V}^{T}$.
4.2. Tensor extrapolation methods applied to TTSVD sequences in ill-posed problems. Let $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ be given and consider its TSVD. That is let us apply Algorithm 1 for $\mathcal{A}$. In the case that the matrix $\operatorname{bcirc}(\mathcal{A})$ has too many singular values being close to zero, the tensor $\mathcal{A}$ is called ill-determined rank.

In this subsection, we discuss solutions of tensor equations in the following form

$$
\begin{equation*}
\mathcal{A} * \widetilde{X}=\widetilde{\mathcal{B}} \tag{4.8}
\end{equation*}
$$

```
Algorithm 2 Matlab pseudocode for TTSVD and the corresponding approximation of Moore-
Penrose inverse
Input. \(\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}\) and \(k \leq \min \left(n_{1}, n_{2}\right)\);
Step 1. Set \(\mathcal{D}=\mathrm{fft}(\mathcal{A},[], 3)\);
Step 2. for \(i=1,2, \ldots, n_{3}\)
\([U, S, V]=\operatorname{svd}(\mathcal{D}(:,:, i)) ;\)
\(\mathcal{U}(:,:, i)=U(:, 1: k) ; \mathcal{V}(:,:, i)=V(:, 1: k)\);
    \(\mathcal{S}(:,:, i)=S(1: k, 1: k) ; \mathcal{S}^{\dagger}(:,:, i)=\operatorname{pinv}(S(1: k, 1: k))\);
end
Step 3. \(\mathcal{U}_{(k)}=\operatorname{ifft}(\mathcal{U},[], 3) ; \mathcal{V}_{(k)}=\operatorname{ifft}(\mathcal{V},[], 3) ; \mathcal{S}_{(k)}=\operatorname{ifft}(\mathcal{S},[], 3) ; \mathcal{S}_{(k)}^{\dagger}=\operatorname{ifft}\left(\mathcal{S}^{\dagger},[], 3\right) ;\)
Output. \(\mathcal{A}_{k}=\mathcal{U}_{(k)} * \mathcal{S}_{(k)} * \mathcal{V}_{(k)}^{T}\) and \(\mathcal{A}_{k}^{\dagger}=\mathcal{V}_{(k)} * \mathcal{S}_{(k)}^{\dagger} * \mathcal{U}_{(k)}^{T}\).
```

where the ill-determined rank tensor $\mathcal{A} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ and right-hand side $\widetilde{\mathcal{B}} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ are given and $\mathcal{X} \in \mathbb{R}^{n_{2} \times n_{2} \times n_{3}}$ is an unknown tensor to be determined when the values of $n_{2}$ and $n_{3}$ are small or moderate. Our goal here is to find an approximate solution for the above tensor equation. Systems of tensors equations (4.8) with a tensor of ill-determined rank often are referred to as linear discrete tensor ill-posed problems. They arise in several areas in science and engineering, such as the restoration of color and multispectral images [4, 27, 40], blind source separation [29], when one seeks to determine the cause of an observed effect. In these applications, the right-hand side $\widetilde{\mathcal{B}} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}$ is typically contaminated by an error $\mathcal{E}$, i.e., $\widetilde{\mathcal{B}}=\tilde{\bar{B}}+\varepsilon$ where $\tilde{\overline{\mathcal{B}}}$ denotes the unknown error-free right-hand side. We are interested in determining an accurate approximation of the (least-squares) solution $\tilde{\bar{X}}$ of the following (in)consistent tensor equation

$$
\mathcal{A} * \tilde{\bar{x}}=\widetilde{\overline{\mathcal{B}}}
$$

with error-free right-hand side by solving (4.8). From Theorem 4.4, it is known that the exact solution is given by

$$
\tilde{\tilde{X}}=\mathcal{A}^{\dagger} * \tilde{\overline{\mathcal{B}}} .
$$

In the matrix case, when the size of the rank deficient matrix $A$ is moderate, a popular method for computing an approximation for the (least-squares) solution of (in)consistent linear system of equations $A x=b$ consist in using the Truncated SVD which replaces the matrix $A^{\dagger}$ by a low-rank approximation; see, e.g., Golub and Van Loan [12] or Hansen [14]. Following the same idea, we approximate $\widetilde{X}$ by using the expression (4.4). In view of Theorem 4.3, we approximate $\mathcal{A}^{\dagger} * \overline{\mathcal{B}}$ by $\widetilde{X}_{k}$ given as follow:

$$
\widetilde{X}_{k}=\sum_{j=1}^{k} \bar{V}_{j} * d_{j}^{\dagger} * \bar{U}_{j}^{T} * \widetilde{\mathcal{B}} .
$$

