Certified Newton schemes for the evaluation of low-genus theta functions

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Abstract

Theta functions and theta constants in low genus, especially genus 1 and 2, can be evaluated at any given point in quasi-linear time in the required precision using Newton schemes based on Borchardt sequences. Our goal in this paper is to provide the necessary tools to implement these algorithms in a provably correct way. In particular, we obtain uniform and explicit convergence results in the case of theta constants in genus 1 and 2, and theta functions in genus 1: the associated Newton schemes will converge starting from approximations to N bits of precision for $N=60,\,300,\,$ and 1600 respectively, for all suitably reduced arguments. We also describe a uniform quasi-linear time algorithm to evaluate genus 2 theta constants on the Siegel fundamental domain. Our main tool is a detailed study of Borchardt means as multivariate analytic functions.

1 Introduction

Let $g \geq 1$ be an integer, and let \mathcal{H}_g be the Siegel upper half space of degree g, which consists of all symmetric $g \times g$ complex matrices with positive definite imaginary part. Let $a, b \in \{0, 1\}^g$. Then the theta function of genus g and characteristic (a, b) is defined on $\mathbb{C}^g \times \mathcal{H}_g$ by the following exponential series:

$$\theta_{a,b}(z,\tau) = \sum_{m \in \mathbb{Z}^g} \exp\left(i\pi(m + \frac{a}{2})^t \tau(m + \frac{a}{2}) + 2i\pi(m + \frac{a}{2})^t (z + \frac{b}{2})\right). \tag{1}$$

Theta functions appear in many areas of mathematics, from partial differential equations to arithmetic geometry; an overview is given in [16, 23, 24]. They have symmetries with respect to the action of the modular group $\operatorname{Sp}_{2q}(\mathbb{Z})$ [23, §II.5], and they also

satisfy the Riemann relations, a broad generalization of the well-known duplication formula [23, §II.6]. Theta constants are the values of these functions taken at z = 0, and are of interest in number theory. Each theta constant is a Siegel modular form, and every Siegel modular form for $\operatorname{Sp}_{2g}(\mathbb{Z})$ has an expression as a rational fraction in terms of theta constants [15]; even a polynomial, if $g \leq 3$ [15, 12].

In this paper, we are interested in algorithms to evaluate theta constants at a given point $\tau \in \mathcal{H}_g$, or more generally theta functions at a given point (z,τ) , to precision N for some integer $N \geq 0$. In the whole paper, we consider absolute precision: the output will be a finitely encodable (for instance, dyadic) complex number x such that $|\theta(z,\tau)-x| \leq 2^{-N}$.

Two main approaches to computing theta functions exist. The first one, sometimes called the *naive algorithm*, consists in computing partial sums of the series (1) and obtaining an upper bound on the modulus of its tail [4, 9, 11, 1]. The resulting algorithm can be applied in any genus; its complexity is $O(\mathcal{M}(N)N^{g/2})$ if (z,τ) is fixed [22, Prop. 4.2], and can be made uniform in (z,τ) if this input is suitably reduced [4, Thm. 3 and Thm. 8].

The second approach was first described by Dupont [6, 5] in the case of theta constants of genus $g \leq 2$. It combines the arithmetic-geometric mean (AGM), and higher-dimensional analogues of the AGM called Borchardt means, with Newton iterations, and claims a complexity of $O(\mathcal{M}(N)\log N)$ binary operations. Extensions to theta functions in genus $g \leq 2$, as well as higher genera, were then described in [21, 22]. In practice, these algorithms beat the naive method for precisions greater than a few hundred thousand bits for g = 1, and a few thousand bits for g = 2. This improvement is especially welcome in number-theoretic applications, where huge precisions are often necessary to recognize rational numbers from their complex approximations [8, 7, 10, 18], although the naive method remains superior for g = 1 in the current range of practical applications.

In order to prove the correctness of an algorithm based on Newton's method, and establish an upper bound on its complexity, the first step is usually to show that the linearized system that Newton's method uses is actually invertible. A proof of this fact is currently missing for $g \geq 2$ [5, §10.2], [22, Conj. 3.6]. For g = 1, the invertibility of this linear system was proved [6, Prop. 11], [21, Prop. 4.4], but the rate of convergence of the resulting Newton scheme was not made explicit. This makes these algorithms difficult to implement in a provably correct way.

The purpose of the present paper is to turn the quasi-linear time algorithms for theta constants in genus 1 and 2, as well as theta functions in genus 1, into provably correct algorithms. This is done by giving explicit upper bounds on derivatives of certain analytic functions derived from Borchardt sequences on explicit polydisk neighborhoods of the points where Newton's method is applied. In the case of genus 2 theta constants, we also show how to combine Newton's method with the naive algorithm to obtain a uniform quasi-linear complexity on the Siegel fundamental domain, thus generalizing earlier constructions in genus 1 [6, Thm. 5], [21, §4.2]. In the case of theta functions in genus 2, and higher genera, we are no longer able to prove that Newton's method will succeed for all inputs. However, if it succeeds, then the same methods can be applied to certify the correctness of the result.

The paper is organized as follows. In Section 2, we give a general result of explicit convergence for Newton schemes involving multivariate analytic functions. We study Borchardt means as analytic functions in detail in Section 3. In Section 4, we review the existing Newton schemes for the computation of theta functions; then, we obtain explicit values for the magnitudes and radii of convergence of the analytic functions defining them, and thus explicit convergence results. Finally, we present the uniform algorithm to compute genus 2 theta constants in Section 5.

2 Certified multivariate Newton iterations

In this section, we are interested in designing provably correct Newton schemes for multivariate analytic functions, assuming that the system is linearized using finite differences at each step. More precisely, let \mathcal{U} be an open set in \mathbb{C}^r , let $f: \mathcal{U} \to \mathbb{C}^r$ be an analytic function, and let $x_0 \in \mathcal{U}$; assuming that $f(x_0)$ is known and that f can be evaluated at any point, we are interested in building a Newton scheme to compute x_0 itself.

First, we give an explicit convergence result provided that the first and second derivatives of f are locally bounded around x_0 , and that $df(x_0)$ is invertible. Using Cauchy's formula, we also obtain explicit convergence estimates if we simply assume that f is bounded on a certain polydisk around x_0 . Finally, we translate these theoretical results into the concrete world of finite-precision arithmetic. All these results are certainly well-known in spirit, but we were unfortunately unable to find sufficiently explicit results in the literature.

Let us introduce some notation. We always consider \mathbb{C}^r as a normed vector space for the L^{∞} norm, denoted simply by $\|\cdot\|$: in terms of coordinates, we have

$$||(x_1,\ldots,x_r)|| = \max_{1 \le j \le r} |x_j|.$$

If $\rho > 0$ and $x \in \mathbb{C}^r$, we denote by $\mathcal{D}_{\rho}(z)$ the open ball (i.e. the polydisk) centered in z of radius ρ . We also denote the induced norm of (multi-)linear operators by $\|\cdot\|$.

Let (e_i) be the canonical basis of \mathbb{C}^r , and denote the coordinates by x_1, \ldots, x_r . If $x \in \mathcal{U}$, then we have

$$df(x) = \sum_{i=1}^{r} \frac{\partial f}{\partial x_i}(x) dx_i,$$

where dx_i is seen as the linear form $x \mapsto x_i$. For $\eta > 0$ such that $\mathcal{D}_{\eta}(x) \subset \mathcal{U}$, we also define

$$FD_{\eta} f(x) = \sum_{i=1}^{r} \frac{f(x + \eta e_i) - f(x)}{\eta} dx_i.$$

This linear operator is an approximation of df(x) using finite differences.

Assume we already know $x \in \mathcal{U}$ such that $||x - x_0|| \le \varepsilon$ for some $\varepsilon > 0$. Then we can formulate a Newton iteration step to refine the approximation x of x_0 as follows: simply replace x by x + h, where

$$h = df(x)^{-1}(f(x_0) - f(x)).$$

In the finite differences version, we take instead:

$$h = FD_n f(x)^{-1} (f(x_0) - f(x)),$$

where $\eta > 0$ is a suitably chosen small parameter. Then, provided that ε is small enough, $||x + h - x_0||$ will be of the order of ε^2 , ensuring quadratic convergence of the Newton iteration.

Proposition 2.1. Let $\mathcal{U} \subset \mathbb{C}^r$ be an open set, let $f: \mathcal{U} \to \mathbb{C}^r$ be an analytic function, and let $x_0 \in \mathcal{U}$. Let $\rho > 0$ and $B_1, B_2, B_3 \geq 1$ be real numbers such that $\mathcal{D}_{\rho}(x_0) \subset \mathcal{U}$ and the following inequalities are satisfied:

- 1. $||df(x)|| \le B_1 \text{ and } ||d^2f(x)|| \le B_2 \text{ for all } x \in \mathcal{D}_{\rho}(x_0);$
- 2. $df(x_0)$ is invertible and $||df(x_0)^{-1}|| \le B_3$.

Let $\varepsilon, \eta > 0$ be such that

$$\varepsilon \le \min\left\{\frac{\rho}{2}, \frac{1}{2B_2B_3}\right\} \quad and \quad \eta \le \frac{\varepsilon}{4rB_1B_3}.$$

Then, for each $x \in \mathbb{C}^r$ such that $||x - x_0|| \le \varepsilon$, if we set

$$h = FD_{\eta} f(x)^{-1} (f(x_0) - f(x)),$$

we will have

$$||x + h - x_0|| \le 2B_2 B_3 \varepsilon^2.$$

Proof. First, note that

$$||df(x) - df(x_0)|| \le B_2 ||x - x_0|| \le B_2 \varepsilon \le \frac{1}{2||df(x_0)^{-1}||},$$

so df(x) is also invertible, with $||df(x)^{-1}|| \le 2B_3$. We can now study the "usual" Newton scheme. Let us write

$$f(x_0) = f(x) + df(x)(x_0 - x) + v,$$

for some vector v such that $||v|| \leq \frac{1}{2}B_2\varepsilon^2$. Let $h_0 = df(x)^{-1}(f(x_0) - f(x))$. Then

$$||x + h_0 - x_0|| = ||df(x)^{-1}v|| \le B_2 B_3 \varepsilon^2.$$
(2)

Finally, we show that h is close to h_0 . Since $\mathcal{D}_{\eta}(x) \subset \mathcal{D}_{\rho}(x_0)$ (because $\eta \leq \varepsilon \leq \rho/2$), we have for each $1 \leq j \leq r$:

$$\left| \frac{f(x + \eta e_i) - f(x)}{\eta} - \frac{\partial f}{\partial x_i}(x) \right| \le \frac{1}{2} B_2 \eta.$$

Therefore,

$$\|\operatorname{FD}_{\eta} f(x) - df(x)\| \le \frac{r}{2} B_2 \eta \le \frac{1}{4B_3} \le \frac{1}{2\|df(x)^{-1}\|},$$

so that

$$\|\operatorname{FD}_{\eta} f(x)^{-1} - df(x)^{-1}\| \le 2\|df(x)^{-1}\|^2 \cdot \|\operatorname{FD}_{\eta} f(x) - df(x)\| \le 4rB_2B_3^2\eta,$$

and

$$||h - h_0|| \le 4rB_2B_3^2\eta ||f(x) - f(x_0)|| \le 4rB_1B_2B_3^2\eta\varepsilon \le B_2B_3\varepsilon^2.$$
 (3)

We obtain the result from (2), (3), and the triangle inequality.

