

Uniform Stability for a Spatially-Discrete, Subdiffusive Fokker–Planck Equation*

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Abstract

We prove stability estimates for the spatially discrete, Galerkin solution of a fractional Fokker–Planck equation, improving on previous results in several respects. Our main goal is to establish that the stability constants are bounded uniformly in the fractional diffusion exponent $\alpha \in (0, 1]$. In addition, we account for the presence of an inhomogeneous term and show a stability estimate for the gradient of the Galerkin solution. As a by-product, the proofs of error bounds for a standard finite element approximation are simplified.

1 Introduction

We consider the stability of semidiscrete Galerkin methods for the time-fractional Fokker–Planck equation [1, 2]

$$\begin{aligned} \partial_t u - \nabla \cdot (\partial_t^{1-\alpha} \kappa \nabla u - \vec{F} \partial_t^{1-\alpha} u) &= g & \text{for } \vec{x} \in \Omega \text{ and } 0 < t \leq T, \\ u &= u_0(\vec{x}) & \text{for } \vec{x} \in \Omega \text{ when } t = 0, \\ u &= 0 & \text{for } \vec{x} \in \partial\Omega \text{ and } 0 < t \leq T. \end{aligned} \quad (1)$$

Here, Ω is a bounded Lipschitz domain in \mathbb{R}^d ($d \geq 1$), the fractional exponent satisfies $0 < \alpha \leq 1$ and the fractional time derivative is understood in the Riemann–Liouville sense: $\partial_t^{1-\alpha} = \partial_t \mathcal{I}^\alpha$ where the fractional integration operator \mathcal{I}^α is defined as usual in (9) below. The diffusivity $\kappa \in L_\infty(\Omega)$ is assumed independent of time, positive and bounded below: $\kappa(\vec{x}) \geq \kappa_{\min} > 0$ for $\vec{x} \in \Omega$. The forcing vector \vec{F} may depend both on \vec{x} and t , and we assume that \vec{F} , $\partial_t \vec{F}$, $\nabla \cdot \vec{F}$ and $\nabla \cdot \partial_t \vec{F}$ are bounded on $\Omega \times [0, T]$. Note that if $\alpha = 1$ then $\partial_t^{1-\alpha} \phi = \phi$ so the governing equation in (1) reduces to a classical Fokker–Planck equation.

If \vec{F} is independent of t , then by applying $\mathcal{I}^{1-\alpha}$ to both sides of the governing equation we find that (1) is equivalent to

$${}^C \partial_t^\alpha u - \nabla \cdot (\kappa \nabla u - \vec{F} u) = \mathcal{I}^{1-\alpha} g, \quad (2)$$

where ${}^C \partial_t^\alpha u = \mathcal{I}^{1-\alpha} \partial_t u$ is the Caputo fractional derivative of order α . In this form, numerous authors have studied the numerical solution of the problem, mostly for a 1D spatial domain $\Omega = (0, L)$ and with $g \equiv 0$. For instance, Deng [4] considered the method of lines, Jiang and Xu [8] proposed a finite volume method, Yang et al. [21] a spectral collocation method, and Duong and Jin [5] a Wasserstein gradient flow formulation.

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For both continuous and discrete solutions to fractional PDEs, it is natural to expect stability constants to remain bounded as $\alpha \rightarrow 1$ if the limiting classical problem is stable. In applications, the value of α is typically estimated from measurements, and this process might be treated as an inverse problem. It would then be desirable that simulations of the forward problem are uniformly stable in α , particularly if the diffusion turns out to be classical ($\alpha = 1$) or only slightly subdiffusive (α close to 1). Perhaps for these reasons, interest in the question of α -uniform stability and convergence seems to be growing. Note that growth in the stability constant as $\alpha \rightarrow 0$ is of less concern, since very small values of α have not been observed in real physical systems.

In the special case $\vec{F} \equiv \vec{0}$ of fractional diffusion, Chen and Stynes [3] showed that as α tends to 1 the solution of (2) tends to the solution of the classical diffusion problem, uniformly in x and t . They also discussed several examples of numerical schemes for which the error analysis leads to constants that remain bounded as $\alpha \rightarrow 1$ (said to be α -robust bounds), as well as several for which the constants blow up (α -nonrobust bounds). Recent examples in the former category include Jin et al. [9, see Remark 4], Huang et al. [6, see Section 5] and Mustapha [17, Lemma 3.1, Theorem 3.5].

We work with the weak solution $u : (0, T] \rightarrow H_0^1(\Omega)$ of (1) characterized by

$$\langle u', v \rangle + \langle \partial_t^{1-\alpha} \kappa \nabla u, \nabla v \rangle - \langle \vec{F} \partial_t^{1-\alpha} u, \nabla v \rangle = \langle g, v \rangle \quad \text{for } v \in H_0^1(\Omega) \quad (3)$$

and $0 < t \leq T$, with $u(0) = u_0$, where $u' = \partial_t u$, $\langle u, v \rangle = \int_{\Omega} uv$ and $\langle \vec{u}, \vec{v} \rangle = \int_{\Omega} \vec{u} \cdot \vec{v}$. Strictly speaking, to allow minimal assumptions on the regularity of the data u_0 and g , we define the solution u by requiring that it satisfy the time-integrated equation

$$\langle u, v \rangle + \langle \mathcal{I}^\alpha \kappa \nabla u, \nabla v \rangle - \langle \vec{\mathcal{B}}_1 u, \nabla v \rangle = \langle f, v \rangle, \quad (4)$$

where

$$(\vec{\mathcal{B}}_1 \phi)(t) = \int_0^t (\vec{F} \partial_t^{1-\alpha} \phi)(s) ds \quad \text{and} \quad f(t) = u_0 + \int_0^t g(s) ds. \quad (5)$$

In previous work, we have established that this problem is well-posed [12, 16].

For a fixed, finite dimensional subspace $\mathbb{X} \subseteq H_0^1(\Omega)$, the semidiscrete Galerkin solution $u_{\mathbb{X}} : [0, T] \rightarrow \mathbb{X}$ is given by

$$\langle u_{\mathbb{X}}, \chi \rangle + \langle \mathcal{I}^\alpha \kappa \nabla u_{\mathbb{X}}, \nabla \chi \rangle - \langle \vec{\mathcal{B}}_1 u_{\mathbb{X}}, \nabla \chi \rangle = \langle f_{\mathbb{X}}, \chi \rangle \quad \text{for } \chi \in \mathbb{X}, \quad (6)$$

with $f_{\mathbb{X}}(t) = u_{0\mathbb{X}} + \int_0^t g(s) ds$ and with $u_{\mathbb{X}}(0) = u_{0\mathbb{X}} \in \mathbb{X}$ a suitable approximation to u_0 . Previously, we studied this problem in the particular case when $\mathbb{X} = S_h$ is a space of continuous, piecewise-linear finite element functions corresponding to a conforming triangulation of Ω with maximum element size $h > 0$. We showed that the Galerkin finite element solution $u_h(t)$ is stable in the norm of $L_2(\Omega)$ when $g(t) \equiv 0$, satisfying the bound [10, Theorem 4.5]

$$\|u_h(t)\| \leq C_\alpha \|u_{0h}\| \quad \text{for } 0 \leq t \leq T \text{ and } 0 < \alpha < 1, \quad (7)$$

where $u_h(0) = u_{0h} \in S_h$ approximates u_0 . The method of proof relied on estimates for fractional integrals [10, Lemmas 3.2–3.4] involving powers of $(1 - \alpha)^{-1}$, leading to a stability constant C_α that blows up as $\alpha \rightarrow 1$. However, in the limiting case $\alpha = 1$ the semidiscrete finite element method is easily seen to be stable [10, Remark 4.7], that is, (7) holds with $C_1 < \infty$. In the absence of forcing, that is, in the simple case $\vec{F} \equiv \vec{0}$ of fractional diffusion, the stability constant equals one: $\|u_h(t)\| \leq \|u_{0h}\|$ for $0 < \alpha \leq 1$.

Our primary aim in what follows is to improve the results of our earlier paper [10] via a new analysis of (6) that yields a uniform stability constant for $0 < \alpha \leq 1$, as well as allowing a non-zero source term g . Recently, Huang et al. [6] have addressed the same question using

a different analysis that requires $1/2 < \alpha \leq 1$ and $u_0 \in H^1(\Omega)$, with a stability constant that blows up as $\alpha \rightarrow 1/2$.

Throughout the paper, C denotes a generic constant that may depend of T , Ω , κ and \vec{F} . Dependence on any other parameters will be shown explicitly, and in particular, we write C_α to show that the constant may also depend on α . After citing some technical lemmas in Section 2, we present the stability proof in Section 3, stating our main result as Theorem 3.4. Using similar arguments, Section 4 establishes an estimate for the gradient of $u_{\mathbb{X}}$ in Theorem 4.4. Combining these results gives, for $0 \leq t \leq T$ and $0 < \alpha \leq 1$,

$$\begin{aligned} \|u_{\mathbb{X}}(t)\| + t^{\alpha/2} \|\nabla u_{\mathbb{X}}(t)\| &\leq C \left(\|u_{0\mathbb{X}}\| + \int_0^t \|g(s)\| ds \right) + C \left(\frac{1}{t} \int_0^t \|sg(s)\|^2 ds \right)^{1/2} \\ &\leq C \left(\|u_{0\mathbb{X}}\| + t^{1/2} \int_0^t \|g(s)\|^2 ds \right). \end{aligned} \quad (8)$$

(Here, the second bound follows from the first by Lemma 3.5 with $\eta = 1$.) At the end of Section 4, in Remark 4.5, we note that the exact solution u has the same uniform stability property as the semidiscrete Galerkin approximation $u_{\mathbb{X}}$, that is, (8) holds with u and u_0 replacing $u_{\mathbb{X}}$ and $u_{0\mathbb{X}}$. Also, in Remark 4.6, we discuss briefly the implications of enforcing a zero-flux boundary condition instead of the homogeneous Dirichlet one in problem (1). Section 5 applies our new stability analysis to the piecewise linear Galerkin finite element solution u_h , showing in Theorem 5.4 that $\|u_h - u\| = O(t^{-\alpha(2-r)/2} h^2)$ and $\|\nabla u_h - \nabla u\| = O(t^{-\alpha(2-r)/2} h)$. These error bounds rely on a regularity assumption involving the parameter $r \in [0, 2]$, which we justify in Theorem 5.3. Finally, Section 6 considers the time discretization of (1) in the special case $\vec{F} \equiv \vec{0}$ of plain fractional diffusion using the discontinuous Galerkin method, proving a fully-discrete stability result in Theorem 6.1.

