

Finite element method for singularly perturbed problems with two parameters on a Bakhvalov-type mesh in 2D *

Jin Zhang^{a,*}, Yanhui Lv^{a,1}

^a*School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China*

Abstract

For a singularly perturbed elliptic model problem with two small parameters, we analyze finite element methods of any order on a Bakhvalov-type mesh. For convergence analysis, we construct a new interpolation by using the characteristics of layers. Besides, a more subtle analysis of the mesh scale near the exponential layer is carried out. Based on the interpolation and new analysis of the mesh scale, we prove the optimal convergence order.

Keywords: Singular perturbation, Convection–diffusion equation, Finite element method, Bakhvalov-type mesh, Two parameters

1. Introduction

In this paper, we reconsider the singularly perturbed elliptic problem in [12]

$$Lu := -\varepsilon_1 \Delta u + \varepsilon_2 b(x)u_x + c(x)u = f(x, y) \quad \text{in } \Omega := (0, 1) \times (0, 1), \quad (1)$$

$$u|_{\partial\Omega} = 0,$$

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*Corresponding author: jinhangalex@hotmail.com

¹Email: yanhuilv@hotmail.com

with

$$b(x) \geq \lambda > 0, \quad c(x) \geq \beta > 0 \quad \text{for } x \in [0, 1], \quad (2)$$

$$c(x) - \frac{1}{2}\varepsilon_2 b'(x) \geq \gamma > 0, \quad (3)$$

$$f(0, 0) = f(0, 1) = f(1, 1) = f(1, 0) = 0, \quad (4)$$

where b , c , and f are sufficiently smooth functions and λ , β , γ are constants. Here we only discuss the case of $0 < \varepsilon_1, \varepsilon_2 \ll 1$ (see [7]), and if you are interested in the case of $\varepsilon_2 = 0$ and $\varepsilon_2 = 1$, you can refer to [12] and its references. For the sake of analysis, we use $b(x)$ and $c(x)$ in our problem instead of $b(x, y)$ and $c(x, y)$, because that doesn't affect the properties of the true solution(see[10]), and this is also the case in the equations of [11], [12], and [4].

Moreover, the conditions (2) and (3) ensure that there exists a unique solution $u \in C^{3,\alpha_0}(\Omega)$ with $\alpha_0 \in (0, 1)$, which is characterized by exponential layers at $x = 0$ and $x = 1$, parabolic layers at $y = 0$ and $y = 1$, and corner layers at four corners of the domain. For the treatment of boundary layer, researchers usually use a class of special meshes which are very fine on the layer region of the solution. Compared to the quasi-uniform mesh, this kind of mesh can capture the change of layers better. Among those the most representative ones are Shishkin mesh [9] and Bakhvalov mesh [2], and experiments show that the convergence order of numerical solutions on Bakhvalov mesh is better.

In fact, up to now there are few articles about finite element method on Bakhvalov mesh, because directly applying the Lagrangian interpolation commonly used in numerical analysis to the Bakhvalov mesh is not workable. In [8], Roos clearly stated the difficulty of convergence analysis on Bakhvalov-type mesh in 1D, and obtained the optimal convergence order by using quasi-interpolation. Brdar and Zarin analyzed a singularly perturbed problem with two-parameter in 1D using the same method in [3]. However, Roos' method is powerless in the face of higher-order finite element methods or higher-dimensional problems. Recently, Zhang and Liu proposed a new interpolation which is much simpler to construct and analyze for the one-dimensional one-parameter prob-

lem in [14], and can be directly extended to higher-order cases. But, when the idea is applied to two-dimension, the boundary conditions need to be properly corrected to meet the homogeneous Dirichlet boundary conditions. Standard analysis is not successful for these corrections (see [13]).

In this paper, we define a new interpolation according to the characteristics of the layers for convergence analysis of optimal order. In view of the difficulties brought by the corrections on the boundary, we make use of a new estimation of the mesh scale near the Bakhvalov-type transition point and take the structure of the mesh near the boundary into account. Furthermore, we use different techniques to handle error estimations in different subdomains and then obtain the optimal convergence order on the Bakhvalov-type mesh.

The rest of this article is organized as follows. In the section 2, the prior estimation of the solution of the continuous problem is given. In the section 3, we will construct the Bakhvalov-type mesh, and give some mesh properties, and finally establish the finite element method. The new interpolation will appear in the section 4, and we will also prove some results of Lagrangian interpolation error. The convergence analysis is carried out in section 5. Finally, numerical experiments are given in section 6 to verify our conclusion.

Throughout the paper, we shall use C to denote a generic positive constant independent of ε_1 , ε_2 and N , which can take different values at different places. For any domain D of Ω , we use the standard notation for Banach spaces $L^p(D)$, Sobolev spaces $W^{k,p}(D)$, $H^k(D) = W^{k,2}(D)$. Define $\|\cdot\|_{\infty,D}$ to be $\|\cdot\|_{L^\infty(D)}$, $\|\cdot\|_D$ to be $\|\cdot\|_{L^2(D)}$, and $|\cdot|_D$ to be the seminorms of $\|\cdot\|_{H^1(D)}$; The scalar product in $L^2(D)$ is denoted with $(\cdot, \cdot)_D$. And we will drop the subscript D from the notation for simplicity when $D = \Omega$.

2. A priori estimates of solution of the continuous problem

Compared with the parabolic layer, the exponential layer changes more dramatically. So in order to describe the exponential layers at $x = 0$ and $x = 1$, we

introduce the characteristic equation as following

$$-\varepsilon_1 g^2(x) + \varepsilon_2 b(x)g(x) + c(x) = 0.$$

This equation defines two continuous functions $g_0, g_1 : [0, 1] \rightarrow \mathbb{R}$ with $g_0(x) < 0$, $g_1(x) > 0$. Let

$$\mu_0 = -\max_{0 \leq x \leq 1} g_0(x), \quad \mu_1 = \min_{0 \leq x \leq 1} g_1(x).$$

For the sake of simplicity, we take

$$\mu_0 = \frac{-\varepsilon_2 b_* + \sqrt{\varepsilon_2^2 b_*^2 + 4\varepsilon_1 \beta}}{2\varepsilon_1}, \quad \mu_1 = \frac{\varepsilon_2 \lambda + \sqrt{\varepsilon_2^2 \lambda^2 + 4\varepsilon_1 \beta}}{2\varepsilon_1},$$

with $b_* = \max_{0 \leq x \leq 1} b(x)$, which is the same as [12, (6)]. Then we give some properties of μ_0 and μ_1 (see [11]):

$$\mu_0 \leq \mu_1, \quad \max\{\mu_0^{-1}, \varepsilon_1 \mu_1\} \leq C(\varepsilon_2 + \varepsilon_1^{\frac{1}{2}}), \quad (5)$$

$$\varepsilon_2 \mu_0 \leq \lambda^{-1} \|c\|_\infty, \quad \varepsilon_2 (\varepsilon_1 \mu_1)^{-\frac{1}{2}} \leq C \varepsilon_2^{\frac{1}{2}}. \quad (6)$$

These properties will play an important role in the subsequent analysis.

In this paper we assume that

$$\mu_1^{-1} \leq \mu_0^{-1} \leq N^{-1}, \quad (7)$$

and it is worth noting that there is no such limitation in practice. By direct computations of (7), we can obtain

$$\varepsilon_1 \leq c_0 N^{-2}, \quad \varepsilon_2 \leq c_1 N^{-1}, \quad (8)$$

with $c_0 = \beta$ and $c_1 = \frac{\beta}{b_*}$.

On the basis of the prior estimation of the solution given in [11], we make the following assumption about the decomposition of the solution and the prior estimation of each component. In the subsequent analysis, k is a fixed positive integer and $k \geq 1$.

Assumption 1. *Let there be given elliptic problem (1) on the unit square $\bar{\Omega}$ satisfying conditions (2)-(4), and let $p \in (0, 1)$ and $k_0 \in (0, \frac{1}{2})$ be arbitrary.*

Assume that

$$2\|b'\|_\infty \varepsilon_2 \leq k_0(1-p)\beta.$$

Furthermore, let δ be a positive constant satisfying

$$\delta^2 \leq \frac{(1-p)\beta}{2}.$$

Then the solution u of problem (1) can be decomposed as

$$u = S + E_{10} + E_{11} + E_{20} + E_{21} + E_{31} + E_{32} + E_{33} + E_{34},$$

where for all $(x, y) \in \bar{\Omega}$ and $0 \leq i+j \leq k+1$, the regular part S satisfies

$$\left| \frac{\partial^{i+j} S}{\partial x^i \partial y^j} \right| \leq C,$$

the exponential and parabolic layer components satisfy

$$\left| \frac{\partial^{i+j} E_{10}}{\partial x^i \partial y^j} \right| \leq C \mu_0^i e^{-p\mu_0 x}, \quad (9)$$

$$\left| \frac{\partial^{i+j} E_{11}}{\partial x^i \partial y^j} \right| \leq C \mu_1^i e^{-p\mu_1(1-x)}, \quad (10)$$

$$\left| \frac{\partial^{i+j} E_{20}}{\partial x^i \partial y^j} \right| \leq C \varepsilon_1^{-\frac{j}{2}} e^{-\frac{\delta y}{\sqrt{\varepsilon_1}}},$$

$$\left| \frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j} \right| \leq C \varepsilon_1^{-\frac{j}{2}} e^{-\frac{\delta(1-y)}{\sqrt{\varepsilon_1}}},$$

while the corner layer components satisfy the following estimates

$$\left| \frac{\partial^{i+j} E_{31}}{\partial x^i \partial y^j} \right| \leq C \varepsilon_1^{-\frac{j}{2}} \mu_0^i e^{-p\mu_0 x} e^{-\frac{\delta y}{\sqrt{\varepsilon_1}}}, \quad (11)$$

$$\left| \frac{\partial^{i+j} E_{32}}{\partial x^i \partial y^j} \right| \leq C \varepsilon_1^{-\frac{j}{2}} \mu_1^i e^{-p\mu_1(1-x)} e^{-\frac{\delta y}{\sqrt{\varepsilon_1}}}, \quad (12)$$

$$\left| \frac{\partial^{i+j} E_{33}}{\partial x^i \partial y^j} \right| \leq C \varepsilon_1^{-\frac{j}{2}} \mu_1^i e^{-p\mu_1(1-x)} e^{-\frac{\delta(1-y)}{\sqrt{\varepsilon_1}}},$$

$$\left| \frac{\partial^{i+j} E_{34}}{\partial x^i \partial y^j} \right| \leq C \varepsilon_1^{-\frac{j}{2}} \mu_0^i e^{-p\mu_0 x} e^{-\frac{\delta(1-y)}{\sqrt{\varepsilon_1}}}.$$

