# New subspace minimization conjugate gradient methods based on regularization model for unconstrained optimization 

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#### Abstract

In this paper, two new subspace minimization conjugate gradient methods based on $p$-regularization models are proposed, where a special scaled norm in $p$-regularization model is analyzed. Different choices for special scaled norm lead to different solutions to the $p$-regularized subproblem. Based on the analyses of the solutions in a two-dimensional subspace, we derive new directions satisfying the sufficient descent condition. With a modified nonmonotone line search, we establish the global convergence of the proposed methods under mild assumptions. $R$-linear convergence of the proposed methods are also analyzed. Numerical results show that, for the CUTEr library, the proposed methods are superior to four conjugate gradient methods, which were proposed by Hager and Zhang (SIAM J Optim 16(1):170-192, 2005), Dai and Kou (SIAM J Optim 23(1):296-320, 2013), Liu and Liu (J Optim Theory Appl 180(3):879-906, 2019) and Li et al. (Comput Appl Math 38(1): 2019), respectively.


Keywords Conjugate gradient method • $p$-regularization model • Subspace technique • Nonmonotone line search • Unconstrained optimization

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## 1 Introduction

Conjugate gradient (CG) methods are of great importance for solving the large-scale unconstrained optimization problem

$$
\begin{equation*}
\min _{x \in R^{n}} f(x) \tag{1}
\end{equation*}
$$

where $f: R^{n} \rightarrow R$ is a continuously differentiable function. The key features of CG methods are that they do not require matrix storage. The iterations $\left\{x_{n}\right\}$ satisfy the iterative form

$$
\begin{equation*}
x_{k+1}=x_{k}+\alpha_{k} d_{k} \tag{2}
\end{equation*}
$$

where $\alpha_{k}$ is the stepsize and $d_{k}$ is the search direction defined by

$$
d_{k+1}= \begin{cases}-g_{k+1}, & \text { if } k=0  \tag{3}\\ -g_{k+1}+\beta_{k} d_{k}, & \text { if } k>0\end{cases}
$$

where $g_{k+1}=\nabla f\left(x_{k+1}\right)$ and $\beta_{k} \in R$ is called the CG parameter.
For general nonlinear functions, various choices of $\beta_{k}$ cause different CG methods. Some well-known options for $\beta_{k}$ are called FR [19], HS [27], PRP [38], DY [13] and HZ [24] formula, and are given by

$$
\beta_{k}^{F R}=\frac{\left\|g_{k+1}\right\|^{2}}{\left\|g_{k}\right\|^{2}}, \beta_{k}^{H S}=\frac{g_{k+1}^{T} y_{k}}{d_{k}^{T} y_{k}}, \beta_{k}^{P R P}=\frac{g_{k+1}^{T} y_{k}}{\left\|g_{k}\right\|^{2}}, \beta_{k}^{D Y}=\frac{\left\|g_{k+1}\right\|^{2}}{d_{k}^{T} y_{k}}
$$

and

$$
\beta_{k}^{H Z}=\frac{1}{d_{k}^{T} y_{k}}\left(y_{k}-2 d_{k} \frac{\left\|y_{k}\right\|^{2}}{d_{k}^{T} y_{k}}\right)^{T} g_{k+1}
$$

where $y_{k}=g_{k+1}-g_{k}$ and $\|$.$\| denotes the Euclidean norm. Recently, other efficient CG methods have been$ proposed by different ideas, which can be seen in [12, 18, 24, 25, 28, 39, 40, 47].

With the increasing scale of optimization problems, subspace methods have become a class of very efficient numerical methods because it is not necessary to solve large-scale subproblems at each iteration [48]. Yuan and Stoer [46] first put forward the subspace minimization conjugate gradient (SMCG) method, the search direction of which is computed by solving the following problem:

$$
\begin{equation*}
\min _{d \in \Omega_{k+1}} m_{k+1}(d)=g_{k+1}^{T} d+\frac{1}{2} d^{T} B_{k+1} d \tag{4}
\end{equation*}
$$

where $\Omega_{k+1}=\left\{g_{k+1}, s_{k}\right\}$ and the direction $d$ is given by

$$
\begin{equation*}
d=\mu g_{k+1}+v s_{k} \tag{5}
\end{equation*}
$$

where $B_{k+1}$ is an approximation to Hessian matrix, $\mu$ and $v$ are parameters and $s_{k}=x_{k+1}-x_{k}$. The detailed information of subspace technique can be referred to [1, 26, 29, 43, 49]. The SMCG method can be considered as a generalization of CG method and it reduces to the linear CG method when it uses the exact line search condition and objective function is convex quadratic function. Based on the analysis of the SMCG method, Dai and Kou [15] made a theoretical analysis of the BBCG by combining the BarzilaiBorwein (BB) idea [3] with the SMCG. Liu and Liu [31] presented an efficient Barzilai-Borwein conjugate gradient method (SMCG_BB) with the generalized Wolfe line search for unconstrained optimization. Li, Liu and Liu [30] deliver a subspace minimization conjugate gradient method based on conic model for unconstrained optimization (SMCG_Conic).

Generally, the iterative methods are often based on a quadratic model because the quadratic model can approximate the objective function well at a small neighborhood of the minimizer. However, when iterative point is far from the minimizer, the quadratic model might not work well if the objective function possesses high non-linearity $[41,45]$. In theory, the successive gradients generated by the conjugate gradient method applied to a quadratic function should be orthogonal. However, for some ill-conditioned problems, orthogonality is quickly lost due to the rounding errors, and the convergence is much slower than expected [26]. There are many methods to deal with ill-conditioned problems, among which regularization method is one of the effective methods. Recently, $p$-regularized subproblem plays an important role in more regularization approaches $[10,23,35]$ and some $p$-regularization algorithms for unconstrained optimization enjoy a growing interest $[4,7,11,10]$. The idea is to incorporate a local quadratic approximation of the objective function with a weighted regularization term $\left(\sigma_{k} / p\right)\|x\|^{p}, p>2$, and then globally minimize it at each iteration. Interestingly, Cartis et al. $[10,11]$ proved that, under suitable assumptions, $p$-regularization algorithmic scheme is able to achieve superlinear convergence. The most common choice to regularize the quadratic approximation is $p$-regularization with $p=3$, which is known as the cubic regularization, since functions of this form are used as local models (to be minimized) in many algorithmic frameworks for unconstrained optimization $[5,6,7,8,9,10,11,17,21,23,35,37,42]$. The cubic regularization was first introduced by Griewank [23] and was later considered by many authors with global convergence and complexity analysis, see $[11,35,42]$.

Recently, how to approximate the $p-$ regularized subproblem solution has become a hot research topic. Practical approaches to get an approximate solution are proposed in [7,22], where the solution of the secular equation is typically approximated over specific evolving subspaces using Krylov methods. The main drawback of such approaches is the large amount of calculation, because they may need to solve multiple linear systems in turn.

In this paper, motivated by [2] and [44], the $p$-regularization with a special scaled norm is analyzed and solutions of the new $p$-regularization that arise in unconstrained optimization are considered. Based on [2] we propose a method to solve it by using a special scaled norm in the $p$-regularized subproblem. According to the advantages of the new $p$-regularization method with SMCG method, we propose two new subspace minimization conjugate gradient methods. In our algorithms, if the objective function is close to a quadratic, we use a quadratic approximation model in a two-dimensional subspace to generate the direction; otherwise, $p$-regularization model is considered. We prove that the search direction possesses the sufficient descent property and the proposed methods satisfy the global convergence under mild conditions. We present some numerical results, which show that the proposed methods are very promising.

The remainder of this paper is organized as follows. In Section 2, we will state the form of $p$-regularized subproblem and provide how to solve the $p$-regularization problem based on the special $p$-regularization model. Four choices of search direction by minimizing the approximate models including $p$-regularization and quadratic model on certain subspace are presented in Section 3. In Section 4, we describe two algorithms and discuss some important properties of the search direction in detail. In Section 5, we establish the convergence of the proposed methods under mild conditions. Some performances of the proposed methods are reported in Section 6. Conclusions and discussions are presented in the last section.

## 2 The $p$-regularized Subproblem

In this section, we will briefly introduce several forms of the $p$-regularized subproblem by using a special scaled norm and provide the solutions of the resulting problems in the whole space and the two-dimensional subspace, respectively. The chosen scaled norm is of the form $\|x\|_{A}=\sqrt{x^{T} A x}$, where $A$ is a symmetric positive definite matrix. After analysis, we will mainly consider two special cases: (I) $A$ is the Hessian matrix. In this case, the $p$-regularized subproblem has the unique solution; (II) $A$ is the identity matrix. In this case, the $p$-regularized subproblem is the same as the general form.

### 2.1 The Form in the Whole Space

The general form of the $p$-regularized subproblem is:

$$
\begin{equation*}
\min _{x \in R^{n}} h(x)=c^{T} x+\frac{1}{2} x^{T} H x+\frac{\sigma}{p}\|x\|^{p}, \tag{6}
\end{equation*}
$$

where $p>2, c \in R^{n}, \sigma>0$ and $H \in R^{n \times n}$ is a symmetric matrix.
As for how to solve the above problem, the following theorem is given.
Theorem 2.1 [ [44], Thm.1.1] The point $x^{*}$ is a global minimizer of (6) if and only if

$$
\begin{equation*}
\left(H+\sigma\left\|x^{*}\right\|^{p-2} I\right) x^{*}=-c, \quad H+\sigma\left\|x^{*}\right\|^{p-2} I \succeq 0 \tag{7}
\end{equation*}
$$

Moreover, the $l_{2}$ norms of all the global minimizers are equal.
Now, we give another form of the $p$-regularized subproblem with a special scaled norm:

$$
\begin{equation*}
\min _{x \in R^{n}} h(x)=c^{T} x+\frac{1}{2} x^{T} H x+\frac{\sigma}{p}\|x\|_{A}^{p} \tag{8}
\end{equation*}
$$

where $A \in R^{n \times n}$ is a symmetric positive definite matrix.
By setting $y=A^{\frac{1}{2}} x$, (8) can be arranged as follows:

$$
\begin{equation*}
\min _{y \in R^{n}} h(y)=\left(A^{-\frac{1}{2}} c\right)^{T} y+\frac{1}{2} y^{T} A^{-\frac{1}{2}} H A^{-\frac{1}{2}} y+\frac{\sigma}{p}\|y\|^{p} . \tag{9}
\end{equation*}
$$

According to Theorem 2.1, we know that the point $y^{*}$ is a global minimizer of (9) if and only if

$$
\begin{gather*}
\left(A^{-\frac{1}{2}} H A^{-\frac{1}{2}}+\sigma\left\|y^{*}\right\|^{p-2} I\right) y^{*}=-A^{-\frac{1}{2}} c,  \tag{10}\\
A^{-\frac{1}{2}} H A^{-\frac{1}{2}}+\sigma\left\|y^{*}\right\|^{p-2} I \succeq 0 . \tag{11}
\end{gather*}
$$

Let $V \in R^{n \times n}$ be an orthogonal matrix such that

$$
V^{T}\left(A^{-\frac{1}{2}} H A^{-\frac{1}{2}}\right) V=Q,
$$

where $Q=\operatorname{diag}_{i=1, \cdots, n}\left\{\mu_{i}\right\}$ and $\mu_{1} \leq \mu_{2} \leq \cdots \leq \mu_{n}$ are the eigenvalues of $A^{-\frac{1}{2}} H A^{-\frac{1}{2}}$. Now we can introduce the vector $a \in R^{n}$ such that

$$
\begin{equation*}
y=V a . \tag{12}
\end{equation*}
$$

Denote $z=\|y\|$ and pre-multiplying (10) by $V^{T}$, we get

$$
\begin{equation*}
\left(Q+\sigma z^{p-2} I\right) a=-\beta \tag{13}
\end{equation*}
$$

where $\beta=V^{T}\left(A^{-\frac{1}{2}} c\right)$.
The expression (13) can be equivalently written as

$$
a_{i}=\frac{-\beta_{i}}{\mu_{i}+\sigma z^{p-2}}, i=1,2, \cdots, n,
$$

where $a_{i}$ and $\beta_{i}$ are the components of vectors $a$ and $\beta$, respectively. By the way, if $\mu_{i}+\sigma z^{p-2}=0$, it means $\beta=0$ from (13).