2 Preliminaries

This brief section introduces notations and gathers together results from the literature that we will use in our subsequent analysis. Denote the fractional integral operator of order $\mu > 0$ by

$$(\mathcal{I}^\mu \phi)(t) = \int_0^t \omega_\mu(t-s) \phi(s) ds \quad \text{for } t > 0, \text{ where } \omega_\mu(t) = \frac{t^{\mu-1}}{\Gamma(\mu)}, \quad (9)$$

with $\mathcal{I}^0 \phi = \phi$, and observe that $(\mathcal{I}^1 \phi)(t) = \int_0^t \phi(s) ds$. If we denote the Laplace transform of ϕ by $\hat{\phi}(z) = \int_0^\infty e^{-zt} \phi(t) dt$ then $(\widehat{\mathcal{I}^\mu \phi})(z) = \hat{\omega}_\mu(z) \hat{\phi}(z)$ and $\hat{\omega}_\mu(z) = z^{-\mu}$. Hence, assuming ϕ is real-valued with (say) compact support in $[0, \infty)$, we find that

$$\int_0^t \langle \phi, \mathcal{I}^\mu \phi \rangle ds = \frac{\pi\mu/2}{\pi} \int_0^\infty y^{-\mu} \|\hat{\phi}(iy)\|^2 dy \geq 0 \quad \text{if } 0 < \mu < 1. \quad (10)$$

Our analysis relies on properties of \mathcal{I}^μ stated in the next three lemmas.

Lemma 2.1. *If $0 \leq \mu \leq \nu \leq 1$, then for $t > 0$ and $\phi \in L_2((0, t), L_2(\Omega))$,*

$$\int_0^t \|(\mathcal{I}^\nu \phi)(s)\|^2 ds \leq 2t^{2(\nu-\mu)} \int_0^t \|(\mathcal{I}^\mu \phi)(s)\|^2 ds.$$

Proof. See Le et al. [10, Lemma 3.1]. □

Lemma 2.2. *If $0 < \mu \leq 1$, then for $t > 0$ and $\phi \in L_2((0, t), L_2(\Omega))$,*

$$\int_0^t \|(\mathcal{I}^\mu \phi)(s)\|^2 ds \leq 2 \int_0^t \omega_\mu(t-s) \int_0^s \langle \phi(q), (\mathcal{I}^\mu \phi)(q) \rangle dq.$$

Proof. See McLean et al. [16, Lemma 2.2]. \square

Lemma 2.3. *Let $0 \leq \mu < \nu \leq 1$. If $\phi : [0, T] \rightarrow L_2(\Omega)$ is continuous with $\phi(0) = 0$, and if its restriction to $(0, T]$ is differentiable with $\|\phi'(t)\| \leq Ct^{-\mu}$ for $0 < t \leq T$, then*

$$\|\phi(t)\|^2 \leq 2\omega_{2-\nu}(t) \int_0^t \langle \phi'(s), (\mathcal{I}^\nu \phi')(s) \rangle ds.$$

Proof. See McLean et al. [16, Lemma 2.3]. \square

We will also require the following fractional Gronwall inequality involving the Mittag-Leffler function $E_\mu(z) = \sum_{n=0}^{\infty} z^n / \Gamma(1 + n\mu)$.

Lemma 2.4. *Let $\mu > 0$ and assume that a and b are non-negative and non-decreasing functions on the interval $[0, T]$. If $y : [0, T] \rightarrow \mathbb{R}$ is an integrable function satisfying*

$$0 \leq y(t) \leq a(t) + b(t) \int_0^t \omega_\mu(t-s)y(s) ds \quad \text{for } 0 \leq t \leq T,$$

then

$$y(t) \leq a(t)E_\mu(b(t)t^\mu) \quad \text{for } 0 \leq t \leq T.$$

Proof. See Dixon and McKee [7, Theorem 3.1] and Ye, Gao and Ding [22, Corollary 2]. \square

We recall the definition of the linear operator $\vec{\mathcal{B}}_1$ in (5), and introduce two other linear operators \mathcal{M} and $\vec{\mathcal{B}}_2$, defined by

$$(\mathcal{M}\phi)(t) = t\phi(t) \quad \text{and} \quad \vec{\mathcal{B}}_2\phi = (\mathcal{M}\vec{\mathcal{B}}_1\phi)'.$$

With the help of the identity

$$\mathcal{M}\mathcal{I}^\alpha - \mathcal{I}^\alpha\mathcal{M} = \alpha\mathcal{I}^{\alpha+1} \tag{11}$$

and using our assumption that \vec{F} and \vec{F}' are bounded, one can show the following technical estimates.

Lemma 2.5. *Let $\mu > 0$. If $\phi : [0, T] \rightarrow L_2(\Omega)$ is continuous, and if its restriction to $(0, T]$ is differentiable with $\|\phi'(t)\| \leq Ct^{\mu-1}$ for $0 < t \leq T$, then*

$$\int_0^t \|\vec{\mathcal{B}}_1\phi\|^2 ds \leq C \int_0^t \|\mathcal{I}^\alpha\phi\|^2 ds$$

and

$$\int_0^t \|\vec{\mathcal{B}}_2\phi\|^2 ds \leq C \int_0^t (\|\mathcal{I}^\alpha(\mathcal{M}\phi)'\|^2 + \|\mathcal{I}^\alpha\phi\|^2) ds.$$

Proof. The estimate for $\vec{\mathcal{B}}_1$ was proved by Le et al. [10, Lemma 4.1], who also proved the bound for $\vec{\mathcal{B}}_2$ (denoted there by B_3) but with the extra term $C \int_0^t \|\mathcal{I}^\alpha\mathcal{M}\phi\|^2 ds$. However, we may omit this extra term because it is bounded by $Ct^2 \int_0^t \|\mathcal{I}^\alpha\phi\|^2 ds$, as one sees from Lemma 2.1 and (11). \square

3 Stability analysis

To simplify the error analysis of Section 5 we will include an additional term on the right-hand side of (6) and study the stability of $u_{\mathbb{X}} : [0, T] \rightarrow \mathbb{X}$ satisfying

$$\langle u_{\mathbb{X}}, \chi \rangle + \langle \kappa \mathcal{I}^\alpha \nabla u_{\mathbb{X}}, \nabla \chi \rangle - \langle \vec{\mathcal{B}}_1 u_{\mathbb{X}}, \nabla \chi \rangle = \langle f_1, \chi \rangle + \langle \vec{f}_2, \nabla \chi \rangle \quad \text{for } \chi \in \mathbb{X}, \quad (12)$$

with $u_{\mathbb{X}}(0) = u_{0\mathbb{X}}$; the semidiscrete Galerkin solution is then given by the special case $f_1 = f_{\mathbb{X}}$ and $\vec{f}_2 = \vec{0}$. Our goal is to bound $\|u_{\mathbb{X}}(t)\|$ pointwise in t , and for this purpose our overall strategy is to apply Lemma 2.3 with $\phi = \mathcal{M}u_{\mathbb{X}}$, which in turn requires an estimate for $\int_0^t \langle \mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})', (\mathcal{M}u_{\mathbb{X}})' \rangle ds$. The technical details are worked out in Lemmas 3.1 and 3.2 below, and the stability estimate itself is then obtained in Lemma 3.3 for (12), and in Theorem 3.4 for the original semidiscrete Galerkin problem (6).

Our method of proof begins by multiplying both sides of (12) by t and using the identity (11) to obtain

$$\langle \mathcal{M}u_{\mathbb{X}}, \chi \rangle + \langle \kappa \mathcal{I}^\alpha (\mathcal{M} \nabla u_{\mathbb{X}}) + \alpha \kappa \mathcal{I}^{\alpha+1} \nabla u_{\mathbb{X}} - \mathcal{M} \vec{\mathcal{B}}_1 u_{\mathbb{X}}, \nabla \chi \rangle = \langle \mathcal{M}f_1, \chi \rangle + \langle \mathcal{M} \vec{f}_2, \nabla \chi \rangle.$$

Differentiating this equation with respect to time then yields

$$\begin{aligned} \langle (\mathcal{M}u_{\mathbb{X}})', \chi \rangle + \langle \kappa \partial_t^{1-\alpha} (\mathcal{M} \nabla u_{\mathbb{X}}) + \alpha \kappa \mathcal{I}^\alpha \nabla u_{\mathbb{X}} - \vec{\mathcal{B}}_2 u_{\mathbb{X}}, \nabla \chi \rangle \\ = \langle (\mathcal{M}f_1)', \chi \rangle + \langle (\mathcal{M} \vec{f}_2)', \nabla \chi \rangle. \end{aligned} \quad (13)$$

Lemma 3.1. *For $i \in \{0, 1\}$ and $0 \leq t \leq T$, the solution of (12) satisfies*

$$\int_0^t \left(\|\mathcal{I}^\alpha (\mathcal{M}^i u_{\mathbb{X}})\|^2 + t^\alpha \|\mathcal{I}^\alpha (\nabla \mathcal{M}^i u_{\mathbb{X}})\|^2 \right) ds \leq C t^{2(\alpha+i)} \int_0^t (\|f_1\|^2 + t^{-\alpha} \|\vec{f}_2\|^2) ds.$$

Proof. Choose $\chi = (\mathcal{I}^\alpha u_{\mathbb{X}})(t)$ in (12) so that

$$\begin{aligned} \langle u_{\mathbb{X}}, \mathcal{I}^\alpha u_{\mathbb{X}} \rangle + \kappa_{\min} \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2 &\leq \langle \vec{f}_2 + \vec{\mathcal{B}}_1 u_{\mathbb{X}}, \mathcal{I}^\alpha \nabla u_{\mathbb{X}} \rangle + \langle f_1, \mathcal{I}^\alpha u_{\mathbb{X}} \rangle \\ &\leq \kappa_{\min}^{-1} (\|\vec{f}_2\|^2 + \|\vec{\mathcal{B}}_1 u_{\mathbb{X}}\|^2) + \frac{1}{2} \kappa_{\min} \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2 + \|f_1\| \|\mathcal{I}^\alpha u_{\mathbb{X}}\|. \end{aligned}$$

