

A simple proof of the Jamiolkowski criterion for complete positivity of linear maps

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Abstract

We give a simple direct proof of the Jamiolkowski criterion to check whether a linear map between matrix algebras is completely positive or not. This proof is more accesible for physicists than others found in the literature and provides a systematic method to give *any* set of Kraus matrices of its Kraus decomposition.

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I. INTRODUCTION

Complete positivity for linear maps defined upon operator algebras is a mathematical property which arises as a natural generalization of positivity of linear functionals on vector spaces [1, 2, 3, 4]. In Quantum Physics, it has been playing a remarkable role in several disciplines, especially in the analysis of entanglement and separability of composite systems (cf. [5, 6, 7, 8] and references therein), in the dynamics of open systems [9, 10, 11, 12, 13, 14] and in the Foundations of Quantum Mechanics [15, 16]. This property is used in different ways in these contexts, but in any case it is vital to have criteria to check whether a given linear map between operator algebras is completely positive (CP hereafter) or not.

Here we focus on a concrete one based on the Jamiołkowski isomorphism [17, 18] and in particular we give a very accessible proof of this criterion for matrix algebras (i.e. for finite quantum systems) which, in the positive case, allows us to find *any* set of Kraus operators of its Kraus decomposition [15, 19]. A barely accessible for physicists and more involved proof of this criterion can be found in [20].

The paper is divided as follows. In section II we give a brief reminder of complete positivity (definition and main properties); in section III we provide the above mentioned proof of the Jamiołkowski criterion, which is illustrated with useful and well-known examples in section IV. We end the paper in section V with a few conclusions.

II. COMPLETE POSITIVITY: A BRIEF REMINDER

The rigorous and abstract definition of complete positivity is the following.

Definition II.1. *Let \mathcal{A} be a C^* algebra and $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ an endomorphism. α is CP if the endomorphism $\alpha_N \equiv \alpha \otimes I_N$ on $\mathcal{A} \otimes \mathcal{M}_N(\mathbb{C})$ is positive for all $N \geq 1$, where I_N denotes the identity map on the algebra $\mathcal{M}_N(\mathbb{C})$ of complex matrices of dimension N .*

Though a direct application of this definition is always possible to check whether a given map is CP or not, the main tool in this sense is the famous Stinespring's theorem [1]

Theorem II.1 (Stinespring's Theorem). *For a map $\alpha : \mathcal{A} \rightarrow \mathcal{B}(\mathfrak{H})$ from a C^* algebra to the space of bounded linear operators on the Hilbert space \mathfrak{H} , a sufficient and necessary condition for α to be CP is that there exists a representation $\pi : \mathcal{A} \rightarrow \mathfrak{K}$, \mathfrak{K} a Hilbert space,*

such that

$$\alpha(X) = V\pi(X)V^* \quad \forall X \in \mathcal{A} \text{ and } V \in \mathcal{B}(\mathfrak{K}) \quad (\text{II.1})$$

This theorem is the base of the Kraus decomposition of a CP map [15, 19], which is mostly used by physicists, and which we divide for future purposes into the cases of finite complex-matrices algebras and algebras of bounded linear operators on a Hilbert space.

Theorem II.2 (Kraus decomposition). For finite algebras: Let $\alpha : \mathcal{M}_N(\mathbb{C}) \rightarrow \mathcal{M}_N(\mathbb{C})$ be a linear map. Then α is CP if and only if there exist M_1, \dots, M_K elements in $\mathcal{M}_N(\mathbb{C})$ such that

$$\alpha(X) = \sum_{p=1}^K M_p X M_p^\dagger \quad \forall X \in \mathcal{M}_N(\mathbb{C}) \quad (\text{II.2})$$

For infinite algebras: Let $\alpha : \mathcal{B}(\mathfrak{H}) \rightarrow \mathcal{B}(\mathfrak{H})$ be a linear map. Then α is CP if and only if there exists a sequence of operators M_1, M_2, \dots , each one in $\mathcal{B}(\mathfrak{H})$ such that

$$\alpha(X) = \sum_{p=1}^{\infty} M_p X M_p^\dagger \quad \forall X \in \mathcal{B}(\mathfrak{H}) \quad (\text{II.3})$$

The sum converges in the strong topology of $\mathcal{B}(\mathfrak{H})$.

