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An ensemble scheme for the numerical solution of a random transient heat equation with uncertain inputs

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Abstract

An ensemble-based time stepping scheme is applied to solving a transient heat equation with random Robin boundary and diffusion coefficients. By introducing two ensemble means of Robin boundary and diffusion coefficients, we propose a new ensemble Monte Carlo (EMC) scheme for the a transient heat equation. The EMC scheme solves a single linear system including several right-side vectors at each time step. Stability analysis and error estimates are derived. Two numerical examples verify the theoretical results and the validity of the EMC method.

Keywords: Robin boundary coefficient, Diffusion coefficient, Ensemble, Monte Carlo method, Transient heat equation

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1 Introduction

We consider the numerical simulations to a transient heat problem with random diffusion coefficients and Robin coefficients: Seek y satisfying almost surely (a.s.),

$$\begin{cases} y_t - \nabla \cdot [a(t, \mathbf{x}, \omega)\nabla y] = f(t, \mathbf{x}, \omega), \quad \text{in } [0, T] \times D \times \Omega, \\ a\nabla y(t, \mathbf{x}, \omega) \cdot \mathbf{n} = 0, \quad \text{on } [0, T] \times \partial D_0 \times \Omega, \\ a\nabla y(t, \mathbf{x}, \omega) \cdot \mathbf{n} = \alpha(t, \mathbf{x}, \omega)(u(t, \mathbf{x}, \omega) - y(t, \mathbf{x}, \omega)), \text{ on } [0, T] \times \partial D_1 \times \Omega, \\ y(0, \mathbf{x}) = y^0(\mathbf{x}, \omega)), \quad \text{in } D \times \Omega, \end{cases} \tag{1}$$

here $D \subseteq \mathbb{R}^d(d = 2, 3)$ is a Lipschitz domain. The boundary ∂D is divided into two disjoint parts ∂D_0 and ∂D_1, i.e. $\partial D = \partial D_0 \bigcup \partial D_1$. $\mathbf{n}$ is the unit outward normal vector to ∂D. $(\Omega, \mathcal{F}, P)$ stands for a complete probability space, where Ω is the sample space, $\mathcal{F} \subset 2^\Omega$ is the σ-algebra of events, and $P : \mathcal{F} \to [0, 1]$ is a probability measure.

Uncertainty widely exists in many problems of physics or engineering. The process of heat and mass transfer is affected by many uncertain factors, such as random ambient temperature, random initial temperature, random material characteristics, random thermal conductivity (diffusion coefficient), random convective heat transfer coefficient (Robin coefficient), or random geometry. The problem (1) is random diffusion coefficient and Robin coefficient, which can been seen in [3, 4]. The model (1) also appears in other problems, such as [21, 22].

There are many numerical algorithms for solving PDEs with random coefficients (see, e.g., [1, 7, 9, 18, 23, 25, 26]). In addition to polynomial chaos method, stochastic collocation method and stochastic finite element method, Monte Carlo (MC) method is a very important method (see, e.g., [6, 8, 10, 20]). MC method is non-intrusive and the convergence is not rely on the dimension of the random model parameters. It is easy to implement for the MC method. Once MC method is adopted, independent sampling is needed first, and then independent numerical simulation is carried out for each sample. These simulations are influenced by independent initial and boundary conditions, physical forces, diffusion coefficients and Robin coefficients. Assume the simulations involve J independent members, where the j-th member holds

$$\begin{cases} y_{j,t} - \nabla \cdot [a_j(t, \mathbf{x})\nabla y_j] = f_j(t, \mathbf{x}), \quad \text{in } (0, T) \times D, \\ a_j \nabla y_j(t, \mathbf{x}) \cdot \mathbf{n} = 0, \quad \text{on } (0, T) \times \partial D_0, \\ a_j \nabla y_j(t, \mathbf{x}) \cdot \mathbf{n} = \alpha_j(t, \mathbf{x})(u_j(t, \mathbf{x}) - y_j(t, \mathbf{x})), \quad \text{on } (0, T) \times \partial D_1, \\ y_j(0, \mathbf{x}) = y_j^0(\mathbf{x}), \quad \text{in } D, \end{cases} \tag{2}$$

$j \in \{1, 2, \ldots, J\}$. Here we can think $a_j(t, \mathbf{x}) = a(t, \mathbf{x}, \omega_j)$, $\alpha_j(t, \mathbf{x}) = \alpha(t, \mathbf{x}, \omega_j)$. The others are similar.

In the numerical simulation of problem (2), a large number of linear equations $\mathbf{A}_j\mathbf{z}_j = \mathbf{b}_j, j \in \{1, 2, \ldots, J\}$ need to be solved. The calculation cost is very high since the number J is relatively large. In order to improve computing efficiency, ensemble method are widely used (see, e.g., [11, 15, 19, 20]. The ensemble method change solving $\mathbf{A}_j\mathbf{z}_j = \mathbf{b}_j, j \in \{1, 2, \ldots, J\}$ to solving $\tilde{\mathbf{A}}\mathbf{z}_j = \tilde{\mathbf{b}}_j, j \in \{1, 2, \ldots, J\}$. In [19], the authors studied the parabolic problem with random coefficients by using ensemble method, and obtained an error estimate. But the error estimate therein is not optimal with respect to (w.r.t.) space. In view of this problem, the authors of [16] combined the ensemble with HDG method to obtain an optimal error estimate in space. For the heat conduction problem with random Robin coefficient, we have not found relevant results with the ensemble method.

We study the numerical approximation of (1) by ensemble method in this work. Before this, we consider the numerical simulation of model (2). Denote the ensemble average for the diffusion coefficient and the Robin coefficient by

$$\bar{a}(t, \mathbf{x}) := \frac{1}{J}\sum_{j=1}^{J} a_j\left(t, \mathbf{x}\right), \tag{3}$$

and

$$\bar{\alpha}(t, \mathbf{x}) := \frac{1}{J}\sum_{j=1}^{J} \alpha_j\left(t, \mathbf{x}\right), \tag{4}$$

respectively. We use an isometric time division on $[0, T]$ with $t_n = n\Delta t$, where Δt represents the step size. Let $y_j^n, f_j^n, u_j^n, a_j^n$, α_j^n, $\bar{a}^n$ and $\bar{\alpha}^n$ be the values of functions y_j, f_j, u_j, a_j, α_j, $\bar{a}$ and $\bar{\alpha}$ at $t = t_n$. The ensemble-based time stepping scheme describes as

$$\frac{y_j^{n+1} - y_j^n}{\Delta t} - \nabla \cdot \left(\bar{a}^{n+1}\nabla y_j^{n+1}\right) - \nabla \cdot \left[\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla y_j^n\right] = f_j^{n+1}, j = 1, \ldots, J,$$

with the same boundary and initial conditions of (2). In order to appear Robin coefficient in the scheme, we develop a variational ensemble scheme reads, for all $v \in H^1(\bar{D})$,

$$\begin{aligned}
&\left(\frac{y_j^{n+1} - y_j^n}{\Delta t}, v\right) + \left(\bar{a}^{n+1}\nabla y_j^{n+1}, \nabla v\right) + \left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla y_j^n, \nabla v\right)\\
&+ \left(\bar{\alpha}^{n+1} y_j^{n+1}, v\right)_{\partial D_1} + \left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right) y_j^n, v\right)_{\partial D_1}\\
=& \left(f_j^{n+1}, v\right) + \left(\alpha_j^{n+1} u_j^{n+1}, v\right)_{\partial D_1}, j = 1, \ldots, J.
\end{aligned} \tag{5}$$

After rearrangement, we obtain, for $j = 1, \ldots, J$,

$$\begin{aligned}
&\left(\frac{1}{\Delta t} y_j^{n+1}, v\right) + \left(\bar{a}^{n+1} \nabla y_j^{n+1}, \nabla v\right) + \left(\bar{\alpha}^{n+1} y_j^{n+1}, v\right)_{\partial D_1} \\
&= \left(f_j^{n+1}, v\right) + \left(\alpha_j^{n+1} u_j^{n+1}, v\right)_{\partial D_1} + \left(\frac{1}{\Delta t} y_j^n, v\right) \\
&\quad - \left(\left(a_j^{n+1} - \bar{a}^{n+1}\right) \nabla y_j^n, \nabla v\right) - \left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right) y_j^n, v\right)_{\partial D_1}.
\end{aligned} \tag{6}$$

Through a spatial discretization, we observe that the coefficient matrix of the resulting linear system will be independent of j. The discrete systems share a common coefficient matrix, and the right-hand-side (RHS) vectors vary among the ensemble members. Then, if the scale of the problem is small, the solution of the group can be obtained by LU decomposition of the coefficient matrix only once (see, e.g., [19, 20]). While the case is of a large-scale, block Krylov subspace iteration method will be used to compute efficiently (see, e.g., [5, 14, 24]).

The following describes the structure of this article. In Section 2, some notations and preliminaries are introduced. The full discretization ensemble scheme for (5) is given in Section 3 with its stability and convergence analysis. The random transient heat equation and its stability as well as error analysis are discussed in Section 4. In Section 5, two numerical tests are presented. Some conclusions are given in Section 6.

2 Basic preliminaries.

In this section, we will give some notations. For simplicity, $d\mathbf{x}, d\mathbf{s}$ and dt in some expression will be omitted when there is no confusion. The boundaries ∂D_0 and ∂D_1 concern to the experimentally accessible and inaccessible parts, respectively.

Let $\|\cdot\|$ and $(\cdot,\cdot)$ be the $L^2(D)$ norm as well as inner product, respectively. Simultaneously $\|\cdot\|_{\partial D}$ and $(\cdot,\cdot)_{\partial D}$ stand for corresponding the $L^2(\partial D)$ norm as well as inner product. The Sobolev space $W^{s,q}(D)$ with the norm $\|v\|_{W^{s,q}(D)}$, here $s \in N^+$ (positive integer set) and $1 \leq q \leq +\infty$. We denote $H^s(D) = W^{s,2}(D)$. Particularly, $H^1(D)$ is equipped the norm $\|\cdot\|_1 = \|\cdot\|_{1,D}$, which is defined by

$$\|y\|_{1,D} = \left(\|y\|^2 + \|\nabla y\|^2\right)^{\frac{1}{2}}.$$

Let $H^{-s}(D)$ is the dual space of bounded linear functions on $H^s(D)$, with norm $\|f\|_{-s} = \sup_{0 \neq v \in H^s(D)} (f, v)/\|v\|_s$. The norm $\|\cdot\|_{1,\partial D_1}$ is defined by

$$\|y\|_{1,\partial D_1}^2 = \int_D |\nabla y|^2 + \int_{\partial D_1} y^2,$$

which is equivalent to the standard norm $\|\cdot\|_1$, cf. [12, 13]. In the later statements, it will not differentiate the norm $\|\cdot\|_{1,\partial D_1}$ and $\|\cdot\|_1 = \|\cdot\|_{H^1(D)}$.