From (12), we have an equation about $z$ :

$$
\begin{equation*}
z^{2}=y^{T} y=a^{T} a=\sum_{i=1}^{n} \frac{\beta_{i}^{2}}{\left(\mu_{i}+\sigma z^{p-2}\right)^{2}} \tag{14}
\end{equation*}
$$

Denote

$$
\phi(z)=\sum_{i=1}^{n} \frac{\beta_{i}^{2}}{\left(\mu_{i}+\sigma z^{p-2}\right)^{2}}-z^{2} .
$$

We can easily obtain

$$
\phi^{\prime}(z)=\sum_{i=1}^{n} \frac{-2 \sigma(p-2) \beta_{i}^{2} z^{p-3}\left(\mu_{i}+\sigma z^{p-2}\right)}{\left(\mu_{i}+\sigma z^{p-2}\right)^{4}}-2 z .
$$

It follows from $p>2, z>0$ and $\sigma>0$ that $\phi^{\prime}(z)<0$, which indicates that $\phi(z)$ is monotonically decreasing in the interval $[0,+\infty)$. Moreover, we can observe that $\phi(0)>0$, when $\beta \neq 0$, and $\lim _{z \rightarrow \infty} \phi(z)=-\infty$. So, there exists a unique positive solution to (14) when $\beta \neq 0$. On the other hand, if $\beta=0, z=0$ is the only solution of (14) in which means $x^{*}=0$ is the only global minimizer of (8).

Based on the above derivation and analysis, we can get the following theorem.
Theorem 2.2 The point $x^{*}$ is a global minimizer of (8) if and only if

$$
\begin{gather*}
\left(H+\sigma\left(z^{*}\right)^{p-2} A\right) x^{*}=-c,  \tag{15}\\
H+\sigma\left(z^{*}\right)^{p-2} A \succeq 0, \tag{16}
\end{gather*}
$$

where $z^{*}$ is the unique non-negative root of the equation

$$
\begin{equation*}
z^{2}=\sum_{i=1}^{n} \frac{\beta_{i}^{2}}{\left(\mu_{i}+\sigma z^{p-2}\right)^{2}} . \tag{17}
\end{equation*}
$$

Moreover, the $l_{A}$ norms of all the global minimizers are equal.
Now, let us consider a special case that $H \succ 0$ and $A=H$. It is clear that $H+\sigma z^{p-2} H$ is always a positive definite matrix since $\sigma>0$ and $z \geq 0$. So, the global minimizer of (8) is unique.
Inference 2.3 Let $H \succ 0, A=H$, then the point $x^{*}=\frac{-1}{1+\sigma\left(z^{*}\right)^{p-2}} H^{-1} c$ is the only global minimizer of (8) and $z^{*}$ is the unique non-negative solution to the equation

$$
\begin{equation*}
\sigma z^{p-1}+z-\sqrt{c^{T} H^{-1} c}=0 . \tag{18}
\end{equation*}
$$

Remark 1 i) $c=0$. It is obvious that the equation (18) becomes

$$
\sigma z^{p-1}+z=0
$$

that is

$$
z\left(\sigma z^{p-2}+1\right)=0
$$

From $\sigma>0$, we know $z^{*}=0$ is the unique non-negative solution to the equation (18).
ii) $c \neq 0$. Denote

$$
\begin{equation*}
\psi(z)=\sigma z^{p-1}+z-\sqrt{c^{T} H^{-1} c} \tag{19}
\end{equation*}
$$

We can easily obtain

$$
\psi^{\prime}(z)=\sigma(p-1) z^{p-2}+1>0
$$

which indicates that the $\psi(z)$ is monotonically increasing. From $\psi(0)<0$ and $\psi\left(\sqrt{c^{T} H^{-1} c}\right)>0$, we know that $z^{*}$ is the unique positive solution to the equation (18).

### 2.2 The Form in the Two-Dimensional Space

Let $g$ and $s$ be two linearly independent vectors. Denote $\Omega=\{d \mid d=\mu g+\nu s, \mu, \nu \in R\}$. In this part, we suppose that $H$ is symmetric and positive definite and $y=H s$.

We consider the following problem

$$
\begin{equation*}
\min _{d \in \Omega} h(d)=c^{T} d+\frac{1}{2} d^{T} H d+\frac{\sigma}{p}\|d\|_{A}^{p} . \tag{20}
\end{equation*}
$$

Obviously, when $A=H$, problem (20) can be translated into

$$
\begin{equation*}
\min _{\mu, \nu \in R}\binom{g^{T} c}{s^{T} c}^{T}\binom{\mu}{\nu}+\frac{1}{2}\binom{\mu}{\nu}^{T} B\binom{\mu}{\nu}+\frac{\sigma}{p}\left\|\binom{\mu}{\nu}\right\|_{B}^{p} \tag{21}
\end{equation*}
$$

where $\rho=g^{T} H g$, and $B=\left(\begin{array}{cc}\rho & g^{T} y \\ g^{T} & y\end{array} y^{T} s\right)$ is a symmetric and positive definite matrix since the $H$ is a symmetric positive definite matrix and the two vectors $g$ and $s$ are linearly independent.

By the Inference 2.3, we can obtain the unique solution of (21):

$$
\begin{equation*}
\binom{\mu^{*}}{\nu^{*}}=\frac{-1}{1+\sigma\left(z^{*}\right)^{p-2}} B^{-1}\binom{g^{T} c}{s^{T} c} \tag{22}
\end{equation*}
$$

where $z^{*}$ is the unique non-negative solution to $\sigma z^{p-1}+z-\sqrt{\binom{g^{T} c}{s^{T} c}^{T} B^{-1}\binom{g^{T} c}{s^{T} c}}=0$.
When $A=I$, we obtain from (20) that

$$
\begin{equation*}
\min _{\mu, \nu \in R}\binom{g^{T} c}{s^{T} c}^{T}\binom{\mu}{\nu}+\frac{1}{2}\binom{\mu}{\nu}^{T} B\binom{\mu}{\nu}+\frac{\sigma}{p}\left\|\binom{\mu}{\nu}\right\|_{E}^{p} \tag{23}
\end{equation*}
$$

where $E=\left(\begin{array}{cc}\|g\|^{2} & g^{T} s \\ g^{T} s & \|s\|^{2}\end{array}\right)$ is positive definite due to the linear independence of vectors $g$ and $s$.
By the Theorem 2.2, we can gain the unique solution to (23):

$$
\begin{equation*}
\binom{\mu^{*}}{\nu^{*}}=-\left(B+\sigma\left(z^{*}\right)^{p-2} E\right)^{-1}\binom{g^{T} c}{s^{T} c} \tag{24}
\end{equation*}
$$

where $z^{*}$ is the unique non-negative solution to (17) in which $0<\mu_{1} \leq \mu_{2}$ are the eigenvalues of $E^{-\frac{1}{2}} B E^{-\frac{1}{2}}$, $\beta=V^{T}\left(E^{-\frac{1}{2}}\binom{g^{T} c}{s^{T} c}\right)$.

## 3 The Search Direction and The Initial Stepsize

In this section, based on the different choices of special scaled norm, we derive two new directions by minimizing the two $p$-regularization models of the objective function on the subspace $\Omega_{k}=\operatorname{span}\left\{g_{k}, s_{k-1}\right\}$. The selection criteria for how to choose the initial stepsize is given. For the rest, we assume that $s_{k}^{T} y_{k}>0$ guaranteed by the condition (49).

### 3.1 Derivation of The New Search Direction

The parameter $t_{k}$ by Yuan [47] is used describe how $f(x)$ is close to a quadratic function on the line segment between $x_{k-1}$ and $x_{k}$, and defined by

$$
\begin{equation*}
t_{k}=\left|\frac{2\left(f_{k-1}-f_{k}+g_{k}^{T} s_{k-1}\right)}{s_{k-1}^{T} y_{k-1}}-1\right| . \tag{25}
\end{equation*}
$$

On the other hand, the ratio

$$
\begin{equation*}
\theta_{k}=\frac{f_{k-1}-f_{k}}{0.5 s_{k-1}^{T} y_{k-1}-g_{k}^{T} s_{k-1}} \tag{26}
\end{equation*}
$$

shows difference between the actual reduction and the predicted reduction for the quadratic model.
If the following condition [33] holds, namely,

$$
\begin{equation*}
t_{k} \leq c_{1} \text { or }\left(t_{k} \leq c_{2} \text { and } t_{k-1} \leq c_{2}\right) \tag{27}
\end{equation*}
$$

or

$$
\begin{equation*}
\left|\theta_{k}-1\right|<\gamma, \tag{28}
\end{equation*}
$$

where $c_{1}, c_{2}$ and $\gamma$ are small positive constants, then $f(x)$ might be very close to a quadratic on the line segment between $x_{k-1}$ and $x_{k}$. We choose the quadratic model.

Moreover, if the conditions [30]

$$
\begin{equation*}
\left(s_{k}^{T} y_{k}\right)^{2} \leq 10^{-5}\left\|s_{k}\right\|^{2}\left\|y_{k}\right\|^{2} \text { and }\left(f_{k+1}-f_{k}-0.5\left(g_{k}^{T} s_{k}+g_{k+1}^{T} s_{k}\right)\right)^{2} \leq 10^{-6}\left\|s_{k}\right\|^{2}\left\|y_{k}\right\|^{2} \tag{29}
\end{equation*}
$$

hold, then the problem might have very large condition number, which seems to be ill-conditioned. And the current iterative point is far away from the minimizer of problem. At this point, the information might be inaccurate, then we also choose the quadratic model to derive a search direction.

General iterative methods, which are often based on a quadratic model, have been quite successful for solving unconstrained optimization problems, since the quadratic model can approximate the objective function $f(x)$ well at a small neighborhood of $x_{k}$ in many cases. Consequently, when the condition (27), (28) or (29) holds, the quadratic approximation model (4) is preferable. However, when the conditions (27), (28) and (29) do not hold, the iterative point is far away from the minimizer, the quadratic model may not very well approximate the original problem. Thus in this case, we select the $p$-regularization model which could include more useful information of the objective function to approximate the original problem.