3. Bakhvalov-type mesh and finite element method

3.1. Bakhvalov-type mesh

Let $N \in \mathbb{N}, N \geq 8$, can be divisible 4. Define

$$\sigma_{x,i} := \frac{\tau}{p\mu_j} \ln \mu_j \quad i = 0, 1, \quad \text{and} \quad \sigma_y := \frac{\tau}{\delta} \sqrt{\varepsilon_1} \ln \frac{1}{\sqrt{\varepsilon_1}},$$

where $\tau \geq k + 1$ is a user-chosen parameter and $p \in (0, 1)$ is the parameter from Assumption 1. On x -axis, we set $\sigma_{x,0}$ and $1 - \sigma_{x,1}$ as transition points, where the mesh changes from fine to coarse and viceversa. On y -axis, we set σ_y and $1 - \sigma_y$ as transition points. For technical reasons, we also assume

$$\sigma_{x,i} \leq \frac{1}{4} \quad i = 0, 1, \dots \quad \text{and} \quad \sigma_y \leq \frac{1}{4}. \quad (13)$$

Now we define a Bakhvalov-type mesh for problem (1), which is introduced in [8]. The mesh points x_i , $i = 0, 1, \dots, N$, are defined by

$$x_i = \begin{cases} \frac{\tau}{p\mu_0} \varphi_0(t_i) & i = 0, 1, \dots, \frac{N}{4}, \\ \sigma_{x,0} + 2(t_i - \frac{1}{4})(1 - \sigma_{x,0} - \sigma_{x,1}) & i = \frac{N}{4}, \frac{N}{4} + 1, \dots, \frac{3N}{4}, \\ 1 - \frac{\tau}{p\mu_1} \varphi_1(t_i) & i = \frac{3N}{4}, \frac{3N}{4} + 1, \dots, N, \end{cases} \quad (14)$$

where $t_i = \frac{i}{N}$, $i = 0, 1, \dots, N$ and

$$\varphi_0(t) = -\ln(1 - 4(1 - \mu_0^{-1})t), \quad \varphi_1(t) = -\ln(1 - 4(1 - \mu_1^{-1})(1 - t)).$$

It can be seen from (14) that the mesh is graded on $[x_0, \sigma_{x,0}]$ and $[1 - \sigma_{x,1}, x_N]$, and the mesh is uniform on $[\sigma_{x,0}, 1 - \sigma_{x,1}]$. The mesh points y_j , $j = 0, 1, \dots, N$, are defined by

$$y_j = \begin{cases} \frac{\tau}{\delta} \sqrt{\varepsilon_1} \phi_0(t_j) & j = 0, 1, \dots, \frac{N}{4}, \\ \sigma_y + 2(t_j - \frac{1}{4})(1 - 2\sigma_y) & j = \frac{N}{4}, \frac{N}{4} + 1, \dots, \frac{3N}{4}, \\ 1 - \frac{\tau}{\delta} \sqrt{\varepsilon_1} \phi_1(t_j) & j = \frac{3N}{4}, \frac{3N}{4} + 1, \dots, N, \end{cases} \quad (15)$$

where $t_j = \frac{j}{N}$, $j = 0, 1, \dots, N$ and

$$\phi_0(t) = -\ln(1 - 4(1 - \sqrt{\varepsilon_1})t), \quad \phi_1(t) = -\ln(1 - 4(1 - \sqrt{\varepsilon_1})(1 - t)).$$

Similar to the x -direction mesh layout, from (15) we could see that the mesh is graded on $[y_0, \sigma_y]$ and $[1 - \sigma_y, y_N]$, and the mesh is uniform on $[\sigma_y, 1 - \sigma_y]$. With mesh points $\{(x_i, y_j)\}$, we obtain a tensor-product rectangular mesh \mathcal{T} .

Set $h_{x,i} := x_{i+1} - x_i$ and $h_{y,j} := y_{j+1} - y_j$ are the mesh sizes in the x and y directions, respectively. Also set $\mathcal{T}_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ be any mesh rectangle in \mathcal{T} .

According to [14, Lemmas 2 and 3], we have the following three lemmas.

Lemma 1. From (14), we can get the mesh size in the x direction as follows

$$\begin{aligned}
C\mu_0^{-1}N^{-1} &\leq h_{x,0} \leq h_{x,1} \leq \cdots \leq h_{x,\frac{N}{4}-2}, \\
\frac{\tau}{4p}\mu_0^{-1} &\leq h_{x,\frac{N}{4}-2} \leq \frac{\tau}{p}\mu_0^{-1}, \\
\frac{\tau}{2p}\mu_0^{-1} &\leq h_{x,\frac{N}{4}-1} \leq \frac{4\tau}{p}N^{-1}, \\
N^{-1} &\leq h_{x,i} \leq 2N^{-1} \quad \frac{N}{4} \leq i \leq \frac{3N}{4} - 1, \\
\frac{\tau}{2p}\mu_1^{-1} &\leq h_{x,\frac{3N}{4}} \leq \frac{4\tau}{p}N^{-1}, \\
\frac{\tau}{4p}\mu_1^{-1} &\leq h_{x,\frac{3N}{4}+1} \leq \frac{\tau}{p}\mu_1^{-1}, \\
h_{x,\frac{3N}{4}+1} &\geq h_{x,\frac{3N}{4}+2} \geq \cdots \geq h_{x,N-1} \geq C\mu_1^{-1}N^{-1}, \\
1 - x_{\frac{3N}{4}+2} &\leq C\mu_1^{-1} \ln N.
\end{aligned}$$

Lemma 2. From (15), we can get the mesh size in the y direction as follows

$$\begin{aligned}
C\sqrt{\varepsilon_1}N^{-1} &\leq h_{y,0} \leq h_{y,1} \leq \cdots \leq h_{y,\frac{N}{4}-2}, \\
C_1\sqrt{\varepsilon_1} &\leq h_{y,\frac{N}{4}-2} \leq \frac{\tau}{p}\sqrt{\varepsilon_1}, \\
C_2\sqrt{\varepsilon_1} &\leq h_{y,\frac{N}{4}-1} \leq \frac{4\tau}{p}N^{-1}, \\
N^{-1} &\leq h_{y,j} \leq 2N^{-1} \quad \frac{N}{4} \leq j \leq \frac{3N}{4} - 1, \\
C_2\sqrt{\varepsilon_1} &\leq h_{y,\frac{3N}{4}} \leq \frac{4\tau}{p}N^{-1}, \\
C_1\sqrt{\varepsilon_1} &\leq h_{y,\frac{3N}{4}+1} \leq \frac{\tau}{p}\sqrt{\varepsilon_1}, \\
h_{y,\frac{3N}{4}+1} &\geq h_{y,\frac{3N}{4}+2} \geq \cdots \geq h_{y,N-1} \geq C\sqrt{\varepsilon_1}N^{-1},
\end{aligned}$$

where $C_1 = \frac{\tau}{\delta(\sqrt{c_0}+8)}$, $C_2 = \frac{\tau}{\delta(\sqrt{c_0}+4)}$.

Lemma 3. For $0 \leq i \leq \frac{N}{4} - 2$ and $0 \leq m \leq \tau$, one has

$$h_{x,i}^m e^{-p\mu_0 x_i} \leq C\mu_0^{-m}N^{-m}. \quad (16)$$

For $\frac{3N}{4} + 1 \leq i \leq N - 1$ and $0 \leq m \leq \tau$, one has

$$h_{x,i}^m e^{-p\mu_1(1-x_{i+1})} \leq C\mu_1^{-m}N^{-m}. \quad (17)$$

For $0 \leq j \leq \frac{N}{4} - 2$ and $0 \leq m \leq \tau$, one has

$$h_{y,j}^m e^{-\frac{\delta y_j}{\sqrt{\varepsilon_1}}} \leq C \varepsilon_1^{\frac{m}{2}} N^{-m}. \quad (18)$$

For $\frac{3N}{4} + 1 \leq j \leq N - 1$ and $0 \leq m \leq \tau$, one has

$$h_{y,j}^m e^{-\frac{\delta(1-y_j+1)}{\sqrt{\varepsilon_1}}} \leq C \varepsilon_1^{\frac{m}{2}} N^{-m}.$$

Also we need to re-estimate $h_{x,\frac{3N}{4}}$ for our convergence analysis.

Lemma 4. For any fixed $\eta \in (0, 1]$, one has

$$h_{x,\frac{3N}{4}} \leq C \mu_1^{\eta-1} N^{-\eta}.$$

Proof. For any fixed $\eta \in (0, 1]$, standard arguments show

$$\ln x \leq \frac{x^\eta}{\eta} \quad x \in [1, +\infty).$$

Combine (14) to get

$$\begin{aligned} h_{x,\frac{3N}{4}} &= \frac{\tau}{p\mu_1} \ln \frac{\mu_1^{-1} + 4(1 - \mu_1^{-1})N^{-1}}{\mu_1^{-1}} \\ &\leq \frac{\tau}{p\mu_1} \ln(N^{-1}\mu_1) \leq \frac{1}{\eta} \frac{\tau}{p\mu_1} N^{-\eta} \mu_1^\eta \\ &\leq C \mu_1^{\eta-1} N^{-\eta}. \end{aligned}$$

□

3.2. Finite element method

The weak form of problem (1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \quad (19)$$

where

$$a(u, v) := \varepsilon_1(\nabla u, \nabla v) + (\varepsilon_2 b u_x + c u, v) \quad (20)$$

and (\cdot, \cdot) denotes the standard scalar product in $L^2(\Omega)$.

Define the finite element space on the Bakhvalov-type mesh

$$V^N = \{w \in C(\bar{\Omega}) : w|_{\partial\Omega} = 0, w|_{\mathcal{T}} \in \mathcal{Q}_k(\mathcal{T}) \quad \forall \mathcal{T} \in \mathcal{T}\},$$

where $\mathcal{Q}_k(\mathcal{T}) = \sum_{0 \leq i,j \leq k} \alpha_{ij} x^i y^j$ with constants $\alpha_{ij} \in \mathbb{R}$.

The finite element method for (19) is to find $u^N \in V^N$ such that

$$a(u^N, v^N) = (f, v^N) \quad \forall v^N \in V^N. \quad (21)$$

The energy norm associated with $a(\cdot, \cdot)$ is defined by

$$\|v\|_E^2 := \varepsilon_1 |v|_1^2 + \|v\|^2 \quad \forall v \in H^1(\Omega).$$

Using (3), it's easy to prove coercivity

$$a(v^N, v^N) \geq C \|v^N\|_E^2 \quad \text{for all } v^N \in V^N. \quad (22)$$

It follows that u^N is well defined by (21) (see [5] and references therein).

4. Interpolation errors

In this section we will introduce a new interpolation. The structure of this interpolation is similar to one in [14]. Set $x_i^s := x_i + \frac{s}{k} h_{x,i}$ and $y_j^t := y_j + \frac{t}{k} h_{y,j}$ for $i, j = 0, 1, \dots, N-1$ and $s, t = 1, 2, \dots, k$.