Cauchy's integration formula [14, Thm. 2.2.1] provides uniform upper bounds on ||df(x)|| and $||d^2f(x)||$ for $x \in \mathcal{D}_{\rho}(x_0)$ whenever a uniform upper bound on ||f|| on a slightly larger polydisk is known; this makes the necessary data in Proposition 2.1 easier to collect.

Proposition 2.2. Let $r, s \ge 1$, let $x_0 \in \mathbb{C}^r$, let $\rho > 0$, and let $f : \mathcal{D}_{\rho}(x_0) \to \mathbb{C}^s$ be an analytic function. Let $M \ge 0$ such that $||f(x)|| \le M$ for all $x \in D_{\rho}(x_0)$. Then for every $n \ge 0$ and every $x \in \mathcal{D}_{\rho/2}(x_0)$, we have

$$||d^n f(x)|| \le \frac{2^n n!}{\rho^n} \binom{n+r}{r} M.$$

Proof. It is enough to prove that

$$||d^n f(x_0)|| \le \frac{n!}{\rho^n} {n+r \choose r} M$$

for all n; afterwards, we simply note that $D_{\rho/2}(x) \subset \mathcal{D}_{\rho}(x_0)$ for each $x \in \mathcal{D}_{\rho/2}(x_0)$. Write $x_0 = (z_1, \dots, z_r)$. We compute the Taylor expansion of f at x_0 using Cauchy's formula. For each $\zeta = (\zeta_1, \dots, \zeta_r) \in \mathcal{D}_{\rho/2}(x_0)$, we have

$$f(\zeta) = \sum_{n=(n_1,...,n_r) \in \mathbb{N}^r} a_n(f) \prod_{j=1}^r (\zeta_j - z_j)^{n_j},$$

where the Taylor coefficients $a_n(f) \in \mathbb{C}^s$ are computed as follows:

$$a_n(f) = \frac{1}{(2\pi i)^r} \int_{\partial \mathcal{D}_o(z_1)} \cdots \int_{\partial \mathcal{D}_o(z_r)} \frac{f(x_1, \dots, x_r)}{\prod_{j=1}^r (x_j - z_j)^{n_j + 1}} dx_1 \cdots dx_r.$$

In particular,

$$||a_n(f)|| \le \frac{M}{\rho^{\sum_j n_j}}.$$

Now, for each $v \in \mathbb{C}^r$, the value of $d^n f(x_0)(v, \dots, v) \in \mathbb{C}^s$ is given by all terms of total degree n in the Taylor expansion, up to a factor of n!:

$$d^{n} f(x_{0})(v, \dots, v) = n! \sum_{m \in \mathbb{N}^{r}, \sum m_{j} = n} a_{m}(f) \prod_{j=1}^{r} v_{j}^{m_{j}}.$$

There are exactly $\binom{n+r}{r}$ terms in the sum. Since $d^n f(x_0)$ is a symmetric operator, the result follows easily.

In order to run certified Newton iterations on a computer, showing a theoretical convergence result is not enough: we also have to consider precision losses, which will for instance prevent us from choosing η too close to zero. Thankfully, Newton iterations are self-correcting, and precision losses can be controlled by taking an additional, explicit safety margin.

We adopt the following computational model for complex numbers. Dyadic elements of \mathbb{C}^r (i.e. elements of $2^{-N}\mathbb{Z}[i]^r$ for some $N \in \mathbb{Z}$) are represented exactly; and for a general $z \in \mathbb{C}^r$, we call an approximation of z to precision N a dyadic z' such that $||z-z'|| \leq 2^{-N}$. Elementary operations on approximations of complex numbers can be carried out using ball arithmetic [27]. Recall that a function $C: \mathbb{Z}_{\geq 1} \to \mathbb{R}_{\geq 0}$ is called superlinear if $C(m+n) \geq C(m) + C(n)$ for all $m, n \in \mathbb{Z}_{\geq 1}$.

Theorem 2.3. Let $\mathcal{U} \subset \mathbb{C}^r$ be an open set, let $f: \mathcal{U} \to \mathbb{C}^r$ be an analytic function, and let $x_0 \in \mathcal{U}$. Let $\rho \leq 1, M \geq 1$, and $B_3 \geq 1$ be real numbers such that $D_{\rho}(x_0) \subset \mathcal{U}$, $||f(x)|| \leq M$ for each $x \in \mathcal{D}_{\rho}(x_0)$, and $||df(x_0)^{-1}|| \leq B_3$. Let $C: \mathbb{Z}_{\geq 1} \to \mathbb{R}$ be a superlinear function such that the following holds:

- there exists an algorithm A which, given an exact $x \in \mathcal{D}_{\rho}(x_0)$ and $N \geq 0$, computes an approximation of f(x) to precision N in C(N) binary operations;
- two N-bit integers can be multiplied in C(N) binary operations;
- we have $C(2N) \leq KC(N)$ for some $K \geq 1$ and for all N sufficiently large.

Then, given $N \geq 0$, an approximation of $f(x_0)$ to precision N, and an approximation of x_0 to precision

$$n_0 = 2\lceil \log_2(2(r+1)M/\rho) \rceil + 2\lceil \log_2(B_3) \rceil + 4,$$

Algorithm 2.4 below computes an approximation of x_0 to precision $N - \lceil \log_2(B_3) \rceil - 1$ in O(C(N)) binary operations; the hidden constant in this complexity bound depends only on r, ρ, M, B_3 , and K.

We now describe the algorithm. Let

$$B_1 = \frac{2(r+1)M}{\rho}$$
 and $B_2 = \frac{2(r+1)(r+2)M}{\rho^2}$.

By Proposition 2.2, the real numbers $\rho/2$, B_1 , B_2 , B_3 meet the conditions of Proposition 2.1. Up to decreasing ρ and increasing B_1 , B_2 , B_3 , we may assume that they are all powers of 2. Denote the given dyadic approximation of $f(x_0)$ by z_0 .

Algorithm 2.4 (Certified Newton iterations for analytic functions).

- 1. Let $n = n_0$, and let x be the given dyadic approximation of x_0 to precision n.
- 2. While n < N, do:
 - (a) Let $m = n + \log_2(B_1) + \log_2(B_3) + \lceil \log_2(r) \rceil + 2$, and $\eta = 2^{-m}$;
 - (b) Using algorithm \mathcal{A} , compute approximations of f(x) and $f(x + \eta e_j)$ for all $1 \le j \le r$ to precision $p = 2n + 2\lceil \log_2(r) \rceil + 2\log_2(B_1) + 2\log_2(B_3) + 9$;
 - (c) Compute an approximation of the $r \times r$ matrix M_1 whose jth column contains the finite difference $\frac{1}{\eta}(f(x+\eta e_j)-f(x))$, for all j, to precision $p-\log_2(1/\eta)-1$ (entrywise);

- (d) Compute an approximation of the $r \times r$ matrix $M_2 = M_1^{-1}$ to precision $p' = p \log_2(1/\eta) 2\log_2(B_3) 7;$
- (e) Compute an approximation of the vector $h = M_2(z_0 f(x))$ to precision $p' + n 1 \log_2(B_1) \lceil \log_2(r) \rceil$;
- (f) Let $n' = 2n \log(B_2) \log(B_3) 2$; replace x by a dyadic approximation of x + h to precision n' + 1, and replace n by n'.

3. Return x.

Proof of Theorem 2.3. We will show that the different quantities appearing in Algorithm 2.4 can be computed to the claimed precisions, and that x remains an approximation of $f^{-1}(z_0)$ to precision n. Since $||df(x)^{-1}|| \leq 2B_3$ for all $x \in \mathcal{D}_{\rho}(x_0)$, the result will be an approximation of x_0 to precision $N - \log_2(B_3) - 1$, as claimed.

At the beginning of each loop, x is dyadic, and so are the $x+\eta e_j$ for each $1 \leq j \leq r$. Therefore, each entry of M_1 can be computed to precision $p - \log_2(1/\eta) - 1$. Note that $\|\operatorname{FD}_{\eta} f(x)^{-1}\| \leq 4B_3$ as a linear operator. Let M'_1 be a dyadic approximation of M_1 to precision $p - \log_2(1/\eta)$; then we have

$$||M_1 - M_1'|| \le \frac{1}{2||M_1^{-1}||},$$

so that $||M_1^{-1} - M_1'^{-1}|| \le 2||M_1^{-1}||^2||M_1 - M_1'|| \le 32B_3^2 2^{-p}/\eta$. This shows that M_1^{-1} can be computed to the required precision p' in step (2d). In step (2e), we perform the matrix-vector product using the schoolbook formula. The entries of M_2 have modulus at most $4B_3$, and are known up to precision p'; the entries of $z_0 - f(x)$ have modulus at most $2^{-n}B_1$, and are known up to precision p-1. The total error on the product can be bounded above by

$$r(4B_32^{-p+1} + 2^{-n}B_12^{-p'} + 2^{-p'-p-1}) \le 2^{-n+1}rB_12^{-p'}.$$

The precision p was chosen in such a way that we obtain, at the end of the loop, an approximation of x + h to precision $2n - \log(B_3) - 1 \ge n' + 1$. By Proposition 2.1, the result is also an approximation of $f^{-1}(z_0)$ to precision n'.

The initial value of n_0 ensures that n' > 3n/2, so that number of steps in the loop is $O(\log N)$. Each loop involves a finite number of elementary operations with complex numbers of modulus O(1) at precision 2n+O(1), where the hidden constants depend only on r, ρ, M , and B_3 ; the cost of these computations is O(C(n)) binary operations. Since C is superlinear, the cost of the last loop dominates the cost of the whole algorithm, a well-known feature of Newton's method.

3 Borchardt means as analytic functions

The existing Newton schemes for the computation of theta functions [6, 5, 21, 22] are based on *Borchardt means*, a higher-dimensional analogue of the classical arithmetic-geometric mean (AGM) [3]. Additional references for the study of Borchardt means, especially in genus 2, are [2, 17]. Our goal in this section is to study Borchardt means as analytic functions in detail, obtaining explicit bounds on their magnitudes and radii of convergence.

3.1 Borchardt sequences

Fix $g \ge 1$, and let $\mathcal{I}_g = (\mathbb{Z}/2\mathbb{Z})^g$. A Borchardt sequence of genus g is by definition a sequence of complex numbers

$$s = \left(s_b^{(n)}\right)_{b \in \mathcal{I}_g, n \ge 0}$$

that satisfy the following recurrence relation: for every $n \geq 0$, there exists a choice of square roots $(t_b^{(n)})_{b \in \mathcal{I}_q}$ of $(s_b^{(n)})_{b \in \mathcal{I}_q}$ such that for all $b \in \mathcal{I}_g$, we have

$$s_b^{(n+1)} = \frac{1}{2^g} \sum_{b_1 + b_2 = b} t_{b_1}^{(n)} t_{b_2}^{(n)}. \tag{4}$$

We say that $(s_b^{(n+1)})_{b\in\mathcal{I}_g}$ is the result of a Borchardt step given by the choice of square roots $(t_b^{(n)})_{b\in\mathcal{I}_g}$ at the n^{th} term. This recurrence relation emulates the duplication formula satisfied by theta constants [23, p. 214], after identifying $\{0,1\}^g$ with \mathcal{I}_g in the natural way: for every $\tau \in \mathcal{H}_g$, the sequence of squared theta constants

$$\left(\theta_{0,b}^2(0,2^n\tau)\right)_{b\in\mathcal{I}_q,n\geq 0}\tag{5}$$

is a Borchardt sequence.