For general functions, if the condition

$$
\begin{equation*}
\xi_{1} \leq \frac{s_{k-1}^{T} y_{k-1}}{\left\|s_{k-1}\right\|^{2}} \leq \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}} \leq \xi_{2} \tag{30}
\end{equation*}
$$

holds, where $\xi_{1}$ and $\xi_{2}$ are positive constants, then the condition number of the Hessian matrix might be not very large. In this case, we consider the quadratic approximation model or the $p$-regularization model.

Now we divide it into following four cases to derive the search direction.
Case 1. When the condition (30) holds and any of the conditions (27, 28, 29) do not hold, we consider the following $p$-regularized subproblem

$$
\begin{equation*}
\min _{d_{k} \in \Omega_{k}} m_{k}\left(d_{k}\right)=d_{k}^{T} g_{k}+\frac{1}{2} d_{k}^{T} H_{k} d_{k}+\frac{1}{p} \sigma_{k}\left\|d_{k}\right\|_{A_{k}}^{p} \tag{31}
\end{equation*}
$$

where $H_{k}$ is a symmetric and positive definite approximation to Hessian matrix satisfying the equation $H_{k} s_{k-1}=y_{k-1}, A_{k}$ is a symmetric positive definite matrix, $\sigma_{k}$ is a dynamic non-negative regularization parameter and $\Omega_{k}=\operatorname{span}\left\{g_{k}, s_{k-1}\right\}$.

Denote

$$
\begin{equation*}
d_{k}=\mu_{k} g_{k}+\nu_{k} s_{k-1} \tag{32}
\end{equation*}
$$

where $\mu_{k}$ and $\nu_{k}$ are parameters to be determined.
In the following, we will discuss that $A_{k}=H_{k}$ and $A_{k}=I$ in two parts.
(I) $A_{k}=H_{k}$

It is easy to see the problem (31) is similar to the problem (21), we obtain

$$
\begin{equation*}
\min _{\mu_{k}, \nu_{k} \in R}\binom{\left\|g_{k}\right\|^{2}}{g_{k}^{T} s_{k-1}}^{T}\binom{\mu_{k}}{\nu_{k}}+\frac{1}{2}\binom{\mu_{k}}{\nu_{k}}^{T} B_{k}\binom{\mu_{k}}{\nu_{k}}+\frac{\sigma_{k}}{p}\left\|\binom{\mu_{k}}{\nu_{k}}\right\|_{B_{k}}^{p} \tag{33}
\end{equation*}
$$

where $\rho_{k} \approx g_{k}^{T} H_{k} g_{k}$ and $B_{k}=\left(\begin{array}{cc}\rho_{k} & g_{k}^{T} y_{k-1} \\ g_{k}^{T} y_{k-1} s_{k-1}^{T} y_{k-1}\end{array}\right)$.
It is very important for how to choose the two parameters $\rho_{k}$ and $\sigma_{k}$ in (33).
Motivated by the Barzilai-Borwein method, Dai and Kou [15] proposed a BBCG3 method with the very efficient parameter $\rho_{k}^{B B C G 3}=\frac{3}{2} \frac{\left\|y_{k-1}\right\|^{2}}{S_{k-1}^{T} y_{k-1}}\left\|g_{k}\right\|^{2}$ and considered it a good estimation of the $g_{k}^{T} H_{k} g_{k}$. So in this paper, we choose $\rho_{k}=\rho_{k}^{B B C G 3}$ in the above function that will make $B_{k}$ positive, which guarantees definite the unique solution to (33).

There are many ways $[10,21]$ to get the value of $\sigma_{k}$, and the interpolation condition is one of them. Here, we use interpolation condition to get it. By imposing the following interpolation condition:

$$
f_{k-1}=f_{k}-g_{k}^{T} s_{k-1}+\frac{1}{2} s_{k-1}^{T} y_{k-1}+\frac{\sigma_{k}}{p}\left(s_{k-1}^{T} y_{k-1}\right)^{\frac{p}{2}}
$$

we obtain

$$
\sigma_{k}=\frac{p\left(f_{k-1}-f_{k}+g_{k}^{T} s_{k-1}-\frac{1}{2} s_{k-1}^{T} y_{k-1}\right)}{\left(s_{k-1}^{T} y_{k-1}\right)^{\frac{p}{2}}}
$$

In order to ensure that $\sigma_{k} \geq 0$, we set

$$
\sigma_{k}=\frac{p\left|f_{k-1}-f_{k}+g_{k}^{T} s_{k-1}-\frac{1}{2} s_{k-1}^{T} y_{k-1}\right|}{\left(s_{k-1}^{T} y_{k-1}\right)^{\frac{p}{2}}}
$$

From (22), we can get the unique solution to (33):

$$
\begin{equation*}
\mu_{k}=\frac{1}{\left(1+\sigma_{k}\left(z^{*}\right)^{p-2}\right) \Delta_{k}}\left(g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1}-s_{k-1}^{T} y_{k-1}\left\|g_{k}\right\|^{2}\right) \tag{34}
\end{equation*}
$$

$$
\begin{equation*}
\nu_{k}=\frac{1}{\left(1+\sigma_{k}\left(z^{*}\right)^{p-2}\right) \Delta_{k}}\left(g_{k}^{T} y_{k-1}\left\|g_{k}\right\|^{2}-\rho_{k} g_{k}^{T} s_{k-1}\right), \tag{35}
\end{equation*}
$$

where $\Delta_{k}=\left|\begin{array}{cc}\rho_{k} & g_{k}^{T} y_{k-1} \\ g_{k}^{T} y_{k-1} & s_{k-1}^{T} y_{k-1}\end{array}\right|=\rho_{k} s_{k-1}^{T} y_{k-1}-\left(g_{k}^{T} y_{k-1}\right)^{2}>0$ and $z^{*}$ is the unique positive solution to

$$
\begin{equation*}
\sigma_{k} z^{p-1}+z-\sqrt{\binom{\left\|g_{k}\right\|^{2}}{g_{k}^{T} s_{k-1}}^{T} B_{k}^{-1}\binom{\left\|g_{k}\right\|^{2}}{g_{k}^{T} s_{k-1}}}=0 . \tag{36}
\end{equation*}
$$

We denote $\tilde{q}=\sqrt{\binom{\left\|g_{k}\right\|^{2}}{g_{k}^{T} s_{k-1}} B_{k}^{-1}\binom{\left\|g_{k}\right\|^{2}}{g_{k}^{T} s_{k-1}}}$. Substituting $\tilde{q}$ into (36), we get

$$
\begin{equation*}
\sigma_{k} z^{p-1}+z-\tilde{q}=0 \tag{37}
\end{equation*}
$$

Since it is difficult to obtain the exact root of (37) when $p$ is large, we only consider $p=3$ and $p=4$ for simplicity.
(i) $p=3$. It is not difficult to know the unique positive solution to (37)

$$
\begin{equation*}
z^{*}=\frac{2 \tilde{q}}{1+\sqrt{1+4 \sigma_{k} \tilde{q}}} . \tag{38}
\end{equation*}
$$

(ii) $p=4$. According to the formula of extracting roots on cubic equation and $z>0$, the unique positive solution to (37) can be obtained

$$
\begin{equation*}
z^{*}=\sqrt[3]{\frac{\tilde{q}}{2 \sigma_{k}}+\sqrt{\frac{\tilde{q}^{2}}{4 \sigma_{k}^{2}}+\left(\frac{1}{3 \sigma_{k}}\right)^{3}}}+\sqrt[3]{\frac{\tilde{q}}{2 \sigma_{k}}-\sqrt{\frac{\tilde{q}^{2}}{4 \sigma_{k}^{2}}+\left(\frac{1}{3 \sigma_{k}}\right)^{3}}} . \tag{39}
\end{equation*}
$$

For ensuring the sufficient descent condition of the direction produced by (34) and (35), if $\sigma_{k}\left(z^{*}\right)^{p-2}>1$, we set $\sigma_{k}\left(z^{*}\right)^{p-2}=1$, where $z^{*}$ is determined by (38) or (39).
(II) $A_{k}=I$

Based on the analysis of (I), we can get the following problem similarly:

$$
\begin{equation*}
\min _{\mu_{k}, \nu_{k} \in R}\binom{\left\|g_{k}\right\|^{2}}{g_{k}^{T} s_{k-1}}^{T}\binom{\mu_{k}}{\nu_{k}}+\frac{1}{2}\binom{\mu_{k}}{\nu_{k}}^{T} B_{k}\binom{\mu_{k}}{\nu_{k}}+\frac{\sigma_{k}}{p}\left\|\binom{\mu_{k}}{\nu_{k}}\right\|_{E_{k}}^{p} \tag{40}
\end{equation*}
$$

where $E_{k}=\left(\begin{array}{cc}\left\|g_{k}\right\|^{2} & g_{k}^{T} s_{k-1} \\ g_{k}^{T} s_{k-1} & \left\|s_{k-1}\right\|^{2}\end{array}\right)$ and $\rho_{k}, B_{k}$ are the same as those in problem (33).
Similarly, we still use the interpolation condition to determine $\sigma_{k}$ :

$$
f_{k-1}=f_{k}-g_{k}^{T} s_{k-1}+\frac{1}{2} s_{k-1}^{T} y_{k-1}+\frac{\sigma_{k}}{p}\left\|s_{k-1}\right\|^{\frac{p}{2}},
$$

we get

$$
\sigma_{k}=\frac{p\left|f_{k-1}-f_{k}+g_{k}^{T} s_{k-1}-\frac{1}{2} s_{k-1}^{T} y_{k-1}\right|}{\left\|s_{k-1}\right\|^{\frac{p}{2}}} .
$$

According to (24), the unique solution to (40) can be obtained:

$$
\begin{equation*}
\hat{\mu}_{k}=\frac{1}{\overline{\Delta_{k}}}\left(g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1}-s_{k-1}^{T} y_{k-1}\left\|g_{k}\right\|^{2}+\lambda\left(g_{k}^{T} s_{k-1}\right)^{2}-\lambda\left\|s_{k-1}\right\|^{2}\left\|g_{k}\right\|^{2}\right) \tag{41}
\end{equation*}
$$

$$
\begin{equation*}
\hat{\nu}_{k}=\frac{1}{\bar{\Delta}_{k}}\left(g_{k}^{T} y_{k-1}\left\|g_{k}\right\|^{2}-\rho_{k} g_{k}^{T} s_{k-1}\right) \tag{42}
\end{equation*}
$$

where

$$
\begin{gather*}
\bar{\Delta}_{k}=\left(\rho_{k}+\lambda\left\|g_{k}\right\|^{2}\right)\left(s_{k-1}^{T} y_{k-1}+\lambda\left\|s_{k-1}\right\|^{2}\right)-\left(g_{k}^{T} y_{k-1}+\lambda g_{k}^{T} s_{k-1}\right)^{2}  \tag{43}\\
\lambda=\sigma_{k}\left(z^{*}\right)^{p-2}
\end{gather*}
$$

and $z^{*}$ satisfies the equation (17), which can be solved by tangent method [36]. For ensuring the sufficient descent of the direction produced by (41) and (42), if $\sigma_{k}\left(z^{*}\right)^{p-2}>\frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}$, we set $\lambda=\frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}$.
Remark 2 It is worth emphasizing that in the process of finding the direction, $\binom{\left\|g_{k}\right\|^{2}}{g_{k}^{T} s_{k-1}} \neq 0$, which is equivalent to the problem (8) in which $c \neq 0$.
Case 2. When the condition (30) holds and one of the conditions (27, 28, 29) at least holds, we choose the quadratic model which corresponds to (33) with $\sigma_{k}=0$. So the parameters in (32) are generated by solving (34) and (35) with $\sigma_{k}=0$ :

$$
\begin{gather*}
\bar{\mu}_{k}=\frac{1}{\Delta_{k}}\left(g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1}-s_{k-1}^{T} y_{k-1}\left\|g_{k}\right\|^{2}\right),  \tag{44}\\
\bar{\nu}_{k}=\frac{1}{\Delta_{k}}\left(g_{k}^{T} y_{k-1}\left\|g_{k}\right\|^{2}-\rho_{k} g_{k}^{T} s_{k-1}\right) . \tag{45}
\end{gather*}
$$

Case 3. If the exact line search is used, the direction in Case 2 is parallel to the HS direction with convex quadratic functions. It is known that the conjugate condition, namely, $d_{k+1}^{T} y_{k}=0$, still holds whether the line search is exact or not for HS conjugate gradient method.