For any $v \in C^0(\bar{\Omega})$ its Lagrange interpolation $v^I \in V^N$ on the Bakhvalov-type mesh is defined by

$$\begin{aligned} v^I(x, y) &= \sum_{i=0}^{N-1} \sum_{s=0}^{k-1} \left(\sum_{j=0}^{N-1} \sum_{t=0}^{k-1} v(x_i^s, y_j^t) \theta_{i,j}^{s,t}(x, y) + v(x_i^s, y_N^0) \theta_{i,N}^{s,0}(x, y) \right) \\ &\quad + \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} v(x_N^0, y_j^t) \theta_{N,j}^{0,t}(x, y) + v(x_N^0, y_N^0) \theta_{N,N}^{0,0}(x, y), \end{aligned}$$

where $\theta_{i,j}^{s,t}(x, y) \in V^N$ is the piecewise k th-order Lagrange basis function satisfying the well-known delta properties associated with the nodes (x_i^s, y_j^t) . We define the interpolation Πu to the solution u by

$$\Pi u := S^I + E_{10}^I + \pi_{11} E_{11} + E_{20}^I + E_{21}^I + E_{31}^I + \pi_{32} E_{32} + \pi_{33} E_{33} + E_{34}^I, \quad (23)$$

where

$$\pi_i E_i(x, y) = E_i^I - P E_i + \Theta E_i \quad \text{for } i = 11, 32, 33 \quad (24)$$

with

$$(PE_a)(x, y) = \sum_{i=\frac{3N}{4}}^k \sum_{s=1}^k \left(\sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_a(x_i^s, y_j^t) \theta_{i,j}^{s,t}(x, y) + E_a(x_i^s, y_N^0) \theta_{i,N}^{s,0}(x, y) \right)$$

$$(\Theta E_a)(x, y) = \sum_{s=1}^k E_a(x_{\frac{3N}{4}}^s, y_0^0) \theta_{\frac{3N}{4},0}^{s,0} + \sum_{s=1}^k E_a(x_{\frac{3N}{4}}^s, y_N^0) \theta_{\frac{3N}{4},N}^{s,0} \quad a = 11, 32, 33.$$

From (23) and (24) we can easily get $\Pi u \in V^N$ and

$$\Pi u = u^I - \sum_{i=11,32,33} (PE_i - \Theta E_i). \quad (25)$$

Next, we will prove the Lagrange interpolation estimation. From [1, Theorem 2.7], we have the following anisotropic interpolation results.

Lemma 5. *Let $\mathcal{T} \in \mathcal{T}$ and $v \in H^{k+1}(\mathcal{T})$. Then there exists a constant C such that Lagrange interpolation v^I satisfies*

$$\|v - v^I\|_{\mathcal{T}} \leq C \sum_{i+j=k+1} h_{x,\mathcal{T}}^i h_{y,\mathcal{T}}^j \left\| \frac{\partial^{k+1} v}{\partial x^i \partial y^j} \right\|_{\mathcal{T}},$$

$$\|(v - v^I)_x\|_{\mathcal{T}} \leq C \sum_{i+j=k} h_{x,\mathcal{T}}^i h_{y,\mathcal{T}}^j \left\| \frac{\partial^{k+1} v}{\partial x^{i+1} \partial y^j} \right\|_{\mathcal{T}},$$

$$\|(v - v^I)_y\|_{\mathcal{T}} \leq C \sum_{i+j=k} h_{x,\mathcal{T}}^i h_{y,\mathcal{T}}^j \left\| \frac{\partial^{k+1} v}{\partial x^i \partial y^{j+1}} \right\|_{\mathcal{T}},$$

where $h_{x,\mathcal{T}}$ and $h_{y,\mathcal{T}}$ are respectively the mesh size in x direction and y direction on the rectangular interval \mathcal{T} .

Lemma 6. *Assume $\tau \geq k + 1$. On Bakhvalov-type mesh \mathcal{T} , one has*

$$\|E_i - E_i^I\| \leq CN^{-(k+1)} \quad i = 10, 11, 20, 21, 31, 32, 33, 34.$$

Proof. To consider $\|E_{10} - E_{10}^I\|$, we decompose it as follows

$$\begin{aligned} \|E_{10} - E_{10}^I\|^2 &= \|E_{10} - E_{10}^I\|_{[x_0, x_{\frac{N}{4}-1}] \times [0,1]}^2 + \|E_{10} - E_{10}^I\|_{[x_{\frac{N}{4}-1}, x_N] \times [0,1]}^2 \\ &=: A_1 + A_2. \end{aligned}$$

Using (9), Lemmas 1, 2, 5 and (16) with $m = l$ we obtain

$$\begin{aligned}
A_1 &= \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \|E_{10} - E_{10}^I\|_{\mathcal{T}_{i,j}}^2 \leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \sum_{l+r=k+1} h_{x,i}^{2l} h_{y,j}^{2r} \left\| \frac{\partial^{k+1} E_{10}}{\partial x^l \partial y^r} \right\|_{\mathcal{T}_{i,j}}^2 \\
&\leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \sum_{l+r=k+1} h_{x,i}^{2l} h_{y,j}^{2r} (\mu_0^{2l} e^{-2p\mu_0 x_i} h_{x,i} h_{y,j}) \\
&\leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \sum_{l+r=k+1} (\mu_0^{-2l} N^{-2l}) \mu_0^{2l} h_{x,i} h_{y,j}^{2r+1} \\
&\leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \mu_0^{-1} N^{-2(k+1)-1} \\
&\leq C \mu_0^{-1} N^{-(2k+1)},
\end{aligned} \tag{26}$$

and after a simple calculation, we get $|E_{10}(x, y)|_{[x_{\frac{N}{4}-1}, x_N] \times [0, 1]} \leq CN^{-\tau}$. Then the triangle inequality yields

$$\begin{aligned}
A_2 &\leq C(\|E_{10}\|_{[x_{\frac{N}{4}-1}, x_N] \times [0, 1]}^2 + \|E_{10}^I\|_{[x_{\frac{N}{4}-1}, x_N] \times [0, 1]}^2) \\
&\leq C(\|E_{10}\|_{\infty, [x_{\frac{N}{4}-1}, x_N] \times [0, 1]}^2 + \|E_{10}^I\|_{\infty, [x_{\frac{N}{4}-1}, x_N] \times [0, 1]}^2) \\
&\leq C\|E_{10}\|_{\infty, [x_{\frac{N}{4}-1}, x_N] \times [0, 1]}^2 \\
&\leq CN^{-2\tau}.
\end{aligned} \tag{27}$$

From (7), (26) and (27) we could prove our conclusion. Using the same method we could get the estimates of $\|E_i - E_i^I\|$ with $i = 11, 20, 21, 31, 32, 33, 34$. For the cases of $i = 31, 32, 33, 34$, we divide the whole interval into three pieces not two pieces in the case of $E_{10} - E_{10}^I$. For example, for $\|E_{31} - E_{31}^I\|$, we can break it down into

$$\begin{aligned}
\|E_{31} - E_{31}^I\|^2 &= \|E_{31} - E_{31}^I\|_{[x_0, x_{\frac{N}{4}-1}] \times [y_0, y_{\frac{N}{4}-1}]}^2 \\
&\quad + \|E_{31} - E_{31}^I\|_{[x_0, x_{\frac{N}{4}-1}] \times [y_{\frac{N}{4}-1}, y_N]}^2 + \|E_{31} - E_{31}^I\|_{[x_{\frac{N}{4}-1}, x_N] \times [0, 1]}^2 \\
&=: B_1 + B_2 + B_3.
\end{aligned}$$

Similar to (26), we get

$$B_1 \leq C\varepsilon_1^{\frac{1}{2}} \mu_0^{-1} N^{-2k}.$$

And similar to (27), one has

$$B_2 + B_3 \leq CN^{-2\tau}.$$

□

Lemma 7. Assume $\tau \geq k+1$. On Bakhvalov-type mesh \mathcal{T} , one has

$$\begin{aligned} \|E_i - E_i^I\|_E &\leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)} \quad i = 10, 11, 20, 21, \\ \|PE_{11}\|_E &\leq CN^{-\tau - \frac{1}{2}}, \\ \|\Theta E_{11}\|_E &\leq C\varepsilon_1^{\frac{1}{4}} N^{-\tau}. \end{aligned}$$

Proof. We only consider $\|E_{10} - E_{10}^I\|_E$, because the remaining terms could be analyzed in a similar way. Clearly, one has

$$|E_{10} - E_{10}^I|^2 = \|(E_{10} - E_{10}^I)_x\|^2 + \|(E_{10} - E_{10}^I)_y\|^2.$$

From (9), Lemmas 2, 5 and (16) with $m = l + \frac{1}{2}$ we could obtain

$$\begin{aligned} \|(E_{10} - E_{10}^I)_x\|_{[x_0, x_{\frac{N}{4}-1}] \times [0, 1]}^2 &= \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \|(E_{10} - E_{10}^I)_x\|_{\mathcal{T}_{i,j}}^2 \\ &\leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \sum_{l+r=k} h_{x,i}^{2l} h_{y,j}^{2r} \left\| \frac{\partial^{k+1} E_{10}}{\partial x^{l+1} \partial y^r} \right\|_{\mathcal{T}_{i,j}}^2 \\ &\leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \sum_{l+r=k} h_{x,i}^{2l} h_{y,j}^{2r} (\mu_0^{2(l+1)} e^{-2p\mu_0 x_i} h_{x,i} h_{y,j}) \\ &\leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \sum_{l+r=k} (\mu_0^{-2l-1} N^{-2l-1}) \mu_0^{2(l+1)} h_{y,j}^{2r+1} \\ &\leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{N-1} \mu_0 N^{-2(k+1)} \\ &\leq C \mu_0 N^{-2k}. \end{aligned} \tag{28}$$