The convergence behavior of Borchardt sequences is similar to that of the classical AGM [5, §7.2]. Let us define a set of complex numbers to be *in good position* if it is included in an open quarter plane seen from the origin, i.e. a set of the form

$$\left\{r\exp(i\theta)\colon r>0, \alpha<\theta<\alpha+\frac{\pi}{2}\right\}$$

for some angle $\alpha \in \mathbb{R}$. We say that the n^{th} step of a Borchardt sequence is given by $good\ sign\ choices$ (or for short, is good) if the square roots $\left(t_b^{(n)}\right)_{b\in\mathcal{I}_g}$ are in good position; otherwise, we say that this step is bad. Then a Borchardt sequence s will

converge to (0, ..., 0) if and only if s contains infinitely many bad steps. On the other hand, a Borchardt sequence s in which all steps are good after a while converges to a limit of the form $(\mu, ..., \mu)$ for some $\mu \neq 0$, and the speed of convergence is quadratic; we call $\mu = \mu(s)$ the Borchardt mean of the sequence. Borchardt sequences given by theta functions as in (5) are of this second type: see for instance [5, Prop. 6.1].

A related kind of recurrent sequence is used in the context of computing theta functions. Let us call an *extended Borchardt sequence* of genus g a pair (u, s) of sequence of complex numbers

$$u = (u_b^{(n)})_{b \in \mathcal{I}_g, n \ge 0}, \qquad s = (s_b^{(n)})_{b \in \mathcal{I}_g, n \ge 0}$$

satisfying the following recurrence relation: for every $n \geq 0$, there exists a choice of square roots $(v_b^{(n)})_{b \in \mathcal{I}_g}$ of $(u_b^{(n)})_{b \in \mathcal{I}_g}$ and $(t_b^{(n)})_{b \in \mathcal{I}_g}$ of $(s_b^{(n)})_{b \in \mathcal{I}_g}$ such that for all b,

$$u_b^{(n+1)} = \frac{1}{2^g} \sum_{b_1 + b_2 = b} v_{b_1}^{(n)} t_{b_2}^{(n)} \quad \text{and} \quad s_b^{(n+1)} = \frac{1}{2^g} \sum_{b_1 + b_2 = b} t_{b_1}^{(n)} t_{b_2}^{(n)}. \tag{6}$$

In particular, s is a regular Borchardt sequence. We say that the n^{th} step in (u,s) is good if both of the sets $(v_b^{(n)})_{b\in\mathcal{I}_g}$ and $(t_b^{(n)})_{b\in\mathcal{I}_g}$ are independently in good position, and bad otherwise. For each $\tau\in\mathcal{H}_g$ and $z\in\mathbb{C}^g$, the duplication formula for theta functions implies that the sequence

$$(\theta_{0,b}^2(z,2^n\tau),\theta_{0,b}^2(0,2^n\tau))_{b\in\mathcal{I}_g,n\geq 0}$$

is an extended Borchardt sequence; it contains only finitely many bad steps as well.

It is not true in general that an extended Borchardt sequence containing finitely many bad steps converges quadratically. Instead, following [22], we define the extended Borchardt mean of such a sequence (u, s) to be

$$\lambda(u,s) = \mu(s) \cdot \lim_{n \to +\infty} \left(\frac{u_0^{(n)}}{s_0^{(n)}} \right)^{2^n} = \mu(s) \cdot \lim_{n \to +\infty} \left(\frac{u_0^{(n)}}{\mu(s)} \right)^{2^n}. \tag{7}$$

These associated sequences do converge quadratically [22, Prop. 3.7].

Assume that we are given a Borchardt sequence s containing finitely many bad steps. Then we may try to construct a function μ_s , defined at any point $x = (x_b)_{b \in \mathcal{I}_g}$ in some neighborhood of $(s_b^{(0)})_{b \in \mathcal{I}_g}$, by the following procedure: "construct a modified Borchardt sequence whose first term is (x_b) that follows same choices of square roots as in s, and take its Borchardt mean". The Newton schemes we want to study are precisely built around this kind of functions μ_s , and their analogues for extended Borchardt means. In the rest of this section, we show that these functions indeed exist as analytic functions defined on explicit polydisks, provided that all terms in the relevant Borchardt sequences are bounded away from zero.

3.2 The case of good sign choices

Let s be a Borchardt sequence containing good steps only. Then we can find real numbers $0 < m_0 < M_0$ and α such that such that the first term $(s_b^{(0)})_{b \in \mathcal{I}_g}$ of s lies in the open set $\mathcal{U}_q(m_0, M_0)$ of \mathbb{C}^{2^g} defined as follows:

$$\mathcal{U}_g(m_0, M_0) = \bigcup_{\alpha \in [0, 2\pi]} \mathcal{U}_g(m_0, M_0, \alpha),$$

where

$$\mathcal{U}_g(m_0, M_0, \alpha) = \{(x_b)_{b \in \mathcal{I}_g} \colon \forall b \in \mathcal{I}_g, m_0 < \operatorname{Re}(e^{-i\alpha}x_b) < M_0 \}.$$

Proposition 3.1. Let $0 < m_0 < M_0$ be real numbers. Then there exists a unique analytic function $\mu \colon \mathcal{U}_g(m_0, M_0) \to \mathbb{C}$ with the following property: for every point $x = (x_b)_{b \in \mathcal{I}_g} \in \mathcal{U}_g(m_0, M_0)$, the value of μ at x is the Borchardt mean of the unique Borchardt sequence with first term x given by good steps only. Moreover, the inequalities $m_0 \le |\mu(x)| \le M_0$ hold for all $x \in \mathcal{U}_g(m_0, M_0)$.

Proof. For each $x \in \mathcal{U}_g(m_0, M_0, \alpha)$, there is a unique way of making a good Borchardt step starting from x; moreover the result of this Borchardt step still lands in $\mathcal{U}_g(m_0, M_0, \alpha)$ by [5, Lem. 7.3]. Therefore we may define $\mu(x)$ as the limit of the resulting Borchardt sequence; we have $m_0 \leq |\mu(x)| \leq M_0$. Since there exists an analytic square root function on $\mathcal{U}_1(m_0, M_0, \alpha)$, the function μ on $\mathcal{U}_g(m_0, M_0, \alpha)$ is the pointwise limit of a sequence of analytic functions. The convergence is uniform on compact sets by [5, Prop. 7.2], so μ is analytic on the whole of $\mathcal{U}_g(m_0, M_0)$.

We now consider the case of extended Borchardt means given by good choices of square roots only. This case is easier to analyse if we assume that the truly Borchardt part of the sequence already starts in the quadratic convergence area. By [5, Prop. 7.1], if we have

$$\left| s_b^{(n)} - s_0^{(n)} \right| < \frac{\varepsilon}{4} \left| s_0^{(n)} \right| \tag{8}$$

for some $\varepsilon \leq 1/2$, then we have

$$\left| s_b^{(n+k)} - s_0^{(n+k)} \right| \le \frac{2}{7} \left(\frac{7\varepsilon}{8} \right)^{2^k} \cdot \max_{b \in \mathcal{I}_g} \left| s_b^{(n)} \right|$$

for all $k \geq 0$ and $b \in \mathcal{I}_g$. If we assume that the first term of s lies in a ball of the form $\mathcal{D}_{\rho}(z_0)$ for some $z_0 \in \mathbb{C}^{\times}$ and $0 < \rho < \frac{1}{17}|z_0|$, then inequality (8) will be satisfied with $\varepsilon = \frac{1}{2}$ at n = 0.

Proposition 3.2. Let $0 < m_0 < M_0$ be real numbers, fix a nonzero $z_0 \in \mathbb{C}$, and let $0 < \rho < \frac{1}{17}|z_0|$. Then there exists a unique analytic function

$$\lambda : \mathcal{U}_g(m_0, M_0) \times \mathcal{D}_{\rho}(z_0)^{2^g} \to \mathbb{C}$$

with the following property: for every (x,y) in this open set, $\lambda(x,y)$ is equal to the extended Borchardt mean of any extended Borchardt sequence with first term (x,y) given by good steps only. Moreover, we have

$$\exp\left(-28\log^2(4M/m)\right) \le |\lambda(x,y)| \le \exp\left(20\log^2(4M/m)\right)$$

where $M = \max\{|z_0| + \rho, M_0, 1\}$ and $m = \min\{|z_0| - \rho, m_0, 1\}$.

Proof. We follow the proof of [21, Thm. 3.10], and hints on how to generalize it to higher genera given in [22, Prop. 3.7]. We may fix $\alpha \in \mathbb{R}$ and restrict our attention to $\mathcal{U}_q(m_0, M_0, \alpha)$.

First of all, by the proof of [21, Lem. 3.8], each $(x,y) \in \mathcal{U}_g(m_0, M_0, \alpha) \times \mathcal{D}_{\rho}(z_0)^{2^g}$ is the starting point of at least one extended Borchardt sequence (u,s) with good sign choices at all steps. Any two such sequences differ at the n^{th} term by global multiplication $(u_b^{(n)})_{b \in \mathcal{I}_g}$ by a 2^n -th root of unity; therefore, their extended Borchardt means are equal. Note that M (resp. m) is an upper (resp. lower) bound on the modulus of all complex numbers appearing in these extended Borchardt sequences. In the rest of this proof, we fix $\theta_0 \in \mathbb{R}$ such that $\theta_0 = \arg(z_0) \mod 2\pi$, and consider the unique such sequence (u,s) whose n^{th} term lies in

$$\mathcal{U}_g\left(m, M, \frac{\alpha + (2^n - 1)\theta_0}{2^n}\right) \times \mathcal{U}_g(m, M, 0).$$

Each term of (u, s) is an analytic function of its starting point (x, y).