If the condition (30) does not hold and the conditions

$$
\begin{equation*}
\frac{\left|g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1}\right|}{s_{k-1}^{T} y_{k-1}\left\|g_{k}\right\|^{2}} \leq \xi_{3} \text { and } \xi_{1} \leq \frac{s_{k-1}^{T} y_{k-1}}{\left\|s_{k-1}\right\|^{2}} \tag{46}
\end{equation*}
$$

hold, where $0 \leq \xi_{3} \leq 1$, then $\bar{\mu}_{k}$ in Case 2 is close to -1 , then we use the HS conjugate gradient direction. Besides, with the finite-termination property of the HS method for exact convex quadratic programming, such choice of the direction might lead to a rapid convergence rate of our algorithm.
Case 4. If the condition (30) does not hold and the condition (46) does not hold, then we choose the negative gradient as the search direction, namely,

$$
\begin{equation*}
d_{k}=-g_{k} \tag{47}
\end{equation*}
$$

In conclusion, the new search direction can be stated as

$$
d_{k}=\left\{\begin{array}{cl}
\mu_{k} g_{k}+\nu_{k} s_{k-1}, & \text { if (30) holds and any of }(27,28,29) \text { do not hold, } \\
\bar{\mu}_{k} g_{k}+\bar{\nu}_{k} s_{k-1}, & \text { if (30) holds and one of }(27,28,29) \text { at least holds, } \\
-g_{k}+\beta_{k}^{H S} d_{k-1}, & \text { if (30) does not hold and } \\
-g_{k}, & \text { if }(30) \text { does not hold and }
\end{array}(46) \text { holds, } \quad \text { does not hold, }, ~\right.
$$

where $\mu_{k}, \nu_{k}$ are given by (34), (35) or (41), (42) and $\bar{\mu}_{k}, \bar{\nu}_{k}$ are given by (44), (45), respectively.

### 3.2 Choices of The Initial Stepsize and The Wolfe Line Search

It is universally acknowledged that the choice of the initial stepsize and the Wolfe line search are of great importance for an optimization method. In this section, we introduce a strategy to choose the initial stepsize and develop a modified nonmonotone Wolfe line search.

### 3.2.1 Choices of The Initial Stepsize

Denote

$$
\phi_{k}(\alpha)=f\left(x_{k}+\alpha d_{k}\right), \alpha \geq 0
$$

(i) The initial stepsize for the search directions in Case1.-Case3. in Section 3.1.

Similar to [31], we choose the initial stepsize as

$$
\alpha_{k}^{0}= \begin{cases}\hat{\alpha}_{k}, & \text { if }(27) \text { holds and } \bar{\alpha}_{k}>0 \\ 1, & \text { otherwise }\end{cases}
$$

where

$$
\bar{\alpha}_{k}=\min q\left(\phi_{k}(0), \phi_{k}^{\prime}(0), \phi_{k}(1)\right), \quad \hat{\alpha}_{k}=\min \left\{\max \left\{\bar{\alpha}_{k}, \lambda_{\min }\right\}, \lambda_{\max }\right\} \text { and } \lambda_{\max }>\lambda_{\min }>0
$$

In the above formula, $q\left(\phi_{k}(0), \phi_{k}^{\prime}(0), \phi_{k}(1)\right)$ denotes the interpolation function for the three values $\phi_{k}(0)$, $\phi^{\prime}{ }_{k}(0)$, and $\phi_{k}(1)$. And $\lambda_{\max }$ and $\lambda_{\min }$ represent two positive parameters.
(ii) The initial stepsize for the negative gradient direction (47).

As we all know, the gradient method with the adaptive BB stepsize [51] is very efficient for strictly convex quadratic minimization, especially when the condition number is large. In this paper we choose the strategy in [31]:

$$
\alpha_{k}^{0}= \begin{cases}\min \left\{\max \left\{\tilde{\tilde{\alpha}}_{k}, \lambda_{\min }\right\}, \lambda_{\max }\right\}, & \text { if }(27) \text { holds, } d_{k-1} \neq-g_{k-1},\left\|g_{k}\right\|^{2} \leq 1 \text { and } \tilde{\tilde{\alpha}}_{k}>0 \\ \overline{\bar{\alpha}}_{k}, & \text { otherwise }\end{cases}
$$

where

$$
\overline{\bar{\alpha}}_{k}=\left\{\begin{array}{lll}
\left\{\min \left\{\lambda_{k} \alpha_{k}^{B B_{2}}, \lambda_{\max }\right\}, \lambda_{\min }\right\}, & \text { if } g_{k}^{T} s_{k-1}>0, \\
\left\{\min \left\{\lambda_{k} \alpha_{k}^{B B_{1}}, \lambda_{\max }\right\}, \lambda_{\min }\right\}, & \text { if } g_{k}^{T} s_{k-1} \leq 0,
\end{array}, \quad \tilde{\tilde{\alpha}}_{k}=\min q\left(\phi_{k}(0), \phi_{k}{ }^{\prime}(0), \phi_{k}\left(\overline{\bar{\alpha}}_{k}\right)\right)\right.
$$

$\lambda_{k}$ is a scaling parameter given by $\lambda_{k}= \begin{cases}0.999, & \text { if } n>10 \text { and Numgra }>12, \\ 1, & \text { otherwise },\end{cases}$ where Numgra denotes the number of the successive use of the negative gradient direction.

### 3.2.2 Choice of The Wolfe Line Search

The line search is an important factor for the overall efficiency of most optimization algorithms. In this paper, we pay attention to the nonmonotone line search proposed by Zhang and Hager [50] (ZH line search)

$$
\begin{gather*}
f\left(x_{k}+\alpha_{k} d_{k}\right) \leq C_{k}+\delta \alpha_{k} \nabla f\left(x_{k}\right)^{T} d_{k}  \tag{48}\\
\nabla f\left(x_{k}+\alpha_{k} d_{k}\right)^{T} d_{k} \geq \sigma \nabla f\left(x_{k}\right)^{T} d_{k} \tag{49}
\end{gather*}
$$

where $0<\delta<\sigma<1, C_{0}=f_{0}, Q_{0}=1$, and $C_{k}$ and $Q_{k}$ are updated by

$$
\begin{equation*}
Q_{k+1}=\eta_{k} Q_{k}+1, C_{k+1}=\frac{\eta_{k} Q_{k} C_{k}+f\left(x_{k+1}\right)}{Q_{k+1}} \tag{50}
\end{equation*}
$$

where $\eta_{k} \in[0,1]$.
It is worth mentioning that some improvements have been made to ZH line search to find a more suitable stepsize and obtain a better convergence result. Specially,

$$
\begin{equation*}
C_{1}=\min \left\{C_{0}, f_{1}+1.0\right\}, Q_{1}=2.0, \tag{51}
\end{equation*}
$$

when $k \geq 1, C_{k+1}$ and $Q_{k+1}$ are updated by (50), where $\eta_{k}$ is taken as

$$
\eta_{k}= \begin{cases}\eta, & \text { if } \bmod (k, l)=0  \tag{52}\\ 1, & \text { if } \bmod (k, l) \neq 0\end{cases}
$$

where $l=\max (20, n), \bmod (k, l)$ denotes the remainder for $k$ modulo $l$ and $\eta=0.7$ when $C_{k}-f_{k+1}>$ $0.999\left|C_{k}\right|$, otherwise $\eta=0.999$. Such choice of $\eta_{k}$ can be used to control nonmonotonicity dynamically, referred to [32].

## 4 Algorithms

In this section, according to the different choices of special scaled norm, we will introduce two new subspace minimization conjugate gradient algorithms based on the $p$-regularization and analyze some theoretical properties of the direction $d_{k}$.

Denote

$$
r_{k-1}=\left|\frac{f_{k}}{f_{k-1}+0.5\left(g_{k-1}^{T} s_{k-1}+g_{k}^{T} s_{k-1}\right)}-1\right|, \quad \bar{r}_{k-1}=\left|f_{k}-f_{k-1}-0.5\left(g_{k-1}^{T} s_{k-1}+g_{k}^{T} s_{k-1}\right)\right| .
$$

If $r_{k-1}$ or $\bar{r}_{k-1}$ is close to 0 , then the function might be close to a quadratic function. If there are continuously many iterations such that $r_{k-1} \leq \xi_{4}$ or $\bar{r}_{k-1} \leq \xi_{5}$, where $\xi_{4}, \xi_{5}>0$, we restart the method with $-g_{k}$. In addition, if the number of the successive use of CG direction reaches to the threshold MaxRestart, we also restart the method with $-g_{k}$.

Firstly, we describe the subspace minimization conjugate gradient method in which the direction of the regularization model is generated by the problem (33), which is called SMCG_PR1.

```
Algorithm 1 SMCG method with \(p\)-regularization (SMCG_PR1)
Step 0. Given \(x_{0} \in R^{n}, \quad \varepsilon>0, \quad 0<\delta<\sigma<1, \quad \xi_{1}, \quad \xi_{2}, \quad \xi_{3}, \quad \xi_{4}, \quad \xi_{5}, \quad c_{1}, \quad c_{2}, \gamma \in(0,1), \alpha_{0}^{(0)}\). Let \(C_{0}=f_{0}, Q_{0}=1\),
    \(d_{0}=-g_{0}\) and \(k:=0\). Set IterRestart \(:=0\), Numgrad \(:=0\), IterQuad \(:=0\), Isnotgra \(=0\), MaxRestart, MinQuad.
```

Step 1. If $\left\|g_{k}\right\|_{\infty} \leq \varepsilon$, then stop.
Step 2. Compute a stepsize $\alpha_{k}>0$ satisfying (48) and (49). Let $x_{k+1}=x_{k}+\alpha_{k} d_{k}$. If $\left\|g_{k}\right\|_{\infty} \leq \varepsilon$, then stop. Otherwise,
set IterRestart:=IterRestart+1. If $r_{k-1} \leq \xi_{4}$ or $\bar{r}_{k-1} \leq \xi_{5}$, then IterQuad $:=$ IterQuad+1, else IterQuad $:=0$.