Denote

$$L^\infty(D) = \{v \mid v \text{ is a measurable functions and } \mid v\mid_\infty < +\infty\},$$

where $\mid v \mid_\infty = \text{ess } \sup_{\mathbf{x}\in D} \mid v \mid$.

$(\Omega, \mathcal{F}, P)$ is a complete probability space. $Z \in L^1_P(\Omega)$ is a random variable. Denote the expected value of Z by

$$\mathbb{E}[Z] = \int_\Omega Z(\omega) dP(\omega).$$

Let $\delta = (\delta_1, \ldots, \delta_d)$ be a d-tuple with the length $\mid \delta \mid = \sum_{i=1}^d \delta_i, \delta_i \in N^+$. The stochastic Sobolev spaces $\widetilde{W}^{s,q}(D) = L^q_P\left(\Omega, W^{s,q}(D)\right)$ contains stochastic function, $v : \Omega \times D \to R$, which is measurable w.r.t. the product σ-algebra $\mathcal{F} \otimes B(D)$. The norm of $\widetilde{W}^{s,q}(D)$ is defined by

$$\|v\|_{\widetilde{W}^{s,q}(D)} = \left(\mathbb{E}\left[\|v\|^q_{W^{s,q}(D)}\right]\right)^{1/q} = \left(\mathbb{E}\left[\sum_{|\delta|\le s}\int_D \mid \partial^\delta v \mid^q\right]\right)^{1/q},$$

where $1 \le q < +\infty$. Let $\widetilde{H}^s(D) = \widetilde{W}^{s,2}(D) \simeq L^2_P(\Omega) \otimes H^s(D)$.

We will use the Hilbert space

$$X = \widetilde{L}^2\left(0, T;\ H^1(D)\right) \simeq L^2_P\left(0, T;\ H^1(D);\ \Omega\right)$$

with its inner product

$$(v, u)_X \equiv \mathbb{E}\left[\int_0^T \int_D (\nabla v \cdot \nabla u + vu)\right].$$

The induced norm is given by

$$\|v\|_X = \left(\mathbb{E}\left[\int_0^T \int_D \left(\mid \nabla v \mid^2 + v^2\right)\right]\right)^{1/2}.$$

Suppose $\mathcal{T}_h$ is a quasi-uniform triangulation of the domain D, such that $\bar{D} = \bigcup_{K\in\mathcal{T}_h} \bar{K}$. Let h_K be the diameter of the element K. Define $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathbf{P}_l$ be the set of polynomials of degree l. Denote the finite element (FE) space V_h of the domain D by

$$V_h = span\,\{\varphi_k\}_1^M \subset \left\{v \in H^1(D) \cap H^{l+1}(D) : v\mid_K \in \mathbf{P}_l, \forall K \in \mathcal{T}_h\right\}. \quad (7)$$

The FEs space S_h of the boundary ∂D by

$$S_h = span\,\{\phi_i\}_1^{M_{\partial D}} \subset \left\{v \in H^1(\partial D) \cap H^{l+1}(\partial D) : v\mid_{\partial D\cap K} \in \mathbf{P}_l, \forall K \in \mathcal{T}_h\right\}.$$

Throughout this work C is a positive constant, it has different values in different places and does not rely on time step Δt, sample size J and mesh size h.

3 A variational ensemble scheme for deterministic transient heat equations

In this section, we first give some assumptions and a full discretization ensemble scheme for (5). Then the stability and error estimate results are presented.

Suppose the following two hypotheses (**H1**) and (**H2**) hold.

(**H1**) There exists positive constants λ , μ_{min} and μ_{max}, such that for $j \in \{1, 2, \dots, J\}$,

$$\min_{\mathbf{x}\in\bar{D}} a_j(t,\mathbf{x}) \geq \lambda, \quad \forall t \in [0,T], \tag{8}$$

and

$$\mu_{max} \geq \max_{\mathbf{x}\in\partial D_1} \alpha_j(t,\mathbf{x}) \geq \alpha_j(t,\mathbf{x}) \geq \min_{\mathbf{x}\in\partial D_1} \alpha_j(t,\mathbf{x}) \geq \mu_{min}, \quad \forall t \in [0,T]. \tag{9}$$

(**H2**) There exists positive constants $\lambda_-, \lambda_+, \mu_-$ and μ_+, such that for $j \in \{1, 2, \dots, J\}$,

$$\lambda_- \leq\mid a_j(t,\mathbf{x}) - \bar{a}(t,\mathbf{x}) \mid_\infty \leq \lambda_+, \quad \forall t \in [0,T], \tag{10}$$

and

$$\mu_- \leq\mid \alpha_j(t,\mathbf{x}) - \bar{\alpha}(t,\mathbf{x}) \mid_\infty \leq \mu_+, \quad \forall t \in [0,T]. \tag{11}$$

Evidently, hypothesis (**H1**) means the problem is uniformly coercivity. Hypothesis (**H2**) shows that the difference between $a_j(t,\mathbf{x})$ and $\bar{a}(t,\mathbf{x})$ is uniformly bounded, and so does the Robin coefficient $\alpha_j(t,\mathbf{x})$.

Under the assumption that the isometric time division on $[0,T]$, the full-discrete scheme for (5) is that: seek $y_{j,h}^{n+1} \in V_h \cup S_h$ such that, for all $v_h \in V_h \cup S_h$,

$$\begin{aligned}
&\left(\frac{y_{j,h}^{n+1} - y_{j,h}^{n}}{\Delta t}, v_h\right) + \left(\bar{a}^{n+1}\nabla y_{j,h}^{n+1}, \nabla v_h\right) + \left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla y_{j,h}^{n}, \nabla v_h\right) \\
&\quad + \left(\bar{\alpha}^{n+1} y_{j,h}^{n+1}, v_h\right)_{\partial D_1} + \left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right) y_{j,h}^{n}, v_h\right)_{\partial D_1} \\
&= \left(f_j^{n+1}, v_h\right) + \left(\alpha_j^{n+1} u_j^{n+1}, v_h\right)_{\partial D_1}, j = 1, \cdots, J, n = 0, \dots, N-1,
\end{aligned} \tag{12}$$

here $N = T/\Delta t$, the initial value $y_{j,h}^0 \in V_h \cup S_h$, $\left(y_{j,h}^0, v_h\right) = \left(y_j^0, v_h\right)$.

3.1 Stability.

The stability of the ensemble scheme (12) has the following result.

Theorem 1 *Assume that* $f_j \in L^2\left(0,T;\ H^{-1}(D)\right)$, $u_j \in L^2\left(0,T;\ L^2(\partial D_1)\right)$, *and hypotheses* (**H1**) *and* (**H2**) *are held. Then the ensemble scheme* (5) *is stable if*

$$\lambda - \lambda_+ > 0 \qquad \text{and} \qquad \mu_{min} - \mu_+ > 0. \tag{13}$$

Furthermore, the numerical solution to (12) *satisfies*

$$\begin{aligned}
&\left\|y_{j,h}^N\right\|^2 + \lambda_-\Delta t\left\|\nabla y_{j,h}^N\right\|^2 + (\lambda-\lambda_+)\,\Delta t\sum_{n=0}^{N-1}\left\|\nabla y_{j,h}^{n+1}\right\|^2\\
&+\mu_-\Delta t\left\|y_{j,h}^N\right\|_{\partial D_1}^2 + (\mu_{min}-\mu_+)\,\Delta t\sum_{n=0}^{N-1}\left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2\\
\leq& C\Delta t\Big[\sum_{n=0}^{N-1}\left\|f_j^{n+1}\right\|_{-1}^2 + \sum_{n=0}^{N-1}\left\|u_j^{n+1}\right\|_{\partial D_1}^2 + \left\|\nabla y_{j,h}^0\right\|^2\\
&+\left\|y_{j,h}^0\right\|_{\partial D_1}^2 + \left\|y_{j,h}^0\right\|^2\Big].
\end{aligned} \tag{14}$$

Proof Choosing $v_h = y_{j,h}^{n+1}$ in (12), we have

$$\begin{aligned}
&\frac{1}{\Delta t}\left(y_{j,h}^{n+1}-y_{j,h}^n, y_{j,h}^{n+1}\right) + \left(\bar{a}^{n+1}\nabla y_{j,h}^{n+1}, \nabla y_{j,h}^{n+1}\right)\\
&+\left(\left(a_j^{n+1}-\bar{a}^{n+1}\right)\nabla y_{j,h}^n, \nabla y_{j,h}^{n+1}\right) + \left(\bar{\alpha}^{n+1}y_{j,h}^{n+1}, y_{j,h}^{n+1}\right)_{\partial D_1}\\
&+\left(\left(\alpha_j^{n+1}-\bar{\alpha}^{n+1}\right)y_{j,h}^n, y_{j,h}^{n+1}\right)_{\partial D_1} = \left(f_j^{n+1}, y_{j,h}^{n+1}\right) + \left(\alpha_j^{n+1}u_j^{n+1}, y_{j,h}^{n+1}\right)_{\partial D_1}.
\end{aligned}$$