By construction, we have for all $n \geq 0$:

$$\left| s_b^{(n)} - s_0^{(n)} \right| < 2^{-2^n} |z_0|.$$

Let μ be the Borchardt mean of s. For $n \geq 1$, write

$$q_n = \frac{(u_0^{(n)})^2}{u_0^{(n-1)}\mu},$$

so that for all $k \geq 0$, we have

$$\lambda(x,y) = \left(\frac{u_0^{(k)}}{\mu}\right)^{2^k} \prod_{n \ge k} q_{n+1}^{2^n}.$$

These complex numbers q_n converge quadratically fast to 1. To be more explicit, we have for all $n \geq 1$:

$$|u_0^{(n+1)} - v_0^{(n)} t_0^{(n)}| \le \frac{\sqrt{M}}{2^g} \sum_{b \in \mathcal{I}_g} \left(\left| t_b^{(n)} - t_0^{(n)} \right| + \left| v_b^{(n)} - v_0^{(n)} \right| \right)$$

$$\le \frac{\sqrt{M}}{2^g \cdot 2\sqrt{m}} \sum_{b \in \mathcal{I}_g} \left(\left| s_b^{(n)} - s_0^{(n)} \right| + \left| u_b^{(n)} - u_0^{(n)} \right| \right)$$

$$\le \frac{\sqrt{M}}{2\sqrt{m}} \left(2^{-2^n} |z_0| + \frac{1}{2^g} \sum_{b \in \mathcal{I}_g} \left| u_b^{(n)} - u_0^{(n)} \right| \right).$$

To bound the remaining sum, we write

$$\begin{aligned} \left| u_b^{(n)} - u_0^{(n)} \right| &\leq \frac{\sqrt{M}}{2^g} \sum_{b' \in \mathcal{I}_g} \left| t_{b+b'}^{(n-1)} - t_b^{(n-1)} \right| \\ &\leq \frac{\sqrt{M}}{2^g \cdot 2\sqrt{m}} \sum_{b' \in \mathcal{I}_g} \left| s_{b+b'}^{(n-1)} - s_b^{(n-1)} \right| &\leq \frac{\sqrt{M}}{\sqrt{m}} \, 2^{-2^{n-1}} |z_0|. \end{aligned}$$

Therefore, we have for all $n \geq 1$

$$\left| u_0^{(n+1)} - v_0^{(n)} t_0^{(n)} \right| \le \frac{5}{4} \sqrt{\frac{M}{m}} \ 2^{-2^{n-1}} |z_0| =: B \cdot 2^{-2^{n-1}}.$$

We deduce as in [21, Thm. 3.10] that

$$|q_{n+1} - 1| \le B' \cdot 2^{-2^{n-1}}$$

where

$$B' = 2|z_0| + \frac{1}{m^2}(2MB + B^2) \le \frac{5M^3}{m^3}.$$

Let $k \geq 1$ be minimal such that $B' \cdot 2^{-2^{k-1}} \leq \frac{1}{2}$. Then we have

$$\sum_{n\geq k} 2^n \log |q_{n+1}| \leq \sum_{n\geq k} 2^n \cdot \frac{1}{2} \cdot 2^{2^{k-1}} \cdot 2^{-2^{n-1}} \leq 2^k.$$

This proves that the sequence (7) converges; since our estimates are uniform, λ must be analytic. Moreover,

$$|\lambda(x,y)| = \left|\frac{u_0^{(k)}}{\mu}\right|^{2^k} \prod_{n\geq k} |q_{n+1}|^{2^n} \leq \exp\left(2^k \left(1 + \log(M/m)\right)\right).$$

We obtain the final upper bound on $|\lambda(x,y)|$ from the inequality $2^k \leq 4(1+\log_2(B'))$, after some further simplifications. The lower bound comes from the inequality

$$\sum_{n \ge k} 2^n \log |q_{n+1}| \ge -2^k \cdot 2 \log(2)$$

in a similar way.

3.3 The general case

Let s be a Borchardt sequence containing finitely many bad steps. We now construct the "Borchardt mean following s" in a neighborhood of the first term of s as an analytic function, provided that s contains no zero value. To make things explicit, we introduce the following quantities:

- a real number $M_0 > 0$ such that $|s_b^{(0)}| < M_0$ for all $b \in \mathcal{I}_g$;
- an integer n_0 such that all steps in s of index $n \ge n_0$ are good;
- a real number $m_{\infty} > 0$ such that $\left(s_b^{(n_0)}\right)_{b \in \mathcal{I}_g} \in \mathcal{U}_g(m_{\infty}, M_0)$ in the notation of §3.2;
- for each $0 \le n \le n_0 1$, a real number $m_n > 0$ such that $\left| s_b^{(n)} \right| > m_n$ for all $b \in \mathcal{I}_a$.

For each $n \leq n_0 - 1$, we also let $(t_b^{(n)})_{b \in \mathcal{I}_g}$ be a collection of square roots of $(s_b^{(n)})_{b \in \mathcal{I}_g}$ such that the $n + 1^{\text{st}}$ term of s is given by the recurrence relation (4).

It will be useful to introduce Borchardt steps as analytic maps, besides the case of good sign choices. Let $z=(z_b)_{b\in\mathcal{I}_g}\in\mathbb{C}^{2^g}$; assume that 0< m< M are real numbers such that $m<|z_b|^2< M$ for all b. Then for each $b\in\mathcal{I}_g$, there exists a unique analytic square root map $\operatorname{sqrt}_{z_b}$ on the disk $\mathcal{D}_{m/2}(z_b^2)$ which maps z_b^2 to z_b . Thus, we have a well-defined analytic map

$$BStep_z: \prod_{b \in \mathcal{I}_g} \mathcal{D}_{m/2}(z_b^2) \to \mathbb{C}^{2^g}$$

A quick calculation shows that $||dBStep_z|| \leq \sqrt{(2M+m)/m}$ uniformly on its open set of definition.

Lemma 3.3. Given s and the quantities listed above, let

$$\rho = \min \left\{ \frac{m_0}{2}, \frac{m_1}{2} \sqrt{\frac{m_0}{2M_0 + m_0}}, \cdots, \frac{m_\infty}{2} \prod_{j=0}^{n_0 - 1} \sqrt{\frac{m_j}{2M_0 + m_j}} \right\}. \tag{9}$$

Let $s^{(0)} = (s_b^{(0)})_{b \in \mathcal{I}_g}$ be the first term of s, and let $x \in \mathcal{D}_{\rho}(s^{(0)})$. Then there exists a unique Borchardt sequence s' with the following properties:

- 1. the first term of s' is x;
- 2. for all $0 \le n \le n_0 1$ and all $b \in \mathcal{I}_g$, we have $\left| s'_b^{(n)} s_b^{(n)} \right| < \frac{1}{2}m_n$; moreover the $n+1^{\text{st}}$ term of s' is the result of a Borchardt step with choice of square roots $\operatorname{sqrt}_{t_b^{(n)}}(s'_b^{(n)})$ for all $b \in \mathcal{I}_g$;
- 3. for all $n \ge n_0$, the $n + 1^{st}$ term of s' is the result of a Borchardt step from the previous term with good sign choices.

Proof. We proceed by induction, using the above estimate on derivatives of Borchardt steps for $n \le n_0 - 1$.

Proposition 3.4. Given s and the quantities listed above, let $s^{(0)} = (s_b^{(0)})_{b \in \mathcal{I}_g}$ be the first term of s, and define $\rho > 0$ as in (9). Then there exists a unique analytic function $\mu_s \colon \mathcal{D}_{\rho}(s^{(0)}) \to \mathbb{C}$ with the following property: for each $x \in \mathcal{D}_{\rho}(s^{(0)})$, the value of μ_s at x is the Borchardt mean of the sequence defined in Lemma 3.3. We have $\frac{1}{2}m_{\infty} \leq |\mu_s(x)| \leq M_0 + \rho$ for all $x \in \mathcal{D}_{\rho}(s^{(0)})$.

Proof. By Lemma 3.3, the function μ_s is obtained as the composition of a finite number of analytic Borchardt steps, followed by an analytic Borchardt mean as defined in Proposition 3.1. The upper bound on $|\mu_s(x)|$ comes from the fact that $||x|| \leq M_0 + \rho$. For the lower bound, we remark that the n_0^{th} term of the Borchardt sequence of Lemma 3.3 lands in $\mathcal{U}_g(\frac{1}{2}m_\infty, M_0 + \rho)$.

We extend this result to the case of extended Borchardt means. Let (u, s) be an extended Borchardt sequence containing finitely many bad steps. Assume that we are given:

- a disk $\mathcal{D}_{\rho}(z_0) \subset \mathbb{C}$ such that $\rho < \frac{1}{17}|z_0|$ (for instance, z_0 and ρ may be dyadic);
- An integer n_0 such that all values in $s_b^{(n_0)}$ lie in $\mathcal{D}_{\rho}(z_0)$, and after which all sign choices in (u, s) are good;

- A real number $M_0 > 1$ such that $\left| s_b^{(0)} \right| < M_0$ and $\left| u_b^{(0)} \right| < M_0$ for all $b \in \mathcal{I}_g$, and $M_0 > |z_0| + \rho$;
- a real number $0 < m_{\infty} < 1$ such that the n_0^{th} term of u lies in $\mathcal{U}_g(m_{\infty}, M_0)$, and $m_{\infty} < |z_0| \rho$;
- For each $0 \le n \le n_0 1$, a real number $m_n > 0$ such that $|s_b^{(n)}| > m_n$ and $|u_b^{(n)}| > m_n$ for all $b \in \mathcal{I}_g$.

For each $n \leq n_0 - 1$, we also let $(t_b^{(n)})_{b \in \mathcal{I}_g}$ and $(v_b^{(n)})_{b \in \mathcal{I}_g}$ be collections of square roots of $(s_b^{(n)})_{b \in \mathcal{I}_g}$ and $(u_b^{(n)})_{b \in \mathcal{I}_g}$ respectively such that the $n + 1^{\text{st}}$ term of (u, s) is given by the recurrence relation (6).

The following lemma and proposition are proved by the same methods we used for regular Borchardt means, and we omit their proofs.

Lemma 3.5. Given (u, s) and the quantities listed above, let

$$\rho = \min_{0 \le n \le n_0} \left(\frac{m_n}{2} \prod_{j=0}^{n-1} \sqrt{\frac{m_n}{2M_0 + m_n}} \right), \tag{10}$$

with the convention that $m_{n_0} = m_{\infty}$. Let $(u^{(0)}, s^{(0)})$ be the first term of (u, s), and let $(x, y) \in \mathcal{D}_{\rho}((u^{(0)}, s^{(0)}))$. Then there exist extended Borchardt sequences (u', s') with the following properties:

- 1. the first term of (u', s') is (x, y);
- 2. for each $0 \le n \le n_0 1$ and each $b \in \mathcal{I}_q$, we have

$$\left| s_b^{\prime (n)} - s_b^{(n)} \right| < \frac{1}{2} m_n \quad and \quad \left| u_b^{\prime (n)} - u_b^{(n)} \right| < \frac{1}{2} m_n;$$

moreover the $n+1^{\text{st}}$ term of (u',s') is the result of an extended Borchardt step with choices of square roots $\operatorname{sqrt}_{t_i^{(n)}}(s'_b^{(n)})$ and $\operatorname{sqrt}_{v_i^{(n)}}(u'_b^{(n)})$ for all $b \in \mathcal{I}_g$;

3. for all $n \ge n_0$, the $n + 1^{st}$ term of (u, s) is obtained from the previous one by an extended Borchadt step with good sign choices.

These extended Borchardt sequences coincide up to their n_0 th terms, and their extended Borchardt means are equal.