After cancelling $\frac{1}{2} \kappa_{\min} \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2$, integrating in time and applying Lemma 2.5, we deduce that

$$\begin{aligned} \int_0^t \left(\langle u_{\mathbb{X}}, \mathcal{I}^\alpha u_{\mathbb{X}} \rangle + \frac{1}{2} \kappa_{\min} \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2 \right) ds \\ \leq C_0 t^{-\alpha} \int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds + \int_0^t (t^\alpha \|f_1\|^2 + \kappa_{\min}^{-1} \|\vec{f}_2\|^2) ds, \end{aligned} \quad (14)$$

where C_0 is a fixed constant depending on T , κ_{\min} and \vec{F} . Apply \mathcal{I}^α to both sides of (12) and choose $\chi = \mathcal{I}^\alpha u_{\mathbb{X}}(t)$ to obtain, for any $\eta > 0$,

$$\begin{aligned} \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 + \langle \kappa \mathcal{I}^\alpha (\mathcal{I}^\alpha \nabla u_{\mathbb{X}}), \mathcal{I}^\alpha \nabla u_{\mathbb{X}} \rangle &= \langle \mathcal{I}^\alpha (\vec{f}_2 + \vec{\mathcal{B}}_1 u_{\mathbb{X}}), \mathcal{I}^\alpha \nabla u_{\mathbb{X}} \rangle + \langle \mathcal{I}^\alpha f_1, \mathcal{I}^\alpha u_{\mathbb{X}} \rangle \\ &\leq \eta (\|\mathcal{I}^\alpha \vec{f}_2\|^2 + \|\mathcal{I}^\alpha (\vec{\mathcal{B}}_1 u_{\mathbb{X}})\|^2) + \frac{1}{2} \eta^{-1} \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2 + \frac{1}{2} \|\mathcal{I}^\alpha f_1\|^2 + \frac{1}{2} \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2. \end{aligned}$$

Simplifying, integrating in time, and noting $\int_0^t \langle \kappa \mathcal{I}^\alpha (\mathcal{I}^\alpha \nabla u_{\mathbb{X}}), \mathcal{I}^\alpha \nabla u_{\mathbb{X}} \rangle ds \geq 0$ by (10), we observe that

$$\int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds \leq \int_0^t \|\mathcal{I}^\alpha f_1\|^2 ds + 2\eta \int_0^t (\|\mathcal{I}^\alpha \vec{f}_2\|^2 + \|\mathcal{I}^\alpha (\vec{\mathcal{B}}_1 u_{\mathbb{X}})\|^2) ds + \frac{1}{\eta} \int_0^t \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2 ds.$$

By Lemmas 2.1 and 2.5,

$$\int_0^t \|\mathcal{I}^\alpha(\vec{\mathcal{B}}_1 u_{\mathbb{X}})\|^2 ds \leq 2t^{2\alpha} \int_0^t \|\vec{\mathcal{B}}_1 u_{\mathbb{X}}\|^2 ds \leq Ct^{2\alpha} \int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds,$$

and so, again with the help of Lemma 2.1,

$$\begin{aligned} \int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds &\leq 2t^{2\alpha} \int_0^t (\|f_1\|^2 + 2\eta\|\vec{f}_2\|^2) ds + C\eta t^{2\alpha} \int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds \\ &\quad + \frac{1}{\eta} \int_0^t \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2 ds. \end{aligned} \quad (15)$$

However, by (14),

$$\int_0^t \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2 ds \leq \frac{2C_0}{\kappa_{\min}} t^{-\alpha} \int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds + C \int_0^t (t^\alpha \|f_1\|^2 + \|\vec{f}_2\|^2) ds,$$

and by Lemma 2.2,

$$\int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds \leq 2 \int_0^t \omega_\alpha(t-s) \int_0^s \langle u_{\mathbb{X}}(q), (\mathcal{I}^\alpha u_{\mathbb{X}})(q) \rangle dq ds. \quad (16)$$

Inserting these two estimates in the right-hand side of (15) and choosing $\eta = 4C_0 t^{-\alpha} / \kappa_{\min}$ yields

$$\int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds \leq Ct^\alpha \int_0^t (t^\alpha \|f_1\|^2 + \|\vec{f}_2\|^2) ds + Ct^\alpha \int_0^t \omega_\alpha(t-s) \int_0^s \langle u_{\mathbb{X}}, \mathcal{I}^\alpha u_{\mathbb{X}} \rangle dq ds. \quad (17)$$

Let $y(t) = \int_0^t (\langle u_{\mathbb{X}}, \mathcal{I}^\alpha u_{\mathbb{X}} \rangle + \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2) ds$, and deduce from (14) and (17) that

$$y(t) \leq C \int_0^t (t^\alpha \|f_1\|^2 + \|\vec{f}_2\|^2) ds + C \int_0^t \omega_\alpha(t-s) y(s) ds. \quad (18)$$

It follows by Lemma 2.4 that, for $0 \leq t \leq T$,

$$y(t) \leq CE_\alpha(Ct^\alpha) \int_0^t (t^\alpha \|f_1\|^2 + \|\vec{f}_2\|^2) ds \leq C \int_0^t (t^\alpha \|f_1\|^2 + \|\vec{f}_2\|^2) ds. \quad (19)$$

Together, (16) and (19) imply

$$\int_0^t (\|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 + t^\alpha \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2) ds \leq Ct^\alpha \int_0^t (t^\alpha \|f_1\|^2 + \|\vec{f}_2\|^2) ds + Ct^\alpha \int_0^t \omega_\alpha(t-s) y(s) ds,$$

and using (19) a second time gives

$$\begin{aligned} \int_0^t \omega_\alpha(t-s) y(s) ds &\leq C \left(\int_0^t \omega_\alpha(t-s) ds \right) \int_0^t (t^\alpha \|f_1\|^2 + \|\vec{f}_2\|^2) ds \\ &\leq Ct^\alpha \int_0^t (t^\alpha \|f_1\|^2 + \|\vec{f}_2\|^2) ds, \end{aligned}$$

completing the proof for the case $i = 0$.

The identity (11) implies that

$$\|(\mathcal{I}^\alpha \mathcal{M} u_{\mathbb{X}})(t)\|^2 \leq 2t^2 \|(\mathcal{I}^\alpha u_{\mathbb{X}})(t)\|^2 + 2\alpha^2 \|(\mathcal{I}^{\alpha+1} u_{\mathbb{X}})(t)\|^2$$

so, using Lemma 2.1,

$$\int_0^t \|\mathcal{I}^\alpha \mathcal{M}u_{\mathbb{X}}\|^2 ds \leq 2t^2 \int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds + 4\alpha^2 t^2 \int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds \leq 6t^2 \int_0^t \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 ds.$$

Similarly,

$$\int_0^t \|\mathcal{I}^\alpha \mathcal{M}\nabla u_{\mathbb{X}}\|^2 ds \leq 6t^2 \int_0^t \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2 ds,$$

so the case $i = 1$ follows from the already proven case $i = 0$. \square

The next lemma makes use of the identity

$$(\partial_t^{1-\alpha} \phi)(t) = (\mathcal{I}^\alpha \phi)'(t) = \phi(0)\omega_\alpha(t) + (\mathcal{I}^\alpha \phi')(t). \quad (20)$$

Lemma 3.2. *For $0 \leq t \leq T$, the solution of (12) satisfies*

$$\begin{aligned} \int_0^t \left(\langle (\mathcal{M}u_{\mathbb{X}})', \mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})' \rangle + \|\mathcal{I}^\alpha (\mathcal{M}\nabla u_{\mathbb{X}})'\|^2 \right) ds \\ \leq Ct^\alpha \int_0^t (\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2) ds + C \int_0^t (\|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2) ds. \end{aligned}$$

Proof. Rewriting (13) as

$$\langle (\mathcal{M}u_{\mathbb{X}})', \chi \rangle + \langle \kappa \partial_t^{1-\alpha} (\mathcal{M}\nabla u_{\mathbb{X}}), \nabla \chi \rangle = \langle (\mathcal{M}\vec{f}_2)' + \vec{\mathcal{B}}_2 u_{\mathbb{X}} - \alpha \kappa \mathcal{I}^\alpha \nabla u_{\mathbb{X}}, \nabla \chi \rangle + \langle (\mathcal{M}f_1)', \chi \rangle, \quad (21)$$

we note that the first term on the right is bounded by

$$\frac{1}{2} \kappa_{\min} \|\nabla \chi\|^2 + \frac{3}{2} \kappa_{\min}^{-1} (\|(\mathcal{M}\vec{f}_2)'\|^2 + \|\vec{\mathcal{B}}_2 u_{\mathbb{X}}\|^2 + \alpha^2 \|\kappa \mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2).$$

Choose $\chi = \partial_t^{1-\alpha} (\mathcal{M}u_{\mathbb{X}})(t) = (\mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}}))'(t)$ and note that, since $(\mathcal{M}u_{\mathbb{X}})(0) = 0$, the identity (20) implies $\chi = \mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})'$ so

$$\begin{aligned} \langle (\mathcal{M}u_{\mathbb{X}})', \mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})' \rangle + \frac{1}{2} \kappa_{\min} \|\mathcal{I}^\alpha (\mathcal{M}\nabla u_{\mathbb{X}})'\|^2 \\ \leq \|(\mathcal{M}f_1)'\| \|\mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})'\| + C \|(\mathcal{M}\vec{f}_2)'\|^2 + C \|\vec{\mathcal{B}}_2 u_{\mathbb{X}}\|^2 + C \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2. \end{aligned}$$

Thus, by Lemma 2.5,

$$\begin{aligned} y(t) &\equiv \int_0^t \left(\langle (\mathcal{M}u_{\mathbb{X}})', \mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})' \rangle + \|\mathcal{I}^\alpha (\mathcal{M}\nabla u_{\mathbb{X}})'\|^2 \right) ds \\ &\leq C \int_0^t (t^\alpha \|(\mathcal{M}f_1)'\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2) ds \\ &\quad + C \int_0^t (\|\mathcal{I}^\alpha (\nabla u_{\mathbb{X}})\|^2 + \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2) ds + C_0 t^{-\alpha} \int_0^t \|\mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})'\|^2 ds. \end{aligned}$$