Applying the polarization identity and the coercivity of $\bar{a}^{n+1}$ and $\bar{\alpha}^{n+1}$, multiplying both sides by Δt, we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|y_{j,h}^{n+1}\right\|^2 - \frac{1}{2}\left\|y_{j,h}^n\right\|^2 + \frac{1}{2}\left\|y_{j,h}^{n+1}-y_{j,h}^n\right\|^2 + \Delta t\lambda\left\|\nabla y_{j,h}^{n+1}\right\|^2 + \Delta t\mu_{min}\left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2\\
\leq& -\Delta t\left(\left(a_j^{n+1}-\bar{a}^{n+1}\right)\nabla y_{j,h}^n, \nabla y_{j,h}^{n+1}\right) + \Delta t\left(f_j^{n+1}, y_{j,h}^{n+1}\right)\\
&-\Delta t\left(\left(\alpha_j^{n+1}-\bar{\alpha}^{n+1}\right)y_{j,h}^n, y_{j,h}^{n+1}\right)_{\partial D_1} + \Delta t\left(\alpha_j^{n+1}u_j^{n+1}, y_{j,h}^{n+1}\right)_{\partial D_1}.
\end{aligned} \tag{15}$$

For some $\eta, \xi, \theta, \rho > 0$, according to Cauchy-Schwarz and Young inequalities, one obtains

$$\begin{aligned}
&\Delta t\mid\left(\left(a_j^{n+1}-\bar{a}^{n+1}\right)\nabla y_{j,h}^n, \nabla y_{j,h}^{n+1}\right)\mid\\
\leq&\Delta t\mid a_j^{n+1}-\bar{a}^{n+1}\mid_\infty \left\|\nabla y_{j,h}^n\right\|\left\|\nabla y_{j,h}^{n+1}\right\|\\
\leq&\Delta t\mid a_j^{n+1}-\bar{a}^{n+1}\mid_\infty\left(\frac{1}{2\eta}\left\|\nabla y_{j,h}^n\right\|^2 + \frac{\eta}{2}\left\|\nabla y_{j,h}^{n+1}\right\|^2\right),
\end{aligned} \tag{16}$$

$$\begin{aligned}
\Delta t\mid\left(f_j^{n+1}, y_{j,h}^{n+1}\right)\mid \leq& \Delta t\left\|f_j^{n+1}\right\|_{-1}\left\|y_{j,h}^{n+1}\right\|_1\\
\leq& \Delta t\left\|f_j^{n+1}\right\|_{-1}C\left(\left\|\nabla y_{j,h}^{n+1}\right\|^2 + \left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2\right)^{\frac{1}{2}}\\
=& \frac{C\Delta t}{4\xi}\left\|f_j^{n+1}\right\|_{-1}^2 + \xi\Delta t\left(\left\|\nabla y_{j,h}^{n+1}\right\|^2 + \left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2\right),
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\Delta t \mid \left(\left(\alpha_j^{n+1} - \bar{a}^{n+1}\right) \nabla y_{j,h}^n, \nabla y_{j,h}^{n+1}\right) \mid_{\partial D_1} \\
&\leq \Delta t \mid \alpha_j^{n+1} - \bar{\alpha}^{n+1} \mid_\infty \|y_{j,h}^n\|_{\partial D_1} \left\|y_{j,h}^{n+1}\right\|_{\partial D_1} \\
&\leq \Delta t \mid \alpha_j^{n+1} - \bar{\alpha}^{n+1} \mid_\infty \left(\frac{1}{2\theta} \|y_{j,h}^n\|_{\partial D_1}^2 + \frac{\theta}{2} \left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2\right),
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\Delta t \left(\alpha_j^{n+1} u_j^{n+1}, y_{j,h}^{n+1}\right)_{\partial D_1} &\leq \Delta t \mid \alpha_j^{n+1} \mid_\infty \left\|u_j^{n+1}\right\|_{\partial D_1} \left\|y_{j,h}^{n+1}\right\|_{\partial D_1} \\
&\leq \Delta t \mu_{max} \left(\frac{1}{4\rho} \left\|u_j^{n+1}\right\|_{\partial D_1}^2 + \rho \left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2\right).
\end{aligned} \tag{19}$$

Substituting (16), (17), (18) and (19) into (15), and dropping the non-negative term $\frac{1}{2} \left\|y_{j,h}^{n+1} - y_{j,h}^n\right\|^2$, we get

$$\begin{aligned}
&\frac{1}{2}\left(\left\|y_{j,h}^{n+1}\right\|^2 - \|y_{j,h}^n\|^2\right) + \Delta t \left[\lambda - \xi - \left(\frac{\eta}{2} + \frac{1}{2\eta}\right) \mid a_j^{n+1} - \bar{a}^{n+1} \mid_\infty\right] \left\|\nabla y_{j,h}^{n+1}\right\|^2 \\
&+ \frac{\Delta t}{2\eta} \mid a_j^{n+1} - \bar{a}^{n+1} \mid_\infty \left(\left\|\nabla y_{j,h}^{n+1}\right\|^2 - \|\nabla y_h^n\|^2\right) \\
&+ \Delta t \left[\mu_{min} - \xi - \rho\mu_{max} - \left(\frac{\theta}{2} + \frac{1}{2\theta}\right) \mid \alpha_j^{n+1} - \bar{\alpha}^{n+1} \mid_\infty\right] \left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2 \\
&+ \frac{\Delta t}{2\theta} \mid \alpha_j^{n+1} - \bar{\alpha}^{n+1} \mid_\infty \left(\left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2 - \|y_{j,h}^n\|_{\partial D_1}^2\right) \\
&\leq \frac{C\Delta t}{4\xi} \left\|f_j^{n+1}\right\|_{-1}^2 + \frac{\Delta t \mu_{max}}{4\rho} \left\|u_j^{n+1}\right\|_{\partial D_1}^2.
\end{aligned}$$

Amounting n from 0 to $N-1$, multiplying both sides by 2, taking $\eta = 1$ and $\theta = 1$, we have

$$\begin{aligned}
&\left\|y_{j,h}^N\right\|^2 - \left\|y_{j,h}^0\right\|^2 + 2\Delta t \sum_{n=0}^{N-1} \left(\lambda - \xi - \mid a_j^{n+1} - \bar{a}^{n+1} \mid_\infty\right) \left\|\nabla y_{j,h}^{n+1}\right\|^2 \\
&+ \Delta t \sum_{n=0}^{N-1} \mid a_j^{n+1} - \bar{a}^{n+1} \mid_\infty \left(\left\|\nabla y_{j,h}^{n+1}\right\|^2 - \|\nabla y_{j,h}^n\|^2\right) \\
&+ 2\Delta t \sum_{n=0}^{N-1} \left[\mu_{min} - \xi - \rho\mu_{max} - \mid \alpha_j^{n+1} - \bar{\alpha}^{n+1} \mid_\infty\right] \left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2 \\
&+ \Delta t \sum_{n=0}^{N-1} \mid \alpha_j^{n+1} - \bar{\alpha}^{n+1} \mid_\infty \left(\left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2 - \|y_{j,h}^n\|_{\partial D_1}^2\right) \\
&\leq \frac{C\Delta t}{2\xi} \sum_{n=0}^{N-1} \left\|f_j^{n+1}\right\|_{-1}^2 + \frac{\Delta t \mu_{max}}{2\rho} \sum_{n=0}^{N-1} \left\|u_j^{n+1}\right\|_{\partial D_1}^2.
\end{aligned} \tag{20}$$

Choosing

$$\begin{aligned}
&\xi = min\left\{\frac{\lambda - \mid a_j^{n+1} - \bar{a}^{n+1} \mid_\infty}{2}, \frac{\mu_{min} - \mid \alpha_j^{n+1} - \bar{\alpha}^{n+1} \mid_\infty}{4}\right\}, \\
&\rho = \frac{\mu_{min} - \mid \alpha_j^{n+1} - \bar{\alpha}^{n+1} \mid_\infty}{4\mu_{max}},
\end{aligned}$$

and using the conditions (8)-(11) and (13), we can obtain

$$\begin{aligned}
&\left\|y_{j,h}^{N}\right\|^{2}+\lambda_{-}\Delta t\left\|\nabla y_{j,h}^{N}\right\|^{2}+\left(\lambda-\lambda_{+}\right)\Delta t\sum_{n=0}^{N-1}\left\|\nabla y_{j,h}^{n+1}\right\|^{2}\\
&+\mu_{-}\Delta t\left\|y_{j,h}^{N}\right\|_{\partial D_1}^{2}+\left(\mu_{min}-\mu_{+}\right)\Delta t\sum_{n=0}^{N-1}\left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^{2}\\
\leq&\frac{C\Delta t}{min\left\{\lambda-\lambda_{+},\frac{\mu_{min}-\mu_{+}}{2}\right\}}\sum_{n=0}^{N-1}\left\|f_j^{n+1}\right\|_{-1}^{2}+\frac{2\Delta t\mu_{max}^{2}}{\mu_{min}-\mu_{+}}\sum_{n=0}^{N-1}\left\|u_j^{n+1}\right\|_{\partial D_1}^{2}\\
&+\lambda_{-}\Delta t\left\|\nabla y_{j,h}^{0}\right\|^{2}+\mu_{-}\Delta t\left\|y_{j,h}^{0}\right\|_{\partial D_1}^{2}+\left\|y_{j,h}^{0}\right\|^{2}.
\end{aligned}$$

One gets the desired conclusion immediately. □

Remark 1 The stability condition (13) requests, for $\left\{a_j\right\}_{j=1}^{J}$, the difference between a_j and its mean $\bar{a}$ is smaller than the coercivity constant λ. Similar requirement holds for $\left\{\alpha_j\right\}_{j=1}^{J}$. Without such case, one partition the ensemble into smaller groups and applies the ensemble algorithm to each smaller groups, in this process one must maintain the stability condition in these smaller groups.

3.2 Error analysis.

In this subsection, the approximation error of the ensemble scheme (12) is estimated. Suppose the transient heat equation's solution is smooth enough, particularly,

$$y_j\in L^2\left(0,T;\ H^1(D)\cap H^{l+1}(D)\right)\cap H^1\left(0,T;\ H^{l+1}(D)\right)\cap H^2\left(0,T;\ L^2(D)\right),$$

and similar requirement holds on boundary ∂D_1. We have the following result for the error of the numerical approximation of (12).