Proposition 3.6. Given (u, s) and the quantities listed above, let $(u^{(0)}, s^{(0)})$ be the first term of (u, s), and define $\rho > 0$ as in (10). Then there exists a unique analytic function $\lambda_{(u,s)} \colon \mathcal{D}_{\rho}(z_0) \to \mathbb{C}$ with the following property: for each $(x, y) \in \mathcal{D}_{\rho}(z)$, the value of $\lambda_{(u,s)}$ at x is the extended Borchardt mean of any of the extended Borchardt sequences defined in Lemma 3.5. Moreover, we have

$$\exp(-28\log^2(4M/m)) \le |\lambda_{u,s}(x,y)| \le \exp(20\log^2(4M/m))$$

where $m = \frac{1}{2}m_{\infty}$ and $M = M_0 + \rho$.

Remark 3.7. In [6, §6.1], [5, §7.4.2], [21, §3.4], and [22, Prop. 3.7] it is shown that the analytic functions μ , λ , μ_s and $\lambda_{(u,s)}$ that we just defined can be evaluated at any given complex point in quasi-linear time $O(\mathcal{M}(N)\log N)$ in the required precision, where $\mathcal{M}(N)$ denotes the cost of multiplying N-bit integers. In fact, these proofs show that these analytic functions can be evaluated in *uniform* quasi-linear time. In the case of μ_s and $\lambda_{(u,s)}$, the implied constant only depends on the auxiliary data listed in this section, not on the Borchardt sequences themselves.

4 Newton schemes for theta functions

In this section, we present the different Newton schemes used for the computation of theta constants and theta functions in genus 1 and 2 as well as possible extensions to higher genera, following [6, 5, 21, 22]. We formulate them in terms of the analytic Borchardt functions introduced in §3. In the three cases of theta functions in genus 1 and theta constants in genus 1 and 2, we are able to write down the inverse of the analytic function $\mathbb{C}^r \to \mathbb{C}^r$ used in the Newton scheme in an explicit way. This provides us with all the necessary data to apply the results of §2 and obtain explicit convergence results for these Newton schemes.

4.1 General picture

The Newton schemes we consider to compute theta constants at a given point $\tau \in \mathcal{H}_g$ use increasingly better approximations of the point

$$\Theta(\tau) = \left(\frac{\theta_{0,b}(0,\tau/2)}{\theta_{0,0}(0,\tau/2)}\right)_{b \in \mathcal{I}_q \setminus \{0\}} \in \mathbb{C}^{2^g - 1}.$$
 (11)

From this input, computing certain Borchardt means will provide approximations of the quantities $\theta_{0,b}^2(0,N\tau)$, for any symplectic matrix $N \in \operatorname{Sp}_{2g}(\mathbb{Z})$ that we might

choose. Recall that a matrix $N \in \operatorname{Sp}_{2g}(\mathbb{Z})$ with $g \times g$ blocks $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on \mathcal{H}_g as $N\tau = (a\tau + b)(c\tau + d)^{-1}$, and on $\mathbb{C}^g \times \mathcal{H}_g$ as $N \cdot (z,\tau) = ((c\tau + d)^{-t}z, N\tau)$, where $^{-t}$ denotes inverse transposition. The next proposition, derived from the works mentioned above, is key.

Proposition 4.1. Let $\tau \in \mathcal{H}_q$, let $z \in \mathbb{C}^g$, and let $\lambda, \mu \in \mathbb{C}^{\times}$. Then

1. The sequence

$$\left(\frac{\theta_{0,b}^2(0,2^n\tau)}{\mu}\right)_{b\in\mathcal{I}_a,n\geq 0}$$
(12)

is a Borchardt sequence with Borchardt mean $1/\mu$, obtained from the choice of square roots

$$\left(\frac{\theta_{0,b}(0,2^n\tau)}{\sqrt{\mu}}\right)_{b\in\mathcal{I}_g}$$

for some choice of $\sqrt{\mu}$, at each step.

2. All sequences of the form

$$\left(\frac{\theta_{0,b}^{2}(z,2^{n}\tau)}{\lambda^{2^{-n}}\mu^{1-2^{-n}}},\frac{\theta_{0,b}^{2}(0,2^{n}\tau)}{\mu}\right)_{b\in\mathcal{I}_{q},n\geq0}$$
(13)

with compatible choices of 2^{-n} -th roots (i.e. such that $(\lambda^{2^{-n-1}})^2 = \lambda^{2^{-n}}$ and $(\mu^{1-2^{-n-1}})^2 = \mu \cdot \mu^{1-2^{-n}}$ for all n) are extended Borchardt sequences with extended Borchardt mean $1/\lambda$; they precisely are the sequences obtained from choices of square roots of the form

$$\left(\frac{\theta_{0,b}(z,2^n\tau)}{\lambda^{2^{-n-1}}\mu^{(1-2^{-n})/2}},\frac{\theta_{0,b}(0,2^n\tau)}{\sqrt{\mu}}\right)_{b\in\mathcal{I}_g}$$

for some choice of square roots of $\mu, \lambda^{2^{-n}}$ and $\mu^{1-2^{-n}}$, at each step.

Consider first the case of theta constants. From the theta quotients (11), one can compute all squared theta quotients of the form $\theta_{a,b}^2(0,\tau)/\theta_{0,0}^2(0,\tau/2)$ using the duplication formula. Then, applying the transformation formulas under $\operatorname{Sp}_{2g}(\mathbb{Z})$ [23, §II.5] allows us to compute all theta quotients of the form $\theta_{0,b}^2(0,N\tau)/\theta_{0,0}^2(0,N\tau)$ for $b \in \mathcal{I}_g$. Finally, applying Proposition 4.1, (1) gives us access to $\mu = \theta_{0,0}^2(0,N\tau)$, so that we can recover all $\theta_{0,b}^2(0,N\tau)$, as promised. At the end of the algorithm, we apply the transformation formulas once more: the relations between squared theta values $\theta_{a,b}^2(0,\tau)$ and $\theta_{a,b'}^2(0,N\tau)$ involve a factor $\det(C\tau+D)$ where C,D are the

lower $g \times g$ blocks of N. These determinants are simple functions of the entries of τ , and we use this feedback in a Newton scheme to compute a better approximation of the initial theta quotients (11). When an appropriate precision is reached, we repeat the above process one last time to return approximations of the squared theta values $\theta_{a,b}^2(0,\tau)$.

In the case of theta functions, we consider the following larger set of theta quotients:

$$\Theta'(\tau) = \left(\frac{\theta_{0,b}(0,\tau/2)}{\theta_{0,0}(0,\tau/2)}, \frac{\theta_{0,b}(z,\tau/2)}{\theta_{0,b}(z,\tau/2)}\right)_{b \in \mathcal{I}_a \setminus \{0\}} \in \mathbb{C}^{2^{g+1}-2}.$$
 (14)

We obtain the theta quotients $\theta_{0,b}^2 \left(N \cdot (z,\tau) \right) / \theta_{0,0}^2 \left(N \cdot (z,\tau) \right)$ from the transformation formulas, and Proposition 4.1, (2) allows us to compute $\lambda = \theta_{0,0}^2 \left(N \cdot (z,\tau) \right)$. The feedback is again provided by transformation formulas, and involves simple functions (determinants and exponentials) in the entries of z and τ .

In order to run this algorithm, one has to make the correct choices of square roots each time Proposition 4.1 is applied. At the end of the loop, when using feedback on z and τ to obtain theta values at a higher precision, one assumes that the Jacobian matrix of the system is well-defined and invertible; in particular, it must be a square matrix. In practice, one computes an approximation of this Jacobian matrix using finite differences; the resulting Newton scheme is of the type studied in §2.

We close this presentation with a discussion on argument reduction. Before attempting to run these Newton schemes, one should reduce the input (z,τ) using symmetries of theta functions. Performing this reduction is necessary to even hope for algorithms with uniform complexities in (z,τ) . If $g \leq 2$, it is possible to use the action of $\operatorname{Sp}_{2g}(\mathbb{Z})$ on τ to reduce it to the Siegel fundamental domain $\mathcal{F}_g \subset \mathcal{H}_g$ defined by the following conditions [20, §I.3]:

- $\operatorname{Im}(\tau)$ is Minkowski-reduced;
- $|\operatorname{Re}(\tau_{i,j})| \leq 1/2$ for all $1 \leq i, j \leq g$;
- $|\det(C\tau + D)| \ge 1$ for all $g \times g$ matrices C, D forming the lower blocks of a symplectic matrix $N \in \operatorname{Sp}_{2g}(\mathbb{Z})$; in particular we have $|\tau_{i,i}| \ge 1$ for all $1 \le i \le g$, so that $\operatorname{Im}(\tau_{i,i}) \ge \sqrt{3}/2$.

The reduction algorithm is described in [26, §6].

In fact, it is possible to obtain useful information on values of theta functions, and to study the Newton schemes described above, without assuming that all the conditions defining \mathcal{F}_g hold: see for instance [26, Prop. 7.6]. On the other hand, we will additionally assume that the imaginary part of τ is bounded; this assumption is

necessary to show that the Newton schemes converge uniformly. Other inputs can be handled using duplication formulas and the naive algorithm: see [6, §6.3] and [21, §4.2] in the genus 1 case. We will adapt this strategy to obtain a uniform algorithm for genus 2 theta constants in §5.

The argument z can be reduced as well. By periodicity of the theta function $\theta(\cdot, \tau)$ with respect to the lattice $\mathbb{Z}^g + \tau \mathbb{Z}^g$ [23, §II.1], it is always possible to assume that $|\text{Re}(z_i)| \leq \frac{1}{2}$ for each i, and that

$$\begin{pmatrix} \operatorname{Im}(z_1) \\ \vdots \\ \operatorname{Im}(z_g) \end{pmatrix} = \operatorname{Im}(\tau) \begin{pmatrix} v_1 \\ \vdots \\ v_g \end{pmatrix}$$

for some vector $v \in \mathbb{R}^g$ such that $|v_i| \leq \frac{1}{2}$ for all i. Since duplication formulas relate the values of theta functions at z and 2z, we can in fact assume that z is very close to zero, for instance $|z_i| < 2^{-n}$ for some fixed n.

In the rest of this section, we analyze the Newton systems more closely in the case of theta constants of genus 1 and 2, as well as theta functions in genus 1, for suitably reduced inputs; our goal is to apply Theorem 2.3. We also discuss the situation in higher genera.

4.2 Genus 1 theta constants

In the case of genus 1 theta constants, the Newton system is univariate, and τ is simply a complex number with positive imaginary part. Let $\mathcal{R}_1 \subset \mathcal{H}_1$ be the compact set defined by the following conditions:

- $|\operatorname{Re}(\tau)| \leq \frac{1}{2};$
- $|\tau| > 1$;
- $\operatorname{Im}(\tau) \leq 2$.