For the matrix case, Jbilou et al. [20] proposed the application of the RRE to the sequence of vectors generated by the truncated SVD. In the remainder of this paper, we follow the same idea and consider the application of TRRE to the sequence of tensors generated by the truncated TSVD (TTSVD). To do so, we set

$$
Y_{i}=\Delta^{2} S_{i-1} \quad 1 \leq i \leq k-2,
$$

in which $\left(\bar{S}_{k}\right)_{k \geq 0}$ is the tensor sequence generated by the TTSVD. Thus,

$$
\begin{equation*}
S_{k}=\mathcal{A}_{k}^{\dagger} * \widetilde{\mathcal{B}}=\sum_{j=1}^{k} \bar{V}_{j} * \delta_{j} \tag{4.9}
\end{equation*}
$$

where $\delta_{j}=d_{j}^{\dagger} * \bar{U}_{j}^{T} * \overline{\mathcal{B}}$ and $S_{0}$ is set to be a zero tensor of order $n_{2} \times n_{2} \times n_{3}$. It can be observed that

$$
\begin{equation*}
\Delta S_{k-1}=S_{k}-\bar{S}_{k-1}=\bar{V}_{k} * \delta_{k} \tag{4.10}
\end{equation*}
$$

We assume here that $\delta_{k}^{T} * \delta_{k}$ is invertible and notice that if $\delta_{k}^{T} * \delta_{k}$ is zero, then we can delete the corresponding member from the sequence (4.9) and compute the next one by keeping the same index notation. To overcome the cases $\delta_{k}^{T} * \delta_{k}$ is numerically non invertible, we can use a small shift $\epsilon \mathcal{J}$ (say $\epsilon=1 e-10$ ) on the positive semi definite tensor $\delta_{k}^{T} * \delta_{k}$; see Remark 2.2.

Let $\Delta S_{k}$ and $\Delta^{2} S_{k}$ be 4-mode tensors whose $i$-th frontal slices are given by $\Delta S_{i-1}$ and $\Delta^{2} S_{i-1}$ for $i=1,2, \ldots, k$, respectively, for $i=1,2, \ldots, k$
Notice that $\Delta^{2} S_{j-1}=\bar{V}_{j+1} * \delta_{j+1}-\bar{V}_{j} * \delta_{j}$ for $j=1,2, \ldots k$. Hence,

$$
\begin{aligned}
\left(\Delta^{2} \mathcal{S}_{k} \diamond \Delta \mathcal{S}_{k}\right)_{::: j i} & =\left(\delta_{i+1}^{T} * \bar{V}_{i+1}^{T}-\delta_{i}^{T} * \bar{V}_{i}^{T}\right) * \bar{V}_{j} * \delta_{j} \\
& =\delta_{i+1}^{T} * \bar{V}_{i+1}^{T} * \bar{V}_{j} * \delta_{j}-\delta_{i}^{T} * \bar{V}_{i}^{T} * \bar{V}_{j} * \delta_{j} \quad \text { for } \quad i, j=1,2, \ldots, k
\end{aligned}
$$

Using Remark 4.1, we can deduce that the nonzero frontal slices of $\Delta^{2} \tilde{\mathcal{S}_{k}} \diamond \Delta \mathcal{S}_{k}$ are given by

$$
\left(\Delta^{2} \mathcal{S}_{k} \diamond \Delta \mathcal{S}_{k}\right)_{:::(i+1) i}=\delta_{i+1}^{T} * \delta_{i+1}
$$

and

$$
\left(\Delta^{2} \mathcal{S}_{k} \diamond \Delta \mathcal{S}_{k}\right)_{::: i i}=-\delta_{i}^{T} * \delta_{i}
$$