Theorem 2 *Denote y_j^n and $y_{j,h}^n$ the solutions of systems (2) and (12) at t_n, respectively. Assume $f_j\in L^2\left(0,T;\ H^{-1}(D)\right)$, $u_j\in L^2\left(0,T;\ L^2(\partial D_1)\right)$, and hypothesis (**H1**) and (**H2**) are satisfied. Thus there exists a positive constant C such that*

$$\begin{aligned}
&\left\|y_j^N-y_{j,h}^N\right\|^2+\lambda_{-}\Delta t\left\|\nabla\left(y_j^N-y_{j,h}^N\right)\right\|^2+\mu_{-}\Delta t\left\|y_j^N-y_{j,h}^N\right\|_{\partial D_1}^2\\
&+\left(\lambda-\lambda_{+}\right)\Delta t\sum_{n=1}^{N}\left\|\nabla\left(y_j^n-y_{j,h}^n\right)\right\|^2+\left(\mu_{min}-\mu_{+}\right)\Delta t\sum_{n=1}^{N}\left\|y_j^n-y_{j,h}^n\right\|_{\partial D_1}^2\\
\leq&C\left(\Delta t^2+h^{2l}\right),
\end{aligned}\tag{21}$$

if the stability condition (13) holds, that is, $\lambda-\lambda_{+}>0$ and $\mu_{min}-\mu_{+}>0$.

Proof Firstly, we establish the error equation to calculate the approximation error in (12). Evaluating the system (2) at $t = t_{n+1}$, one can get

$$\begin{aligned} &\frac{1}{\Delta t}\left(y_j^{n+1} - y_j^n, v_h\right) + \left(a_j^{n+1}\nabla y_j^{n+1}, \nabla v_h\right) + \left(\alpha_j^{n+1} y_j^{n+1}, v_h\right)_{\partial D_1} \\ &= \left(f_j^{n+1}, v_h\right) + \left(\alpha_j^{n+1} u_j^{n+1}, v_h\right)_{\partial D_1} - \left(r_j^{n+1}, v_h\right), \quad \forall v_h \in V_h \cup S_h, \end{aligned} \tag{22}$$

where $r_j^{n+1} = y_{j,t}^{n+1} - \frac{y_j^{n+1} - y_j^n}{\Delta t}$. Let $e_j^n := y_j^n - y_{j,h}^n$. Subtracting (12) from (22), one can obtain

$$\begin{aligned} &\frac{1}{\Delta t}\left(e_j^{n+1} - e_j^n, v_h\right) + \left(\bar{a}^{n+1}\nabla e_j^{n+1}, \nabla v_h\right) + \left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla e_j^n, \nabla v_h\right) \\ &+ \left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla\left(y_j^{n+1} - y_j^n\right), \nabla v_h\right) + \left(\bar{\alpha}^{n+1} e_j^{n+1}, v_h\right)_{\partial D_1} \\ &+ \left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right) e_j^n, v_h\right)_{\partial D_1} + \left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right)\left(y_j^{n+1} - y_j^n\right), v_h\right)_{\partial D_1} \\ &+ \left(r_j^{n+1}, v_h\right) = 0. \end{aligned} \tag{23}$$

Divide the error into two parts:

$$e_j^n = \left(y_j^n - Q_h\left(y_j^n\right)\right) - \left(y_{j,h}^n - Q_h\left(y_j^n\right)\right) = \rho_j^n - \phi_{j,h}^n,$$

here $\rho_j^n = y_j^n - Q_h\left(y_j^n\right)$, $\phi_{j,h}^n = y_{j,h}^n - Q_h\left(y_j^n\right)$, $Q_h\left(y_j^n\right)$ is the L^2 projection of y_j^n onto $V_h \cup S_h$, namely, $\left(y_j^n - Q_h(y_j^n), v_h\right) = 0$ for any $v_h \in V_h \cup S_h$. One thus applies the decomposition in (23), and gets

$$\begin{aligned} &\frac{1}{\Delta t}\left(\phi_{j,h}^{n+1} - \phi_{j,h}^n, v_h\right) + \left(\bar{a}^{n+1}\nabla\phi_{j,h}^{n+1}, \nabla v_h\right) + \left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla\phi_{j,h}^n, \nabla v_h\right) \\ &+ \left(\bar{\alpha}^{n+1}\phi_{j,h}^{n+1}, v_h\right)_{\partial D_1} + \left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right)\phi_{j,h}^n, v_h\right)_{\partial D_1} \\ &= \left(\bar{a}^{n+1}\nabla\rho_j^{n+1}, \nabla v_h\right) + \left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla\rho_j^n, \nabla v_h\right) \\ &+ \left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla\left(y_j^{n+1} - y_j^n\right), \nabla v_h\right) \\ &+ \left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right)\left(y_j^{n+1} - y_j^n\right), v_h\right)_{\partial D_1} + \left(r_j^{n+1}, v_h\right). \end{aligned} \tag{24}$$

Letting $v_h = \phi_{j,h}^{n+1}$, using the polarization identity, and coercivity (8) and (9), one has

$$\begin{aligned} &\frac{1}{2\Delta t}\left(\left\|\phi_{j,h}^{n+1}\right\|^2 - \left\|\phi_{j,h}^n\right\|^2 + \left\|\phi_{j,h}^{n+1} - \phi_{j,h}^n\right\|^2\right) + \lambda\left\|\nabla\phi_{j,h}^{n+1}\right\|^2 + \mu_{min}\left\|\phi_{j,h}^{n+1}\right\|_{\partial D_1}^2 \\ &\leq \left|\left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla\phi_{j,h}^n, \nabla\phi_{j,h}^{n+1}\right)\right| + \left|\left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right)\phi_{j,h}^n, \phi_{j,h}^{n+1}\right)_{\partial D_1}\right| \\ &+ \left|\left(\bar{a}^{n+1}\nabla\rho_j^{n+1}, \nabla\phi_{j,h}^{n+1}\right)\right| + \left|\left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla\rho_j^n, \nabla\phi_{j,h}^{n+1}\right)\right| \\ &+ \left|\left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla\left(y_j^{n+1} - y_j^n\right), \nabla\phi_{j,h}^{n+1}\right)\right| \\ &+ \left|\left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right)\left(y_j^{n+1} - y_j^n\right), \phi_{j,h}^{n+1}\right)_{\partial D_1}\right| + \left|\left(r_j^{n+1}, \phi_{j,h}^{n+1}\right)\right|. \end{aligned} \tag{25}$$

For any positive constants $\beta_i (i = 0, \ldots, 4)$, Young's and Cauchy-Schwarz's inequalities imply

$$\left|\left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla\phi_{j,h}^n, \nabla\phi_{j,h}^{n+1}\right)\right| \leq \left| a_j^{n+1} - \bar{a}^{n+1}\right|_\infty \left(\frac{\left\|\nabla\phi_{j,h}^n\right\|^2}{2} + \frac{\left\|\nabla\phi_{j,h}^{n+1}\right\|^2}{2}\right),$$

$$\begin{aligned}
&\mid \left(\left(\alpha_j^{n+1}-\bar{\alpha}^{n+1}\right)\phi_{j,h}^n, \phi_{j,h}^{n+1}\right)_{\partial D_1} \mid \\
\leq &\mid \alpha_j^{n+1}-\bar{\alpha}^{n+1}\mid_\infty \left(\frac{\|\phi_{j,h}^n\|_{\partial D_1}^2}{2}+\frac{\left\|\phi_{j,h}^{n+1}\right\|_{\partial D_1}^2}{2}\right),
\end{aligned}$$

$$\mid \left(\bar{a}^{n+1}\nabla\rho_j^{n+1}, \nabla\phi_{j,h}^{n+1}\right) \mid\leq\mid \bar{a}^{n+1}\mid_\infty \left(\frac{\left\|\nabla\rho_j^{n+1}\right\|^2}{2\beta_0}+\frac{\beta_0\left\|\nabla\phi_{j,h}^{n+1}\right\|^2}{2}\right),$$

$$\mid \left(\left(a_j^{n+1}-\bar{a}^{n+1}\right)\nabla\rho_j^n, \nabla\phi_{j,h}^{n+1}\right) \mid\leq \mid a_j^{n+1}-\bar{a}^{n+1}\mid_\infty \left(\frac{\|\nabla\rho_j^n\|^2}{2\beta_1}+\frac{\beta_1\left\|\nabla\phi_{j,h}^{n+1}\right\|^2}{2}\right),$$

$$\begin{aligned}
&\mid \left(\left(a_j^{n+1}-\bar{a}^{n+1}\right)\nabla\left(y_j^{n+1}-y_j^n\right), \nabla\phi_{j,h}^{n+1}\right) \mid \\
\leq &\mid a_j^{n+1}-\bar{a}^{n+1}\mid_\infty \left(\frac{\left\|\nabla y_j^{n+1}-\nabla y_j^n\right\|^2}{2\beta_2}+\frac{\beta_2\left\|\nabla\phi_{j,h}^{n+1}\right\|^2}{2}\right),
\end{aligned}$$

$$\begin{aligned}
&\mid \left(\left(\alpha_j^{n+1}-\bar{\alpha}^{n+1}\right)\left(y_j^{n+1}-y_j^n\right), \phi_{j,h}^{n+1}\right)_{\partial D_1} \mid \\
\leq &\mid \alpha_j^{n+1}-\bar{\alpha}^{n+1}\mid_\infty \left(\frac{\left\|y_j^{n+1}-y_j^n\right\|_{\partial D_1}^2}{2\beta_3}+\frac{\beta_3\left\|\phi_{j,h}^{n+1}\right\|_{\partial D_1}^2}{2}\right),
\end{aligned}$$

$$\mid \left(r_j^{n+1}, \phi_{j,h}^{n+1}\right) \mid\leq \left(\frac{\left\|r_j^{n+1}\right\|_{-1}^2}{2\beta_4}+\frac{\beta_4\left(\left\|\nabla\phi_{j,h}^{n+1}\right\|^2+\left\|\phi_{j,h}^{n+1}\right\|_{\partial D_1}^2\right)}{2}\right).$$

Dropping the non-negative term $\frac{1}{2\Delta t}\|\phi_{j,h}^{n+1}-\phi_{j,h}^n\|^2$, and using the above inequalities in (25), one can get