Thus, \mathcal{R}_1 is a truncated, closed version of the usual fundamental domain \mathcal{F}_1 . The only matrix in $\operatorname{Sp}_2(\mathbb{Z}) = \operatorname{SL}_2(\mathbb{Z})$ that we consider is

$$N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that $N\tau = -1/\tau$. By [23, §I.7], we have

$$\theta_{0,0}^2(0,N\tau) = -i\tau\theta_{0,0}^2(0,\tau). \tag{15}$$

It turns out that the two Borchardt sequences used in the algorithm, namely

$$s_1 = \left(\frac{\theta_{0,b}^2(0, 2^n \tau)}{\theta_{0,0}^2(0, \tau)}\right)_{b \in \mathbb{Z}/2\mathbb{Z}, n \ge 0} \quad \text{and} \quad s_2 = \left(\frac{\theta_{0,b}^2(0, 2^n N \tau)}{\theta_{0,0}^2(0, N \tau)}\right)_{b \in \mathbb{Z}/2\mathbb{Z}, n \ge 0}$$
(16)

are given by good sign choices only: see [6, Thm. 2] and [3, Lem. 2.9]. Our first aim is to collect the data listed in §3.2 for these sequences. This can be done by looking at the theta series (1) directly; see for instance [21, Lem. 3.3]. We formulate the following result in the more general context of theta functions, since it will also be useful in §4.3.

Lemma 4.2. Let $(z,\tau) \in \mathbb{C} \times \mathcal{H}_1$ be such that that $|\operatorname{Im}(z)| < 2\operatorname{Im}(\tau)$, and write $q = \exp(-\pi\operatorname{Im}(\tau))$. Then we have

$$\left|\theta_{0,b}(z,\tau) - 1\right| < 2q \cosh(2\pi \operatorname{Im}(z)) + \frac{2q^4 \exp(4\pi |\operatorname{Im}(z)|)}{1 - q^5 \exp(2\pi |\operatorname{Im}(z)|)}$$

for all $b \in \mathbb{Z}/2\mathbb{Z}$, and

$$\left| \frac{\theta_{1,0}(z,\tau)}{\exp(\pi i\tau/4)} - \left(\exp(\pi iz) + \exp(-\pi iz) \right) \right| < \frac{2q^2 \exp(3\pi |\operatorname{Im} z|)}{1 - q^4 \exp(2\pi |\operatorname{Im} z|)}.$$

Proof. For the first inequality, write

$$\theta_{0,b}(z,\tau) = 1 + \exp(\pi i \tau + 2\pi i z) + \exp(\pi i \tau - 2\pi i z) + \sum_{n \in \mathbb{Z}, |n| \ge 2} \exp(\pi i n^2 \tau + 2\pi i n z).$$

The modulus of this last sum can be bounded above by

$$2\sum_{n\geq 2} \exp(-\pi n^2 \operatorname{Im}(\tau) + 2\pi n |\operatorname{Im} z|), \tag{17}$$

and we conclude by comparing (17) with the sum of a geometric series matching its first two terms. The proof of the second inequality is similar and omitted.

In particular, for each $\tau \in \mathcal{R}_1$, we have $|\theta_{0,0}(0,\tau/2)-1| < 0.53$, so that $\theta_{0,0}(0,\tau/2)$, which appears as the denominator of $\Theta(\tau)$ in (11), is indeed nonzero. More numerical computations will appear in subsequent proofs; we will only write down the first few digits of all real numbers involved.

Proposition 4.3. Let $\tau \in \mathcal{R}_1$. Then, in the notation of §3.2, the following bounds apply to the Borchardt sequence (16) with $\mu = \theta_{0.0}^2(0, \tau)$:

$$m_0 = 0.56$$
 and $M_0 = 1.7$.

The following bounds apply to the sequence (16) taken at $N\tau$ with $\mu = \theta_{0,0}^2(0, N\tau)$:

$$m_0 = 0.13$$
 and $M_0 = 1.38$.

Proof. For the first sequence, we note that $\exp(-\pi \operatorname{Im}(\tau)) < 0.066$, and conclude using Lemma 4.2. For the second sequence, we invoke the transformation formula: we have

$$\frac{\theta_{0,1}(0,-1/\tau)}{\theta_{0,0}(0,-1/\tau)} = \frac{\theta_{1,0}(0,\tau)}{\theta_{0,0}(0,\tau)}.$$

By Lemma 4.2, the angle between $\theta_{0,0}(0,\tau)$ and $\theta_{1,0}(0,\tau)$ seen from the origin is at most $0.95 < \pi/2$; moreover we have $0.41 < |2 \exp(i\pi\tau/4)| < 1.02$, from which the claimed bounds follow.

Theorem 4.4. Let $\rho = 1.4 \cdot 10^{-4}$, define Θ as in (11) for g = 1, and let

$$\mathcal{V} = \bigcup_{\tau \in \mathcal{R}_1} \mathcal{D}_{\rho} \big(\Theta(\tau) \big).$$

Then the operations described in §4.1, taking good choices of square roots always, combined with eq. (15) define an analytic function $F: \mathcal{V} \to \mathbb{C}$ such that

$$F(\Theta(\tau)) = \tau$$

for each $\tau \in \mathcal{R}_1$. We have $|F(x)| \leq 27$ for all $x \in \mathcal{V}$.

Proof. We backtrack from the result of the previous proposition. Let $\tau \in \mathcal{R}_1$. Then the Borchardt means we take are well-defined as analytic functions on any open set where the theta quotients

$$\frac{\theta_{0,1}^2(\tau)}{\theta_{0,0}^2(\tau)}$$
 and $\frac{\theta_{1,0}^2(\tau)}{\theta_{0,0}^2(\tau)}$ (18)

are perturbed by a complex number of modulus at most m = 0.13. We construct \mathcal{V} in such a way that the maximal perturbation will not exceed m/2. The quantities (18) are obtained as quotients of the form:

$$\frac{\theta_{a,b}^2(\tau)/\theta_{0,0}^2(\tau/2)}{\theta_{0,0}^2(\tau)/\theta_{0,0}^2(\tau/2)}.$$
(19)

By Lemma 4.2, the modulus of the denominator is at least 0.32, the modulus of the numerator is at most 5.7. Hence, each of the individual theta quotients (19) may be perturbed by any complex number of modulus at most $6.2 \cdot 10^{-4}$. In turn, these quotients are obtained from the duplication formula applied to 1 and $\Theta(\tau)$; the modulus of these two complex numbers are at most 2.13, hence they may be perturbed by $\rho = 1.4 \cdot 10^{-4}$. By construction, the value taken by the resulting Borchardt means at any $x \in \mathcal{V}$ has modulus at least 0.066 and at most 1.8, hence the final bound on |F(x)|.

Since the inverse of F is given by theta constants, we easily see that the Jacobian of F is invertible at all the relevant points, in a uniform way.

Proposition 4.5. For each $\tau \in \mathcal{R}_1$, we have

$$||d\Theta(\tau)|| \le 125.$$

Proof. By Lemma 4.2, the denominator of this function has modulus at least 0.47, and its numerator has modulus at most 1.53. The result will then follow from an upper bound on the quantities $\|d\theta_{0,b}(\tau/2)\|$. We can derive such bounds from Proposition 2.2, noting that $\theta_{0,b}$ is an analytic function defined on $\mathcal{D}_{1/4}(\tau/2)$, and has modulus at most 2.34 on this disk by Lemma 4.2.

Combining Theorem 4.4 and Proposition 4.5 with the results of §2, we obtain:

Corollary 4.6. For all $\tau \in \mathcal{R}_1$, the Newton scheme described in §4.1 to compute theta constants at τ will converge starting from approximations of $\Theta(\tau)$ to 60 bits of precision.

4.3 Genus 1 theta functions

In the case of genus 1 theta functions, Newton iterations are performed using two complex variables. As in §4.2, we only use the symplectic matrix

$$N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Newton scheme involves the extended Borchardt mean of the sequence (13) with $\lambda = \theta_{0,0}^2(z,\tau)$ and $\mu = \theta_{0,0}^2(0,\tau)$, as well as the analogous sequence taken at $N \cdot (z,\tau) = (z/\tau,-1/\tau)$ instead of (z,τ) . Feedback is then provided by the two following equalities [23, §I.7]:

$$\theta_{0,0}^{2}(0,N\tau) = -i\tau\theta_{0,0}^{2}(0,\tau),$$

$$\theta_{0,0}^{2}(N\cdot(z,\tau)) = -i\tau\exp(2\pi iz^{2}/\tau)\theta_{0,0}^{2}(z,\tau).$$
(20)

Both extended Borchardt sequences are given by good sign choices only [21, Prop. 4.1], provided that the following reductions are met: $\tau \in \mathcal{R}_1$, and the inequalities

$$|\operatorname{Im}(z)| \le \frac{1}{8} \operatorname{Im}(\tau), \quad |\operatorname{Re}(z)| \le \frac{1}{8}$$
 (21)

are satisfied. Let $S_1 \subset \mathbb{C} \times \mathcal{H}_1$ be the compact set of such (z, τ) . Our first goal is to collect the necessary data to apply Proposition 3.6 in this context.

Lemma 4.7. Let $(z,\tau) \in \mathcal{S}_1$, and let $a,b \in \mathbb{Z}/2\mathbb{Z}$. Then the following inequalities hold:

$$|\theta_{0,b}(z,\tau/2) - 1| < 0.68,$$

 $|\theta_{a,b}(z,\tau)| < 1.17,$ and
 $|\theta_{1,0}(z,\tau)| > 0.37.$

Proof. These inequalities are direct consequences of Lemma 4.2. Let us only detail the lower bound on $|\theta_{1,0}(z,\tau)|$. Since $|\operatorname{Re} z| \leq \frac{1}{8}$, we have

$$\operatorname{Re}(\exp(\pi i z) + \exp(-\pi i z)) \ge 2\cos(\pi/8)\cosh(\pi\operatorname{Im}(z)) > 1.84.$$

Therefore,

$$|\theta_{1,0}(z,\tau)| > \exp(-\pi \operatorname{Im}(\tau)/4)(1.84 - 0.025) > 0.37.$$

Proposition 4.8. Let $(z,\tau) \in S_1$. Then, in the notation of §3.3, the following bounds apply to the extended Borchardt sequence (13), where $\lambda = \theta_{0,0}^2(z,\tau)$ and $\mu = \theta_{0,0}^2(0,\tau)$:

$$n_0 = 1$$
, $M_0 = 1.94$, $m_0 = 0.51$ and $m_\infty = 0.72$.

The following bounds apply to the extended Borchardt sequence (13) taken at $N \cdot (z, \tau)$, with $\lambda = \theta_{0,0}^2 (N \cdot (z, \tau))$ and $\mu = \theta_{0,0}^2 (0, N\tau)$:

$$n_0 = 1$$
, $M_0 = 1.69$, $m_0 = 0.1$, and $m_\infty = 0.51$.