Note $\|(E_{10})_x\|_{[x_{\frac{N}{4}-1}, x_N] \times [0,1]} \leq C\mu_0^{\frac{1}{2}} N^{-\tau}$. Then one has

$$\begin{aligned}
& \|(E_{10} - E_{10}^I)_x\|_{[x_{\frac{N}{4}-1}, x_N] \times [0,1]}^2 \leq C\|(E_{10})_x\|_{[x_{\frac{N}{4}-1}, x_N] \times [0,1]}^2 + C\|(E_{10}^I)_x\|_{[x_{\frac{N}{4}-1}, x_N] \times [0,1]}^2 \\
& \leq C\|(E_{10})_x\|_{[x_{\frac{N}{4}-1}, x_N] \times [0,1]}^2 + C \sum_{i=\frac{N}{4}-1}^{N-1} \sum_{j=0}^{N-1} \|(E_{10}^I)_x\|_{\mathcal{T}_{i,j}}^2 \\
& \leq C\mu_0 N^{-2\tau} + C\mu_1 N^{2-2\tau} \\
& \leq C\mu_1 N^{2-2\tau},
\end{aligned} \tag{29}$$

where inverse inequality [6, Theorem 3.2.6], (9), Lemmas 1 and 2 yield

$$\begin{aligned}
& \sum_{i=\frac{N}{4}-1}^{N-1} \sum_{j=0}^{N-1} \|(E_{10}^I)_x\|_{\mathcal{T}_{i,j}}^2 \leq C \sum_{i=\frac{N}{4}-1}^{N-1} \sum_{j=0}^{N-1} h_{x,i}^{-2} \|E_{10}^I\|_{\mathcal{T}_{i,j}}^2 \\
& \leq C \sum_{i=\frac{N}{4}-1}^{N-1} \sum_{j=0}^{N-1} h_{x,i}^{-2} \|E_{10}^I\|_{\infty, \mathcal{T}_{i,j}}^2 h_{x,i} h_{y,j} \\
& \leq C\mu_1 N^{2-2\tau}.
\end{aligned}$$

Similar to (28), we can get

$$\|(E_{10} - E_{10}^I)_y\|_{[x_0, x_{\frac{N}{4}-1}] \times [0,1]}^2 \leq C\mu_0^{-1} N^{-2k}. \tag{30}$$

Similar to (29), one has

$$\|(E_{10} - E_{10}^I)_y\|_{[x_{\frac{N}{4}-1}, x_N] \times [0,1]}^2 \leq C\varepsilon_1^{-\frac{1}{2}} N^{2-2\tau}. \tag{31}$$

From (28)–(31), (5) and Lemma 6 we can easily obtain

$$\begin{aligned}
\|E_{10} - E_{10}^I\|_E^2 & \leq C(\varepsilon_1 \mu_1 N^{-2(k+1)} N^2 + \varepsilon_1^{\frac{1}{2}} N^{-2(k+1)} N^2) + CN^{-2(k+1)} \\
& \leq C(\varepsilon_1 \mu_1 N^{-2k} + \varepsilon_1^{\frac{1}{2}} N^{-2k}) + CN^{-2(k+1)} \\
& \leq C((\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-2k} + \varepsilon_1^{\frac{1}{2}} N^{-2k}) + CN^{-2(k+1)} \\
& \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-2k} + CN^{-2(k+1)},
\end{aligned}$$

i.e.

$$\|E_{10} - E_{10}^I\|_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)}.$$

For $\|PE_{11}\|_E$ and $\|\Theta E_{11}\|_E$, Lemmas 1, 2 , (5) and (8) yield

$$\begin{aligned}
\|PE_{11}\|_E^2 &\leq CN^{-2\tau} \sum_{s=1}^k \left(\sum_{j=0}^{N-1} \sum_{t=0}^{k-1} \|\theta_{\frac{3N}{4},j}^{s,t}\|_E^2 + \|\theta_{\frac{3N}{4},N}^{s,0}\|_E^2 \right) \\
&\leq CN^{-2\tau} \sum_{j=0}^{N-1} (\varepsilon_1 h_{x,\frac{3N}{4}}^{-1} h_{y,j} + \varepsilon_1 h_{x,\frac{3N}{4}} h_{y,j}^{-1} + h_{x,\frac{3N}{4}} h_{y,j}) \\
&\leq CN^{-2\tau} (\varepsilon_1 \mu_1 + \varepsilon_1^{\frac{1}{2}} N + N^{-1}), \\
&\leq CN^{-2\tau} (\varepsilon_2 + \varepsilon_1^{\frac{1}{2}} N + N^{-1}) \\
&\leq CN^{-2\tau-1},
\end{aligned}$$

and

$$\begin{aligned}
\|\Theta E_{11}\|_E^2 &\leq CN^{-2\tau} \left(\sum_{s=1}^k \|\theta_{\frac{3N}{4},0}^{s,0}\|_E^2 + \sum_{s=1}^k \|\theta_{\frac{3N}{4},N}^{s,0}\|_E^2 \right) \\
&\leq CN^{-2\tau} (\varepsilon_1 h_{x,\frac{3N}{4}}^{-1} h_{y,0} + \varepsilon_1 h_{x,\frac{3N}{4}} h_{y,0}^{-1} + h_{x,\frac{3N}{4}} h_{y,0} \\
&\quad + \varepsilon_1 h_{x,\frac{3N}{4}}^{-1} h_{y,N-1} + \varepsilon_1 h_{x,\frac{3N}{4}} h_{y,N-1}^{-1} + h_{x,\frac{3N}{4}} h_{y,N-1}) \\
&\leq CN^{-2\tau} (\varepsilon_1 \mu_1 \varepsilon_1^{\frac{1}{2}} + \varepsilon_1 N^{-1} \varepsilon_1^{-\frac{1}{2}} N + N^{-1} \varepsilon_1^{\frac{1}{2}}) \\
&\leq CN^{-2\tau} (\varepsilon_1^{\frac{1}{2}} (\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) + \varepsilon_1^{\frac{1}{2}}) \\
&\leq C\varepsilon_1^{\frac{1}{2}} N^{-2\tau}.
\end{aligned}$$

□

Lemma 8. *For interpolation error estimates of corner layers we have*

$$\begin{aligned}
\|E_i - E_i^I\|_E &\leq CN^{-(k+1)} \quad i = 31, 32, 33, 34, \\
\|PE_j\|_E &\leq CN^{-\tau-\frac{1}{2}} \quad j = 32, 33, \\
\|\Theta E_j\|_E &\leq C\varepsilon_1^{\frac{1}{2}} N^{-\tau} \quad j = 32, 33.
\end{aligned}$$

Proof. We have omitted the proofs of $\|PE_j\|_E$ and $\|\Theta E_j\|_E$ with $j = 32, 33$ here, because they are similar to ones of $\|PE_{11}\|_E$ and $\|\Theta E_{11}\|_E$, respectively.

In order to analyze $\|(E_{31} - E_{31}^I)\|_E$, we set $D_{0,0} := [x_0, x_{\frac{N}{4}-1}] \times [y_0, y_{\frac{N}{4}-1}]$.

Then

$$\begin{aligned}
\|(E_{31} - E_{31}^I)_x\|_{\Omega \setminus D_{0,0}}^2 &\leq C\|(E_{31})_x\|_{\Omega \setminus D_{0,0}}^2 + C\|(E_{31}^I)_x\|_{\Omega \setminus D_{0,0}}^2 \\
&\leq C\|(E_{31})_x\|_{\Omega \setminus D_{0,0}}^2 + C \sum_{i=\frac{N}{4}-1}^{N-1} \sum_{j=0}^{N-1} \|(E_{31}^I)_x\|_{\mathcal{T}_{i,j}}^2 + C \sum_{i=0}^{\frac{N}{4}-1} \sum_{j=\frac{N}{4}-1}^{N-1} \|(E_{31}^I)_x\|_{\mathcal{T}_{i,j}}^2 \\
&\leq D_1 + D_2 + D_3.
\end{aligned}$$

Inverse inequality, (11), Lemmas 1 and 2 yield

$$\begin{aligned}
D_1 &= \|(E_{31})_x\|_{\Omega \setminus D_{0,0}}^2 \leq \int_{x_0}^{x_{\frac{N}{4}-1}} \int_{y_{\frac{N}{4}-1}}^{y_N} \mu_0^2 e^{-2p\mu_0 x_i} e^{-\frac{2\delta y_j}{\sqrt{\varepsilon_1}}} dx dy \\
&\quad + \int_{x_{\frac{N}{4}-1}}^{x_N} \int_{y_0}^{y_N} \mu_0^2 e^{-2p\mu_0 x_i} e^{-\frac{2\delta y_j}{\sqrt{\varepsilon_1}}} dx dy \\
&\leq C\varepsilon_1^{\frac{1}{2}} \mu_0 N^{-2\tau}.
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
D_2 &= \sum_{i=\frac{N}{4}-1}^{N-1} \sum_{j=0}^{N-1} \|(E_{31}^I)_x\|_{\mathcal{T}_{i,j}}^2 \leq C \sum_{i=\frac{N}{4}-1}^{N-1} \sum_{j=0}^{N-1} h_{x,i}^{-2} \|E_{31}^I\|_{\mathcal{T}_{i,j}}^2 \\
&\leq C \sum_{i=\frac{N}{4}-1}^{N-1} \sum_{j=0}^{N-1} h_{x,i}^{-2} \|E_{31}^I\|_{\infty, \mathcal{T}_{i,j}}^2 h_{x,i} h_{y,j} \\
&\leq C\mu_1 N^{2-2\tau}.
\end{aligned} \tag{33}$$

Similar to D_2 , we have

$$D_3 \leq C\mu_0 N^{2-2\tau}. \tag{34}$$

Combination of (32), (33), and (34) yields

$$\|(E_{31} - E_{31}^I)_x\|_{\Omega \setminus D_{0,0}}^2 \leq C\mu_1 N^{2-2\tau}. \tag{35}$$

From (11), Lemma 5, (16) with $m = l + \frac{1}{2}$ and (18) with $m = r + \frac{1}{2}$ yield

$$\begin{aligned}
& \| (E_{31} - E_{31}^I)_x \|_{D_{0,0}}^2 = \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{\frac{N}{4}-2} \| (E_{31} - E_{31}^I)_x \|_{\mathcal{T}_{i,j}}^2 \\
& \leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{\frac{N}{4}-2} \sum_{l+r=k} h_{x,i}^{2l} h_{y,j}^{2r} \left\| \frac{\partial^{k+1} E_{31}}{\partial x^{l+1} \partial y^r} \right\|_{\mathcal{T}_{i,j}}^2 \\
& \leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{\frac{N}{4}-2} \sum_{l+r=k} h_{x,i}^{2l} h_{y,j}^{2r} (\varepsilon_1^{-r} \mu_0^{2(l+1)} e^{-2p\mu_0 x_i} e^{-\frac{2\delta y_j}{\sqrt{\varepsilon_1}}} h_{x,i} h_{y,j}) \\
& \leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{\frac{N}{4}-2} \sum_{l+r=k} (\mu_0^{-2l-1} N^{-2l-1} \varepsilon_1^{r+\frac{1}{2}} N^{-2r-1}) \mu_0^{2(l+1)} \varepsilon_1^{-r} \\
& \leq C \sum_{i=0}^{\frac{N}{4}-2} \sum_{j=0}^{\frac{N}{4}-2} \mu_0 \varepsilon_1^{\frac{1}{2}} N^{-2(k+1)} \\
& \leq C \mu_0 \varepsilon_1^{\frac{1}{2}} N^{-2k}.
\end{aligned} \tag{36}$$