The second integral on the right is bounded by $Ct^\alpha \int_0^t (\|f_1\|^2 + t^{-\alpha} \|\vec{f}_2\|^2) ds$ via Lemma 3.1, giving

$$\begin{aligned} y(t) &\leq Ct^\alpha \int_0^t (\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2) ds + C \int_0^t (\|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2) ds \\ &\quad + C_0 t^{-\alpha} \int_0^t \|\mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})'\|^2 ds. \quad (22) \end{aligned}$$

Now apply \mathcal{I}^α to both sides of (21), again with $\chi = \partial_t^{1-\alpha}(\mathcal{M}u_{\mathbb{X}})(t) = \mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'(t)$, to conclude that

$$\begin{aligned} & \|\mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'\|^2 + \langle \kappa \mathcal{I}^\alpha(\mathcal{I}^\alpha(\mathcal{M}\nabla u_{\mathbb{X}})'), \mathcal{I}^\alpha(\mathcal{M}\nabla u_{\mathbb{X}})' \rangle \\ & \leq (\|\mathcal{I}^\alpha(\mathcal{M}\vec{f}_2)'\| + \|\mathcal{I}^\alpha(\vec{\mathcal{B}}_2 u_{\mathbb{X}})\| + \|\mathcal{I}^{2\alpha}\nabla u_{\mathbb{X}}\|) \|\mathcal{I}^\alpha(\mathcal{M}\nabla u_{\mathbb{X}})'\| \\ & \quad + \frac{1}{2}\|\mathcal{I}^\alpha(\mathcal{M}f_1)'\|^2 + \frac{1}{2}\|\mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'\|^2. \end{aligned}$$

After cancelling the last term on the right we have, for any $\eta > 0$,

$$\begin{aligned} & \|\mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'\|^2 + 2\langle \kappa \mathcal{I}^\alpha(\mathcal{I}^\alpha(\mathcal{M}\nabla u_{\mathbb{X}})'), \mathcal{I}^\alpha(\mathcal{M}\nabla u_{\mathbb{X}})' \rangle \\ & \leq 3\eta(\|\mathcal{I}^\alpha(\mathcal{M}\vec{f}_2)'\|^2 + \|\mathcal{I}^\alpha(\vec{\mathcal{B}}_2 u_{\mathbb{X}})\|^2 + \|\mathcal{I}^{2\alpha}\nabla u_{\mathbb{X}}\|^2) + \eta^{-1}\|\mathcal{I}^\alpha(\mathcal{M}\nabla u_{\mathbb{X}})'\|^2 + \|\mathcal{I}^\alpha(\mathcal{M}f_1)'\|^2. \end{aligned}$$

Since the integral over $(0, t)$ of the second term on the left is non-negative, it follows using Lemma 2.1 that

$$\begin{aligned} \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'\|^2 ds & \leq 6\eta t^{2\alpha} \int_0^t (\|(\mathcal{M}\vec{f}_2)'\|^2 + \|\vec{\mathcal{B}}_2 u_{\mathbb{X}}\|^2 + \|\mathcal{I}^\alpha\nabla u_{\mathbb{X}}\|^2) ds \\ & \quad + \eta^{-1} \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}\nabla u_{\mathbb{X}})'\|^2 ds + 2t^{2\alpha} \int_0^t \|(\mathcal{M}f_1)'\|^2 ds. \end{aligned}$$

By Lemmas 2.5 and 3.1,

$$\begin{aligned} & \int_0^t (\|\vec{\mathcal{B}}_2 u_{\mathbb{X}}\|^2 + \|\mathcal{I}^\alpha(\nabla u_{\mathbb{X}})\|^2) ds \leq Ct^\alpha \int_0^t (\|f_1\|^2 + t^{-\alpha}\|\vec{f}_2\|^2) ds \\ & \quad + C \int_0^t (\|\mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'\|^2 + \|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2) ds \\ & \leq C \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'\|^2 ds + C(t^\alpha + t^{2\alpha}) \int_0^t (\|f_1\|^2 + t^{-\alpha}\|\vec{f}_2\|^2) ds, \end{aligned}$$

and consequently,

$$\begin{aligned} & C_0 t^{-\alpha} \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'\|^2 ds \leq Ct^\alpha \int_0^t (\eta t^\alpha \|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2) ds \\ & + C\eta t^\alpha \int_0^t (\|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2) ds + C\eta t^\alpha \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'\|^2 ds + \frac{C_0 t^{-\alpha}}{\eta} \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}\nabla u_{\mathbb{X}})'\|^2 ds. \end{aligned}$$

Choosing $\eta = 2C_0 t^{-\alpha}$, we see from (22) that

$$\begin{aligned} y(t) & \leq Ct^\alpha \int_0^t (\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2) ds + C \int_0^t (\|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2) ds \\ & \quad + C \int_0^t \|\mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})'\|^2 ds. \end{aligned}$$

The desired estimate follows after applying Lemma 2.2 to bound the last integral on the right in terms of y , and then applying Lemma 2.4. \square

Lemma 3.3. *For $0 < t \leq T$, the solution of (12) satisfies*

$$\|u_{\mathbb{X}}(t)\|^2 \leq \frac{C}{t} \int_0^t (\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2) ds + \frac{C}{t^{1+\alpha}} \int_0^t (\|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2) ds.$$

Proof. Apply Lemma 2.3, with $\phi = \mathcal{M}u_{\mathbb{X}}$, followed by Lemma 3.2, to conclude that

$$\begin{aligned} t^2 \|u_{\mathbb{X}}(t)\|^2 &= \|\mathcal{M}u_{\mathbb{X}}(t)\|^2 \leq \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^t \langle \mathcal{I}^\alpha(\mathcal{M}u_{\mathbb{X}})', (\mathcal{M}u_{\mathbb{X}})' \rangle ds \\ &\leq Ct \int_0^t (\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2) ds + Ct^{1-\alpha} \int_0^t (\|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2) ds, \end{aligned}$$

and then divide by t^2 . □

Theorem 3.4. *The semidiscrete Galerkin solution, defined by (6), satisfies*

$$\|u_{\mathbb{X}}(t)\| \leq C \left(\|u_{0\mathbb{X}}\| + \int_0^t \|g(s)\| ds \right) + C \left(\frac{1}{t} \int_0^t \|sg(s)\|^2 ds \right)^{1/2}$$

for $0 < t \leq T$, where the stability constant C depends on T , Ω , κ and \vec{F} , but not on $\alpha \in (0, 1]$ or the subspace \mathbb{X} .

Proof. Apply Lemma 3.3 with $f_1 = f_{\mathbb{X}}$ and $\vec{f}_2 = \vec{0}$, noting that

$$\frac{1}{t} \int_0^t \|f_{\mathbb{X}}\|^2 ds \leq \max_{0 \leq s \leq t} \|f_{\mathbb{X}}(s)\|^2 \leq \left(\|u_{0\mathbb{X}}\| + \int_0^t \|g(s)\| ds \right)^2$$

and $(\mathcal{M}f_{\mathbb{X}})' = f_{\mathbb{X}} + \mathcal{M}g$. □

The terms in g from the above estimate can be bounded as follows. In particular, by choosing $\eta = 1$ we see that $\|u_{\mathbb{X}}(t)\| \leq C\|u_{0\mathbb{X}}\| + Ct^{1/2}\|g\|_{L_2((0,T);L_2(\Omega))}$.

Lemma 3.5. *For $0 < t \leq T$ and $0 < \eta \leq 1$,*

$$\left(\int_0^t \|g(s)\| ds \right)^2 + \frac{1}{t} \int_0^t \|sg(s)\|^2 ds \leq (1 + \eta^{-1})t^\eta \int_0^t s^{1-\eta} \|g(s)\|^2 ds.$$

Proof. Using the Cauchy–Schwarz inequality,

$$\begin{aligned} \left(\int_0^t \|g(s)\| ds \right)^2 &= \left(\int_0^t s^{-(1-\eta)/2} s^{(1-\eta)/2} \|g(s)\| ds \right)^2 \\ &\leq \int_0^t s^{\eta-1} ds \int_0^t s^{1-\eta} \|g(s)\|^2 ds = \frac{t^\eta}{\eta} \int_0^t s^{1-\eta} \|g(s)\|^2 ds, \end{aligned}$$

and furthermore,

$$\frac{1}{t} \int_0^t \|sg(s)\|^2 ds \leq \int_0^t s \|g(s)\|^2 ds \leq t^\eta \int_0^t s^{1-\eta} \|g(s)\|^2 ds.$$

□

4 Gradient bounds

A strategy similar to the one used in Section 3 will allow us to bound $\|\nabla u_{\mathbb{X}}(t)\|$ pointwise in t : we once again apply Lemma 2.3, this time with $\phi = \mathcal{M}\nabla u_{\mathbb{X}}$. The key result is stated as Lemma 4.3 for the generalized problem (12), and as Theorem 4.4 for the semidiscrete Galerkin equation (6). The proofs rely on the following estimates; cf. Lemma 2.5.