Step 3. (Calculation of the direction)
3.1. If Isnotgra $=$ MaxRestart or (IterQuad $=$ MinQuad and IterRestart $\neq$ IterQuad), then set $d_{k+1}=-g_{k+1}$. Set Numgrad $:=$ Numgrad +1 , Isnotgra $:=0$ and IterRestart $:=0$, and go to Step 4. If the condition (30) holds, go to 3.2; otherwise go to 3.3 .
3.2. If the condition (27) or (28) or (29) holds, compute the search direction $d_{k+1}$ by (32) with (44) and (45). Set Isnotgra:=Isnotgra+1 and go to Step 4; otherwise, compute the search direction $d_{k+1}$ by (32) with (34) and (35). Set Isnotgra:=Isnotgra +1 and go to Step 4.
3.3. If the condition (46) holds, compute the search direction $d_{k+1}$ by (3) where $\beta_{k}=\beta_{k}^{H S}$. Set Isnotgra: $=\operatorname{Isnotgra}+1$ and go to Step 4; otherwise, compute the search direction $d_{k+1}$ by (47). Set Numgrad $:=$ Numgrad+1, Isnotgra $:=0$ and IterRestart $:=0$, and go to Step 4.
Step 4. Update $Q_{k+1}$ and $C_{k+1}$ using (51) and (50) with (52).
Step 5. Set $k:=k+1$, and go to Step 1 .

Remark 3 In Algorithm 1, Numgrad denotes the number of the successive use of the negative gradient direction; Isnotgra denotes the number of the successive use of the CG direction; MaxRestart represents a quantification and when the Isnotgra reaches this value, we restart the method with $-g_{k}$; MinQuad also represents a quantification and when the IterQuad reaches this value, we restart the method with $-g_{k}$. These parameters are related to the restart of the algorithm, which has an important impact on the numerical performance of the CG.

Secondly, we describe the subspace minimization conjugate gradient method in which the direction of the regularization model is generated by the problem (40).

If the condition

$$
\begin{equation*}
\left(g_{k}^{T} s_{k-1}\right)^{2}>\left(1-10^{-5}\right)\left\|g_{k}\right\|^{2}\left\|s_{k-1}\right\|^{2} \tag{53}
\end{equation*}
$$

holds, the value of $\frac{\left(g_{k}^{T} s_{k-1}\right)^{2}}{\left\|g_{k}\right\|^{2}\left\|s_{k-1}\right\|^{2}}$ is close to 1 , which means that vectors $g_{k}$ and $s_{k-1}$ may be linearly correlated. So the positive definiteness of the matrix $E_{k}$ in (40) might not be guaranteed. Therefore, we choose the quadratic model to derive a search direction.

We may consider to use "3.2. If the condition (27) or (28) or (29) holds, compute the search direction $d_{k+1}$ by (32) with (44) and (45). Set Isnotgra:=Isnotgra+1 and go to Step 4; otherwise, if the condition (53) holds, compute the search direction $d_{k+1}$ by (32), (41) and (42) with $\lambda=0$, otherwise, compute the search direction $d_{k+1}$ by (32) with (41) and (42). Set Isnotgra:=Isnotgra+1 and go to Step 4." to replace the Step 3.2 in Algorithm 1. The resulting method is called SMCG_PR2. We use SMCG_PR to denote either SMCG_PR1 or SMCG_PR2.

The following two Lemmas show some properties of the direction $d_{k}$, which are essential to the convergence of SMCG_PR.
Lemma 4.1 Suppose the direction $d_{k}$ is calculated by SMCG_PR. Then, there exists a constant $c_{1}$ such that

$$
\begin{equation*}
g_{k}^{T} d_{k} \leq-c_{1}\left\|g_{k}\right\|^{2} \tag{54}
\end{equation*}
$$

Proof. We divide the proof into four cases.
Case 1. The direction $d_{k}$ is given by (32) with (34) and (35), as in SMCG_PR1. Denote $T=\frac{1}{1+\sigma_{k}\left(z^{*}\right)^{p-2}}$. Obviously, in this case,

$$
\mu_{k}=T \bar{\mu}_{k}, \quad \nu_{k}=T \bar{\nu}_{k} .
$$

If $\sigma_{k}\left(z^{*}\right)^{p-2}>1$, we have $T=\frac{1}{2}$ from the first line after(39). Moreover, $\sigma_{k}\left(z^{*}\right) \geq 0$. So we can establish that $\frac{1}{2} \leq T \leq 1$. From (3.31) and (3.32) of [15], we can get that

$$
\begin{equation*}
g_{k}^{T} d_{k}=T g_{k}^{T}\left(\bar{\mu}_{k} g_{k}+\bar{\nu}_{k} s_{k-1}\right) \leq-T \frac{\left\|g_{k}\right\|^{4}}{\rho_{k}} \leq-\frac{\left\|g_{k}\right\|^{4}}{2 \rho_{k}} . \tag{55}
\end{equation*}
$$

Substituting $\rho_{k}=\frac{3}{2} \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}\left\|g_{k}\right\|^{2}$ into (55), we deduce that $g_{k}^{T} d_{k} \leq-\frac{\left\|g_{k}\right\|^{4}}{2 \rho_{k}}=-\frac{1}{3} \frac{s_{k-1}^{T} y_{k-1}}{\left\|y_{k-1}\right\|^{2}}\left\|g_{k}\right\|^{2}$. From (30), we konw $-\frac{1}{\xi_{1}} \leq-\frac{s_{k-1}^{T} y_{k-1}}{\left\|y_{k-1}\right\|^{2}} \leq-\frac{1}{\xi_{2}}$. Therefore, we get

$$
\begin{equation*}
g_{k}^{T} d_{k} \leq-\frac{\left\|g_{k}\right\|^{4}}{2 \rho_{k}}=-\frac{1}{3} \frac{s_{k-1}^{T} y_{k-1}}{\left\|y_{k-1}\right\|^{2}}\left\|g_{k}\right\|^{2} \leq-\frac{1}{3 \xi_{2}}\left\|g_{k}\right\|^{2} \tag{56}
\end{equation*}
$$

On the other hand, if the direction $d_{k}$ is given by (32) with (41) and (42), which in SMCG_PR2. We have that by direct calculation

$$
\begin{aligned}
g_{k}^{T} d_{k} & =\hat{\mu}_{k}\left\|g_{k}\right\|^{2}+\hat{\nu}_{k} g_{k}^{T} s_{k-1} \\
& =-\frac{\left\|g_{k}\right\|^{4}}{\Delta_{k}}\left(s_{k-1}^{T} y_{k-1}-2 g_{k}^{T} y_{k-1} \frac{g_{k}^{T} s_{k-1}}{\left\|g_{k}\right\|^{2}}+\rho_{k}\left(\frac{g_{k}^{T} s_{k-1}}{\left\|g_{k}\right\|^{2}}\right)^{2}-\lambda g_{k}^{T} s_{k-1} \frac{g_{k}^{T} s_{k-1}}{\left\|g_{k}\right\|^{2}}+\lambda\left\|s_{k-1}\right\|^{2}\right) \\
& =-\frac{\left\|g_{k}\right\|^{4}}{\Delta_{k}}\left(\left(\rho_{k}+\lambda\left\|g_{k}\right\|^{2}\right)\left(\frac{g_{k}^{T} s_{k-1}}{\left\|g_{k}\right\|^{2}}\right)^{2}-\left(2 g_{k}^{T} y_{k-1}+2 \lambda g_{k}^{T} s_{k-1}\right) \frac{g_{k}^{T} s_{k-1}}{\left\|g_{k}\right\|^{2}}+s_{k-1}^{T} y_{k-1}+\lambda\left\|s_{k-1}\right\|^{2}\right) \\
& \leq-\frac{\left\|g_{k}\right\|^{4}}{\Delta_{k}} \frac{\bar{\Delta}_{k}}{\rho_{k}+\lambda\left\|g_{k}\right\|^{2}} \\
& =\frac{-\left\|g_{k}\right\|^{2}}{\frac{3}{2} \frac{\left\|y_{k-1}\right\|^{T}}{s_{k-1}}+\lambda} \\
& \leq-\frac{2}{5 \xi_{k}}\left\|g_{k}\right\|^{2} .
\end{aligned}
$$

Due to $0 \leq \lambda \leq \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}$, we have $\frac{3}{2} \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}} \leq \frac{3}{2} \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}+\lambda \leq \frac{5}{2} \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}$. So, $-\frac{2}{3} \frac{s_{k-1}^{T} y_{k-1}}{\left\|y_{k-1}\right\|^{2}} \leq \frac{-1}{\frac{3}{2} \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}+\lambda} \leq$ $-\frac{2}{5} \frac{s_{k-1}^{T} y_{k-1}}{\left\|y_{k-1}\right\|^{2}}$. From (30), we konw $-\frac{1}{\xi_{1}} \leq-\frac{s_{k-1}^{T} y_{k-1}}{\left\|y_{k-1}\right\|^{2}} \leq-\frac{1}{\xi_{2}}$. Therefore, the last inequality is established.

Case 2. $d_{k}=\bar{\mu}_{k} g_{k}+\bar{\nu}_{k} s_{k-1}$, where $\bar{\mu}_{k}$ and $\bar{\nu}_{k}$ are calculated by (44) and (45), respectively. From (55) and (56), we can get that

$$
\begin{equation*}
g_{k}^{T} d_{k}=g_{k}^{T}\left(\bar{\mu}_{k} g_{k}+\bar{\nu}_{k} s_{k-1}\right) \leq-\frac{2}{3 \xi_{2}}\left\|g_{k}\right\|^{2} . \tag{57}
\end{equation*}
$$

Case 3. If the direction $d_{k}$ is given by (3) where $\beta_{k}=\beta_{k}^{H S}$, (54) is satisfied by setting $c_{1}=1-\xi_{3}$. The proof is similar to Lemma 3 in [29].