For $\| (E_{31} - E_{31}^I)_y \|$, we use the same processing technique as $\| (E_{31} - E_{31}^I)_x \|$ to obtain

$$\| (E_{31} - E_{31}^I)_y \|_{D_{0,0}}^2 \leq C \mu_0^{-1} \varepsilon_1^{-\frac{1}{2}} N^{-2k}, \tag{37}$$

$$\| (E_{31} - E_{31}^I)_y \|_{\Omega \setminus D_{0,0}}^2 \leq C \varepsilon_1^{-\frac{1}{2}} N^{2-2\tau}. \tag{38}$$

From (35)–(38) we can easily obtain

$$|E_{31} - E_{31}^I|_1^2 \leq C \mu_1 N^{-2k} + \varepsilon_1^{-\frac{1}{2}} N^{-2k}.$$

By combining Lemma 6 and (5) we get

$$\begin{aligned}
\| E_{31} - E_{31}^I \|_E^2 & \leq C \varepsilon_1 (\mu_1 N^{-2k} + \varepsilon_1^{-\frac{1}{2}} N^{-2k}) + C N^{-2(k+1)} \\
& \leq C (\varepsilon_1 \mu_1 N^{-2k} + \varepsilon_1^{\frac{1}{2}} N^{-2k}) + C N^{-2(k+1)} \\
& \leq C ((\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-2k} + \varepsilon_1^{\frac{1}{2}} N^{-2k}) + C N^{-2(k+1)} \\
& \leq C (\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-2k} + C N^{-2(k+1)}.
\end{aligned}$$

Similarly, we have

$$\| E_i - E_i^I \|_E^2 \leq C (\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-2k} + C N^{-2(k+1)} \quad i = 32, 33, 34.$$

□

When calculating the interpolation error of $\|(E_i - E_i^I)_x\|$ ($i = 10, 20, 21, 31, 34$), we use a different technique.

Lemma 9. *Assume $\tau \geq k + 1$. On Bakhvalov-type mesh \mathcal{T} , one has*

$$\begin{aligned} \|(E_{10} - E_{10}^I)_x\| &\leq C\mu_0 N^{-(k+1)}, \\ \|(E_i - E_i^I)_x\| &\leq C\varepsilon_1^{\frac{1}{4}}N^{-k} + CN^{-(k+1)} \quad i = 20, 21, \\ \|(E_j - E_j^I)_x\| &\leq C\mu_0^{\frac{1}{2}}\varepsilon_1^{\frac{1}{4}}N^{-k} + C\mu_0 N^{-(k+1)} \quad j = 31, 34. \end{aligned}$$

Proof. Here we only prove the conclusion of the boundary layer at $x = 0$, because the proof for other boundary layers is similar. The analysis of the two corner layers is also similar, so we only present the proof of one of them.

For $\|(E_{10} - E_{10}^I)_x\|$, on the interval $[x_0, x_{\frac{N}{4}-1}] \times [0, 1]$, using (28) to get

$$\|(E_{10} - E_{10}^I)_x\|_{[x_0, x_{\frac{N}{4}-1}] \times [0, 1]}^2 \leq C\mu_0 N^{-2k}. \quad (39)$$

But, on the interval $[x_{\frac{N}{4}-1}, x_N] \times [0, 1]$, instead of using the inverse inequality in (29), we use the triangle inequality and (9) yield

$$\begin{aligned} &\|(E_{10} - E_{10}^I)_x\|_{[x_{\frac{N}{4}-1}, x_N] \times [0, 1]} \\ &\leq \|(E_{10})_x\|_{\infty, [x_{\frac{N}{4}-1}, x_N] \times [0, 1]} + \|(E_{10}^I)_x\|_{\infty, [x_{\frac{N}{4}-1}, x_N] \times [0, 1]} \\ &\leq C\|(E_{10})_x\|_{\infty, [x_{\frac{N}{4}-1}, x_N] \times [0, 1]} \\ &\leq C\mu_0 N^{-\tau}. \end{aligned} \quad (40)$$

From (39) and (40) we get $\|(E_{10} - E_{10}^I)_x\| \leq C\mu_0 N^{-(k+1)}$.

For $\|(E_{31} - E_{31}^I)_x\|$, we decompose it as follows

$$\begin{aligned} \|(E_{31} - E_{31}^I)_x\| &\leq \|(E_{31} - E_{31}^I)_x\|_{[x_0, x_{\frac{N}{4}-1}] \times [y_0, y_{\frac{N}{4}-1}]} \\ &+ \|(E_{31} - E_{31}^I)_x\|_{[x_0, x_{\frac{N}{4}-1}] \times [y_{\frac{N}{4}-1}, y_N]} + \|(E_{31} - E_{31}^I)_x\|_{[x_{\frac{N}{4}-1}, x_N] \times [0, 1]} \\ &\leq C\mu_0^{\frac{1}{2}}\varepsilon_1^{\frac{1}{4}}N^{-k} + C\mu_0 N^{-(k+1)}, \end{aligned}$$

where similar to (36), we have

$$\|(E_{31} - E_{31}^I)_x\|_{[x_0, x_{\frac{N}{4}-1}] \times [y_0, y_{\frac{N}{4}-1}]}^2 \leq \mu_0\varepsilon_1^{\frac{1}{2}}N^{-2k},$$

and similar to (40) we obtain

$$\|(E_{31} - E_{31}^I)_x\|_{[x_0, x_{\frac{N}{4}-1}] \times [y_{\frac{N}{4}-1}, y_N]} + \|(E_{31} - E_{31}^I)_x\|_{[x_{\frac{N}{4}-1}, x_N] \times [0, 1]} \leq \mu_0 N^{-\tau}.$$

□

Theorem 1. Assume $\tau \geq k+1$. On the Bakhvalov-type mesh \mathcal{T} , one has

$$\begin{aligned} \sum_i \|\pi_i E_i - E_i\| &\leq CN^{-(k+1)}, \quad i = 11, 32, 33, \\ \|u - u^I\|_E + \|u - \Pi u\|_E &\leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + N^{-(k+1)}. \end{aligned}$$

Proof. From Lemma 6 and the proof of Lemma 7 we could obtain

$$\|\pi E_{11} - E_{11}\| \leq \|E_{11} - E_{11}\| + \|PE_{11}\| + \|\Theta E_{11}\| \leq CN^{-(k+1)}.$$

Similarly we could get estimates for $\|\pi E_i - E_i\|$ with $i = 32, 33$.

By a simple calculation we get $\|S - S^I\| \leq CN^{-(k+1)}$ and $|S - S^I|_1 \leq CN^{-k}$. Then by combining Lemmas 6, 7 and 8 we prove $\|u - u^I\|_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + N^{-(k+1)}$. Finally using (25), we have $\|u - \Pi u\|_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + N^{-(k+1)}$. □

5. Uniform convergence

Set $\chi := \Pi u - u^N$. Using (20), (22), (23), integration by parts and Galerkin's orthogonality we have

$$\begin{aligned} \alpha \|\chi\|_E^2 &\leq a(\chi, \chi) = a(\Pi u - u, \chi) \\ &= \varepsilon_1 \int_{\Omega} \nabla(\Pi u - u) \nabla \chi dx dy + \varepsilon_2 \int_{\Omega} (S^I - S)_x \chi dx dy \\ &+ \sum_{l=10, 20, 21, 31, 34} \varepsilon_2 \int_{\Omega} (E_l^I - E_l)_x \chi dx dy - \varepsilon_2 \int_{\Omega} (\pi_{11} E_{11} - E_{11}) b \chi_x dx dy \\ &- \sum_{i=32, 33} \varepsilon_2 \int_{\Omega} (\pi_i E_i - E_i) b \chi_x dx dy - \sum_{j=11, 32, 33} \varepsilon_2 \int_{\Omega} (\pi_j E_j - E_j) b_x \chi dx dy + \int_{\Omega} c(\Pi u - u) \chi dx dy \\ &=: I + II + III + IV + V + VI + VII. \end{aligned}$$

Theorem 1 yields

$$\begin{aligned} |(I + VII) + VI| &\leq C\|\Pi u - u\|_E \|\chi\|_E + \sum_{j=11,32,33} \|\pi_j E_j - E_j\| \|\chi\| \\ &\leq C((\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + N^{-(k+1)}) \|\chi\|_E. \end{aligned} \quad (41)$$

Using (6), Lemma 9 and Hölder inequality we can get

$$\begin{aligned} |II + III| &\leq C(\varepsilon_2 \|(S^I - S)_x\| \|\chi\| + \varepsilon_2 \sum_{l=10,20,21,31,34} \|(E_l^I - E_l)_x\| \|\chi\|) \\ &\leq C\varepsilon_2(N^{-k} + \mu_0 N^{-(k+1)} + \mu_0^{\frac{1}{2}} \varepsilon_1^{\frac{1}{4}} N^{-k}) \|\chi\| \\ &\leq C(\varepsilon_2 N^{-k} + N^{-(k+1)} + \varepsilon_2^{\frac{1}{2}} \varepsilon_1^{\frac{1}{4}} N^{-k}) \|\chi\| \\ &\leq C(\varepsilon_2 N^{-k} + N^{-(k+1)} + \varepsilon_2^{\frac{1}{2}} \varepsilon_1^{\frac{1}{4}} N^{-k}) \|\chi\|_E. \end{aligned} \quad (42)$$

For IV and V , we have the following two lemmas.

Lemma 10. *Assuming that $\tau \geq k + 1$, on the Bakhvalov-type mesh \mathcal{T} , one has*

$$|IV| \leq C(\varepsilon_2 N^{-k} + \varepsilon_2^{\frac{1}{2}} N^{-(k+1)}) \|\chi\|_E.$$

Proof. After analysis, we do the following decomposition

$$\begin{aligned} \int_{\Omega} (\pi_{11} E_{11} - E_{11}) b \chi_x dx dy &= \int_{x_0}^{x_{\frac{3N}{4}}} \int_0^1 b(E_{11}^I - E_{11}) \chi_x dx dy \\ &+ \int_{x_{\frac{3N}{4}}}^{x_{\frac{3N}{4}+1}} \int_0^1 b(\pi_{11} E_{11} - E_{11}) \chi_x dx dy + \int_{x_{\frac{3N}{4}+1}}^{x_{\frac{3N}{4}+2}} \int_0^1 b(\pi_{11} E_{11} - E_{11}) \chi_x dx dy \\ &+ \int_{x_{\frac{3N}{4}+2}}^{x_N} \int_0^1 b(E_{11}^I - E_{11}) \chi_x dx dy =: F_1 + F_2 + F_3 + F_4. \end{aligned} \quad (43)$$