Lemma 4.1. Let $\mu > 0$. If $\phi : [0, T] \rightarrow H^1(\Omega)$ is continuous, and if its restriction to $(0, T]$ is differentiable with $\|\phi'(t)\|_{H^1(\Omega)} \leq Ct^{\mu-1}$ for $0 < t \leq T$, then

$$\int_0^t \|\nabla \cdot (\vec{\mathcal{B}}_1 \phi)\|^2 ds \leq C \int_0^t (\|\mathcal{I}^\alpha \phi\|^2 + \|\mathcal{I}^\alpha \nabla \phi\|^2) ds$$

and

$$\int_0^t \|\nabla \cdot (\vec{\mathcal{B}}_2 \phi)\|^2 ds \leq C \int_0^t (\|\mathcal{I}^\alpha \phi\|^2 + \|\mathcal{I}^\alpha (\mathcal{M}\phi)'\|^2 + \|\mathcal{I}^\alpha \nabla \phi\|^2 + \|\mathcal{I}^\alpha (\mathcal{M}\nabla \phi)'\|^2) ds.$$

Proof. Integration by parts gives $(\vec{\mathcal{B}}_1 \phi)(t) = \vec{F}(\mathcal{I}^\alpha \phi)(t) - \int_0^t \vec{F}' \mathcal{I}^\alpha \phi ds$, so

$$\nabla \cdot (\vec{\mathcal{B}}_1 \phi) = (\nabla \cdot \vec{F})(\mathcal{I}^\alpha \phi) + \vec{F} \cdot (\mathcal{I}^\alpha \nabla \phi) - \int_0^t \left((\nabla \cdot \vec{F})'(\mathcal{I}^\alpha \phi) + \vec{F}' \cdot (\mathcal{I}^\alpha \nabla \phi) \right) ds,$$

implying the first estimate. Furthermore,

$$\vec{\mathcal{B}}_2 \phi = \vec{F}'(\mathcal{I}^\alpha \mathcal{M}\phi + \alpha \mathcal{I}^{\alpha+1} \phi) + \vec{F}(\mathcal{I}^\alpha (\mathcal{M}\phi)' + \alpha \mathcal{I}^\alpha \phi) - \mathcal{I}^1(\vec{F}' \mathcal{I}^\alpha \phi) - \mathcal{M} \vec{F}' \mathcal{I}^\alpha \phi,$$

which implies the second estimate. \square

The next result builds on the estimates of Lemmas 3.1 and 3.2.

Lemma 4.2. The solution of (12) satisfies, for $0 \leq t \leq T$,

$$\begin{aligned} \int_0^t (\|(\mathcal{M}u_{\mathbb{X}})'\|^2 + \langle \kappa \mathcal{I}^\alpha (\mathcal{M}\nabla u_{\mathbb{X}})', (\mathcal{M}\nabla u_{\mathbb{X}})' \rangle) ds &\leq C \int_0^t \|u_{\mathbb{X}}\|^2 ds \\ &+ C \int_0^t \left(\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2 + \|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2 + \|\nabla \cdot \vec{f}_2\|^2 + \|(\mathcal{M}\nabla \cdot \vec{f}_2)'\|^2 \right) ds. \end{aligned}$$

Proof. Using the first Green identity, we deduce from (21) that

$$\begin{aligned} \langle (\mathcal{M}u_{\mathbb{X}})', \chi \rangle + \langle \kappa (\mathcal{I}^\alpha \mathcal{M}\nabla u_{\mathbb{X}})', \nabla \chi \rangle &= \langle (\mathcal{M}f_1)' - (\mathcal{M}\nabla \cdot \vec{f}_2)' - \nabla \cdot (\vec{\mathcal{B}}_2 u_{\mathbb{X}}), \chi \rangle \\ &\quad - \alpha \langle \kappa \mathcal{I}^\alpha \nabla u_{\mathbb{X}}, \nabla \chi \rangle, \end{aligned}$$

and from (12) that $\langle \kappa \mathcal{I}^\alpha \nabla u_{\mathbb{X}}, \nabla \chi \rangle = \langle f_1 - \nabla \cdot (\vec{\mathcal{B}}_1 u_{\mathbb{X}}) - \nabla \cdot \vec{f}_2 - u_{\mathbb{X}}, \chi \rangle$, so

$$\begin{aligned} \langle (\mathcal{M}u_{\mathbb{X}})', \chi \rangle + \langle \kappa (\mathcal{I}^\alpha \mathcal{M}\nabla u_{\mathbb{X}})', \nabla \chi \rangle &= \langle f_3 + \alpha \nabla \cdot (\vec{\mathcal{B}}_1 u_{\mathbb{X}}) - \nabla \cdot (\vec{\mathcal{B}}_2 u_{\mathbb{X}}) + \alpha u_{\mathbb{X}}, \chi \rangle \\ &\leq \frac{1}{2} \|\chi\|^2 + 2(\|f_3\|^2 + \|\nabla \cdot (\vec{\mathcal{B}}_1 u_{\mathbb{X}})\|^2 + \|\nabla \cdot (\vec{\mathcal{B}}_2 u_{\mathbb{X}})\|^2 + \|u_{\mathbb{X}}\|^2), \end{aligned}$$

where $f_3 = (\mathcal{M}f_1)' - \alpha f_1 - (\mathcal{M}\nabla \cdot \vec{f}_2)' + \alpha \nabla \cdot \vec{f}_2$. Choose $\chi = (\mathcal{M}u_{\mathbb{X}})'$, cancel the term $\frac{1}{2} \|\chi\|^2$, and integrate in time to obtain

$$\int_0^t (\|(\mathcal{M}u_{\mathbb{X}})'\|^2 + \langle \kappa \mathcal{I}^\alpha (\mathcal{M}\nabla u_{\mathbb{X}})', (\mathcal{M}\nabla u_{\mathbb{X}})' \rangle) ds \leq C J(t) + C \int_0^t \|f_3\|^2 ds \quad (23)$$

where, by Lemma 4.1,

$$J(t) = \int_0^t \left(\|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 + \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2 + \|\mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})'\|^2 + \|\mathcal{I}^\alpha (\mathcal{M}\nabla u_{\mathbb{X}})'\|^2 + \|u_{\mathbb{X}}\|^2 \right) ds.$$

If we let $y(t) = \int_0^t \langle (\mathcal{M}u_{\mathbb{X}})', \mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})' \rangle ds$ then, by Lemma 2.2,

$$\int_0^t \|\mathcal{I}^\alpha (\mathcal{M}u_{\mathbb{X}})'\|^2 ds \leq 2 \int_0^t \omega_\alpha(t-s) y(s) ds \leq 2\omega_{\alpha+1}(t) \max_{0 \leq s \leq t} y(s),$$

and so, using Lemmas 3.1 and 3.2,

$$J(t) \leq Ct^\alpha \int_0^t (\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2) ds + C \int_0^t (\|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2) ds.$$

Since

$$\int_0^t \|f_3\|^2 ds \leq 4 \int_0^t (\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2 + \|\nabla \cdot \vec{f}_2\|^2 + \|(\mathcal{M}\nabla \cdot \vec{f}_2)'\|^2) ds,$$

the desired estimate now follows from (23). \square

Lemma 4.3. For $0 < t \leq T$,

$$\begin{aligned} t^\alpha \|\nabla u_{\mathbb{X}}(t)\|^2 &\leq \frac{C}{t} \int_0^t (\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2) ds \\ &\quad + \frac{C}{t} \int_0^t (\|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2 + \|\nabla \cdot \vec{f}_2\|^2 + \|(\mathcal{M}\nabla \cdot \vec{f}_2)'\|^2) ds. \end{aligned}$$

Proof. Since $t^\alpha \|\nabla u_{\mathbb{X}}(t)\|^2 = t^{\alpha-2} \|\mathcal{M}\nabla u_{\mathbb{X}}(t)\|^2$, Lemmas 2.3 and 4.2 imply that

$$\begin{aligned} t^\alpha \|\nabla u_{\mathbb{X}}(t)\|^2 &\leq \frac{2\omega_{2-\alpha}(t)}{t^{2-\alpha}} \int_0^t \langle \mathcal{I}^\alpha(\mathcal{M}\nabla u_{\mathbb{X}})', (\mathcal{M}\nabla u_{\mathbb{X}})' \rangle ds \leq \frac{C}{t} \int_0^t \|u_{\mathbb{X}}\|^2 ds \\ &\quad + \frac{C}{t} \int_0^t (\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2 + \|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2 + \|\nabla \cdot \vec{f}_2\|^2 + \|(\mathcal{M}\nabla \cdot \vec{f}_2)'\|^2) ds, \end{aligned}$$

and it suffices to estimate $\int_0^t \|u_{\mathbb{X}}\|^2 ds$. Choose $\chi = u_{\mathbb{X}}(t)$ in (12) and use the first Green identity to deduce that

$$\begin{aligned} \|u_{\mathbb{X}}\|^2 + \langle \kappa \mathcal{I}^\alpha \nabla u_{\mathbb{X}}, \nabla u_{\mathbb{X}} \rangle &= \langle f_1 - \nabla \cdot \vec{f}_2 - \nabla \cdot \vec{\mathcal{B}}_1 u_{\mathbb{X}}, u_{\mathbb{X}} \rangle \\ &\leq \frac{1}{2} \|u_{\mathbb{X}}\|^2 + \frac{3}{2} (\|f_1\|^2 + \|\nabla \cdot \vec{f}_2\|^2 + \|\nabla \cdot \vec{\mathcal{B}}_1 u_{\mathbb{X}}\|^2). \end{aligned}$$

After cancelling $\frac{1}{2} \|u_{\mathbb{X}}\|^2$, integrating in time and using (10), we have

$$\int_0^t \|u_{\mathbb{X}}\|^2 ds \leq 3 \int_0^t (\|f_1\|^2 + \|\nabla \cdot \vec{f}_2\|^2) ds + 3 \int_0^t \|\nabla \cdot (\vec{\mathcal{B}}_1 u_{\mathbb{X}})\|^2 ds,$$

and by Lemmas 3.1 and 4.1,

$$\int_0^t \|\nabla \cdot (\vec{\mathcal{B}}_1 u_{\mathbb{X}})\|^2 ds \leq C \int_0^t (\|\mathcal{I}^\alpha u_{\mathbb{X}}\|^2 + \|\mathcal{I}^\alpha \nabla u_{\mathbb{X}}\|^2) ds \leq Ct^\alpha \int_0^t (\|f_1\|^2 + t^{-\alpha} \|\vec{f}_2\|^2) ds,$$

which completes the proof. \square

The main result for this section now follows easily; once again, the terms in g may be estimated using Lemma 3.5.

Theorem 4.4. The semidiscrete Galerkin solution, defined by (6), satisfies

$$t^{\alpha/2} \|\nabla u_{\mathbb{X}}(t)\| \leq C \left(\|u_{0\mathbb{X}}\| + \int_0^t \|g(s)\| ds \right) + C \left(\frac{1}{t} \int_0^t \|sg(s)\|^2 ds \right)^{1/2}$$

for $0 < t \leq T$, where the constant C depends on T , Ω , κ and \vec{F} , but not on $\alpha \in (0, 1]$ or the subspace \mathbb{X} .