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\left\|\phi_{j,h}^{n+1}\right\|^2-\|\phi_{j,h}^n\|^2\right)+\frac{\mid a_j^{n+1}-\bar{a}^{n+1}\mid_\infty}{2}\left(\left\|\nabla\phi_{j,h}^{n+1}\right\|^2-\|\nabla\phi_{j,h}^n\|^2\right) \\
&+\frac{\mid \alpha_j^{n+1}-\bar{\alpha}^{n+1}\mid_\infty}{2}\left(\left\|\phi_{j,h}^{n+1}\right\|_{\partial D_1}^2-\|\phi_{j,h}^n\|_{\partial D_1}^2\right) \\
&+\left(\lambda-\frac{\beta_0}{2}\mid \bar{a}^{n+1}\mid_\infty-\frac{\beta_1+\beta_2+2}{2}\mid a_j^{n+1}-\bar{a}^{n+1}\mid_\infty-\frac{\beta_4}{2}\right)\left\|\nabla\phi_{j,h}^{n+1}\right\|^2 \\
&+\left(\mu_{min}-\frac{\beta_3+2}{2}\mid \alpha_j^{n+1}-\bar{\alpha}^{n+1}\mid_\infty-\frac{\beta_4}{2}\right)\left\|\phi_{j,h}^{n+1}\right\|_{\partial D_1}^2 \\
\leq&\frac{\mid \bar{a}^{n+1}\mid_\infty}{2\beta_0}\left\|\nabla\rho_j^{n+1}\right\|^2+\frac{\mid a_j^{n+1}-\bar{a}^{n+1}\mid_\infty}{2\beta_1}\|\nabla\rho_j^n\|^2 \\
&+\frac{\mid a_j^{n+1}-\bar{a}^{n+1}\mid_\infty}{2\beta_2}\left\|\nabla y_j^{n+1}-\nabla y_j^n\right\|^2 \\
&+\frac{\mid \alpha_j^{n+1}-\bar{\alpha}^{n+1}\mid_\infty}{2\beta_3}\left\|y_j^{n+1}-y_j^n\right\|_{\partial D_1}^2+\frac{\left\|r_j^{n+1}\right\|_{-1}^2}{2\beta_4}.
\end{aligned}$$

Selecting $\beta_0 = \frac{\delta|a_j^{n+1}-\bar{a}^{n+1}|_\infty}{2|\bar{a}^{n+1}|_\infty}$, $\beta_1 = \beta_2 = \frac{\delta}{2}$, $\beta_3 = \frac{\delta|a_j^{n+1}-\bar{a}^{n+1}|_\infty}{2|\alpha_j^{n+1}-\bar{\alpha}^{n+1}|_\infty}$, and $\beta_4 = \frac{\delta|a_j^{n+1}-\bar{a}^{n+1}|_\infty}{2}$ for some positive δ, yields that

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\left\|\phi_{j,h}^{n+1}\right\|^2 - \|\phi_{j,h}^{n}\|^2\right) + \frac{|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty}{2}\left(\left\|\nabla\phi_{j,h}^{n+1}\right\|^2 - \|\nabla\phi_{j,h}^{n}\|^2\right)\\
&+\frac{|\, \alpha_j^{n+1} - \bar{\alpha}^{n+1}\,|_\infty}{2}\left(\left\|\phi_{j,h}^{n+1}\right\|^2 - \|\phi_{j,h}^{n}\|^2_{\partial D_1}\right)\\
&+\left[\lambda - (1+\delta)\,|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty\right]\left\|\nabla\phi_{j,h}^{n+1}\right\|^2\\
&+\left[\mu_{min} - \delta\,|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty - |\, \alpha_j^{n+1} - \bar{\alpha}^{n+1}\,|_\infty\right]\left\|\phi_{j,h}^{n+1}\right\|^2_{\partial D_1}\\
\leq&\frac{|\,\bar{a}^{n+1}\,|^2_\infty}{\delta\,|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty}\left\|\nabla\rho_j^{n+1}\right\|^2 + \frac{|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty}{\delta}\left\|\nabla\rho_j^{n}\right\|^2\\
&+\frac{\left\|r_j^{n+1}\right\|^2_{-1}}{\delta\,|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty} + \frac{|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty}{\delta}\left\|\nabla y_j^{n+1} - \nabla y_j^{n}\right\|^2\\
&+\frac{|\, \alpha_j^{n+1} - \bar{\alpha}^{n+1}\,|^2_\infty}{\delta\,|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty}\left\|y_j^{n+1} - y_j^{n}\right\|^2_{\partial D_1}.
\end{aligned} \tag{26}$$

Taking $\delta = min\left\{\frac{\lambda-\lambda_+}{2\lambda_+}, \frac{\mu_{min}-\mu_+}{2\lambda_+}\right\}$, using the stability condition (13), the uniform coercivity (**H1**) and uniform bounded condition (**H2**), we have that

$$\lambda - (1+\delta)\,|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty \geq \frac{\lambda - \lambda_+}{2} > 0,$$

$$\mu_{min} - \delta\,|\, a_j^{n+1} - \bar{a}^{n+1}\,|_\infty - |\, \alpha_j^{n+1} - \bar{\alpha}^{n+1}\,|_\infty \geq \frac{\mu_{min} - \mu_+}{2} > 0.$$

For the last three items on the RHS of (26), it is that

$$\begin{aligned}
&\left\|\nabla y_j^{n+1} - \nabla y_j^{n}\right\|^2 = \int_D |\, \nabla y_j^{n+1} - \nabla y_j^{n}\,|^2 = \int_D \left|\, \int_{t_n}^{t_{n+1}} \left(\nabla y_j\right)_t \,\right|^2\\
\leq&\Delta t\int_{t_n}^{t_{n+1}}\int_D |\, \left(\nabla y_j\right)_t\,|^2 = \Delta t\left\|\nabla y_{j,t}\right\|^2_{L^2(t_n,t_{n+1};\, L^2(D))},
\end{aligned}$$

and

$$\begin{aligned}
&\left\|y_j^{n+1} - y_j^{n}\right\|^2_{\partial D_1} = \int_{\partial D_1} |\, y_j^{n+1} - y_j^{n}\,|^2 = \int_{\partial D_1} \left|\, \int_{t_n}^{t_{n+1}} \left(y_j\right)_t \,\right|^2\\
\leq&\ \Delta t\int_{t_n}^{t_{n+1}}\int_{\partial D_1} |\, \left(y_j\right)_t\,|^2 = \Delta t\left\|y_{j,t}\right\|^2_{L^2(t_n,t_{n+1};\, L^2(\partial D_1))}.
\end{aligned}$$

The integral form of Taylor's theorem,

$$y_j^n = y_j^{n+1} - \Delta t y_{j,t}^{n+1} - \int_{t_n}^{t_{n+1}} y_{j,tt}(.,\tau)\,(t_n - \tau)\,d\tau,$$

implies

$$\begin{aligned}\left\|r_j^{n+1}\right\| &= \frac{1}{\Delta t}\left\|\int_{t_n}^{t_{n+1}} y_{j,tt}(\cdot,\tau)\,(\tau - t_n)\,d\tau\right\| \leq \int_{t_n}^{t_{n+1}} \left\|y_{j,tt}(\cdot,\tau)\right\| \cdot 1 d\tau \\ &\leq \left[\int_{t_n}^{t_{n+1}} \left\|y_{j,tt}(\cdot,\tau)\right\|^2 d\tau\right]^{1/2} \left(\int_{t_n}^{t_{n+1}} 1^2 d\tau\right)^{1/2} \\ &\leq \sqrt{\Delta t}\left\|y_{j,tt}\right\|_{L^2(t_n,t_{n+1};\, L^2(D))},\end{aligned}$$

and

$$\left\|r_j^{n+1}\right\|_{-1}^2 \leq C\left\|r_j^{n+1}\right\|^2 \leq C\Delta t\left\|y_{j,tt}\right\|^2_{L^2(t_n,t_{n+1};\, L^2(D))}.$$

Replacing by these inequalities in (26), using the uniform coercivity (**H1**) and uniform bounded condition (**H2**) , amounting n from 0 to $N-1$, and multiplying both sides of (26) by $2\Delta t$, we obtain

$$\begin{aligned}&\left\|\phi_{j,h}^N\right\|^2 + \lambda_-\Delta t\left\|\nabla\phi_{j,h}^N\right\|^2 + \mu_-\Delta t\left\|\phi_{j,h}^N\right\|^2_{\partial D_1} \\ &+ (\lambda - \lambda_+)\,\Delta t\sum_{n=0}^{N-1}\left\|\nabla\phi_{j,h}^{n+1}\right\|^2 + (\mu_{min} - \mu_+)\,\Delta t\sum_{n=0}^{N-1}\left\|\phi_{j,h}^{n+1}\right\|^2_{\partial D_1} \\ \leq&\frac{2\Delta t}{\min\left\{\frac{\lambda-\lambda_+}{2\lambda_+}, \frac{\mu_{min}-\mu_+}{2\lambda_+}\right\}}\sum_{n=0}^{N-1}\Bigg(\frac{1}{\lambda_-}\mid \bar{a}^{n+1}\mid_\infty^2 \left\|\nabla\rho_j^{n+1}\right\|^2 + \lambda_+\left\|\nabla\rho_j^n\right\|^2 \\ &+ \lambda_+\Delta t\left\|\nabla y_{j,t}\right\|^2_{L^2(t_n,t_{n+1};\, L^2(D))} + \frac{\mu_{max}^2}{\lambda_-}\Delta t\left\|y_{j,t}\right\|^2_{L^2(t_n,t_{n+1};\, L^2(\partial D_1))} \\ &+ \frac{C}{\lambda_-}\Delta t\left\|y_{j,tt}\right\|^2_{L^2(t_n,t_{n+1};\, L^2(D))}\Bigg),\end{aligned}$$

where we used the assumption that $y_{j,h}^0 = Q_h\left(y_j^0\right)$, thus $\left\|\phi_{j,h}^0\right\| = \left\|\nabla\phi_{j,h}^0\right\| = 0$, similar formula holds on boundary ∂D_1. Applying the regularity hypothesis as well as standard FM estimates of the L^2 projection in norm H^1 (see, e.g., Section 4.4 of [2]), that is, $\forall y_j^n \in H^{l+1}(D) \cap H^1(D)$,

$$\left\|\nabla\rho_j^n\right\|^2 = \left\|\nabla\left(Q_h\left(y_j^n\right) - y_j^n\right)\right\|^2 \leq Ch^{2l}\left\|y_j^n\right\|^2_{l+1},$$

we have

$$\begin{aligned}&\left\|\phi_{j,h}^N\right\|^2 + \lambda_-\Delta t\left\|\nabla\phi_{j,h}^N\right\|^2 + \mu_-\Delta t\left\|\phi_{j,h}^N\right\|^2_{\partial D_1} \\ &+ (\lambda - \lambda_+)\,\Delta t\sum_{n=0}^{N-1}\left\|\nabla\phi_{j,h}^{n+1}\right\|^2 + (\mu_{min} - \mu_+)\,\Delta t\sum_{n=0}^{N-1}\left\|\phi_{j,h}^{n+1}\right\|^2_{\partial D_1} \\ \leq&C\left(\Delta t^2 + h^{2l}\right).\end{aligned}$$

Since the triangle inequality, one gets inequality (21). □

The ensemble solution of the transient heat equations about uncertain inputs is investigated in next section.