Proof. These explicit values are also derived from Lemma 4.2. In the case of the second Borchardt sequence, we analyze the first term using the transformation formula for theta functions under $SL_2(\mathbb{Z})$. For the next terms, we use the following inequalities:

$$\operatorname{Im}(-1/\tau) = \frac{\operatorname{Im}(\tau)}{|\tau|^2} \ge \frac{\sqrt{|\tau|^2 - \frac{1}{2}}}{|\tau|^2} \ge 0.46,$$

$$\operatorname{Im}(z/\tau) = \frac{1}{|\tau|^2} \left| \operatorname{Im}(z) \operatorname{Re}(\tau) - \operatorname{Re}(z) \operatorname{Im}(\tau) \right| \le \frac{3}{16} \operatorname{Im}(-1/\tau),$$

so that for instance

$$\left|\theta_{0.0}^2(z/\tau, -1/\tau)\right| < 1.78.$$

Theorem 4.9. Let $\rho = 2.9 \cdot 10^{-5}$, define Θ' as in (14) for g = 1, and let

$$\mathcal{V} = \bigcup_{(z,\tau)\in\mathcal{S}_1} \mathcal{D}_{\rho}(\Theta'(\tau)).$$

Then the operations described in §4.1, taking good choices of square roots always, combined with the formulas (20) define an analytic function $F: \mathcal{V} \to \mathbb{C}^2$ such that

$$F(\Theta'(\tau)) = (\tau, \exp(2\pi i z^2/\tau))$$

for each $(z, \tau) \in \mathcal{S}_1$. We have $||F(x)|| \le 4.3 \cdot 10^{221}$ uniformly on \mathcal{V} .

Proof. We apply Proposition 3.6 with the explicit values provided above; to find an acceptable ρ , we follow a backtracking strategy as in the proof of Theorem 4.4, using the first two inequalities of Lemma 4.7. The upper bound on ||F|| comes from Proposition 3.6.

The upper bound on ||F|| could certainly be improved in this situation, but the above value will be sufficient for our purposes.

This function F admits an analytic reciprocal. Here it is essential that the theta constants $\theta_{0,b}(z,\tau)$ for $b \in \mathbb{Z}/2\mathbb{Z}$ are invariant under $z \mapsto -z$; this implies that they can be rewritten as analytic functions of z^2 .

Proposition 4.10. Let $b \in \mathbb{Z}/2\mathbb{Z}$. Then there exists a unique analytic function

$$\zeta_{0,h} \colon \mathbb{C} \times \mathcal{H}_1 \to \mathbb{C}$$

such that for all $(z,\tau) \in \mathbb{C} \times \mathcal{H}_1$, we have $\theta_{0,b}(z,\tau) = \zeta_{0,b}(z^2,\tau)$.

Proof. Consider the following reorganization of the theta series:

$$\theta_{0,b}(z,\tau) = 1 + \sum_{n \ge 1} (-1)^{nb} \exp(\pi i n^2 \tau) (\exp(2\pi i n z) + \exp(-2\pi i n z)).$$

Each factor $\exp(2\pi inz) + \exp(-2\pi inz)$, as an even entire function, has only powers of z^2 in its Taylor series. We obtain a candidate $\zeta_{0,b}$ as a formal power series, easily seen to converge uniformly on compact sets of $\mathbb{C} \times \mathcal{H}_1$.

Note that for every $(z, \tau) \in \mathcal{S}_1$, we have $\text{Re}(z^2/\tau) < 1/2$; therefore $\exp(2\pi i z^2/\tau)$ lands in the domain of definition of the principal branch of the complex logarithm, denoted by \mathcal{U} . Consider the two following maps:

$$\begin{array}{cccc} \mathcal{H}_{1} \times \mathcal{U} & \to & \mathbb{C} \times \mathcal{H}_{1} & \to & \mathbb{C} \times \mathbb{C} \\ (\tau, x) & \mapsto & \left(\frac{1}{2\pi i} \log(x), \tau\right)\right) \\ & & & (y, \tau) & \mapsto & \left(\frac{\theta_{0,1}(0, \tau/2)}{\theta_{0,0}(0, \tau/2)}, \frac{\zeta_{0,1}(y, \tau/2)}{\zeta_{0,0}(y, \tau/2)}\right). \end{array}$$

Call G their composition; it is well-defined on an open neighborhood of the image of S_1 by $(z, \tau) \mapsto (\tau, \exp(2\pi i z^2/\tau))$, and is the reciprocal of F.

Proposition 4.11. For each $(z, \tau) \in \mathcal{S}_1$, we have

$$\left\| dG\left(\tau, \exp(2\pi i z^2/\tau)\right) \right\| \le 8.6 \cdot 10^4.$$

Proof. Let $x = \exp(2\pi i z^2/\tau)$, and $y = z^2$. We have

$$\left| \operatorname{Im}(z^2/\tau) \right| = \frac{1}{|\tau|^2} \cdot \frac{1}{64} \left(\operatorname{Im}(\tau) + \operatorname{Im}(\tau)^3 + \operatorname{Im}(\tau) \right) \le \frac{1}{16},$$

showing that |x| is close to 1. It only remains to obtain explicit upper bounds on the derivative of $\zeta_{0,1}/\zeta_{0,0}$. To obtain such bounds, we consider the polydisk of radius 1/16 centered in $(y, \tau/2)$; by Lemma 4.2, we have $|\zeta_{0,b}| < 5.8$ on this disk for each $b \in \mathbb{Z}/2\mathbb{Z}$, so that $||d\zeta_{0,b}(y,\tau/2)|| < 277$ by Proposition 2.2. We can conclude using the lower bound on $|\zeta_{0,0}(y,\tau/2)|$ provided by Lemma 4.7.

Corollary 4.12. For all $(z, \tau) \in S_1$, the Newton scheme described in §4.1 to compute theta functions at (z, τ) will converge starting from approximations of $G(\tau)$ to 1600 bits of precision.

4.4 Genus 2 theta constants

In the case of genus 2 theta constants, Newton iterations are performed on three variables, and feedback is provided by the action of three symplectic matrices.

For general g, certain interesting symplectic matrices can be written down explicitly. Denote the elementary $g \times g$ matrices by $E_{i,j}$ for $1 \le i, j \le g$, and let I be the identity matrix. Let M_i and $N_{i,j}$ $(i \ne j)$ be the following symplectic matrices, written in $g \times g$ blocks:

$$M_i = \begin{pmatrix} -I & -E_i \\ E_i & -I + E_i \end{pmatrix}, \quad N_{i,j} = \begin{pmatrix} -I & -E_{i,j} - E_{j,i} \\ E_{i,j} + E_{j,i} & -I + E_{i,i} + E_{j,j} \end{pmatrix}$$

The matrices M_i and $N_{i,j}$ are precisely engineered so that determinants of the form $\det(C\tau + D)$ give us direct access to the entries of τ . In the case of genus 2 theta constants, considering these three matrices M_1, M_2 and $N_{1,2}$ is enough to run the Newton scheme, using the following formulas [10, Prop. 8]:

$$\theta_{00,00}^{2}(0, M_{1}\tau) = -\tau_{1,1}\theta_{01,00}^{2}(0, \tau),$$

$$\theta_{00,00}^{2}(0, M_{2}\tau) = -\tau_{2,2}\theta_{10,00}^{2}(0, \tau), \text{ and}$$

$$\theta_{00,00}^{2}(0, N_{1,2}\tau) = (\tau_{1,2}^{2} - \tau_{1,1}\tau_{2,2})\theta_{00,00}^{2}(0, \tau).$$
(22)

In [19], it is shown that all four Borchardt sequences of the form (12) taken at τ , $M_1\tau$, $M_2\tau$ and $N_{1,2}\tau$ are given by good choices of square roots only, provided that τ satisfies the following conditions:

- $|\operatorname{Re}(\tau_{i,j})| \leq \frac{1}{2}$ for all $1 \leq i, j \leq 2$,
- $2|\operatorname{Im}(\tau_{1,2})| \le \operatorname{Im}(\tau_{1,1}) \le \operatorname{Im}(\tau_{2,2}),$
- $|\tau_{i,j}| \ge 1$ for j = 1, 2.

These inequalities hold in particular whenever τ lies in the Siegel fundamental domain \mathcal{F}_2 . Let \mathcal{R}_2 be the compact set of such matrices τ , with the additional assumption that $\operatorname{Im}(\tau_{1,1}) \leq 2$ and $\operatorname{Im}(\tau_{2,2}) \leq 8$. This choice of upper bounds will be explained by the construction of a uniform algorithm in §5.

As in the previous sections, we will collect the explicit data we need to apply Proposition 3.1 using inequalities satisfied by genus 2 theta constants. Many such inequalities already appear in [20, §9], [5, §6.2.1], [26, §7.2], and [19]; we will use one more.

Lemma 4.13. For each $\tau \in \mathcal{R}_2$, we have $0.44 < |\theta_{0,0}(0,\tau/2)| < 2.66$.

Proof. Let

$$\xi_0(\tau/2) = 1 + 2 \exp(i\pi \operatorname{Im}(\tau_{1,1})/2) + 2 \exp(i\pi \operatorname{Im}(\tau_{2,2})/2).$$

Since $\tau \in \mathcal{R}_2$, the complex number $\xi_0(\tau/2)$ has modulus at least 1 and at most 2.1. By [19, Lem. 4.4], we have $|\theta_{0,0}(0,\tau/2) - \xi_0(\tau/2)| < 0.56$.

Proposition 4.14. Let $\tau \in \mathcal{R}_2$. Then, in the notation of §3.2, the following bounds apply.

1. In the case of the Borchardt sequence (12) with $\lambda = \theta_{0,0}^2(0,\tau)$, we can take $m_0 = 0.069$ and $M_0 = 13$.

- 2. In the case of the Borchardt sequence (12) taken at $M_j \tau$ with $\lambda = \theta_{0,0}^2(0, M_j \tau)$, for each $j \in \{1, 2\}$, we can take $m_0 = 9.7 \cdot 10^{-7}$ and $M_0 = 13$.
- 3. In the case of the Borchardt sequence (12) taken at $N_{1,2}\tau$ with $\lambda = \theta_{0,0}^2(0, N_{1,2}\tau)$, we can take $m_0 = 2.2 \cdot 10^{-9}$ and $M_0 = 13$.

Proof. We only have to analyze the first term of each of these Borchardt sequences. These explicit constants are then derived from the proof in [19] that these complex numbers are in good position. \Box

Theorem 4.15. Let $\rho = 1.9 \cdot 10^{-23}$, define Θ as in (11) for g = 2, and let

$$\mathcal{V} = \bigcup_{\tau \in \mathcal{R}_2} D_{\rho}(\Theta(\tau)) \subset \mathbb{C}^3.$$

Then the operations described in §4.1, taking good choices of square roots always, define an analytic function $F: \mathcal{V} \to \mathbb{C}^2$ such that

$$F(\Theta(\tau)) = (\tau_{1,1}, \tau_{2,2}, \tau_{1,2}^2 - \tau_{1,1}\tau_{2,2})$$

for each $\tau \in \mathcal{R}_2$. We have $||F|| \leq 4.5 \cdot 10^4$ uniformly on \mathcal{V} .

Proof. The first terms of each of the Borchardt sequences analyzed in Proposition 4.14 is obtained as quotients of the quantities

$$\frac{\theta_{a,b}^2(0,\tau)}{\theta_{0,0}^2(0,\tau/2)},$$

for all even theta characteristics (a, b) (i.e. such that $a^tb = 0 \mod 2$), except (11, 11). The numerator and denominator of these quantities is bounded, both above and away from zero, by Lemma 4.13 and [26, Cor. 7.7]. Using these inequalities combined with Propositions 3.1 and 4.14 is sufficient to obtain an explicit value of ρ .