Proof. Choose $f_1 = f_{\mathbb{X}}$ and $f_2 = 0$ in Lemma 4.3, and estimate $f_{\mathbb{X}}$ in terms of $u_{0\mathbb{X}}$ and g using the same steps as in the proof of Theorem 3.4. \square

Remark 4.5. The uniform stability estimate (8) for the semidiscrete Galerkin solution carries over to the weak solution u of the continuous problem (1), that is,

$$\|u(t)\| + t^{\alpha/2} \|\nabla u(t)\| \leq C \left(\|u_0\| + \int_0^t \|g(s)\| ds \right) + C \left(\frac{1}{t} \int_0^t \|sg(s)\|^2 ds \right)^{1/2}.$$

Essentially, it suffices to repeat the steps in an earlier stability proof [16, Theorem 4.1] using (8) as a drop-in replacement for an estimate [16, Theorem 3.3] in which the stability constant was dependent on α .

Remark 4.6. By introducing a flux vector $\vec{Q}u = -\partial_t^{1-\alpha} \kappa \nabla u + \vec{F} \partial_t^{1-\alpha} u$ we can write the fractional Fokker–Planck equation (1) as a conservation law: $\partial_t u + \nabla \cdot \vec{Q}u = g$. It is then natural to consider a zero-flux boundary condition,

$$\vec{n} \cdot \vec{Q}u = 0 \quad \text{for } \vec{x} \in \partial\Omega \text{ and } 0 < t \leq T, \quad (24)$$

where \vec{n} denotes the outward unit normal to Ω . (Notice that this boundary condition is non-local in time.) In this case, the weak solution $u : (0, T] \rightarrow H^1(\Omega)$ is again characterized by (3), and hence satisfies (4), but with the test functions v now taken from the larger space $H^1(\Omega)$. We can then choose a finite dimensional subspace $\mathbb{X} \subseteq H^1(\Omega)$ and again define the Galerkin solution $u_{\mathbb{X}} : [0, T] \rightarrow \mathbb{X}$ by (6). The analysis of Section 3 goes through with no change, and in particular $u_{\mathbb{X}}$ is again stable in $L_2(\Omega)$. However, the first step in the proof of Lemma 4.2 fails because boundary terms are introduced if one integrates by parts in space, so our analysis no longer yields a bound for $t^{\alpha/2} \|\nabla u_{\mathbb{X}}(t)\|$.

5 Error estimates

We now decompose the error in the semidiscrete Galerkin solution as

$$u_{\mathbb{X}} - u = \theta_{\mathbb{X}} - \rho_{\mathbb{X}} \quad \text{where} \quad \theta_{\mathbb{X}} = u_{\mathbb{X}} - R_{\mathbb{X}}u \quad \text{and} \quad \rho_{\mathbb{X}} = u - R_{\mathbb{X}}u,$$

and where $R_{\mathbb{X}}$ denotes the Ritz projector for the (stationary) elliptic problem

$$-\nabla \cdot (\kappa \nabla v) + v = g \quad \text{in } \Omega, \text{ with } v = 0 \text{ on } \partial\Omega. \quad (25)$$

Thus, $R_{\mathbb{X}} : H_0^1(\Omega) \rightarrow \mathbb{X}$ satisfies

$$\langle \kappa \nabla R_{\mathbb{X}}v, \nabla \chi \rangle + \langle R_{\mathbb{X}}v, \chi \rangle = \langle \kappa \nabla v, \nabla \chi \rangle + \langle v, \chi \rangle \quad \text{for } v \in H_0^1(\Omega) \text{ and } \chi \in \mathbb{X}, \quad (26)$$

or in other words, $R_{\mathbb{X}} : v \mapsto v_{\mathbb{X}}$ where $v_{\mathbb{X}} \in \mathbb{X}$ is the Galerkin solution of the elliptic problem (25). Note that, by including the lower-order term v , the Ritz projector $R_{\mathbb{X}} : H^1(\Omega) \rightarrow \mathbb{X}$ would also be well-defined for the zero-flux boundary condition (24).

It follows from (4), (6) and (26) that $\theta_{\mathbb{X}} : [0, T] \rightarrow \mathbb{X}$ satisfies

$$\langle \theta_{\mathbb{X}}(t), \chi \rangle + \langle \mathcal{I}^\alpha(\kappa \nabla \theta_{\mathbb{X}}) - \vec{\mathcal{B}}_1 \theta_{\mathbb{X}}, \nabla \chi \rangle = \langle f_1, \chi \rangle + \langle \vec{f}_2, \nabla \chi \rangle \quad \text{for } \chi \in \mathbb{X}, \quad (27)$$

where

$$f_1 = (u_{0\mathbb{X}} - P_{\mathbb{X}}u_0) + (\rho_{\mathbb{X}} - \mathcal{I}^\alpha \rho_{\mathbb{X}}), \quad \vec{f}_2 = -\vec{\mathcal{B}}_1 \rho_{\mathbb{X}}, \quad (28)$$

and $P_{\mathbb{X}} : L_2(\Omega) \rightarrow \mathbb{X}$ is the orthoprojector given by $\langle P_{\mathbb{X}}v, \chi \rangle = \langle v, \chi \rangle$ for $v \in L_2(\Omega)$ and $\chi \in \mathbb{X}$. If $u_0 \in H_0^1(\Omega)$ so that $R_{\mathbb{X}}u_0$ exists, then $\langle f_1, \chi \rangle = \langle \tilde{f}_1, \chi \rangle$ where

$$\tilde{f}_1 = (u_{0\mathbb{X}} - R_{\mathbb{X}}u_0) + (\rho_{\mathbb{X}} - \rho_{\mathbb{X}}(0)) - \mathcal{I}^\alpha \rho_{\mathbb{X}}. \quad (29)$$

We estimate $\theta_{\mathbb{X}}$ in terms of $\rho_{\mathbb{X}}$ and the error in the discrete initial data $u_{0\mathbb{X}}$, as follows.

Lemma 5.1. For $0 < t \leq T$,

$$\|\theta_{\mathbb{X}}(t)\|^2 \leq C\|u_{0\mathbb{X}} - P_{\mathbb{X}}u_0\|^2 + \frac{C}{t} \int_0^t (\|\rho_X\|^2 + s^2\|\rho'_{\mathbb{X}}\|^2) ds.$$

Proof. Noting that (27) has the same form as (12), with $\theta_{\mathbb{X}}$ playing the role of $u_{\mathbb{X}}$, we may apply Lemma 3.3 and conclude that

$$\|\theta_{\mathbb{X}}(t)\|^2 \leq \frac{C}{t} \int_0^t (\|f_1\|^2 + s^2\|f'_1\|^2) ds + \frac{C}{t^{1+\alpha}} \int_0^t (\|\vec{f}_2\|^2 + s^2\|\vec{f}'_2\|^2) ds.$$

Since $f'_1 = \rho'_{\mathbb{X}} - \partial_t^{1-\alpha}\rho_{\mathbb{X}}$, we find with the help of Lemma 2.1 that

$$\frac{C}{t} \int_0^t (\|f_1\|^2 + s^2\|f'_1\|^2) ds \leq C\|u_{0\mathbb{X}} - P_{\mathbb{X}}u_0\|^2 + \frac{C}{t} \int_0^t (\|\rho_X\|^2 + s^2\|\rho'_{\mathbb{X}}\|^2 + s^2\|\partial_s^{1-\alpha}\rho_{\mathbb{X}}\|^2) ds.$$

Using the identity (11) and noting that $(\mathcal{M}\rho_{\mathbb{X}})(0) = 0$,

$$\begin{aligned} s\partial_s^{1-\alpha}\rho_{\mathbb{X}} &= s\partial_s\mathcal{I}^\alpha\rho_{\mathbb{X}} = \partial_s(\mathcal{M}\mathcal{I}^\alpha\rho_{\mathbb{X}}) - \mathcal{I}^\alpha\rho_{\mathbb{X}} = \partial_s(\mathcal{I}^\alpha\mathcal{M}\rho_{\mathbb{X}} + \alpha\mathcal{I}^{\alpha+1}\rho_{\mathbb{X}}) - \mathcal{I}^\alpha\rho_{\mathbb{X}} \\ &= \mathcal{I}^\alpha(\mathcal{M}\rho_{\mathbb{X}})' + (\alpha - 1)\mathcal{I}^\alpha\rho_{\mathbb{X}} = \mathcal{I}^\alpha(\mathcal{M}\rho'_{\mathbb{X}} + \alpha\rho_{\mathbb{X}}), \end{aligned}$$

so by Lemma 2.1,

$$\int_0^t s^2\|\partial_s^{1-\alpha}\rho_{\mathbb{X}}\|^2 ds \leq 2t^{2\alpha} \int_0^t \|s\rho'_{\mathbb{X}} + \alpha\rho_{\mathbb{X}}\|^2 ds \leq 4t^{2\alpha} \int_0^t (\|\rho_{\mathbb{X}}\|^2 + s^2\|\rho'_{\mathbb{X}}\|^2) ds,$$

and hence

$$\frac{C}{t} \int_0^t (\|f_1\|^2 + s^2\|f'_1\|^2) ds \leq C\|u_{0\mathbb{X}} - P_{\mathbb{X}}u_0\|^2 + \frac{C}{t} \int_0^t (\|\rho_X\|^2 + s^2\|\rho'_{\mathbb{X}}\|^2) ds. \quad (30)$$

Recalling (5), we have $f'_2(t) = -(\vec{F}\partial_t^{1-\alpha}\rho_{\mathbb{X}})(t)$ and therefore by Lemma 2.5,

$$\frac{C}{t^{1+\alpha}} \int_0^t (\|\vec{f}_2\|^2 + s^2\|\vec{f}'_2\|^2) ds \leq \frac{C}{t^{1+\alpha}} \int_0^t \|\mathcal{I}^\alpha\rho_{\mathbb{X}}\|^2 ds + \frac{C}{t^{1+\alpha}} \int_0^t s^2\|\partial_s^{1-\alpha}\rho_{\mathbb{X}}\|^2 ds,$$

which is bounded by the second term on the right-hand side of (30). \square

Two similar bounds hold for $\nabla\theta_{\mathbb{X}}$, but now involving also $\nabla\rho_{\mathbb{X}}$ and $\nabla\rho'_{\mathbb{X}}$.