4 The ensemble scheme for the stochastic transient heat model

Applying the ensemble scheme to stochastic PDEs (1), we first take the MC approach to random sampling. After sampling with independent identically distributed (i.i.d.), we solve (2). Then, the solution for (1) is obtained from the expectation of the solution of (2). An ensemble-based Monte-Carlo algorithm (i.e. EMC algorithm) is proposed to quantifies uncertainty and improve its computational efficiency. This algorithm is comprised of the three steps:

Algorithm 1 EMC Algorithm

Step1. Select a group of stochastic samples for the stochastic diffusion coefficient, Robin coefficient, source term, boundary and initial conditions $a_j \equiv a(\cdot,\cdot,\omega_j)$, $\alpha_j \equiv \alpha(\cdot,\cdot,\omega_j)$, $f_j \equiv f(\cdot,\cdot,\omega_j)$, $u_j \equiv u(\cdot,\cdot,\omega_j)$, and $y_j^0 \equiv y^0(\cdot,\omega_j)$ for the j-th sample, respectively. As a result, the solutions $y(\cdot,\cdot,\omega_j)$ be i.i.d..

Step2. Let $y_j^n = y(t_n,\mathbf{x},\omega_j)$, $\bar{a}^n = \frac{1}{J}\sum_{j=1}^J a(t_n,\mathbf{x},\omega_j)$, and $\bar{\alpha}^n = \frac{1}{J}\sum_{j=1}^J \alpha(t_n,\mathbf{x},\omega_j)$. For the j-th sample, $n = 0,\ldots,N-1$, we seeks y_j^{n+1} such that the algorithm (5). On numerical simulation, the appropriate FE spaces can be selected, one seeks the FE solution $y_h(\cdot,\cdot,\omega_j)$ on FE spaces.

Step3. Approximate the expectation $\mathbb{E}[y]$ by the EMC sample average $\frac{1}{J}\sum_{j=1}^J y_h(\cdot,\cdot,\omega_j)$. A quantity of interest $G(y)$ is given, one is to discuss the outputs for the ensemble systems, $G(y_h(\cdot,\cdot,\omega_1))$, ..., $G(y_h(\cdot,\cdot,\omega_J))$, to obtain the stochastic information.

Next, we give the stability analysis and error estimate.

4.1 Stability.

Paralleling to handle the PDE problem (2), we select the same FE space V_h and S_h mentioned in Section 2. Let $y_{j,h}^n = y_h(t_n,\mathbf{x},\omega_j)$. Given the j-th sample and $n = 0,1,\ldots,N-1$, the EMC finite element method is to seek an approximation solution $y_{j,h}^{n+1} \in V_h \cup S_h$ such that

$$\begin{aligned}
&\left(\frac{y_{j,h}^{n+1} - y_{j,h}^n}{\Delta t}, v_h\right) + \left(\bar{a}^{n+1}\nabla y_{j,h}^{n+1}, \nabla v_h\right) + \left(\left(a_j^{n+1} - \bar{a}^{n+1}\right)\nabla y_{j,h}^n, \nabla v_h\right)\\
&+ \left(\bar{\alpha}^{n+1} y_{j,h}^{n+1}, v_h\right)_{\partial D_1} + \left(\left(\alpha_j^{n+1} - \bar{\alpha}^{n+1}\right) y_{j,h}^n, v_h\right)_{\partial D_1}\\
&= \left(f_j^{n+1}, v_h\right) + \left(\alpha_j^{n+1} u_j^{n+1}, v_h\right)_{\partial D_1}, \forall v_h \in V_h \cup S_h,
\end{aligned} \tag{27}$$

the initial value $y_{j,h}^0 \in V_h \cup S_h$ holds $\left(y_{j,h}^0, v_h\right) = \left(y_j^0, v_h\right)$ for all $v_h \in V_h \cup S_h$.

Note that $y_{j,h}^n$ in (27) is a random variable. To analyze the corresponding stability and error estimate, we suppose the following two hypotheses (**H3**) and (**H4**) are satisfied.

(**H3**) There exist three positive constants λ, μ_{min}, μ_{max}, such that, for $\forall t \in [0, T]$, the probability

$$Pro\left\{\omega_j \in \Omega;\ \min_{x\in\bar{D}} a(t, \mathbf{x}, \omega_j) > \lambda\right\} = 1, \tag{28}$$

and

$$Pro\left\{\omega_j \in \Omega;\ \mu_{max} > \alpha(t, \mathbf{x}, \omega_j) > \mu_{min}\right\} = 1. \tag{29}$$

(**H4**) There exist four positive constants λ_-, λ_+, μ_-, μ_+, such that, for $\forall t \in [0, T]$, the probability

$$Pro\left\{\omega_j \in \Omega;\ \lambda_- \leq \mid a\left(t, \mathbf{x}, \omega_j\right) - \bar{a} \mid_\infty \leq \lambda_+\right\} = 1, \tag{30}$$

and

$$Pro\left\{\omega_j \in \Omega;\ \mu_- \leq \mid \alpha\left(t, \mathbf{x}, \omega_j\right) - \bar{\alpha} \mid_\infty \leq \mu_+\right\} = 1. \tag{31}$$

Hypothesis (**H3**) ensures the uniform coercivity almost surely; hypothesis (**H4**) indicates the uniform bound for $\mid a\left(t, \mathbf{x}, \omega_j\right) - \bar{a}(t, \mathbf{x}) \mid$ a.s.. Similar properties is for the Robin coefficient $\alpha\left(t, \mathbf{x}, \omega_j\right)$.

From Theorem 1 and the property of expectation, we will derive to some stability results of the FE solution $y_{j,h}^n$:

Theorem 3 *Assume $f_j \in \widetilde{L}^2\left(0, T;\ H^{-1}(D)\right)$, $u_j \in \widetilde{L}^2\left(0, T;\ L^2(\partial D_1)\right)$, hypotheses (**H3**) and (**H4**) are held. Then the stability for the FE solution $y_{j,h}^n$ of the algorithm (27) hold if*

$$\lambda - \lambda_+ > 0 \qquad \textit{and} \qquad \mu_{min} - \mu_+ > 0. \tag{32}$$

Particularly, for all $\Delta t > 0$, the FE solution suits to

$$\begin{aligned}
&\mathbb{E}\left[\left\|y_{j,h}^N\right\|^2\right] + \lambda_-\Delta t\mathbb{E}\left[\left\|\nabla y_{j,h}^N\right\|^2\right] + \left(\lambda - \lambda_+\right)\Delta t \sum_{n=0}^{N-1} \mathbb{E}\left[\left\|\nabla y_{j,h}^{n+1}\right\|^2\right] \\
&+ \mu_-\Delta t\mathbb{E}\left[\left\|y_{j,h}^N\right\|_{\partial D_1}^2\right] + \left(\mu_{min} - \mu_+\right)\Delta t \sum_{n=0}^{N-1} \mathbb{E}\left[\left\|y_{j,h}^{n+1}\right\|_{\partial D_1}^2\right] \\
\leq& C\Delta t \sum_{n=0}^{N-1} \mathbb{E}\left[\left\|f_j^{n+1}\right\|_{-1}^2\right] + C\Delta t \sum_{n=0}^{N-1} \mathbb{E}\left[\left\|u_j^{n+1}\right\|_{\partial D_1}^2\right] + C\Delta t\mathbb{E}\left[\left\|\nabla y_{j,h}^0\right\|^2\right] \\
&+ C\Delta t\mathbb{E}\left[\left\|y_{j,h}^0\right\|_{\partial D_1}^2\right] + \mathbb{E}\left[\left\|y_{j,h}^0\right\|^2\right].
\end{aligned} \tag{33}$$

The stability condition (32) confines the difference between diffusion coefficients and the mean. The Robin coefficient is similar. Parallel to the deterministic equation (see Remark 1), If the stability condition false, the ensemble can be divided into smaller groups to maintain the condition (32) that applies to each group, so the EMC scheme will apply to all the smaller groups.

4.2 Convergence analysis.

The EMC approximate solution of the full discretization is defined as $\Psi_h^n \equiv \frac{1}{J}\sum_{j=1}^J y_{j,h}^n$. Thus, one will lead an estimate for $\mathbb{E}\left[y^n\right]-\Psi_h^n$ in some averaged norms. $\mathbb{E}\left[y^n\right]-\Psi_h^n$ can be divide into two terms:

$$\begin{aligned}\mathbb{E}\left[y^n\right]-\Psi_h^n &= \left(\mathbb{E}\left[y_j^n\right]-\mathbb{E}\left[y_{j,h}^n\right]\right)+\left(\mathbb{E}\left[y_{j,h}^n\right]-\Psi_h^n\right)\\ &=\mathcal{E}_h^n+\mathcal{E}_S^n,\end{aligned}$$

here we apply $\mathbb{E}\left[y^n\right]=\mathbb{E}\left[y_j^n\right]$. The first term is corresponding to the FE discretization error, $\mathcal{E}_h^n=\mathbb{E}\left[y_j^n-y_{j,h}^n\right]$, is controlled by the time step as well as mesh size. For the second term that is the statistical error, $\mathcal{E}_S^n=\mathbb{E}\left[y_{j,h}^n\right]-\Psi_h^n$, is controlled by the sample size J. Next, we examine the bounds of $\mathcal{E}_h^n$ and $\mathcal{E}_S^n$. And then we achieve an error estimate for the EMC scheme.