First, using (10), the inverse inequality, Lemmas 1, 2 and 5 we can obtain

$$\begin{aligned}
|F_1| &\leq \sum_{i=0}^{\frac{3N}{4}-1} \sum_{j=0}^{N-1} \|E_{11}^I - E_{11}\|_{\mathcal{T}_{i,j}} \|\chi_x\| \leq C \sum_{i=0}^{\frac{3N}{4}-1} \sum_{j=0}^{N-1} \sum_{l+r=k+1} h_{x,i}^l h_{y,j}^r \left\| \frac{\partial^{k+1} E_{11}}{\partial x^l \partial y^r} \right\|_{\mathcal{T}_{i,j}} \|\chi_x\|_{\mathcal{T}_{i,j}} \\
&\leq C \sum_{i=0}^{\frac{3N}{4}-1} \sum_{j=0}^{N-1} \sum_{l+r=k+1} h_{x,i}^l h_{y,j}^r \mu_1^l e^{-p\mu_1(1-x_{i+1})} h_{x,i}^{\frac{1}{2}} h_{y,j}^{\frac{1}{2}} h_{x,i}^{-1} \|\chi\|_{\mathcal{T}_{i,j}} \\
&\leq C \sum_{i=0}^{\frac{3N}{4}-1} \sum_{j=0}^{N-1} \sum_{l+r=k+1} \mu_1^l \mu_1^{-\tau} h_{x,i}^{l-\frac{1}{2}} h_{y,j}^{r+\frac{1}{2}} \|\chi\|_{\mathcal{T}_{i,j}} \leq C \mu_1^{l-\tau} \sum_{i=0}^{\frac{3N}{4}-1} \sum_{j=0}^{N-1} N^{-(k+1)} \|\chi\|_{\mathcal{T}_{i,j}} \\
&\leq C \mu_1^{l-\tau} \left(\sum_{i=0}^{\frac{3N}{4}-1} \sum_{j=0}^{N-1} N^{-2(k+1)} \right)^{\frac{1}{2}} \left(\sum_{i=0}^{\frac{3N}{4}-1} \sum_{j=0}^{N-1} \|\chi\|_{\mathcal{T}_{i,j}}^2 \right)^{\frac{1}{2}} \\
&\leq C N^{-k} \|\chi\|_{[x_0, x_{\frac{3N}{4}}] \times [0,1]} \leq C N^{-k} \|\chi\|_{E, [x_0, x_{\frac{3N}{4}}] \times [0,1]}.
\end{aligned} \tag{44}$$

Next, using (10), (17) with $m = l$, Lemmas 1, 2 and 5 we can obtain

$$\begin{aligned}
|F_4| &\leq \sum_{i=\frac{3N}{4}+2}^{N-1} \sum_{j=0}^{N-1} \|E_{11}^I - E_{11}\|_{\mathcal{T}_{i,j}} \|\chi_x\|_{\mathcal{T}_{i,j}} \leq C \sum_{i=\frac{3N}{4}+2}^{N-1} \sum_{j=0}^{N-1} \sum_{l+r=k+1} h_{x,i}^l h_{y,j}^r \left\| \frac{\partial^{k+1} E_{11}}{\partial x^l \partial y^r} \right\|_{\mathcal{T}_{i,j}} \|\chi_x\|_{\mathcal{T}_{i,j}} \\
&\leq C \sum_{i=\frac{3N}{4}+2}^{N-1} \sum_{j=0}^{N-1} \sum_{l+r=k+1} h_{x,i}^l h_{y,j}^r \mu_1^l e^{-p\mu_1(1-x_{i+1})} h_{x,i}^{\frac{1}{2}} h_{y,j}^{\frac{1}{2}} \|\chi_x\|_{\mathcal{T}_{i,j}} \\
&\leq C \sum_{i=\frac{3N}{4}+2}^{N-1} \sum_{j=0}^{N-1} \sum_{l+r=k+1} (\mu_1^{-l} N^{-l}) \mu_1^l h_{x,i}^{\frac{1}{2}} h_{y,j}^{r+\frac{1}{2}} \|\chi_x\|_{\mathcal{T}_{i,j}} \leq C \mu_1^{-\frac{1}{2}} \sum_{i=\frac{3N}{4}+2}^{N-1} \sum_{j=0}^{N-1} N^{-(k+1)-\frac{1}{2}} \|\chi_x\|_{\mathcal{T}_{i,j}} \\
&\leq C \mu_1^{-\frac{1}{2}} \left(\sum_{i=\frac{3N}{4}+2}^{N-1} \sum_{j=0}^{N-1} N^{-2(k+1)-1} \right)^{\frac{1}{2}} \left(\sum_{i=\frac{3N}{4}+2}^{N-1} \sum_{j=0}^{N-1} \|\chi_x\|_{\mathcal{T}_{i,j}}^2 \right)^{\frac{1}{2}} \\
&\leq C \mu_1^{-\frac{1}{2}} N^{-(k+\frac{1}{2})} \|\chi_x\|_{[x_{\frac{3N}{4}+2}, x_N] \times [0,1]} \\
&\leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(k+\frac{1}{2})} \|\chi\|_{E, [x_{\frac{3N}{4}+2}, x_N] \times [0,1]}.
\end{aligned} \tag{45}$$

Then, on the interval $[x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0, 1]$, we notice that

$$\pi_{11} E_{11} = E_{11}^I - \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_{11}(x_{\frac{3N}{4}+1}^0, y_j^t) \theta_{\frac{3N}{4}+1,j}^{0,t} - E_{11}(x_{\frac{3N}{4}+1}^0, y_N^0) \theta_{\frac{3N}{4}+1,N}^{0,0}.$$

Thus we have

$$\begin{aligned}
|F_3| &\leq C \sum_{j=0}^{N-1} \|E_{11}^I - E_{11}\|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \\
&+ C \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} |E_{11}(x_{\frac{3N}{4}+1}^0, y_j^t)| \|\theta_{\frac{3N}{4}+1,j}^{0,t}\|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \\
&+ |E_{11}(x_{\frac{3N}{4}+1}^0, y_N^0)| \|\theta_{\frac{3N}{4}+1,N}^{0,0}\|_{\mathcal{T}_{\frac{3N}{4}+1,N-1}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}+1,N-1}} \\
&=: \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 \\
&\leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(k+1)} \|\chi\|_{E, [x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0,1]} \tag{46}
\end{aligned}$$

where same as (45), we get

$$\mathcal{R}_1 \leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(k+1)} \|\chi\|_{E, [x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0,1]},$$

and (10), Lemmas 1 and 2 yield

$$\begin{aligned}
\mathcal{R}_2 &\leq C \sum_{j=0}^{N-1} N^{-\tau} h_{x, \frac{3N}{4}+1}^{\frac{1}{2}} h_{y,j}^{\frac{1}{2}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \leq C \mu_1^{-\frac{1}{2}} \sum_{j=0}^{N-1} N^{-\tau} N^{-\frac{1}{2}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \\
&\leq C \mu_1^{-\frac{1}{2}} \left(\sum_{j=0}^{N-1} N^{-2\tau-1} \right)^{\frac{1}{2}} \left(\sum_{j=0}^{N-1} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}+1,j}}^2 \right)^{\frac{1}{2}} \\
&\leq C \mu_1^{-\frac{1}{2}} N^{-\tau} \|\chi_x\|_{[x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0,1]} \\
&\leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-\tau} \|\chi\|_{E, [x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0,1]}. \tag{47}
\end{aligned}$$

In the same way, one has

$$\mathcal{R}_3 \leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(\tau+\frac{1}{2})} \|\chi\|_{E, \mathcal{T}_{\frac{3N}{4}+1,N-1}}.$$

Last, for F_2 , on the interval $[x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0, 1]$,

$$\pi_{11} E_{11} = \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_{11}(x_{\frac{3N}{4}}^0, y_j^t) \theta_{\frac{3N}{4},j}^{0,t} + \sum_{s=1}^k E_{11}(x_{\frac{3N}{4}}^s, y_0^0) \theta_{\frac{3N}{4},0}^{s,0} + \sum_{s=0}^k E_{11}(x_{\frac{3N}{4}}^s, y_N^0) \theta_{\frac{3N}{4},N}^{s,0}.$$

Thus we have

$$\begin{aligned}
|F_2| &\leq C \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} |E_{11}(x_{\frac{3N}{4}}, y_j^t)| \|\theta_{\frac{3N}{4}, j}^{0,t}\|_{\mathcal{T}_{\frac{3N}{4}, j}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}, j}} \\
&+ C \|E_{11}\|_{[x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0,1]} \|\chi_x\|_{[x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0,1]} \\
&+ C \sum_{s=1}^k |E_{11}(x_{\frac{3N}{4}}, y_0^s)| \|\theta_{\frac{3N}{4}, 0}^{s,0}\|_{\mathcal{T}_{\frac{3N}{4}, 0}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}, 0}} \\
&+ C \sum_{s=0}^k |E_{11}(x_{\frac{3N}{4}}, y_N^s)| \|\theta_{\frac{3N}{4}, N}^{s,0}\|_{\mathcal{T}_{\frac{3N}{4}, N-1}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}, N-1}} \\
&= \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 \\
&\leq C(\varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(k+1)} + C\varepsilon_1^{-\frac{1}{4}} \mu_1^{-\frac{1}{4}} N^{-(\tau+\frac{1}{4})}) \|\chi\|_{E, [x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0,1]}.
\end{aligned} \tag{48}$$

The proof is as follows, same as (47), we get

$$\mathcal{S}_1 + \mathcal{S}_2 \leq C\varepsilon_1^{-\frac{1}{2}} \mu_1^{-\tau} N^{-\frac{1}{2}} \|\chi\|_{E, [x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0,1]}.$$

When dealing with \mathcal{S}_3 and \mathcal{S}_4 , the mesh scale in Lemma 1 is not enough, so we still use the analysis method of (47), but we use the mesh scale of Lemma 4 with $\eta = \frac{1}{2}$, thus we have

$$\mathcal{S}_3 + \mathcal{S}_4 \leq C\varepsilon_1^{-\frac{1}{4}} \mu_1^{-\frac{1}{4}} N^{-(\tau+\frac{1}{4})} \|\chi\|_{E, [x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0,1]}.$$

So, by combining (6), (7), (43), (44), (45), (46) and (48) to obtain

$$\begin{aligned}
|IV| &\leq C(\varepsilon_2 N^{-k} + \varepsilon_2 \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(k+\frac{1}{2})} + C\varepsilon_2 \varepsilon_1^{-\frac{1}{4}} \mu_1^{-\frac{1}{4}} N^{-(k+\frac{5}{4})}) \|\chi\|_E \\
&\leq C(\varepsilon_2 N^{-k} + \varepsilon_2^{\frac{1}{2}} N^{-(k+\frac{1}{2})} + \varepsilon_2^{\frac{3}{4}} N^{-(k+\frac{5}{4})}) \|\chi\|_E \\
&\leq C(\varepsilon_2 N^{-k} + \varepsilon_2^{\frac{1}{2}} N^{-(k+\frac{1}{2})}) \|\chi\|_E.
\end{aligned}$$