Case 4. As $d_{k}=-g_{k}$, we can easily derive $g_{k}^{T} d_{k}=-\left\|g_{k}\right\|^{2}$ which satisfies (54) by setting $c_{1}=\frac{1}{2}$.
To sum up, the sufficient descent condition (54) holds by setting

$$
c_{1}=\min \left\{\frac{1}{2}, 1-\xi_{3}, \frac{2}{3 \xi_{2}}, \frac{1}{3 \xi_{2}}, \frac{2}{5 \xi_{2}}\right\},
$$

which completes the proof.
Lemma 4.2 Suppose the direction $d_{k}$ is calculated by SMCG_PR. Then, there exists a constant $c_{2}>0$ such that

$$
\begin{equation*}
\left\|d_{k}\right\| \leq c_{2}\left\|g_{k}\right\| . \tag{58}
\end{equation*}
$$

Proof. The proof is also divided into four parts.
Case 1. The direction $d_{k}$ is given by (32) with (34) and (35), as in SMCG_PR1. From (3.12) in [29] and $T \leq 1$, we obtain

$$
\left\|d_{k}\right\|=T\left\|\bar{\mu}_{k} g_{k}+\bar{\nu}_{k} s_{k-1}\right\| \leq \frac{20}{\xi_{1}}\left\|g_{k}\right\| .
$$

On the other hand, if the direction $d_{k}$ is given by (32) with (41) and (42), as in SMCG_PR2. At first, we give a lower bound of $\bar{\Delta}_{k}$. From (43), we have

$$
\begin{array}{r}
\bar{\Delta}_{k}=\lambda^{2}\left(\left\|g_{k}\right\|^{2}\left\|s_{k-1}\right\|^{2}-\left(g_{k}^{T} s_{k-1}\right)^{2}\right)+\lambda\left(\rho_{k}\left\|s_{k-1}\right\|^{2}+s_{k-1}^{T} y_{k-1}\left\|g_{k}\right\|^{2}-2 g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1}\right)+ \\
\rho_{k} s_{k-1}^{T} y_{k-1}-\left(g_{k}^{T} y_{k-1}\right)^{2} .
\end{array}
$$

Moreover, using the Cauchy inequality and average inequality, we have

$$
\begin{aligned}
& \rho_{k}\left\|s_{k-1}\right\|^{2}+s_{k-1}^{T} y_{k-1}\left\|g_{k}\right\|^{2}-2 g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1} \\
\geq & \frac{3}{2} \frac{\left\|y_{k-1}\right\|^{2}\left\|s_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}\left\|g_{k}\right\|^{2}+s_{k-1}^{T} y_{k-1}\left\|g_{k}\right\|^{2}-2\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|\left\|g_{k}\right\|^{2} \\
= & \left(\frac{1}{2} \frac{\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|}{s_{k-1}^{T} y_{k-1}}+\frac{\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|}{s_{k-1}^{T} y_{k-1}}+\frac{s_{k-1}^{T} y_{k-1}}{\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|}-2\right)\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|\left\|g_{k}\right\|^{2} \\
\geq & \left(\frac{1}{2} \frac{\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|}{s_{k-1}^{T} y_{k-1}}+2-2\right)\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|\left\|g_{k}\right\|^{2} \\
\geq & \frac{1}{2}\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|\left\|g_{k}\right\|^{2} \geq 0 .
\end{aligned}
$$

It follows from (30) that $s_{k-1}^{T} y_{k-1} \geq \xi_{1}\left\|s_{k-1}\right\|^{2}$. By $\rho_{k}=\frac{3}{2} \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}\left\|g_{k}\right\|^{2}, \quad \lambda \geq 0$ and the Cauchy inequality, we obtain a lower bound of $\bar{\Delta}_{k}$ that

$$
\begin{aligned}
\bar{\Delta}_{k} & \geq \rho_{k} s_{k-1}^{T} y_{k-1}-\left(g_{k}^{T} y_{k-1}\right)^{2}=s_{k-1}^{T} y_{k-1}\left(\rho_{k}-\frac{\left(g_{k}^{T} y_{k-1}\right)^{2}}{s_{k-1}^{T} y_{k-1}}\right) \\
& \geq \xi_{1}\left\|s_{k-1}\right\|^{2}\left(\rho_{k}-\frac{\left(g_{k}^{T} y_{k-1}\right)^{2}}{s_{k-1}^{T} y_{k-1}}\right) \\
& \geq \frac{1}{2} \xi_{1}\left\|s_{k-1}\right\|^{2} \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}\left\|g_{k}\right\|^{2} .
\end{aligned}
$$

Using the triangle inequality, Cauchy inequality, $\rho_{k}=\frac{3}{2} \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}\left\|g_{k}\right\|^{2}, 0 \leq \lambda \leq \frac{\left\|y_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}$ and the last relation, we have

$$
\begin{aligned}
& \left\|d_{k}\right\|=\left\|\hat{\mu}_{k} g_{k}+\hat{\nu}_{k} s_{k-1}\right\| \\
& =\left\|\frac{1}{\Delta_{k}}\left(\left(g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1}-s_{k-1}^{T} y_{k-1}\left\|g_{k}\right\|^{2}+\lambda\left(\left(g_{k}^{T} s_{k-1}\right)^{2}-\left\|s_{k-1}\right\|^{2}\left\|g_{k}\right\|^{2}\right)\right) g_{k}+\left(g_{k}^{T} y_{k-1}\left\|g_{k}\right\|^{2}-\rho_{k} g_{k}^{T} s_{k-1}\right) s_{k-1}\right)\right\| \\
& \leq \frac{1}{\Delta_{k}}\left(\left(\left|g_{k}^{T} y_{k-1} g_{k}^{T} s_{k-1}\right|+\left|s_{k-1}^{T} y_{k-1}\right|\left\|g_{k}\right\|^{2}+\lambda\left|g_{k}^{T} s_{k-1}\right|^{2}+\lambda\left\|s_{k-1}\right\|^{2}\left\|g_{k}\right\|^{2}\right)\left\|g_{k}\right\|+\left|g_{k}^{T} y_{k-1}\left\|g_{k}\right\|^{2}-\rho_{k} g_{k}^{T} s_{k-1}\right|\left\|s_{k-1}\right\|\right) \\
& \leq \frac{1}{\Delta_{k}}\left(\left(2\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|+2 \frac{\left\|y_{k-1}\right\|^{2}\left\|s_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}\right)\left\|g_{k}\right\|^{3}+\left(\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|+\frac{\rho_{k}}{\left\|g_{k}\right\|^{2}}\left\|s_{k-1}\right\|^{2}\right)\left\|g_{k}\right\|^{3}\right) \\
& =\frac{1}{\Delta_{k}}\left(\left(3\left\|s_{k-1}\right\|\left\|y_{k-1}\right\|+\frac{7}{2} \frac{\left\|y_{k-1}\right\|^{2}\left\|s_{k-1}\right\|^{2}}{s_{k-1}^{T} y_{k-1}}\right)\left\|g_{k}\right\|^{3}\right) \\
& \leq \frac{13}{\xi_{1}}\left\|g_{k}\right\| .
\end{aligned}
$$

Case 2. $d_{k}=\bar{\mu}_{k} g_{k}+\bar{\nu}_{k} s_{k-1}$, where $\bar{\mu}_{k}$ and $\bar{\nu}_{k}$ are calculated by (44) and (45), respectively. From (3.12) in [29], we can get (58) is satisfied by setting $c_{2}=\frac{20}{\xi_{1}}$.

Case 3. If the direction $d_{k}$ is given by (3) where $\beta_{k}=\beta_{k}^{H S}$, (58) is satisfied by setting $c_{2}=1+\frac{L}{\xi_{1}}$. The proof is same as Lemma 4 in [29].

Case 4. As $d_{k}=-g_{k}$, we can easily establish that $\left\|d_{k}\right\|=\left\|g_{k}\right\|$.
In summary, we easily obtain the fact that (58) holds by

$$
c_{2}=\max \left\{1,1+\frac{L}{\xi_{1}}, \frac{20}{\xi_{1}}\right\}
$$

which completes the proof.

## 5 Convergence Analysis

In this section, we establish the global convergence and $R$-linear convergence of SMCG_PR. We assume that $\left\|g_{k}\right\| \neq 0$ for each $k$; otherwise, there is a stationary point for some $k$.

At first, we suppose that the objective function $f$ satisfies the following assumptions. Define $\Theta$ as an open neighborhood of the level set $L\left(x_{0}\right)=\left\{x \in R^{n}: f(x) \leq f\left(x_{0}\right)\right\}$, where $x_{0}$ is the initial point.
Assumption $1 f$ is continuously differentiable and bounded from below in $\Theta$.
Assumption 2 The gradient $g$ is Lipchitz continuous in $\Theta$, namely, there exists a constant $L>0$ such that $\|g(x)-g(y)\| \leq L\|x-y\|, \forall x, y \in \Theta$.
Lemma 5.1 Suppose the Assumption 1 holds and the iterative sequence $\left\{x_{k}\right\}$ is generated by the SMCG_PR.
Then, we have $f_{k} \leq C_{k}$ for each $k$.
Proof. Due to (48) and descent direction $d_{k+1}, f_{k+1}<C_{k}$ always holds. Through (51), we can get $C_{1}=C_{0}$ or $C_{1}=f_{1}+1.0$. If $C_{1}=C_{0}$, because of the relations $f_{k+1}<C_{k}$ and $C_{0}=f_{0}$, we know $f_{1} \leq C_{1}$. If $C_{1}=f_{1}+1.0$, we can easily get $f_{1} \leq C_{1}$. When $k \geq 1$, the updated form of $C_{k+1}$ is (50), similar to Lemma 1.1 in [50], we have $f_{k+1} \leq C_{k+1}$. Therefore, $f_{k} \leq C_{k}$ holds for each $k$.

Lemma 5.2 Suppose the Assumption 2 holds and the iterative sequence $\left\{x_{k}\right\}$ is generated by the SMCG_PR. Then,

$$
\begin{equation*}
\alpha_{k} \geq\left(\frac{1-\sigma}{L}\right) \frac{\left|g_{k}^{T} d_{k}\right|}{\left\|d_{k}\right\|^{2}} \tag{59}
\end{equation*}
$$

Proof. By (49) and Assumption 2, we have that

$$
(\sigma-1) g_{k}^{T} d_{k} \leq\left(g_{k+1}-g_{k}\right)^{T} d_{k} \leq \alpha_{k} L\left\|d_{k}\right\|^{2}
$$

Since $d_{k}$ is a descent direction and $\sigma<1$, (59) follows immediately.
Theorem 5.3 Suppose Assumption 1 and 2 hold. If the iterative sequence $\left\{x_{k}\right\}$ is generated by the SMCG_PR, it follows

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|g\left(x_{k}\right)\right\|=0 \tag{60}
\end{equation*}
$$

Proof. By (48), Lemma 5.2, Lemma 4.1, and Lemma 4.2, we get that

$$
f_{k+1} \leq C_{k}-\frac{\delta(1-\sigma)}{L} \frac{\left(g_{k}^{T} d_{k}\right)^{2}}{\left\|d_{k}\right\|^{2}} \leq C_{k}-\frac{\delta(1-\sigma) c_{1}^{2}}{L c_{2}^{2}}\left\|g_{k}\right\|^{2}
$$

In short, set $\beta=\frac{\delta(1-\sigma) c_{1}^{2}}{L c_{2}^{2}}$, we give the fact that

$$
\begin{equation*}
f_{k+1} \leq C_{k}-\beta\left\|g_{k}\right\|^{2} \tag{61}
\end{equation*}
$$