To conclude, we show that the Jacobian of F is uniformly invertible by writing its inverse in terms of theta functions. Since F only recovers the square of $\tau_{1,2}$, we use the fact that each of the fundamental theta constants $\theta_{0,b}(0,\cdot)$ for $b \in \mathcal{I}_2$ is invariant under change of sign of $\tau_{1,2}$.

Lemma 4.16. Let $V \subset \mathbb{C}^3$ be the image of \mathcal{H}_2 under $\tau \mapsto (\tau_{1,1}, \tau_{2,2}, -\det \tau)$. Then, for each $b \in \mathcal{I}_2$, there exists a unique analytic function $\xi_{0,b} \colon V \to \mathbb{C}$ such that

$$\theta_{0,b}(0,\tau/2) = \xi_{0,b}(\tau_{1,1},\tau_{2,2},-\det\tau)$$

for all $\tau \in \mathcal{H}_2$.

Proof. In the theta series (1) for z=0, the only terms involving $\tau_{1,2}$ are those associated with $(n_1, n_2) \in \mathbb{Z}^2$ both nonzero. Write $b=(b_1, b_2)$. Then, the terms associated with (n_1, n_2) and $(n_1, -n_2)$ are

$$\exp(i\pi(\tau_{1,1}n_1^2+\tau_{2,2}n_2^2\pm 2\tau_{1,2}n_1n_2))(-1)^{n_1b_1+n_2b_2},$$

so their sum can be written as a power series in $\tau_{1,2}^2$ only.

Let G be the following analytic function:

$$G(x, y, z) = \left(\frac{\xi_{0,b}(x, y, z)}{\xi_{0,0}(x, y, z)}\right)_{b \in \mathcal{I}_2 \setminus \{0\}}.$$

It is well-defined on a neighborhood of the image of \mathcal{R}_2 by $\tau \mapsto (\tau_{1,1}, \tau_{2,2}, -\det \tau)$, and is the reciprocal of F.

Proposition 4.17. We have $||dG(\tau_{1,1}, \tau_{2,2}, -\det \tau)|| \le 1.3 \cdot 10^4$ for all $\tau \in \mathcal{R}_2$.

Proof. Fix $\rho = 1/4$, and let us compute an upper bound on $|\xi_{0,b}(x,y,z)|$ for each point $(x,y,z) \in \mathcal{D}_{\rho}((\tau_{1,1},\tau_{2,2},-\det\tau))$. Then x,y,z are of the form $(\tau'_{1,1},\tau'_{2,2},-\det\tau')$ for some $\tau' \in \mathcal{H}_2$; the smallest eigenvalue of $\text{Im}(\tau')$ is bounded from below by

$$\frac{\det(\tau')}{\mathrm{Tr}(\tau')} \ge 0.12.$$

By the proof of [19, Lem. 4.7], the function $|\xi_{0,b}|$ is uniformly bounded above by 9.28 on the disk we consider. By Proposition 2.2, we have

$$||d\xi_{0,b}(\tau_{1,1},\tau_{2,2},-\det\tau)|| \le 149$$

for each $b \in \mathcal{I}_2$. The upper bound on ||dG|| then follows from Lemma 4.13.

Corollary 4.18. For all $\tau \in \mathcal{R}_2$, the Newton scheme described in §4.1 to compute theta constants at τ will converge starting from approximations of $\Theta(\tau)$ to 300 bits of precision.

4.5 Higher genera

In higher genera, including the case of genus 2 theta functions, we are no longer able to show that the linear systems appearing in the Newton schemes are invertible, nor a fortiori are we able to give an explicit upper bound on the norm of their inverse Jacobians. Let us shortly explain what the obstacle is.

In order to build a Newton scheme, the linearized system must be square; however, as g grows, the number r of theta quotients (either 2^g-1 in the case of theta constants, or $2^{g+1}-2$ in the case of theta functions) becomes greater than the dimension of \mathcal{H}_g or $\mathbb{C}^g \times \mathcal{H}_g$ respectively. Two ways around this issue are suggested in [22, §3.5]:

- 1. One could keep all theta quotients as variables, and simply consider more symplectic matrices N to provide suitable feedback; or
- 2. One could perform Newton iterations not on the whole of \mathbb{C}^r , but rather on the algebraic subvariety of \mathbb{C}^r obtained as the image of \mathcal{H}_g or $\mathbb{C}^g \times \mathcal{H}_g$ by the fundamental theta quotients (11) or (14).

A fundamental obstacle to the second idea seems to be that the algebraic subvariety of \mathbb{C}^r on which the theta quotients lie is not smooth everywhere in general: consider for instance the Kummer equation [13, §3.1] in the case of genus 2 theta constants. On the other hand, it seems very likely that the first possibility can give rise to suitably invertible systems, since much freedom is allowed in the choice of symplectic matrices. However, the inverse of F will no longer be described completely by theta functions, so the method we employed above to prove the invertibility of the linearized systems no longer applies.

Despite the current lack of a uniform algorithm, the following approach is available to certify the result of Newton's method to evaluate theta constants (resp. functions) at a given $\tau \in \mathcal{H}_g$ (resp. $(z,\tau) \in \mathbb{C}^g \times \mathcal{H}_g$). A finite amount of precomputation, along with the results of Section 3, will allow us to compute real numbers $\rho > 0$ and M > 0 such that the function F appearing in Newton's method is analytic with $|F| \leq M$ on a polydisk of radius ρ around the desired theta values. This gives upper bounds on the norms of derivatives of F on a slightly smaller polydisk; in particular, we can compute a certified approximation of dF^{-1} using finite differences, and check that it is indeed invertible. This provides all the necessary data to run certified Newton iterations.

5 A uniform algorithm for genus 2 theta constants

We have shown in §4.4 that genus 2 theta constants can be evaluated on the compact subset \mathcal{R}_2 of \mathcal{H}_2 in uniform quasi-linear time in the required precision, in a certified way. Using this algorithm as a black box, we now design an algorithm to evaluate genus 2 theta constants on the whole Siegel fundamental domain \mathcal{F}_2 in uniform quasi-linear time, generalizing the strategy presented in [6, Thm. 5], [21, §4.2] in the genus 1 case: we use duplication formulas to replace the input by another period matrix which either lies in \mathcal{R}_2 , or is sufficiently close to the cusp, in which case the naive algorithm can be applied. We will use the following transformations: for every $\tau \in \mathcal{H}_2$, write

$$D_1(\tau) = \frac{\tau}{2}$$
 and $D_2(\tau) = \begin{pmatrix} 2\tau_{1,1} & \tau_{1,2} \\ \tau_{1,2} & \frac{1}{2}\tau_{2,2} \end{pmatrix}$.

Recall that every $\tau \in \mathcal{F}_2$ satisfies the following inequalities:

$$\begin{cases} |\operatorname{Re}(\tau_{i,j})| \leq \frac{1}{2} & \text{for each } 1 \leq i, j \leq 2, \\ 2|\operatorname{Im}(\tau_{1,2})| \leq \operatorname{Im}(\tau_{1,1}) \leq \operatorname{Im}(\tau_{2,2}), \\ |\tau_{i,i}| \geq 1 & \text{for each } i \in \{1, 2\}. \end{cases}$$
(23)

We also define

$$\mathcal{J} = ((00, 00), (00, 01), (10, 00), (10, 01)) \in (\mathcal{I}_2 \times \mathcal{I}_2)^4$$

which is the tuple of theta characteristics corresponding to the indices 0, 2, 4, 6 in Dupont's indexation [5, §6.2]. For each $\tau \in \mathcal{H}_2$, the duplication formula allows us to compute all squares of theta constants at τ given the theta values $\theta_{0,b}(D_1(\tau))$ for all $b \in \mathcal{I}_2$. By applying the theta transformation formula to the symplectic matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we also see that all squares of theta constants at τ can be computed from the theta values $\theta_{a,b}(D_2(\tau))$ for $(a,b) \in \mathcal{J}$. It turns out that these complex numbers are in good position; hence, they are easily determined from their squares up to a harmless global change of sign.

Lemma 5.1. Let $\tau \in \mathcal{H}_2$ be a matrix satisfying (23).

- 1. If $D_1(\tau)$ satisfies (23), then the complex numbers $(\theta_{0,b}(D_1(\tau)))_{b\in\mathcal{I}_2}$ are in good position.
- 2. If $D_2(\tau)$ satisfies (23), except that the real part of $D_2(\tau)_{1,1}$ is allowed to be smaller than 1 instead of $\frac{1}{2}$, then the complex numbers $(\theta_{a,b}(D_2(\tau)))_{(a,b)\in\mathcal{J}}$ are in good position.

Theorem 5.2. There exists an algorithm which, given $\tau \in \mathcal{H}_2$ satisfying (23) and given $N \geq 1$, computes the squares of theta constants at τ to precision N within $O(\mathcal{M}(N) \log N)$ binary operations, uniformly in τ .

Proof. Fix an arbitrary absolute constant C > 0 (for instance 10); in practice, this constant should be adjusted to minimize the algorithm's running time. First, let k_2 be the smallest integer such that

$$2^{k_2}\operatorname{Im}(\tau_{1,1}) \ge \min\{CN, 2^{-k_2-2}\operatorname{Im}(\tau_{2,2})\},\$$

and let τ' be the matrix obtained after applying k_2 times D_2 to τ and reducing the real part at each step. In order to compute theta constants at τ to precision N, we can compute theta constants at τ' to some precision $N' \geq N$, then apply k_2 times the duplication formula; all sign choices are good by Lemma 5.1. We have $k_2 = O(\log N)$, and the total precision loss taken in extracting square roots is O(N) by [26, Prop. 7.7]. Therefore, the total precision loss is O(N) bits, and we can choose N' = C'N where C' is an absolute constant.

Two cases arise now. If $\operatorname{Im}(\tau'_{1,1}) \geq CN$, then we also have $\operatorname{Im}(\tau'_{2,2}) \geq CN$; therefore we can compute theta constants at τ' to precision N' using $O(\mathcal{M}(N))$ operations with the naive algorithm. Otherwise, we have

$$\operatorname{Im}(\tau'_{1,1}) \le \operatorname{Im}(\tau'_{2,2}) \le 4 \operatorname{Im}(\tau'_{1,1}) \le 4CN.$$

Therefore we can find an integer $k_1 = O(\log N)$ such that $\tau'' = D_1^{k_1}(\tau')$ belongs to \mathcal{R}_2 , by definition of this compact set. We will compute theta constants at τ'' to some precision $N'' \geq N'$ using the Newton scheme described in §4, then use the duplication formula k_1 times. Since O(1) bits of precision are lost each time we apply the duplication formula by [26, Prop. 7.7], we can also take N'' = C''N where C'' is an absolute constant. Therefore, the whole algorithm can be executed in $O(\mathcal{M}(N) \log N)$ binary operations.

Remark 5.3. In order to implement this algorithm in a certified way, one could use [26, Prop. 7.7] more explicitly to track down an acceptable value of C''. Another possibility is to start with C'' = 1.1, say, and attempt to run this algorithm using interval arithmetic to obtain real-time upper bounds on the precision losses incurred. If the final precision we obtain is not satisfactory, we may simply double C'' and restart. The resulting algorithm still has a uniform quasi-linear cost.

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