Lemma 5.2. For $0 < t \leq T$,

$$\begin{aligned} t^\alpha\|\nabla\theta_{\mathbb{X}}(t)\|^2 &\leq C\|u_{0\mathbb{X}} - P_{\mathbb{X}}u_0\|^2 + \frac{C}{t} \int_0^t (\|\rho_X\|^2 + s^2\|\rho'_{\mathbb{X}}\|^2) ds \\ &\quad + Ct^{2\alpha-1} \int_0^t (\|\nabla\rho_{\mathbb{X}}\|^2 + s^2\|\nabla\rho'_{\mathbb{X}}\|^2) ds. \end{aligned}$$

If $u_0 \in H_0^1(\Omega)$, then we also have the alternative bound

$$\begin{aligned} t^\alpha\|\nabla\theta_{\mathbb{X}}(t)\|^2 &\leq C\|u_{0\mathbb{X}} - R_{\mathbb{X}}u_0\|^2 + \frac{C}{t} \int_0^t (\|\rho_X - \rho_X(0)\|^2 + s^2\|\rho'_{\mathbb{X}}\|^2) ds \\ &\quad + Ct^{2\alpha-1} \int_0^t (\|\rho_X\|^2 + \|\nabla\rho_{\mathbb{X}}\|^2 + s^2\|\nabla\rho'_{\mathbb{X}}\|^2) ds. \end{aligned}$$

Proof. With f_1 and \vec{f}_2 given by (28), we apply Lemma 4.3 to (27) and bound $t^\alpha \|\nabla \theta_{\mathbb{X}}(t)\|^2$ by

$$\frac{C}{t} \int_0^t \left(\|f_1\|^2 + \|(\mathcal{M}f_1)'\|^2 + \|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2 + \|\nabla \cdot \vec{f}_2\|^2 + \|(\mathcal{M}\nabla \cdot \vec{f}_2)'\|^2 \right) ds.$$

The terms in f_1 can be bounded as in (30), and since $\vec{f}_2 = -\vec{\mathcal{B}}_1 \rho_{\mathbb{X}}$ and $(\mathcal{M}\vec{f}_2)' = -\vec{\mathcal{B}}_2 \rho_{\mathbb{X}}$ we see from Lemma 2.5 followed by Lemma 4.1 and then Lemma 2.1 that

$$\begin{aligned} & \int_0^t \left(\|\vec{f}_2\|^2 + \|(\mathcal{M}\vec{f}_2)'\|^2 + \|\nabla \cdot \vec{f}_2\|^2 + \|(\mathcal{M}\nabla \cdot \vec{f}_2)'\|^2 \right) ds \\ & \leq C \int_0^t \left(\|\mathcal{I}^\alpha \rho_{\mathbb{X}}\|^2 + \|\mathcal{I}^\alpha (\mathcal{M}\rho_{\mathbb{X}})'\|^2 + \|\mathcal{I}^\alpha \nabla \rho_{\mathbb{X}}\|^2 + \|\mathcal{I}^\alpha (\mathcal{M}\nabla \rho_{\mathbb{X}})'\|^2 \right) ds \\ & \leq Ct^{2\alpha} \int_0^t \left(\|\rho_{\mathbb{X}}\|^2 + s^2 \|\rho_{\mathbb{X}}'\|^2 + \|\nabla \rho_{\mathbb{X}}\|^2 + s^2 \|\nabla \rho_{\mathbb{X}}'\|^2 \right) ds, \end{aligned}$$

which completes the proof of the first bound. If $u_0 \in H_0^1(\Omega)$ then we can replace f_1 with \tilde{f}_1 from (29), and since $\tilde{f}_1' = f_1'$ the second bound follows easily via the arguments leading to (30) (with $R_{\mathbb{X}}$ replacing $P_{\mathbb{X}}$). \square

To obtain more explicit error bounds we will use the regularity properties stated in the next theorem. The seminorm $|\cdot|_r$ and norm $\|\cdot\|_r$ in the (fractional-order) Sobolev space $\dot{H}^r(\Omega)$ is defined in the usual way [20] via the Dirichlet eigenfunctions of the Laplacian on Ω , and this spatial domain is assumed convex to ensure H^2 -regularity for the elliptic problem. The proof relies on results [15, Lemma 2, Theorems 11–13] involving constants that blow up as $\alpha \rightarrow 1$. Nevertheless, the estimates (31)–(33) hold in the limiting case $\alpha = 1$, when the problem reduces to the classical Fokker–Planck PDE; see Thomée [20, Lemmas 3.2 and 4.4] for a proof if $M = 0$.

Theorem 5.3. *Assume that Ω is convex, $0 < \alpha < 1$, $0 \leq r \leq 2$ and $\eta > 0$. If $u_0 \in \dot{H}^r(\Omega)$ and if $g : (0, T] \rightarrow L_2(\Omega)$ is continuously differentiable with $\|g(t)\| + t\|g'(t)\| \leq Mt^{\eta-1}$, then the weak solution of (1) satisfies, for $0 < t \leq T$,*

$$\|u(t)\|_1 \leq C_{\alpha, \eta} (\|u_0\|_r t^{-\alpha(1-r)/2} + Mt^{\eta-\alpha/2}) \quad \text{if } r \leq 1, \quad (31)$$

and

$$t^{-\alpha/2} \|u(t) - u_0\|_1 \leq C_{\alpha, \eta} (\|u_0\|_r t^{-\alpha(2-r)/2} + Mt^{\eta-\alpha}) \quad \text{if } r \geq 1, \quad (32)$$

and

$$t^{1-\alpha/2} \|u'(t)\|_1 + \|u(t)\|_2 + t\|u'(t)\|_2 \leq C_{\alpha, \eta} (\|u_0\|_r t^{-\alpha(2-r)/2} + Mt^{\eta-\alpha}). \quad (33)$$

Proof. We showed [15, Theorem 11] that

$$\|u(t)\|_1 \leq C_{\alpha, \eta} (\|u_0\| t^{-\alpha/2} + Mt^{\eta-\alpha/2})$$

and [15, Theorem 12] that

$$\|u(t) - u_0\|_1 + t\|u'(t)\|_1 \leq C_{\alpha, \eta} (\|u_0\|_1 + Mt^{\eta-\alpha/2}).$$

Hence, $\|u(t)\|_1 \leq C(\|u_0\|_1 + Mt^{\eta-\alpha/2})$ and (31) follows by interpolation. The estimates (32) and (33) were proved already [15, Theorems 12 and 13]. \square

Now consider the concrete example in which $\mathbb{X} = S_h$ is the usual continuous piecewise-linear finite element space for a triangulation of $\Omega \subseteq \mathbb{R}^d$ with maximum element diameter h , and use a subscript h instead of \mathbb{X} , writing u_h , θ_h , ρ_h etc. The error in the Ritz projection satisfies

$$\|\rho_h(t)\| + h\|\nabla \rho_h(t)\| \leq Ch^r |u(t)|_r \quad \text{for } r \in \{1, 2\}, \quad (34)$$

allowing us to prove the following error bounds for u_h and ∇u_h . Notice that if $0 < \alpha < 1/2$, then the restriction $\alpha(2-r) < 1$ is satisfied for all $r \in [0, 2]$, but if $1/2 \leq \alpha < 1$ (and hence $0 \leq 2 - \alpha^{-1} < 1$) then we are limited to $r \in (2 - \alpha^{-1}, 2]$.

Theorem 5.4. *Let $0 \leq r \leq 2$ with $\alpha(2-r) < 1$, and assume that the assumptions of Theorem 5.3 are satisfied with $\eta \geq \alpha r/2$. Then, the semidiscrete finite element solution u_h satisfies the error bound*

$$\|u_h(t) - u(t)\| \leq C\|u_{0h} - P_h u_0\| + C_\alpha h^2 \frac{t^{-\alpha(2-r)/2}}{\sqrt{1-\alpha(2-r)}} (\|u_0\|_r + M),$$

and the gradient of u_h satisfies

$$\|\nabla u_h(t) - \nabla u(t)\| \leq C t^{-\alpha/2} \|u_{0h} - Q_{r,h} u_0\| + C_\alpha h \frac{t^{-\alpha(2-r)/2}}{\sqrt{1-\alpha(2-r)}} (\|u_0\|_r + M),$$

where $Q_{r,h}$ is either P_h if $r \leq 1$, or else R_h if $r \geq 1$.

Proof. For brevity, let $K_r = (\|u_0\|_r + M)^2$. Using (34) followed by (33), we have

$$\|\rho_h(s)\|^2 + s^2 \|\rho'_h(s)\|^2 \leq C h^4 (|u(s)|_2^2 + s^2 |u'(s)|_2^2) \leq C_\alpha K_r h^4 s^{-\alpha(2-r)},$$

so, because of the assumption $\alpha(2-r) < 1$,

$$\frac{1}{t} \int_0^t (\|\rho_h(s)\|^2 + s^2 \|\rho'_h(s)\|^2) ds \leq C_\alpha K_r h^4 \frac{t^{-\alpha(2-r)}}{1-\alpha(2-r)}.$$

Since $\|u_h - u\| \leq \|\theta_h\| + \|\rho_h\|$ and $\|\rho_h(t)\|^2 \leq C_\alpha K_r t^{-\alpha(2-r)} h^4$, the error bound for u_h follows by Lemma 5.1.