Straightforward computations together with Remark 4.1 show that the frontal slices of the 4mode tensor ( $\Delta^{2} \mathcal{S}_{k} \diamond \Delta S_{k}$ ) are equal to zero except the last frontal slice being equal to $\delta_{k+1}^{T} * \delta_{k+1}$. For notational simplicity, we define $\Theta_{i+1}=\delta_{i+1}^{T} * \delta_{i+1}$ for $i=0, \ldots k$. In summary, we need to solve the following tensor equation

$$
\left(\Delta^{2} \mathcal{S}_{k} \diamond \Delta \mathcal{S}_{k}\right) \star \beta_{k}=-\left(\Delta^{2} \mathcal{S}_{k} \diamond \Delta S_{k}\right)
$$

where $\beta_{k}$ is a 4-mode tensor with frontal slices $\beta_{0}^{(k)}, \beta_{1}^{(k)}, \ldots, \beta_{k-1}^{(k)}$. Or equivalently, we need to find the solution of the following system of tensor equations:

$$
\left\{\begin{array}{cl}
-\Theta_{1} * \beta_{0}^{(k)}+\Theta_{2} * \beta_{1}^{(k)} & =\mathcal{O}  \tag{4.11}\\
-\Theta_{2} * \beta_{1}^{(k)}+\Theta_{3} * \beta_{2}^{(k)} & =\mathcal{O} \\
\ddots & \\
& -\Theta_{k} * \beta_{k-1}^{(k)}
\end{array}\right)=-\Theta_{k+1} l
$$

here $\mathcal{O}$ stands for zero tensor of order $n_{2} \times n_{2} \times n_{3}$. It is immediate to see that

$$
\begin{equation*}
\beta_{i}^{(k)}=\left(\Theta_{i+1}\right)^{-1} * \Theta_{k+1} \quad 0 \leq i<k \tag{4.12}
\end{equation*}
$$

is a solution of (4.11). Evidently, we have

$$
\begin{equation*}
\sum_{i=0}^{k} \beta_{i}^{(k)}=\sum_{i=0}^{k}\left(\Theta_{i+1}\right)^{-1} * \Theta_{k+1} \tag{4.13}
\end{equation*}
$$

where $\beta_{k}^{(k)}=\mathcal{J}_{n_{2} n_{2} n_{3}}$. By the discussions in Subsection 3.1, from Eq. (3.7), we can derive $\gamma_{j}^{(k)}$ for $j=0,1, \ldots, k-1$ noticing that $\gamma_{k}^{(k)}=\mathcal{J}_{n_{2} n_{2} n_{3}}-\sum_{i=0}^{k-1} \gamma_{i}^{(k)}$. For $i=0,1, \ldots, k-1$, we can further compute $\alpha_{i}^{(k)}$ for $i=0,1, \ldots, k-1$ by (3.10).
Finally, the extrapolated third tensor can be written as follows:

$$
\begin{equation*}
T_{k}=\Delta S_{k} \star \alpha^{(k)} \tag{4.14}
\end{equation*}
$$

where $\Delta \mathcal{S}_{k}$ and $\alpha^{(k)}$ are 4 -mode tensors whose $j$-th frontal slices are receptively given by $\Delta S_{j-1}=S_{j}-S_{j-1}=\bar{V}_{j} * \delta_{j}$ and $\alpha_{j-1}^{(k)}$ for $j=1,2, \ldots, k$.
The generalized residual can be written in the following form

$$
R\left(T_{k}\right)=\Delta \mathcal{S}_{k} \star \gamma^{(k)}=\sum_{i=0}^{k} \bar{V}_{i+1} * \delta_{i+1} * \gamma_{i}^{(k)}
$$

where $\gamma^{(k)}$ is a 4-mode tensor with frontal slices $\gamma_{0}^{(k)}, \gamma_{2}^{(k)}, \ldots, \gamma_{k-1}^{(k)}$. By Remark 4.1, one can derive

$$
\begin{aligned}
R\left(T_{k}\right)^{T} * R\left(T_{k}\right) & =\sum_{i=0}^{k}\left(\gamma_{i}^{(k)}\right)^{T} * \delta_{i+1}^{T} * \delta_{i+1} * \gamma_{i}^{(k)} \\
& =\sum_{i=0}^{k}\left(\gamma_{i}^{(k)}\right)^{T} * \Theta_{i+1} * \gamma_{i}^{(k)}
\end{aligned}
$$

Now by Eqs. (3.7), (4.12) and (4.13), we can observe

$$
R\left(T_{k}\right)^{T} * R\left(T_{k}\right)=\Theta_{k} * \gamma_{k-1}^{(k)}
$$

It is known that (see [25])

$$
\begin{equation*}
\left\|R\left(T_{k}\right)\right\|^{2}=\operatorname{trace}\left(\left(\Theta_{k} * \gamma_{k-1}^{(k)}\right)_{:: 1}\right) \tag{4.15}
\end{equation*}
$$