□

Lemma 11. Assuming that $\tau \geq k + 1$, on the Bakhvalov-type mesh \mathcal{T} , one has

$$|V| \leq C\varepsilon_2^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \|\chi\|_E.$$

Proof. To simplify the analysis, we decompose V as follows

$$\begin{aligned}
V &= \varepsilon_2 \int_{x_0}^{x_{\frac{3N}{4}}} \int_0^1 (E_{32}^I - E_{32}) b \chi_x dx dy + \varepsilon_2 \int_{x_{\frac{3N}{4}}}^{x_{\frac{3N}{4}+1}} \int_0^1 (\pi_{32} E_{32} - E_{32}) b \chi_x dx dy \\
&\quad + \varepsilon_2 \int_{x_{\frac{3N}{4}+1}}^{x_{\frac{3N}{4}+2}} \int_0^1 (\pi_{32} E_{32} - E_{32}) b \chi_x dx dy + \varepsilon_2 \int_{x_{\frac{3N}{4}+2}}^{x_N} \int_0^1 (E_{32}^I - E_{32}) b \chi_x dx dy \\
&=: M_1 + M_2 + M_3 + M_4.
\end{aligned}$$

For M_1 , using triangle inequality, (6) and (12) we can get

$$\begin{aligned}
|M_1| &\leq C\varepsilon_2 (\|E_{32}^I\|_{\infty, [x_0, x_{\frac{3N}{4}}] \times [0, 1]} + \|E_{32}\|_{\infty, [x_0, x_{\frac{3N}{4}}] \times [0, 1]}) \|\chi_x\|_{[x_0, x_{\frac{3N}{4}}] \times [0, 1]} \\
&\leq C\varepsilon_2 \|E_{32}\|_{\infty, [x_0, x_{\frac{3N}{4}}] \times [0, 1]} \|\chi_x\|_{E, [x_0, x_{\frac{3N}{4}}] \times [0, 1]} \\
&\leq C\varepsilon_2 \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\tau} \|\chi\|_{E, [x_0, x_{\frac{3N}{4}}] \times [0, 1]} \\
&\leq C\varepsilon_2^{\frac{1}{2}} \mu_1^{\frac{1}{2}-\tau} \|\chi\|_{E, [x_0, x_{\frac{3N}{4}}] \times [0, 1]}.
\end{aligned} \tag{49}$$

On the interval $[x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0, 1]$, we notice

$$\pi_{32} E_{32} = \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_{32}(x_{\frac{3N}{4}}^0, y_j^t) \theta_{\frac{3N}{4}, j}^{0, t} + \sum_{s=1}^k E_{32}(x_{\frac{3N}{4}}^s, y_0^0) \theta_{\frac{3N}{4}, 0}^{s, 0} + \sum_{s=0}^k E_{32}(x_{\frac{3N}{4}}^s, y_N^0) \theta_{\frac{3N}{4}, N}^{s, 0}.$$

Thus

$$\begin{aligned}
|M_2| &\leq C\varepsilon_2 \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} |E_{32}(x_{\frac{3N}{4}}^0, y_j^t)| \|\theta_{\frac{3N}{4}, j}^{0, t}\|_{\mathcal{T}_{\frac{3N}{4}, j}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}, j}} \\
&\quad + C\varepsilon_2 \sum_{s=1}^k |E_{32}(x_{\frac{3N}{4}}^s, y_0^0)| \|\theta_{\frac{3N}{4}, 0}^{s, 0}\|_{\mathcal{T}_{\frac{3N}{4}, 0}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}, 0}} \\
&\quad + C\varepsilon_2 \sum_{s=0}^k |E_{32}(x_{\frac{3N}{4}}^s, y_N^0)| \|\theta_{\frac{3N}{4}, N}^{s, 0}\|_{\mathcal{T}_{\frac{3N}{4}, N-1}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}, N-1}} \\
&\quad + C\varepsilon_2 \|E_{32}\|_{[x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0, 1]} \|\chi_x\|_{[x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0, 1]} \\
&=: \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4 \\
&\leq C(\varepsilon_2^{\frac{1}{2}} N^{-\tau-\frac{1}{2}} + \varepsilon_2^{\frac{1}{2}} \varepsilon_1^{\frac{1}{2}} N^{-\tau}) \|\chi\|_E,
\end{aligned} \tag{50}$$

where similar to (47), one has $\mathcal{V}_1 \leq C\varepsilon_2 \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} \mu_1^{-\tau+\frac{1}{2}} N^{-1} \|\chi\|_E$, then use (6)

we obtain

$$\mathcal{V}_1 \leq C\varepsilon_2^{\frac{1}{2}} \mu_1^{-\tau+\frac{1}{2}} N^{-1} \|\chi\|_E. \tag{51}$$

In the same way, one has

$$\begin{aligned}\mathcal{V}_2 + \mathcal{V}_3 &\leq C\varepsilon_2 N^{-\tau-\frac{1}{2}} \|\chi\|_E. \\ \mathcal{V}_4 &\leq C\varepsilon_2^{\frac{1}{2}} \varepsilon_1^{\frac{1}{4}} N^{-\tau} \|\chi\|_E.\end{aligned}$$

On the interval $[x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0, 1]$,

$$\pi_{32} E_{32} = E_{32}^I - \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_{32}(x_{\frac{3N}{4}+1}^0, y_j^t) \theta_{\frac{3N}{4}+1,j}^{0,t} - E_{32}(x_{\frac{3N}{4}+1}^0, y_N^0) \theta_{\frac{3N}{4}+1,N}^{0,0}.$$

Thus

$$\begin{aligned}|M_3| &\leq C\varepsilon_2 \sum_{j=0}^{N-1} \|E_{32}^I - E_{32}\|_{\infty, \mathcal{T}_{\frac{3N}{4}+1,j}} h_{x, \frac{3N}{4}+1}^{\frac{1}{2}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \\ &\quad + C\varepsilon_2 \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} |E_{32}(x_{\frac{3N}{4}+1}^0, y_j^t)| \|\theta_{\frac{3N}{4}+1,j}^{0,t}\|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \quad (52) \\ &\quad + C\varepsilon_2 |E_{32}(x_{\frac{3N}{4}+1}^0, y_N^0)| \|\theta_{\frac{3N}{4}+1,N}^{0,0}\|_{\mathcal{T}_{\frac{3N}{4}+1,N-1}} \|\chi_x\|_{\mathcal{T}_{\frac{3N}{4}+1,N-1}} \\ &=: \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 \leq C\varepsilon_2^{\frac{1}{2}} N^{-(k+\frac{1}{2})},\end{aligned}$$

where same as (49), one has

$$\mathcal{W}_1 \leq C\varepsilon_2^{\frac{1}{2}} N^{\frac{1}{2}-\tau} \|\chi\|_E,$$

and in the same way as (51), it can obtain

$$\begin{aligned}\mathcal{W}_2 &\leq C\varepsilon_2^{\frac{1}{2}} N^{-\tau} \|\chi\|_E, \\ \mathcal{W}_3 &\leq C\varepsilon_1^{\frac{1+\tau}{2}} \varepsilon_1^{\frac{1}{4}} N^{-\tau} \|\chi\|_E.\end{aligned}$$

For M_4 , Hölder inequality yields

$$\begin{aligned}|M_4| &\leq C\varepsilon_2 \|E_{32}^I - E_{32}\|_{[x_{\frac{3N}{4}+2}, x_N] \times [y_0, y_{\frac{N}{4}-1}]} \|\chi_x\|_{[x_{\frac{3N}{4}+2}, x_N] \times [y_0, y_{\frac{N}{4}-1}]} \\ &\quad + C\varepsilon_2 \|E_{32}^I - E_{32}\|_{[x_{\frac{3N}{4}+2}, x_N] \times [y_{\frac{N}{4}-1}, y_N]} \|\chi_x\|_{[x_{\frac{3N}{4}+2}, x_N] \times [y_{\frac{N}{4}-1}, y_N]} \quad (53) \\ &=: \mathcal{Z}_1 + \mathcal{Z}_2 \leq C(\varepsilon_2 \mu_1^{-\frac{1}{2}} N^{-k} + \varepsilon_2^{\frac{1}{2}} N^{\frac{1}{2}-\tau}) \|\chi\|_E,\end{aligned}$$

where similar to (45), one has

$$\mathcal{Z}_1 \leq C\varepsilon_2 \mu_1^{-\frac{1}{2}} N^{-k} \|\chi\|_E,$$

and Hölder inequality, (12), Lemma 1 and (6) yield

$$\begin{aligned}
\mathcal{Z}_2 &\leq C\varepsilon_2 \|E_{32}^I - E_{32}\|_{\infty, [x_{\frac{3N}{4}+2}, x_N] \times [y_{\frac{N}{4}-1}, y_N]} (1 - x_{\frac{3N}{4}+2})^{\frac{1}{2}} \|\chi_x\|_{[x_{\frac{3N}{4}+2}, x_N] \times [y_{\frac{N}{4}-1}, y_N]} \\
&\leq C\varepsilon_2 \|E_{32}\|_{\infty, [x_{\frac{3N}{4}+2}, x_N] \times [y_{\frac{N}{4}-1}, y_N]} \mu_1^{-\frac{1}{2}} (\ln N)^{\frac{1}{2}} \|\chi_x\| \\
&\leq C\varepsilon_2 N^\tau \mu_1^{-\frac{1}{2}} N^{\frac{1}{2}} \|\chi_x\| \\
&\leq C\varepsilon_2 \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^\tau N^{\frac{1}{2}} \|\chi\|_E \\
&\leq C\varepsilon_2^{\frac{1}{2}} N^{\frac{1}{2}-\tau} \|\chi\|_E.
\end{aligned}$$

Thus, from (7), (8), (49), (50), (52) and (53) we can obtain our conclusion. \square

Now we present the main conclusions of this paper.

Theorem 2. *Assuming $\tau \geq k + 1$. On the Bakhvalov-type mesh \mathcal{T} , and based on Assumption 1, we have*

$$\begin{aligned}
\|u^I - u^N\|_E + \|\Pi u - u^N\|_E &\leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)}, \\
\|u - u^N\|_E &\leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)}.
\end{aligned}$$

Proof. From (41), (42), Lemmas 10 and 11 we can prove

$$\|\Pi u - u^N\|_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)}.$$

Combination of (25), Lemmas 7 and 8 yields

$$\|u^I - u^N\|_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)}.$$

Finally, using Theorem 1 we prove that

$$\|u - u^N\|_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)}.$$

\square

6. Numerical experiments

The purpose of this section is to verify that our main conclusions are correct. In order to do so, we study the performance of the method when applied to the

test problem

$$\begin{aligned} -\varepsilon_1 \Delta u + \varepsilon_2 (2-x) u_x + u &= f(x, y) \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{54}$$

where choice of the right-hand side satisfies

$$u(x, y) = \frac{1}{4} (1 - e^{-\mu_0 x}) \left(1 - e^{-\mu_1(1-x)}\right) \left(1 - e^{\frac{y}{\sqrt{\varepsilon_1}}}\right) \left(1 - e^{\frac{(1-y)}{\sqrt{\varepsilon_1}}}\right),$$

with

$$\mu_0 = \frac{-\varepsilon_2 + \sqrt{\varepsilon_2^2 + \varepsilon_1}}{\varepsilon_1}, \quad \mu_1 = \frac{\varepsilon_2 + \sqrt{\varepsilon_2^2 + 4\varepsilon_1}}{2\varepsilon_1}.$$

is the exact solution.