To estimate the error in ∇u_h , we apply (34) and (31) to obtain

$$\|\rho_h(s)\|^2 + s^2 \|\rho'_h(s)\|^2 \leq C h^2 (|u(s)|_1^2 + s^2 |u'(s)|_1^2) \leq C_\alpha K_r h^2 s^{-\alpha(1-r)} \quad \text{if } r \leq 1,$$

and (33) to obtain

$$\|\nabla \rho_h(s)\|^2 + s^2 \|\nabla \rho'_h(s)\|^2 \leq C h^2 (|u(s)|_2^2 + s^2 |u'(s)|_2^2) \leq C_\alpha K_r h^2 s^{-\alpha(2-r)},$$

so

$$\frac{1}{t} \int_0^t (\|\rho_h(s)\|^2 + s^2 \|\rho'_h(s)\|^2) ds \leq t^\alpha C_\alpha K_r h^2 \frac{t^{-\alpha(2-r)}}{1-\alpha(1-r)} \quad \text{if } r \leq 1,$$

and

$$t^{2\alpha-1} \int_0^t (\|\nabla \rho_h\|^2 + s^2 \|\nabla \rho'_h\|^2) ds \leq \frac{C_\alpha K_r t^{\alpha r} h^2}{1-\alpha(2-r)} = t^{2\alpha} C_\alpha K_r h^2 \frac{t^{-\alpha(2-r)}}{1-\alpha(2-r)}. \quad (35)$$

Since $\|\nabla u_h(t) - \nabla u(t)\| \leq \|\nabla \theta_h(t)\| + \|\nabla \rho_h(t)\|$, the first estimate of Lemma 5.2 implies that the error bound for ∇u_h holds for the case $r \leq 1$.

If $r \geq 1$, then we see using (31)–(34) that

$$\|\rho_h(s) - \rho_h(0)\|^2 + s^2 \|\rho'_h(s)\|^2 \leq C h^2 (\|u(s) - u(0)\|_1^2 + s^2 \|u'(s)\|_1^2) \leq s^\alpha C_\alpha K_r h^2 s^{-\alpha(2-r)}$$

and $\|\rho_h(s)\|^2 \leq C h^2 \|u(s)\|_1^2 \leq C_\alpha K_1 h^2 \leq C_\alpha K_r h^2$, so

$$\frac{1}{t} \int_0^t (\|\rho_h - \rho_h(0)\|^2 + s^2 \|\rho'_h(s)\|^2) ds + t^{2\alpha-1} \int_0^t \|\rho_h\|^2 ds \leq t^\alpha C_\alpha K_r h^2 \frac{t^{-\alpha(2-r)}}{1-\alpha(2-r)}.$$

Hence, using the second estimate of Lemma 5.2 and (35), the error bound for ∇u_h follows also for the case $r \geq 1$. \square

Remark 5.5. *If $r = 2$ then by choosing $u_{0h} = R_h u_0$ we obtain an error bound that is uniform in time:*

$$\|u_h(t) - u(t)\| + h \|\nabla u_h(t) - \nabla u(t)\| \leq C_\alpha h^2 (\|u_0\|_2 + M) \quad \text{for } 0 < t \leq T.$$

Remark 5.6. *As a consequence of Remark 4.6, if the zero-flux boundary condition (24) is imposed then the proof of the error bound for u_h in Theorem 5.4 remains valid, but not that of the error bound for ∇u_h .*

6 Discontinuous Galerkin time stepping when $\vec{F} \equiv \vec{0}$

We briefly consider a fully-discrete scheme for the fractional diffusion equation, that is, for the problem (1) in the case $\vec{F} \equiv \vec{0}$. For time levels $0 = t_0 < t_1 < \dots < t_N = T$ we denote the n th time interval by $I_n = (t_{n-1}, t_n)$ and the n th step size by $k_n = t_n - t_{n-1}$. We choose an integer $p_n \geq 0$ for each time interval I_n , and define the vector space \mathcal{W} consisting of all functions $X : \bigcup_{n=1}^N I_n \rightarrow \mathbb{X}$ such that the restriction $X|_{I_n}$ is a polynomial in t of degree at most p_n with coefficients in \mathbb{X} . For any $X \in \mathcal{W}$, write

$$X_+^n = \lim_{\epsilon \downarrow 0} X(t_n + \epsilon), \quad X_-^n = \lim_{\epsilon \downarrow 0} X(t_n - \epsilon), \quad \llbracket X \rrbracket^n = X_+^n - X_-^n,$$

then the discontinuous Galerkin (DG) solution $U \in \mathcal{W}$ is defined by requiring that [18]

$$\langle \llbracket U \rrbracket^n, X_+^{n-1} \rangle + \int_{I_n} \langle \partial_t U, X \rangle dt + \int_{I_n} \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla X \rangle dt = \int_{I_n} \langle g, X \rangle dt \quad (36)$$

for $X \in \mathcal{W}$ and $1 \leq n \leq N$, with $U_-^0 = u_{0\mathbb{X}}$ (so that $\llbracket U \rrbracket^0 = U_+^0 - u_{0\mathbb{X}}$). To state a stability estimate for this scheme, let C_Ω denote the constant arising in the Poincaré inequality for Ω ,

$$\|v\|^2 \leq C_\Omega \|\nabla v\|^2 \quad \text{for } v \in H_0^1(\Omega), \quad (37)$$

and define $\Psi : (0, 1] \rightarrow \mathbb{R}$ by

$$\Psi(\alpha) = \frac{1}{\pi^{1-\alpha}} \frac{(2-\alpha)^{2-\alpha}}{(1-\alpha)^{1-\alpha}} \frac{1}{\sin(\frac{1}{2}\pi\alpha)} \quad \text{for } 0 < \alpha < 1.$$

Notice that $\Psi(1) = \lim_{\alpha \rightarrow 1} \Psi(\alpha) = 1$ but $\Psi(\alpha) \sim 8\pi^{-2}\alpha^{-1}$ blows up as $\alpha \rightarrow 0$. We will use the inequality [13, Theorem A.1]

$$\int_0^T \langle \partial_t^{1-\alpha} v, v \rangle dt \geq \frac{T^{1-\alpha}}{\Psi(\alpha)} \int_0^T \|v\|^2 dt. \quad (38)$$

Theorem 6.1. *If $0 < \alpha \leq 1$, then the DG solution of the fractional diffusion problem satisfies*

$$\|U_-^n\|^2 + \sum_{j=1}^{n-1} \|\llbracket U \rrbracket^j\|^2 + \int_0^{t_n} \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla U \rangle dt \leq \|u_{0\mathbb{X}}\|^2 + \frac{C_\Omega \Psi(\alpha)}{\kappa_{\min} t_n^{1-\alpha}} \int_0^{t_n} \|g(t)\|^2 dt$$

for $1 \leq n \leq N$.

Proof. Let $B(U, X)$ denote the bilinear form

$$\langle U_+^0, X_+^0 \rangle + \sum_{n=1}^{N-1} \langle \llbracket U \rrbracket^n, X_+^n \rangle + \sum_{n=1}^N \int_{I_n} \langle \partial_t U, X \rangle dt + \int_0^T \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla X \rangle dt,$$

and observe that the time-stepping equations (36) are equivalent to

$$B(U, X) = \langle u_{0\mathbb{X}}, X_+^0 \rangle + \int_0^T \langle g, X \rangle dt \quad \text{for } X \in \mathcal{W}.$$

Taking $X = U$, we find by arguing as in the proof of Mustapha [18, Theorem 1] that

$$B(U, U) = \frac{1}{2} \left(\|U_+^0\|^2 + \|U_-^N\|^2 + \sum_{n=1}^{N-1} \|\llbracket U \rrbracket^n\|^2 \right) + \int_0^T \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla U \rangle dt,$$

and so

$$\|U_+^0\|^2 + \|U_-^N\|^2 + \sum_{n=1}^{N-1} \|\llbracket U \rrbracket^n\|^2 + 2 \int_0^T \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla U \rangle dt = 2 \langle u_{0\mathbb{X}}, U_+^0 \rangle + 2 \int_0^T \langle g, U \rangle dt.$$

For any constant $M > 0$,

$$2 \langle u_{0\mathbb{X}}, U_+^0 \rangle + 2 \int_0^T \langle g, U \rangle dt \leq \|U_+^0\|^2 + \|u_{0\mathbb{X}}\|^2 + M \int_0^T \|g\|^2 dt + \frac{1}{M} \int_0^T \|U\|^2 dt,$$

and using (37) and (38),

$$\frac{1}{M} \int_0^T \|U\|^2 dt \leq \frac{C_\Omega}{M} \int_0^T \|\nabla U\|^2 dt \leq \frac{C_\Omega \Psi(\alpha)}{M \kappa_{\min} T^{1-\alpha}} \int_0^T \langle \partial_t^{1-\alpha} \kappa \nabla U, \nabla U \rangle dt.$$

Choosing $M = C_\Omega \Psi(\alpha) / (\kappa_{\min} T^{1-\alpha})$ implies the estimate in the case $n = N$, which completes the proof since $T = t_N$. \square

Remark 6.2. If $p_n = 0$ then we have $\|U(t)\| \leq \|U_-^n\|$ for $t \in I_N$, and likewise if $p_n = 1$ then $\|U(t)\| \leq \max(\|U_-^n\|, \|U_+^{n-1}\|) \leq \|U_-^n\| + \|U_-^{n-1}\| + \|\llbracket U \rrbracket^{n-1}\|$ for $t \in I_n$. Thus, for the piecewise constant [14] and piecewise linear [19] DG schemes we can prove stability in $L_\infty(L_2)$, uniformly for α bounded away from zero.

Remark 6.3. For the solution u of the continuous fractional diffusion problem we have the analogous stability property

$$\|u(t)\|^2 + \int_0^t \langle \partial_s^{1-\alpha} \kappa \nabla u, \nabla u \rangle ds \leq \|u_0\|^2 + \frac{C_\Omega \Psi(\alpha)}{\kappa_{\min} t^{1-\alpha}} \int_0^t \|g(s)\|^2 ds \quad \text{for } 0 \leq t \leq T.$$

The proof follows the same lines as above, except that now

$$\int_0^T \langle \partial_t u, u \rangle dt + \int_0^T \langle \partial_t^{1-\alpha} \kappa \nabla u, \nabla u \rangle dt = \int_0^T \langle g, u \rangle dt$$

and

$$\int_0^T \langle \partial_t u, u \rangle dt = \frac{1}{2} (\|u(T)\|^2 - \|u_0\|^2).$$

Remark 6.4. Le et al. [11] proved stability and convergence of the DG scheme with general \vec{F} , but only for the lowest-order ($p_n = 0$) case and with no spatial discretization. Although the constants are bounded as $\alpha \rightarrow 1$, they blow up as $\alpha \rightarrow 1/2$ and thus the fractional exponent is restricted to the range $1/2 < \alpha \leq 1$. Huang et al. [6] proved similar results for a slightly modified scheme.

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