For ill-posed problems, the value of $\left\|R\left(T_{k}\right)\right\|$ decrease when $k$ increases and is sufficiently small. However, the norm of $R\left(T_{k}\right)$ may increase with $k$. Hence, similar to [22], we may need to exploit an alternative stopping criterion when the problem is ill-posed. To this end, we can use

$$
\begin{equation*}
\eta_{k}:=\frac{\left\|T_{k+1}-T_{k}\right\|}{\left\|\widetilde{T}_{k}\right\|}=\frac{\sqrt{\operatorname{trace}\left(\left(\left(T_{k+1}-T_{k}\right)^{T} *\left(\bar{T}_{k+1}-T_{k}\right)\right)_{:: 1}\right)}}{\sqrt{\operatorname{trace}\left(\left(T_{k}^{T} * T_{k}\right)_{:: 1}\right)}} \tag{4.16}
\end{equation*}
$$

Using Remark 4.1 and some computations, one may simplify the above relation in the following way,
$\left(T_{k+1}-T_{k}\right)^{T} *\left(T_{k+1}-T_{k}\right)=\sum_{j=1}^{k}\left(\alpha_{j-1}^{(k+1)}-\alpha_{j-1}^{(k)}\right)^{T} * \Theta_{j} *\left(\alpha_{j-1}^{(k+1)}-\alpha_{j-1}^{(k)}\right)+\alpha_{k}^{(k+1)} * \Theta_{k+1} * \alpha_{k}^{(k+1)}$,
and

$$
T_{k}^{T} * T_{k}=\sum_{j=1}^{k}\left(\alpha_{j-1}^{(k)}\right)^{T} * \Theta_{j} * \alpha_{j-1}^{(k)}
$$

We end this section by summarizing the above discussions in Algorithm 3.

```
Algorithm 3 The TRRE-TSVD Algorithm
Input. \(\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n_{1} \times n_{2} \times n_{3}}\), tolerance \(\epsilon\) and \(k \leq \min \left(n_{1}, n_{3}\right)\) for TTSVD of \(\mathcal{A}\);
Step 1. Run Algorithm 1 to compute an approximation for its Moore-Penrose inverse, i.e.,
\(\mathcal{A}^{\dagger}=\mathcal{V} * \mathcal{S}^{\dagger} * \mathcal{U}^{T}\).
Step 2. Set \(\bar{S}_{0}=\mathcal{O}, \bar{S}_{1}=\bar{V}_{1} * d_{1}^{\dagger} * \bar{U}_{1}^{T} * \overline{\mathcal{B}}, \bar{T}_{1}=\bar{S}_{1}\) and \(k=2\).
Step 3. tol \(=1\).
Step 4. while tol \(<\epsilon\)
    Compute \(\bar{S}_{k}\) from (4.9)
    Compute \(\gamma_{i}^{(k)}\) and \(\alpha_{i}^{(k)}\) for \(i=0, \ldots, k-1\) using (3.7) and (3.10) where
    \(\gamma_{k}^{(k)}=\mathcal{J}_{n_{2} n_{2} n_{3}}-\sum_{i=0}^{k-1} \gamma_{i}^{(k)}\).
    Compute the approximation \(\bar{T}_{k}\) using (4.14).
    Compute \(\left\|\tilde{R}\left(\bar{T}_{k}\right)\right\|\left(\right.\) cf. (4.15)) and \(\eta_{k}\) (cf. (4.16)).
    \(\mathrm{tol}=\min \left(\left\|\tilde{R}\left(\bar{T}_{k}\right)\right\|, \eta_{k}\right)\).
    \(k=k+1\).
    end
```

5. Conclusion. We proposed extrapolation methods in tensor structure. The first class contains the tensor extrapolation polynomial-type methods while for the second class, we introduced the tenasor topological $\epsilon$-transformations. These techniques can be regarded as generalizations of the well-known vector and matrix extrapolation methods. Besides, we introduced some new products between two tensors which can simplify the derivation of extrapolation methods based on tensor format. Some theoretical results were also established including the properties of introduced tensor products and an expression for the minimum norm least-square solution of a tensor equation. Finally, the proposed technique was applied on the sequence of tensors corresponding to the truncated tensor singular value decomposition which can be used for solving tensor ill-posed problems.