Theorem 4 *Assume y_j^n is the solution to stochastic PDE (1) while $\omega=\omega_j$ with $t=t_n$, $y_{j,h}^n$ suits to (27). Let $y_j^0\in\widetilde{L}^2\left(H^1(D)\cap H^{l+1}(D)\right)$, $f_j\in\widetilde{L}^2\left(0,T;\ H^{-1}(D)\right)$, $u_j\in\widetilde{L}^2\left(0,T;\ L^2(\partial D_1)\right)$. Under hypotheses (**H3**) and (**H4**), there is a constant $C>0$ satisfies*

$$\begin{aligned}&\mathbb{E}\left[\left\|y_j^N-y_{j,h}^N\right\|^2\right]+\lambda_-\Delta t\mathbb{E}\left[\left\|\nabla\left(y_j^N-y_{j,h}^N\right)\right\|^2\right]\\ &+\mu_-\Delta t\mathbb{E}\left[\left\|y_j^N-y_{j,h}^N\right\|_{\partial D_1}^2\right]+\left(\lambda-\lambda_+\right)\Delta t\sum_{n=1}^N\mathbb{E}\left[\left\|\nabla\left(y_j^n-y_{j,h}^n\right)\right\|^2\right]\\ &+\left(\mu_{min}-\mu_+\right)\Delta t\sum_{n=1}^N\mathbb{E}\left[\left\|y_j^n-y_{j,h}^n\right\|_{\partial D_1}^2\right]\le C\left(\Delta t^2+h^{2l}\right),\end{aligned}\tag{34}$$

if the stability condition (32) holds.

Proof This result holds by Theorem 2 afterward using the expectation on (21). □

The statistical error $\mathcal{E}_S^n$ can be estimated through applying the standard error calculation of MC approach (e.g., see [17]):

Theorem 5 *Assume statements (**H3**) and (**H4**), the stability condition (32) hold, suppose $f_j\in\widetilde{L}^2\left(0,T;\ H^{-1}(D)\right)$, $u_j\in\widetilde{L}^2\left(0,T;\ L^2(\partial D_1)\right)$, $y_j^0\in$*

$\widetilde{L}^2\left(H^1(D)\cap H^{l+1}(D)\right)$, *thus there is a constant* $C>0$, *satisfies*

$$\begin{aligned}
&\mathbb{E}\left[\left\|\mathcal{E}_S^N\right\|^2\right]+\lambda_-\Delta t\mathbb{E}\left[\left\|\nabla\mathcal{E}_S^N\right\|^2\right]+(\lambda-\lambda_+)\,\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\nabla\mathcal{E}_S^{n+1}\right\|^2\right]\\
&+\mu_-\Delta t\mathbb{E}\left[\left\|\mathcal{E}_S^N\right\|_{\partial D_1}^2\right]+(\mu_{min}-\mu_+)\,\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\mathcal{E}_S^{n+1}\right\|_{\partial D_1}^2\right]\\
\le&\frac{C}{J}\left(\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|f_j^{n+1}\right\|_{-1}^2\right]+\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|u_j^{n+1}\right\|_{\partial D_1}^2\right]+\Delta t\mathbb{E}\left[\left\|\nabla y_{j,h}^0\right\|^2\right]\right.\\
&\left.+\Delta t\mathbb{E}\left[\left\|y_{j,h}^0\right\|_{\partial D_1}^2\right]+\mathbb{E}\left[\left\|y_{j,h}^0\right\|^2\right]\right).
\end{aligned}\tag{35}$$

Proof We first estimate $\mathbb{E}\left[\|\nabla\mathcal{E}_S^n\|\right]$. Define $\langle y_h^n,y_h^n\rangle:=(\nabla y_h^n,\nabla y_h^n)$. It is easily that

$$\begin{aligned}
\mathbb{E}\left[\|\nabla\mathcal{E}_S^n\|^2\right]&=\mathbb{E}\left[\left\langle\frac{1}{J}\sum_{i=1}^J\left(\mathbb{E}\left[y_h^n\right]-y_{i,h}^n\right),\frac{1}{J}\sum_{j=1}^J\left(\mathbb{E}\left[y_h^n\right]-y_{j,h}^n\right)\right\rangle\right]\\
&=\frac{1}{J^2}\sum_{i=1}^J\sum_{j=1}^J\mathbb{E}\left[\left\langle\mathbb{E}\left[y_h^n\right]-y_{i,h}^n,\mathbb{E}\left[y_h^n\right]-y_{j,h}^n\right\rangle\right]\\
&=\frac{1}{J^2}\sum_{j=1}^J\mathbb{E}\left[\left\langle\mathbb{E}\left[y_h^n\right]-y_{j,h}^n,\mathbb{E}\left[y_h^n\right]-y_{j,h}^n\right\rangle\right].
\end{aligned}$$

The last equality is due to the fact that $y_h^n(\omega_1,\cdot),\ldots,y_h^n(\omega_J,\cdot)$ are i.i.d., and thus the expected value of $\left\langle\mathbb{E}[y_h^n]-y_{i,h}^n,\mathbb{E}[y_h^n]-y_{j,h}^n\right\rangle$ is zero for $i\neq j$. We now expand the quantity $\left\langle\mathbb{E}[y_h^n]-y_{j,h}^n,\mathbb{E}[y_h^n]-y_{j,h}^n\right\rangle$, apply the identities $\mathbb{E}[y_h^n]=\mathbb{E}\left[y_{j,h}^n\right]$ as well as $\mathbb{E}\left[(y_h^n)^2\right]=\mathbb{E}\left[\left(y_{j,h}^n\right)^2\right]$. Let $m=y_h^n$ and $\bar{m}=\mathbb{E}[m]$. Noting that

$$\mathbb{E}\left[\langle\bar{m}-m,\bar{m}-m\rangle\right]=\mathbb{E}\left[\bar{m}^2-2\langle\bar{m},m\rangle+m^2\right]=-\mathbb{E}\left[\bar{m}^2\right]+\mathbb{E}\left[m^2\right]\le\mathbb{E}\left[m^2\right].$$

we get

$$\mathbb{E}\left[\|\nabla\mathcal{E}_S^n\|^2\right]\le\frac{1}{J}\mathbb{E}\left[\|\nabla y_{j,h}^n\|^2\right].$$

By Theorem 3, we obtain

$$\begin{aligned}
&(\lambda-\lambda_+)\,\Delta t\sum_{n=1}^N\mathbb{E}\left[\|\nabla y_{j,h}^n\|^2\right]\\
\le&C\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|f_j^{n+1}\right\|_{-1}^2\right]+C\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|u_j^{n+1}\right\|_{\partial D_1}^2\right]+C\Delta t\mathbb{E}\left[\left\|\nabla y_{j,h}^0\right\|^2\right]\\
&+C\Delta t\mathbb{E}\left[\left\|y_{j,h}^0\right\|_{\partial D_1}^2\right]+\mathbb{E}\left[\left\|y_{j,h}^0\right\|^2\right].
\end{aligned}$$

The other terms of (35), including $\mathbb{E}\left[\left\|\mathcal{E}_S^N\right\|^2\right]$, $\mathbb{E}\left[\left\|\nabla\mathcal{E}_S^N\right\|^2\right]$, $\mathbb{E}\left[\left\|\mathcal{E}_S^N\right\|_{\partial D_1}^2\right]$, and $\mathbb{E}\left[\left\|\mathcal{E}_S^{n+1}\right\|_{\partial D_1}^2\right]$, can be processed in the same way. □

Combining the FE error and MC sampling error, we can get the error of EMC finite element method.

Theorem 6 *Suppose* $f_j \in \widetilde{L}^2\left(0,T;\ H^{-1}(D)\right)$, *boundary function* $u_j \in \widetilde{L}^2\left(0,T;\ L^2(\partial D_1)\right)$, *and* $y_j^0 \in \widetilde{L}^2\left(H^1(D)\cap H^{l+1}(D)\right)$. *The hypotheses (***H3***) and (***H4***), the stability condition (32) are satisfied, thus the following formula trues*

$$\begin{aligned}&\mathbb{E}\left[\left\|\mathbb{E}\left[y^N\right]-\Psi_h^N\right\|^2\right]+\lambda_-\Delta t\mathbb{E}\left[\left\|\nabla\left(\mathbb{E}\left[y^N\right]-\Psi_h^N\right)\right\|^2\right]\\&+\left(\lambda-\lambda_+\right)\Delta t\sum_{n=1}^{N}\mathbb{E}\left[\left\|\nabla\left(\mathbb{E}\left[y^n\right]-\Psi_h^n\right)\right\|^2\right]+\mu_-\Delta t\mathbb{E}\left[\left\|\left(\mathbb{E}\left[y^N\right]-\Psi_h^N\right)\right\|_{\partial D_1}^2\right]\\&+\left(\mu_{min}-\mu_+\right)\Delta t\sum_{n=1}^{N}\mathbb{E}\left[\left\|\left(\mathbb{E}\left[y^N\right]-\Psi_h^N\right)\right\|_{\partial D_1}^2\right]\\\leq&\frac{1}{J}\left(\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|f_j^{n+1}\right\|_{-1}^2\right]+C\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|u_j^{n+1}\right\|_{\partial D_1}^2\right]+C\Delta t\mathbb{E}\left[\left\|\nabla y_{j,h}^0\right\|^2\right]\right.\\&\left.+C\Delta t\mathbb{E}\left[\left\|y_{j,h}^0\right\|_{\partial D_1}^2\right]+\mathbb{E}\left[\left\|y_{j,h}^0\right\|^2\right]\right)+C\left(\Delta t^2+h^{2l}\right).\end{aligned}\tag{36}$$

Proof For the first item on the LHS of (36), Young's inequality and triangle inequality imply

$$\mathbb{E}\left[\left\|\mathbb{E}\left[y^N\right]-\Psi_h^N\right\|^2\right]\leq 2\left(\mathbb{E}\left[\left\|\mathbb{E}\left[y^N\right]-\mathbb{E}\left[y_h^N\right]\right\|^2\right]+\mathbb{E}\left[\left\|\mathbb{E}\left[y_h^N\right]-\Psi_h^N\right\|^2\right]\right).$$

Using Jensen inequality, one gets

$$\mathbb{E}\left[\left\|\mathbb{E}\left[y^N\right]-\mathbb{E}\left[y_h^N\right]\right\|^2\right]\leq\mathbb{E}\left[\mathbb{E}\left[\left\|y^N-y_h^N\right\|^2\right]\right]=\mathbb{E}\left[\left\|y^N-y_h^N\right\|^2\right].$$

Thus the result is obtained according to Theorem 4 and Theorem 5. Other items in the LHS of (36) be able to calculate by a similar way. □

5 Some numerical tests

Two numerical examples on the ensemble algorithm are illustrated. Example 1 is deterministic heat transfer model, which is intended to show Theorem 2. Example 2 is random transient heat equation, that is applied to verify Theorem 6 and reveals the effectiveness of the ensemble scheme.