Now, we find a upper bound of $Q_{k+1}$ in (50) with (52). As for $k \geq 1, Q_{k+1}$ can be expressed as [32]

$$
Q_{k+1}=\left\{\begin{array}{lr}
1+(l+1) \sum_{i=1}^{k / l} \eta^{i}, & \bmod (k, l)=0, \\
1+\bmod (k, l)+(l+1) \sum_{i=1}^{\lfloor k / l\rfloor} \eta^{i}, \bmod (k, l) \neq 0,
\end{array}\right.
$$

where $\lfloor$.$\rfloor is the floor function. Then, we obtain$

$$
\begin{align*}
Q_{k+1} & \leq 1+\bmod (k, l)+(l+1) \sum_{i=1}^{\lfloor k / l\rfloor+1} \eta^{i} \\
& \leq 1+(l+1)+(l+1) \sum_{i=1}^{\lfloor k / l\rfloor+1} \eta^{i} \\
& \leq 1+(l+1)+(l+1) \sum_{i=1}^{k+1} \eta^{i}  \tag{62}\\
& =1+(l+1) \sum_{i=0}^{k+1} \eta^{i} \\
& =1+\frac{(l+1)\left(1-\eta^{k+2}\right)}{1-\eta} \\
& \leq 1+\frac{l+1}{1-\eta} .
\end{align*}
$$

Denote $M=1+\frac{l+1}{1-\eta}$, which gives the fact $Q_{k+1} \leq M$.
With the updated form of $C_{k+1}$ in (50), (61) and (62), we obtain

$$
\begin{equation*}
C_{k+1}=C_{k}+\frac{f_{k+1}-C_{k}}{Q_{k+1}} \leq C_{k}-\frac{\beta}{Q_{k+1}}\left\|g_{k}\right\|^{2} \leq C_{k}-\frac{\beta}{M}\left\|g_{k}\right\|^{2} \tag{63}
\end{equation*}
$$

According to (51), we know $C_{1} \leq C_{0}$ which implies that $C_{k}$ is monotonically decreasing. Due to Assumption 1 and Lemma 5.1, we can get $C_{k}$ is bounded from below. Then

$$
\sum_{k=0}^{\infty} \frac{\beta}{M}\left\|g_{k}\right\|^{2}<\infty
$$

therefore,

$$
\lim _{k \rightarrow \infty}\left\|g\left(x_{k}\right)\right\|=0
$$

which completes the proof.
Moreover, $R$-linear convergence of SMCG_PR will be established as followed. In order to establish $R$-linear convergence of SMCG_PR, we introduce Definition 1 and assume that the optimal set $\chi^{*}$ is nonempty.
Definition 1 The continuously differentiable function $f$ has a global error bound on $R^{n}$, if there exists a constant $\kappa_{f}>0$ such that for any $x \in R^{n}$ and $\bar{x}=[x]_{\chi^{*}}$, we have

$$
\begin{equation*}
\|g(x)\| \geq \kappa_{f}\|x-\bar{x}\| \quad \forall x \in R^{n} \tag{64}
\end{equation*}
$$

where $\bar{x}=[x]_{\chi^{*}}$ is the projection of $x$ onto the nonempty solution set $\chi^{*}$. We further denote by $\chi^{*}=$ $\arg \min _{x \in R^{n}} f(x)$ the set of optimal solutions of problem (1).
Remark 4 By Assumption 2 it is $\left\|g(x)-g\left(x^{*}\right)\right\| \leq L\left\|x-x^{*}\right\|$, so that it is also $\|g(x)\| \leq L\left\|x-x^{*}\right\|$, which implies $k_{f} \leq L$.
Remark 5 [34] If $f$ is strongly convex, it must satisfy Definition 1.
Remark 6 If $f$ is a convex function and the optimal solution set is nonempty, the function value at the optimal solution is equal.
Theorem 5.4 Suppose that Assumption 2 holds, $f$ is convex with a minimizer $x^{*}$ and the solution set $\chi^{*}$ is nonempty, and there exists $\bar{\alpha}>0$ such that $\alpha_{k} \leq \bar{\alpha}$ for all $k$. Let $f$ satisfy Definition 1 with constant $\kappa_{f}>0$. In what follows, we only consider the case of $\left\|g_{k}\right\| \neq 0, \forall k \geq 0$. Then there exists $\theta \in(0,1)$ such that

$$
f_{k}-f\left(x^{*}\right) \leq \theta^{k}\left(f_{0}-f\left(x^{*}\right)\right) .
$$

Proof. From Lemma 5.1, we can get $f_{k+1} \leq C_{k+1}$. Due to Remark 6 and $\left\|g_{k}\right\| \neq 0, \forall k \geq 0$, we know $x_{k+1}$ is not the optimal solution. So, we have $f\left(x^{*}\right)<f_{k+1}$. From (63) and $\left\|g_{k}\right\| \neq 0, \forall k \geq 0$, we have that $C_{k+1}<C_{k}$. Therefore, we get $f\left(x^{*}\right)<f_{k+1} \leq C_{k+1}<C_{k}$, which means $f\left(x^{*}\right)<C_{k+1}<C_{k}$. It follows

$$
\begin{equation*}
0<\frac{C_{k+1}-f\left(x^{*}\right)}{C_{k}-f\left(x^{*}\right)}<1, \quad \forall k \geq 0 \tag{65}
\end{equation*}
$$

Set

$$
\begin{equation*}
r=\lim _{k \rightarrow \infty} \sup \frac{C_{k+1}-f\left(x^{*}\right)}{C_{k}-f\left(x^{*}\right)}, \tag{66}
\end{equation*}
$$

then, $0 \leq r \leq 1$.
First of all, we consider the case of $r=1$. According to (66), there exists a subsequence $\left\{x_{k_{j}}\right\}$ such that

$$
\begin{equation*}
\lim _{j \rightarrow \infty} \frac{C_{k_{j}+1}-f\left(x^{*}\right)}{C_{k_{j}}-f\left(x^{*}\right)}=1 . \tag{67}
\end{equation*}
$$

Because of (62), there exists $q>0,0<q \leq \frac{1}{Q_{k_{j}+1}} \leq 1$ holds. Hence, there exists a subsequence of $\left\{x_{k_{j}}\right\}$ such that the corresponding subsequence of $\left\{\frac{1}{Q_{k_{j}+1}}\right\}$ is convergent. Without loss of generality, we assume that

$$
\begin{equation*}
\lim _{j \rightarrow \infty} \frac{1}{Q_{k_{j}+1}}=r_{1} \tag{68}
\end{equation*}
$$

Clearly, $0<r_{1} \leq 1$.
By the updating formula of $C_{k+1}$ in (50), we obtain

$$
\frac{C_{k_{j}+1}-f\left(x^{*}\right)}{C_{k_{j}}-f\left(x^{*}\right)}=\left(1-\frac{1}{Q_{k_{j}+1}}\right)+\frac{1}{Q_{k_{j}+1}} \frac{f_{k_{j}+1}-f\left(x^{*}\right)}{C_{k_{j}}-f\left(x^{*}\right)} .
$$

It follows from (67), (68) and finding the limit of upper formula that

$$
\begin{equation*}
\lim _{j \rightarrow \infty} \frac{f_{k_{j}+1}-f\left(x^{*}\right)}{C_{k_{j}}-f\left(x^{*}\right)}=1 \tag{69}
\end{equation*}
$$

Using convexity of $f$, the solution set $\chi^{*}$ is nonempty and Remark 6, we know $f\left(x^{*}\right)=f(\bar{x})$, where $\bar{x}$ is introduced in Definition 1. So, we have that $f_{k_{j}+1}-f\left(x^{*}\right)=f_{k_{j}+1}-f(\bar{x})$. Through convexity of $f$, we have $f_{k_{j}+1}-f(\bar{x}) \leq\left(\nabla f_{k_{j}+1}, x_{k_{j}+1}-\bar{x}\right)$. According to Definition 1 and Cauchy-Schwarz inequality, then $\left(\nabla f_{k_{j}+1}, x_{k_{j}+1}-\bar{x}\right) \leq \frac{1}{k_{f}}\left\|g_{k_{j}+1}\right\|^{2}$. Therefore, we get

$$
\begin{equation*}
f_{k_{j}+1}-f\left(x^{*}\right)=f_{k_{j}+1}-f(\bar{x}) \leq\left(\nabla f_{k_{j}+1}, x_{k_{j}+1}-\bar{x}\right) \leq \frac{1}{\kappa_{f}}\left\|g_{k_{j}+1}\right\|^{2} \tag{70}
\end{equation*}
$$

According to the Lipschitz continuity of $g, \alpha_{k} \leq \bar{\alpha}$ and (58), we have

$$
\left\|g_{k_{j}+1}\right\| \leq\left\|g_{k_{j}+1}-g_{k_{j}}\right\|+\left\|g_{k_{j}}\right\| \leq L\left\|x_{k_{j}+1}-x_{k_{j}}\right\|+\left\|g_{k_{j}}\right\| \leq\left(1+L \bar{\alpha} c_{2}\right)\left\|g_{k_{j}}\right\|,
$$

together with (70), it implies that

$$
f_{k_{j}+1}-f\left(x^{*}\right) \leq \frac{1}{\kappa_{f}}\left(1+L \bar{\alpha} c_{2}\right)^{2}\left\|g_{k_{j}}\right\|^{2}
$$

Dividing the above inequality by $C_{k_{j}}-f\left(x^{*}\right)$, we have

$$
\begin{equation*}
0<\frac{f_{k_{j}+1}-f\left(x^{*}\right)}{C_{k_{j}}-f\left(x^{*}\right)} \leq \frac{\left(1+L \bar{\alpha} c_{2}\right)^{2}\left\|g_{k_{j}}\right\|^{2}}{\kappa_{f}\left(C_{k_{j}}-f\left(x^{*}\right)\right)} \tag{71}
\end{equation*}
$$

Based on (61)

$$
f_{k_{j}+1}-f\left(x^{*}\right) \leq C_{k_{j}}-f\left(x^{*}\right)-\beta\left\|g_{k_{j}}\right\|^{2} .
$$

Dividing both sides of above inequality by $C_{k_{j}}-f\left(x^{*}\right)$, we get

$$
\frac{f_{k_{j}+1}-f\left(x^{*}\right)}{C_{k_{j}}-f\left(x^{*}\right)} \leq 1-\frac{\beta\left\|g_{k_{j}}\right\|^{2}}{C_{k_{j}}-f\left(x^{*}\right)}
$$

Combining with (69), then

$$
\lim _{j \rightarrow \infty} \frac{\left\|g_{k_{j}}\right\|^{2}}{C_{k_{j}}-f\left(x^{*}\right)}=0
$$

due to (71), it follows

$$
\lim _{j \rightarrow \infty} \frac{f_{k_{j}+1}-f\left(x^{*}\right)}{C_{k_{j}}-f\left(x^{*}\right)}=0
$$

which contradicts with (69). Therefore, the case of $r=1$ does not occur, that is,

$$
\lim _{k \rightarrow \infty} \sup \frac{C_{k+1}-f\left(x^{*}\right)}{C_{k}-f\left(x^{*}\right)}=r<1
$$

Then, there exists an integer $k_{0}>0$ such that

$$
\begin{equation*}
\frac{C_{k+1}-f\left(x^{*}\right)}{C_{k}-f\left(x^{*}\right)} \leq r+\frac{1-r}{2}=\frac{1+r}{2}<1, \quad \forall k>k_{0} . \tag{72}
\end{equation*}
$$

From (65), we know that $0<\max _{0 \leq k \leq k_{0}}\left\{\frac{C_{k+1}-f\left(x^{*}\right)}{C_{k}-f\left(x^{*}\right)}\right\}=\bar{r}<1$. Let $\theta=\max \left\{\frac{1+r}{2}, \bar{r}\right\}$.
Clearly, $0<\theta<1$. It follows from (72) that

$$
C_{k+1}-f\left(x^{*}\right) \leq \theta\left(C_{k}-f\left(x^{*}\right)\right),
$$

which indicates that

$$
C_{k+1}-f\left(x^{*}\right) \leq \theta\left(C_{k}-f\left(x^{*}\right)\right) \leq \theta^{k+1}\left(C_{0}-f\left(x^{*}\right)\right) .
$$

In addition, due to $f_{k+1} \leq C_{k+1}$ in Lemma 5.1 and $C_{0}=f_{0}$, we can deduce that

$$
\left(f_{k}-f\left(x^{*}\right)\right) \leq \theta^{k}\left(f_{0}-f\left(x^{*}\right)\right),
$$

which completes the proof.