## REFERENCES

[1] M. Baburaj and S. N. George, Tensor based approach for inpainting of video containing sparse text, Multimedia Tools and Applications, 78 (2019), 1805-1829.
[2] R. Bouyouli, K. Jbilou, R. Sadaka, H. Sadok, Convergence properties of some block Krylov subspace methods for multiple linear systems, J. Comput. Appl. Math. 196 (2006), 498-511.
[3] K. Braman, Third-order tensors as linear operators on a space of matrices, Linear Algebra Appl. 433 (2010) 1241-1253.
[4] A. H. Bentbib, M. El Guide, K. Jbilou, E. Onunwor and L. Reichel, Solution methods for linear discrete ill-posed problems for color image restoration, BIT Numerical Mathematics, 58 (2018), 555-578.
[5] C. Brezinski, Généralisation de la transformation de Shanks, de la table de Padé et l'epsilon algorithm. Calcolo, 12 (1975), 317-360.
[6] S. Cabay and L. W. Jackson, A polynomial extrapolation method for finding limits and antilimits for vector sequences, SIAM J. Numer. Anal., 13 (1976), 734-752.
[7] T. Chan, Y. Yang and Y. Hsuan, Polar $n$-complex and $n$-bicomplex singular value decomposition and principal component pursuit, IEEE Trans. Signal Process, 64 (2016), 6533-6544.
[8] R. Chan and X. Jin. An Introduction to Iterative Toeplitz Solvers, SIAM, Philadelphia, 2007.
[9] S. Cipolla, M. Redivo-Zaglia and F. Tudisco, Extrapolation Methods for fixed-point Multilinear PageRank computations, arXiv:1906.01494, 2019.
[10] V. P. Eddy, Extrapolation to the limit of a vecteur sequence. In: P. C. C. Wang. (ed.), Information Linkage Between Applied Mathematics and Industry, pp. 387396. Academic Press, New-York, 1979.
[11] G. Ely, S. Aeron, N. Hao and M. E. Kilmer, 5D seismic data completion and denoising using a novel class of tensor decompositions, Geophysics, 80 (2015), V83V95.
[12] G. H. Golub and C. F. Van Loan, Matrix Computations, 3rd ed., Johns Hopkins University Press, Baltimore, 1996.
[13] N. Hao, M. E. Kilmer, K. Braman and R. C. Hoover, Facial recognition using tensor-tensor decompositions, SIAM J. Imaging Sci. 6 (2013), 437-463.
[14] P. C. Hansen, Rank-Deficient and Discrete Ill-Posed Problems, SIAM, Philadelphia, 1998.
[15] W. Hu, D. Tao, W. Zhang, Y. Xie, and Y. Yang, The twist tensor nuclear norm for video completion, IEEE Trans. Neural Netw. Learn. Syst. 28 (2017), 2961-2973.
[16] W. Hu, Y. Yang, W. Zhang and Y. Xie, Moving object detection using tensor-based low-rank and saliently fused-sparse decomposition, IEEE Trans. Image Process. 26 (2017), 724-737.
[17] K. Jbilou and A. Messaoudi, Block extrapolation methods with applications, Applied Numerical Mathematics, 106 (2016), 154-164.
[18] K. Jbilou and H. Sadok, LU-implementation of the modified minimal polynomial extrapolation method. IMA J. Numer. Anal. 19 (1999), 549-561.
[19] K. Jbilou, A. Messaoudi and H. Sadok, Global FOM and GMRES algorithms for matrix equations, Appl. Numer. Math. 31 (1999), 49-63.
[20] K. Jbilou and H. Sadok, Matrix polynomial and epsilon-type extrapolation methods with applications, Numer. Algorithms, 68 (2015), 107-119.
[21] K. Jbilou, A. Messaoudi and K. Tabaa, Some Schur complement identities to matrix extrapolation methods, Linear Algebra Appl. 392 (2004), 195-210.
[22] K. Jbilou, L. Reichel and H. Sadok, Vector extrapolation enhanced TSVD for linear discrete ill-posed problems, Numer. Algorithms, 51 (2009), 195-208.
[23] M. N. Kalyani, F. P. A. Beik and K. Jbilou, On global iterative schemes based on Hessenberg process for (ill-posed) Sylvester tensor equations, J. Comput. Appl. Math. 373 (2020) 112216.
[24] M. E. Kilmer, K. Braman, N. Hao and R. C. Hoover, Third-order tensors as operators on matrices: a theoretical and computational framework with applications in imaging, SIAM J. Matrix Anal. Appl. 34 (2013), 148-172.