5.1 Example 1

The first experiment implements the ensemble scheme for the deterministic heat transfer model (2) and testes the numerical performance of the propose algorithm. $J = 3$. The exact solution is $y_j = (1+\omega_j)\cos(2\pi x_1)\cos(2\pi x_2)\exp(t+1)$, where ω_j is a random perturbation in $[0,1]$,

$t \in [0,1]$ and $(x_1, x_2) \in [0,1]^2$. The diffusion coefficient, Robin coefficient are chosen as $a_j = \alpha_j = 2 + (1+\omega_j)\sin(t)\sin(x_1 x_2)$. The initial condition, Robin boundary function, and source term are selected to match the exact solution.

Table 1 Numerical errors and convergence rates of time

Δt	0.5000	0.2500	0.1250	0.0625
$h = 1/2^9$, $\omega_1 = 0.2$				
$\mathcal{E}_{L^2}^{E,1}$	0.07805	0.03778	0.01852	0.00924
Rate		1.047	1.029	1.004
$h = 1/2^9$, $\omega_2 = 0.7$				
$\mathcal{E}_{L^2}^{E,2}$	0.04854	0.02273	0.01074	0.00508
Rate		1.095	1.081	1.081
$h = 1/2^9$, $\omega_3 = 0.8$				
$\mathcal{E}_{L^2}^{E,3}$	0.0831	0.0388	0.0184	0.0088
Rate		1.100	1.077	1.066

Table 2 Numerical errors and convergence rates of space

h	0.0625	0.03125	0.0156	0.0078
$\Delta t = 1/2^9$, $\omega_1 = 0.2$				
$\mathcal{E}_{L^2}^{E,1}$	0.14636	0.03807	0.00969	0.00251
Rate		1.943	1.975	1.946
$\Delta t = 1/2^9$, $\omega_2 = 0.7$				
$\mathcal{E}_{L^2}^{E,2}$	0.20969	0.05434	0.01365	0.00336
Rate		1.948	1.993	2.022
$\Delta t = 1/2^9$, $\omega_3 = 0.8$				
$\mathcal{E}_{L^2}^{E,3}$	0.2225	0.0576	0.0144	0.0035
Rate		1.949	1.996	2.033

The ensemble scheme (12) is used to simulate the group in this experiment, involves three members with $\omega_1 = 0.2$, $\omega_2 = 0.7$, and $\omega_3 = 0.8$. Define

$$\mathcal{E}_{L^2}^{j} = \max_{1 \le n \le N} \|y_j^n - y_{j,h}^n\|, \quad j = 1, 2, 3.$$

We use linear FEs ($l = 1$) and isometric time partition and uniform space partition. To check the convergence rate in time, we select the time step Δt from $1/2$ to $1/2^4$ with $h = 1/2^9$. The numerical results is listed in Table 1. To check the convergence order in space, we choose space step h from $1/2^4$ to $1/2^7$ with $\Delta t = 1/2^9$. The numerical errors are listed in Table 2.

From Table 1, the convergence order w.r.t. time is about 1. It is in accord with Theorem 2. From Table 1, the rate of convergence w.r.t. space is about 2, which is higher than theoretical findings of Theorem 2. This may be the cause of higher regularity of source term f and boundary function u in this numerical test.

5.2 Example 2

Here we consider transient heat equations (1) with random coefficients on unit square domain $[0,1]^2$. The exact solution, the diffusion coefficient and Robin coefficient are chosen as

$$\begin{aligned} y(t,x,\omega) =& (1+\omega_1+\sum_{i=2}^{5}\omega_i)\cos(2\pi x_1)\cos(2\pi x_2)\exp(t+1), \\ a(x,\omega_1) =& 2+(1+\omega_1)\sin(x_1x_2), \\ \alpha(x,\omega_2,\cdots,\omega_5) =& 10+\exp((\omega_2\cos(\pi x_2)+\omega_3\sin(\pi x_2))\exp(-\frac{3}{4}) \\ &+(\omega_4\cos(\pi x_1)+\omega_5\sin(\pi x_1))\exp(-\frac{3}{4})). \end{aligned}$$

The random variables $\omega_1,\cdots,\omega_5$ are independent. ω_1 is uniformly distributed on $[0,1]$. $\omega_2,\cdots,\omega_5$ are uniformly distributed in the interval $[-\frac{1}{2},\frac{1}{2}]$. The initial condition, Robin boundary condition, and the source term are selected to match the exact solution. Define

$$\mathcal{E}_{L^2} = \max_{1\le n\le N}\sqrt{\frac{1}{J}\sum_{j=1}^{J}\|y_j^n - y_{j,h}^n\|^2}.$$

Two realizations of $\alpha(x,\omega_2,\cdots,\omega_5)$ are depicted in Figure 1.

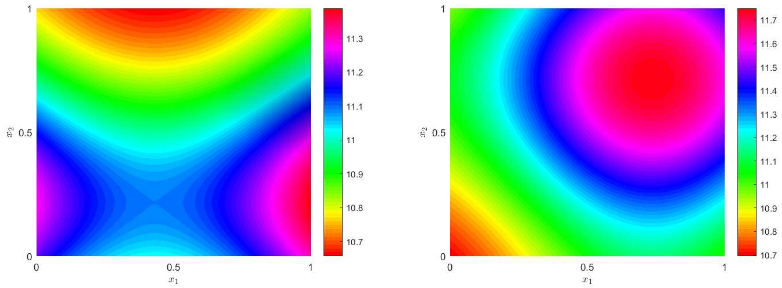

Fig. 1 Two realizations of α. Left: $\omega_2 = 0.47, \omega_3 = -0.09, \omega_4 = 0.38, \omega_5 = -0.41$. Right: $\omega_2 = -0.35, \omega_3 = -0.42, \omega_4 = 0.44, \omega_5 = 0.46$.

Analogy to deterministic case, we use linear FEs ($l = 1$) and isometric time partition and uniform space partition. The corresponding numerical results are listed in Table 3 and Table 4.

From Table 3, the convergence rate of time is agree with theoretical findings of Theorem 4. From Table 4, the convergence order of space is higher than our theoretical findings of Theorem 4. This also may be because the higher regularity of source term f and part boundary function u.

Table 3 Numerical errors and convergence rates in time ($h = 1/2^8, J = 50$)

Δt	0.5000	0.2500	0.1250	0.0625
$\mathcal{E}_{L^2}$	2.400E-03	1.300E-03	6.437E-04	3.286E-04
Rate		0.88	1.01	0.97

Table 4 Numerical errors and convergence rates in space ($\Delta t = 1/2^9, J = 50$)

h	0.0625	0.03125	0.0156	0.0078
$\mathcal{E}_{L^2}$	2.011E-01	5.195E-02	1.308E-02	3.264E-03
Rate		1.95	1.99	2.00

The convergence rate of MC sampling size J is very natural in our theoretical results in Theorem 6. Here we ignore this numerical test.

Choosing the sample size $J = 100$, the mesh size $h = 1/2^6$, $\Delta t = 1/2^4$, we calculate the mean and variance of the solutions at $t = 0.14$. The results are listed in Figure 2. To measure the effectiveness of the EMC scheme, we compare the result with that of individual finite element MC (IFE-MC) solutions using the same realizations. The difference between the mean and the IFE-MC solution is also shown in Figure 2.

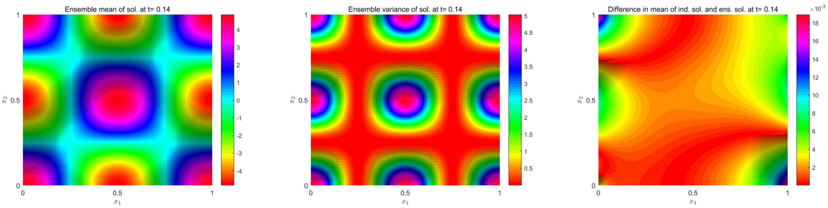

Fig. 2 EMC approximation results. Left: mean. Middle: variance at t = 0.14. Right: difference between the mean and the IFE-MC simulation.

The difference is on the order of 10^{-3}, and the mean $O(1)$. This implies that the EMC scheme can achieve basically accurate approximation as the IFE-MC implements.

To test the efficiency of our proposed ensemble scheme, we list the results of our EMC algorithm and MC method in Table 5 under the same time and space size. Here the discrete system's size is not too large, we apply MATLAB with its LU factorization.

Table 5 CPU time comparisons ($\Delta t = 1/2^3, h = 1/2^8$)

	J	10	50	100	200
EMC	CPU time	22.23	117.671	238.843	490.616
MC	CPU time	57.76	295.998	496.976	1074.91

From Table 5, we can see that our EMC algorithm requires less CPU time than MC method when $J > 1$. The EMC method improves the computational efficiency by about 50%-60%, compared with the non-ensemble scheme. The advantage of EMC becomes more obvious when the ensemble size increases.

6 Conclusion

An ensemble scheme is applied to reform the computational efficiency for numerical solutions to stochastic parabolic equations in this work. The coefficient matrix of linear systems are calculated and stored once (and for all) during a possibly expensive off-line stage, thus enabling a very rapid (and J-independent) assembling of the linear equation during the online stage. We first discuss the ensemble scheme to deterministic transient heat models. Then we establish the ensemble-based MC sampling method for random transient heat models. The effectiveness of both cases are tested.

The EMC algorithm can be applied to nonlinear parabolic equations. Higher order scheme or stochastic Robin boundary control problem are worth investigation next.

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