In our example, we take $k = 1, 2, 3$, $p = 0.5$, $\delta = 0.25$, $N = 2^3, \dots, 2^9$. Besides, we should choose the perturbation parameter range $R(\varepsilon_1, \varepsilon_2)$ that meets the conditions (7), (13) and the mesh is completely in the Bakhvalov-type. Thus, for problem (54), the value range of disturbance parameter $R(\varepsilon_1, \varepsilon_2)$ should be

$$R(\varepsilon_1, \varepsilon_2) = \{(\varepsilon_1, \varepsilon_2) | 0 < \varepsilon_1 \leq 10^{-6}, 0 < \varepsilon_2 \leq 10^{-3}\}.$$

To be more general, we take $\varepsilon_1 = 1, 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}$, $\varepsilon_2 = 1, 10^{-4}, 10^{-8}$.

For any fixed value of k and ε_2 , energy norm error estimation will be calculated by

$$e^N = \|u - u^N\|_E,$$

where u is the exact solution given by (54) and u^N represents its numerical approximation. And its corresponding convergence rate is

$$p^N = \frac{\ln e^N - \ln e^{2N}}{\ln 2}.$$

In Tables 1–3, we give the energy error estimations and convergence orders of $k = 1$ and $\varepsilon_2 = 1, 10^{-4}, 10^{-8}$. At the same time, we present the energy estimations in the cases of $k = 2, \varepsilon_2 = 1, 10^{-4}, 10^{-8}$ and $k = 3, \varepsilon_2 = 1, 10^{-4}, 10^{-8}$ in the figure below. As can be seen from the chart, our conclusion is verified.

Table 1: $\|u - u^N\|_E$ in the case of $\varepsilon_2 = 1$ and $k = 1$

ε_1	N						
	8	16	32	64	128	256	512
1	0.13E-2 1.00	0.66E-3 1.00	0.33E-3 1.00	0.16E-3 1.00	0.82E-4 1.00	0.41E-4 1.00	0.21E-4 —
	0.39E-1 0.98	0.20E-1 1.00	0.99E-2 1.00	0.49E-2 1.00	0.25E-2 1.00	0.12E-2 1.00	0.62E-3 —
10^{-2}	0.46E-1 0.99	0.23E-1 1.00	0.11E-1 1.00	0.57E-2 1.00	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 —
	0.46E-1 0.99	0.23E-1 1.00	0.12E-1 1.00	0.58E-2 1.00	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 —
10^{-4}	0.46E-1 0.99	0.23E-1 1.00	0.12E-1 1.00	0.58E-2 1.00	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 —
	0.46E-1 0.99	0.23E-1 1.00	0.12E-1 1.00	0.58E-2 1.00	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 —
10^{-6}	0.46E-1 0.99	0.23E-1 1.00	0.12E-1 1.00	0.58E-2 1.00	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 —
	0.46E-1 0.99	0.23E-1 1.00	0.12E-1 1.00	0.58E-2 1.00	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 —
10^{-8}	0.46E-1 0.99	0.23E-1 1.00	0.12E-1 1.00	0.58E-2 1.00	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 —
	0.46E-1 0.99	0.23E-1 1.00	0.12E-1 1.00	0.58E-2 1.00	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 —
10^{-10}	0.46E-1 0.99	0.23E-1 1.00	0.12E-1 1.00	0.58E-2 1.00	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 —

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Table 2: $\|u - u^N\|_E$ in the case of $\varepsilon_2 = 10^{-4}$ and $k = 1$

ε_1	N						
	8	16	32	64	128	256	512
1	0.22E-2 1.01	0.11E-2 1.00	0.55E-3 1.00	0.28E-3 1.00	0.014E-3 1.00	0.69E-4 1.00	0.35E-4 —
	0.33E-1 1.00	0.17E-1 1.00	0.84E-2 1.00	0.42E-2 1.00	0.21E-2 1.00	0.11E-2 1.00	0.53E-3 —
10^{-2}	0.26E-1 1.13	0.12E-1 1.03	0.58E-2 1.01	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 1.00	0.36E-3 —
	0.83E-2 1.14	0.38E-2 1.03	0.19E-2 1.01	0.93E-3 1.00	0.46E-3 1.00	0.23E-3 1.00	0.12E-3 —
10^{-4}	0.28E-2 1.18	0.12E-2 1.04	0.60E-3 1.01	0.30E-3 1.00	0.15E-3 1.00	0.74E-4 1.00	0.37E-4 —
	0.17E-2 1.32	0.70E-3 1.08	0.33E-3 1.02	0.16E-3 1.00	0.81E-4 1.00	0.41E-4 1.00	0.20E-4 —

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Table 3: $\|u - u^N\|_E$ in the case of $\varepsilon_2 = 10^{-8}$ and $k = 1$

ε_1	N						
	8	16	32	64	128	256	512
1	0.22E-2 1.01	0.11E-2 1.00	0.55E-3 1.00	0.28E-3 1.00	0.014E-3 1.00	0.69E-4 1.00	0.35E-4 —
	0.33E-1 1.00	0.17E-1 1.00	0.84E-2 1.00	0.42E-2 1.00	0.21E-2 1.00	0.11E-2 1.00	0.53E-3 —
10^{-2}	0.26E-1 1.13	0.12E-1 1.03	0.58E-2 1.01	0.29E-2 1.00	0.14E-2 1.00	0.72E-3 1.00	0.36E-3 —
	0.83E-2 1.13	0.38E-2 1.03	0.19E-2 1.01	0.93E-3 1.00	0.47E-3 1.00	0.23E-3 1.00	0.12E-3 —
10^{-4}	0.26E-2 1.13	0.12E-2 1.03	0.59E-3 1.01	0.29E-3 1.00	0.15E-3 1.00	0.74E-4 1.00	0.37E-4 —
	0.84E-3 1.13	0.38E-3 1.03	0.19E-3 1.01	0.93E-4 1.00	0.47E-4 1.00	0.23E-4 1.00	0.12E-4 —
10^{-6}	0.26E-2 1.13	0.12E-2 1.03	0.59E-3 1.01	0.29E-3 1.00	0.15E-3 1.00	0.74E-4 1.00	0.37E-4 —
	0.84E-3 1.13	0.38E-3 1.03	0.19E-3 1.01	0.93E-4 1.00	0.47E-4 1.00	0.23E-4 1.00	0.12E-4 —
10^{-8}	0.26E-2 1.13	0.12E-2 1.03	0.59E-3 1.01	0.29E-3 1.00	0.15E-3 1.00	0.74E-4 1.00	0.37E-4 —
	0.84E-3 1.13	0.38E-3 1.03	0.19E-3 1.01	0.93E-4 1.00	0.47E-4 1.00	0.23E-4 1.00	0.12E-4 —
10^{-10}	0.26E-2 1.13	0.12E-2 1.03	0.59E-3 1.01	0.29E-3 1.00	0.15E-3 1.00	0.74E-4 1.00	0.37E-4 —
	0.84E-3 1.13	0.38E-3 1.03	0.19E-3 1.01	0.93E-4 1.00	0.47E-4 1.00	0.23E-4 1.00	0.12E-4 —

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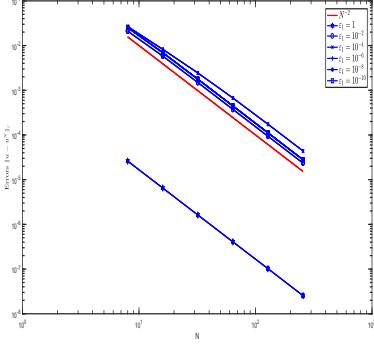


Figure 1: $k = 2, \varepsilon_2 = 1$.

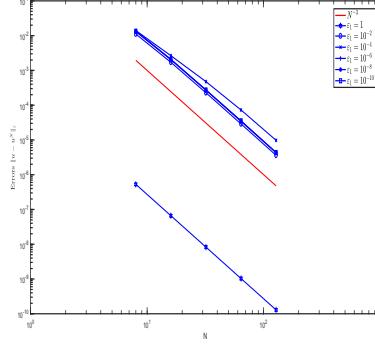


Figure 2: $k = 3, \varepsilon_2 = 1$.

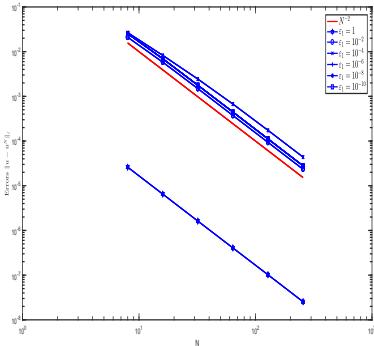


Figure 3: $k = 2, \varepsilon_2 = 10^{-4}$.

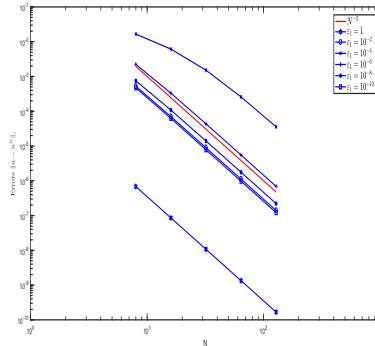


Figure 4: $k = 3, \varepsilon_2 = 10^{-4}$.

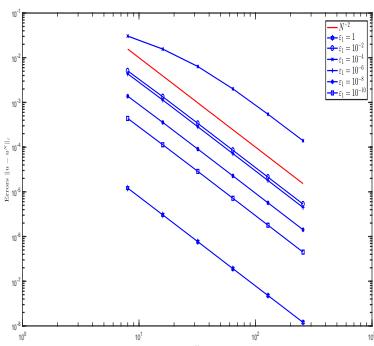


Figure 5: $k = 2, \varepsilon_2 = 10^{-8}$.

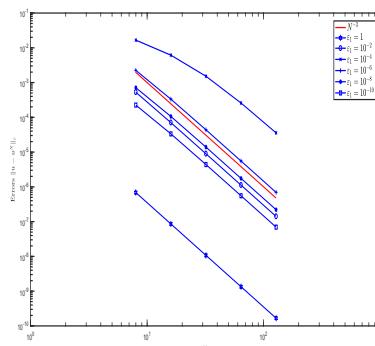


Figure 6: $k = 3, \varepsilon_2 = 10^{-8}$.