[25] M. E. Kilmer and C. D. Martin, Factorization strategies for third-order tensors, Linear Algebra Appl., 435 (2011), 641-658.
[26] T. G. Kolda and B. W. Bader, Tensor Decompositions and Applications. SIAM Rev. 51 (2009), 455-500.
[27] F. Li, M. K. Ng and R. J. Plemmons, Coupled segmentation and denoising/deblurring for hyperspectral material identification, Numer. Linear Algebra Appl. 19 (2012), 153-173.
[28] Y. Liu, L. Chen and C. Zhu, Improved robust tensor principal component analysis via low-rank core matrix, IEEE Journal of Selected Topics in Signal Processing, 12 (2018), 1378-1389.
[29] N. Li, C. Navasca and C. Glenn, Iterative methods for symmetric outer product tensor decomposition, Electron. Trans. Numer. Anal. 44 (2015), 124-139.
[30] X. Liu, S. Aeron, V. Aggarwal, X. Wang and M. Wu, Adaptive sampling of RF fingerprints for fine-grained indoor localization, IEEE Trans. Mobile Computing, 15 (2016), 2411-2423.
[31] Z. Long, Y Liu, L. Chen and C. Zhu, Low rank tensor completion for multiway visual data, Signal Processing, 155 (2019), 301-316.
[32] B. Madathil and S. N. George, Twist tensor total variation regularized-reweighted nuclear norm based tensor completion for video missing area recovery, Information Sciences, 423 (2018), 376-397.
[33] B. Madathil and S. N. George, Dct based weighted adaptive multi-linear data completion and denoising, Neurocomputing, 318 (2018), 120-136.
[34] M. Mesina, Convergence acceleration for the iterative solution of $x=A x+f$, Comput. Methods Appl. Mech. Engrg. 10 (1977), 165-173.
[35] Y. Miao, L. Qi and Y. Wei, Generalized Tensor Function via the Tensor Singular Value Decomposition based on the T-Product, Linear Algebra Appl. 590 (2020) 258-303.
[36] B. P. Pugatchev, Acceleration of the convergence of iterative processes and a method for solving systems of nonlinear equations, USSR Computational Mathematics and Mathematical Physics, 17 (1977), 199-207.
[37] B. Qin, M. Jin, D. Hao, Y. Lv, Q. Liu, Y. Zhu, S. Ding, J. Zhao and B. Fei, Accurate vessel extraction via tensor completion of background layer in X-ray coronary angiograms, Pattern Recognition, 87 (2019), 38-54.
[38] A. Sidi, W.F. Ford and D.A. Smith, Acceleration of convergence of vector sequences, SIAM J. Numer. Anal. 23 (1986), 178-196.
[39] A. Sidi, Convergence and stability of minimal polynomial and reduced rank extrapolation algorithms, SIAM J. Numer. Anal. 23 (1986), 197-209.
[40] M. Signoretto, R. Van de Plas, B. De Moor and J. A. K. Suykens, Tensor versus matrix completion: A comparison with application to spectral data, IEEE Signal Processing Letters, 18 (2011), 403-406
[41] W. Sun, L. Huang, H. C. So and J. Wang, Orthogonal tubal rank-1 tensor pursuit for tensor completion, Signal Processing, 157 (2019), 213-224.
[42] A. Wang, Z. Lai and Z. Jin, Noisy low-tubal-rank tensor completion, Neurocomputing, 330 (2019), 267-279.
[43] P. Wynn, Acceleration technique for iterated vector and matrix problems, Math. Comp. 16 (1962), 301-322
[44] Y. Xie, D. Tao, W. Zhang, Y. Liu, L. Zhang and Y. Qu, On unifying multi-view self-representations for clustering by tensor multi-rank minimization, Int. J. Comput. Vis. 126 (2018), 1157-1179.
[45] M. Yin, J. Gao, S. Xie and Y. Guo, Multiview subspace clustering via tensorial t-product representation, IEEE transactions on neural networks and learning systems, 30 (2018), 851-864.
[46] C. Zhang, W. Hu, T. Jin and Z. Mei, Nonlocal image denoising via adaptive tensor nuclear norm minimization, Neural Computing and Applications, 29 (2018), 3-19.


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