## 6 Numerical Results

In this section, numerical experiments are conducted to show the efficiency of the SMCG_PR with $p=3$ and $p=4$. We compare the performance of SMCG_PR to that of CG_DESCENT (5.3) [24], CGOPT [14], SMCG_BB [31] and SMCG_Conic [30] for the 145 test problems in the CUTEr library [20]. The names and dimensions for the 145 test problems are the same as that of the numerical results in [26]. The codes of CG_DESCENT (5.3), CGOPT and SMCG_BB can be downloaded from http://users.clas.ufl.edu/hager/papers/Software, http://coa.amss.ac.cn/wordpress/?page_id=21 and http://web.xidian.edu.cn/xdliuhongwei/paper.html, respectively.

The following parameters are used in SMCG_PR:

$$
\begin{gathered}
\varepsilon=10^{-6}, \delta=0.0005, \sigma=0.9999, \lambda_{\min }=10^{-30}, \lambda_{\max }=10^{30}, \gamma=10^{-5} \\
\xi_{1}=10^{-7}, \xi_{2}=1.25 \times 10^{4}, \xi_{3}=10^{-5}, \xi_{4}=10^{-9}, \xi_{5}=10^{-11}, c_{1}=10^{-4}, c_{2}=0.080
\end{gathered}
$$

CG_DESCENT (5.3), CGOPT, SMCG_BB and SMCG_Conic use the default parameters in their codes. All test methods are terminated if $\left\|g_{k}\right\|_{\infty} \leq 10^{-6}$ is satisfied or the number of iterations exceeds 200,000.

The performance profiles introduced by Dolan and Moré [16] are used to display the performances of the test methods. We present three groups of the numerical experiments. They all run in Ubuntu 10.04 LTS which is fixed in a VMware Workstation 10.0 installed in Windows 7. In the following Figs. 1-12 and Table 2 , " $N_{\text {iter }}$ ", " $N_{f}$ "," $N_{g}$ " and " $T_{\text {cpu }}$ " represent the number of iterations, the number of function evaluations, the number of gradient evaluations and CPU time(s), respectively.

In the first group of numerical experiments, we compare SMCG_PR1 and SMCG_PR2 with $p=3$ and $p=4$. All these test methods can successfully solve 139 problems. It is observed from Fig.1-Fig. 4 that the SMCG_PR1 with $p=3$ is better than others.


Fig. 1: Performance profile based on $N_{i t e r}$ (CUTEr). Fig. 2: Performance profile based on $N_{f}$ (CUTEr).


Fig. 3: Performance profile based on $N_{g}$ (CUTEr). Fig. 4: Performance profile based on $T_{\text {cpu }}$ (CUTEr).

In the second group of numerical experiments, we compare SMCG_PR1 $(p=3)$ with CG_DECENT (5.3) and CGOPT. SMCG_PR1 successfully solves 139 problems, while CG_DECENT (5.3) and CGOPT successfully solve 144 and 134 problems, respectively.


Fig. 5: Performance profile based on $N_{\text {iter }}$ (CUTEr). Fig. 6: Performance profile based on $N_{f}$ (CUTEr).

Regarding the number of iterations in Fig.5, we observe that SMCG_PR1 is more efficient than CG_DESCENT (5.3) and CGOPT, and it successfully solves about $50.4 \%$ of the test problems with the least number of iterations, while the percentages of solved problems of CG_DESCENT (5.3) and CGOPT are $42.8 \%$ and $23.3 \%$, respectively. As shown in Fig.6, we see that SMCG_PR1 outperforms CG_DESCENT (5.3) and CGOPT for the number of function evaluations.


Fig. 7: Performance profile based on $N_{g}$ (CUTEr). Fig. 8: Performance profile based on $T_{c p u}$ (CUTEr).

Fig. 7 presents the performance profile relative to the number of gradient evaluations. We can observe that the SMCG_PR1 is the top performance and solves about $54.2 \%$ of test problems with the least number of gradient evaluations, and CG_DESCENT (5.3) solves about $31.6 \%$ and CGOPT solves about $21.8 \%$. From Fig.8, we can see that SMCG_PR1 is fastest for about $66.2 \%$ of test problems, while CG_DESCENT (5.3) and CGOPT are fastest for about $8.3 \%$ and $34.6 \%$, respectively. From Figs. 5, 6, 7 and 8, it indicates that SMCG_PR1 outperforms CG_DESCENT (5.3) and CGOPT for the 145 test problems in the CUTEr library.

In the third group of the numerical experiments, we compare SMCG_PR1 $(p=3)$ with SMCG_BB and SMCG_Conic [30]. SMCG_PR1 successfully solves 139 problems, which are 1 problem more than SMCG_Conic, while SMCG_BB successfully solves 140 problems. As shown in Figs. 9, 10, 11 and 12, we
can easily observe that SMCG_PR1 is superior to SMCG_BB and SMCG_Conic for the 145 test problems in the CUTEr library.


Fig. 9: Performance profile based on $N_{\text {iter }}$ (CUTEr). Fig. 10: Performance profile based on $N_{f}($ CUTEr $)$.


Fig. 11: Performance profile based on $N_{g}($ CUTEr $)$. Fig. 12: Performance profile based on $T_{c p u}($ CUTEr $)$.

Due to limited space, we do not list all detailed numerical results. Instead, we present some numerical results about SMCG_PR1 ( $p=3$ ), CG_DESCENT (5.3), CGOPT, SMCG_BB and SMCG_Conic for some ill-conditioned problems. Table 1 illustrates the notations, names and dimensions about the illconditioned problems. Table 2 presents some numerical results about SMCG_PR1 ( $p=3$ ), CG_DESCENT (5.3), CGOPT, SMCG_BB and SMCG_Conic for the problems in Table 1. As shown in Table 2, the most famous CG software packages CGOPT and CG_DESCENT (5.3) both require many iterations, function evaluations and gradient evaluations when solving these ill-conditioned problems, though the dimensions of some of these ill-conditioned problems are small. From Table 2, we observe that SMCG_PR1 $(p=3)$ has significant improvements over the other test methods, especially for CGOPT and CG_DESCENT (5.3). It indicates that SMCG_PR1 $(p=3)$ is relatively competitive for ill-conditioned problems compared to other test methods.

Table 1: Some ill-conditioned problems in CUTEr

| notation | name | dimension | notation | name | dimension |
| :---: | :---: | :---: | :---: | :---: | :---: |
| P1 | EIGENBLS | 2550 | P7 | PALMER1D | 7 |
| P2 | EXTROSNB | 1000 | P8 | PALMER2C | 8 |
| P3 | GROWTHLS | 3 | P9 | PALMER4C | 8 |
| P4 | MARATOSB | 2 | P10 | PALMER6C | 8 |
| P5 | NONCVXU2 | 5000 | P11 | PALMER7C | 8 |
| P6 | PALMER1C | 8 |  |  |  |

Table 2: Numerical results for some ill-conditioned problems in CUTEr

| problem | SMCG_PR1 | CG_DESCENT $(5.3)$ | CGOPT | SMCG_BB | SMCG_Conic |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $N_{\text {iter }} / N_{f} / N_{g}$ | $N_{\text {iter }} / N_{f} / N_{g}$ | $N_{\text {iter }} / N_{f} / N_{g}$ | $N_{\text {iter }} / N_{f} / N_{g}$ | $N_{i t e r} / N_{f} / N_{g}$ |
| P1 | $9190 / 18382 / 9192$ | $16092 / 32185 / 16093$ | $19683 / 39369 / 19686$ | $16040 / 32066 / 16041$ | $12330 / 24654 / 12332$ |
| P2 | $3568 / 6956 / 3574$ | $6879 / 13839 / 6975$ | $9127 / 18465 / 9305$ | $8416 / 16195 / 8426$ | $3733 / 7466 / 3735$ |
| P3 | $1 / 2 / 2$ | $441 / 997 / 596$ | $480 / 1241 / 644$ | $689 / 1512 / 711$ | $1 / 2 / 2$ |
| P4 | $212 / 614 / 389$ | $946 / 2911 / 2191$ | $1411 / 4185 / 2213$ | $1159 / 9592 / 2634$ | $3640 / 13621 / 5632$ |
| P5 | $6096 / 12174 / 6098$ | $7160 / 13436 / 8046$ | $6195 / 12402 / 6207$ | $6722 / 12800 / 6723$ | $6459 / 12816 / 6460$ |
| P6 | $1453 / 2093 / 1546$ | $126827 / 224532 / 378489$ | Failed | $88047 / 135548 / 89509$ | $13007 / 23796 / 13352$ |
| P7 | $445 / 682 / 470$ | $3971 / 5428 / 10036$ | $16490 / 36567 / 19846$ | $2701 / 3703 / 2727$ | $584 / 943 / 635$ |
| P8 | $307 / 440 / 318$ | $21362 / 21455 / 42837$ | $25716 / 61275 / 30492$ | $4894 / 7169 / 5002$ | $695 / 1386 / 697$ |
| P9 | $54 / 107 / 59$ | $44211 / 49913 / 96429$ | $88681 / 197232 / 105736$ | $1064 / 1622 / 1074$ | $1055 / 2025 / 1071$ |
| P10 | $202 / 323 / 213$ | $14174 / 14228 / 28411$ | $29118 / 63118 / 31844$ | $35704 / 58676 / 36281$ | $1458 / 2429 / 1505$ |
| P11 | $6288 / 8757 / 6576$ | $65294 / 78428 / 149585$ | $98699 / 220388 / 119626$ | $46397 / 65692 / 46929 /$ | $502 / 575 / 514$ |

The numerical results indicate that the SMCG_PR method outperforms CG_DESCENT (5.3), CGOPT, SMCG_BB and SMCG_Conic.

## 7 Conclusions

In this paper, we present two new subspace minimization conjugate gradient methods based on the special $p-$ regularization model for $p>2$. In the proposed methods, the search directions satisfy the sufficient descent condition. Under mild conditions, the global convergences of SMCG_PR are established. We also prove that SMCG_PR is $R$-linearly convergent. The numerical experiments show that SMCG_PR is very promising.

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