Note that the Kraus decomposition assures that a dynamical map is CP, but the problem then turns into finding the Kraus decomposition of an arbitrary linear map α . This means that if one is not able to find one, we cannot conclude whether α is CP or not. Some results can be found in the literature [21] which reduces the complete positivity of a given map α to the positivity of the tensor map $\alpha \otimes \alpha$, which turns out to be also a complex question in practice. In the following sections, after providing a simplified proof of the Jamiołkowski criterion for matrix algebras, once proven a map is CP, we give a procedure to build as many Kraus representations as desired.

III. CP LINEAR MAPS ON $\mathcal{M}_N(\mathbb{C})$

We will focus on CP linear maps on complex matrices algebras, representing finite quantum systems. We make use of the following common notation. $\{E_{ij}\}_{i,j=1,\dots,N}$ denotes the Weyl basis of $\mathcal{M}_N(\mathbb{C})$, i.e. those matrices with components $[E_{ij}]_{mn} = \delta_{im}\delta_{jn}$ or in physical notation $E_{ij} \equiv |i\rangle\langle j|$. $\{e_n\}_{n=1,\dots,N}$ will denote the basis on the complex vector space \mathbb{C}^N and $(\cdot, \cdot)_K$ the standard complex scalar product which endows \mathbb{C}^K with the well-known inner

product space structure such that under these conventions we have $(e_m, E_{ij}e_n)_N = \delta_{mi}\delta_{jn}$ for all $i, j, m, n = 1, \dots, N$.

Generically we will denote by $\Lambda : \mathcal{M}_N(\mathbb{C}) \rightarrow \mathcal{M}_N(\mathbb{C})$ a linear map from the algebra of complex matrices to itself. The set of such linear maps will be denoted as usual by $\mathcal{L}(\mathcal{M}_N(\mathbb{C}))$.

Definition III.1. *The map $\mathcal{J}_e : \mathcal{L}(\mathcal{M}(\mathbb{C}^N)) \rightarrow \mathcal{M}_N(\mathbb{C}) \otimes \mathcal{M}_N(\mathbb{C})$ is defined by*

$$\begin{aligned} \mathcal{J}_e : \mathcal{L}(\mathcal{M}_N(\mathbb{C})) &\rightarrow \mathcal{M}_N(\mathbb{C}) \otimes \mathcal{M}_N(\mathbb{C}) \\ \Lambda &\rightarrow \mathcal{J}_e[\Lambda] = \sum_{i,j=1}^N \Lambda[E_{ij}] \otimes E_{ij} \end{aligned} \quad (\text{III.4})$$

This map is known as Jamiołkowski isomorphism.

The subscript e refers to the orthonormal basis in which the Weyl basis $\{E_{ij}\}$ is constructed. The main result is the proof of the criterion based upon this isomorphism, i.e. upon the so-called Jamiołkowski criterion:

Theorem III.1 (Jamiołkowski criterion). *Let $\Lambda : \mathcal{M}_N(\mathbb{C}) \rightarrow \mathcal{M}_N(\mathbb{C})$ be a linear map. Then Λ is CP if, and only if, $\mathcal{J}_e[\Lambda] \geq 0$.*

The proof we provide here rests upon the Kraus decomposition of a linear map.

Proof. Firstly we will prove that if Λ is CP, then $\mathcal{J}_e[\Lambda] \geq 0$. If Λ is CP, then [19] there exists K elements $M_k \in \mathcal{M}_N(\mathbb{C})$ such that

$$\Lambda[A] = \sum_{p=1}^K M_p A M_p^\dagger \quad \forall A \in \mathcal{M}_N(\mathbb{C}) \quad (\text{III.5})$$

In particular, applying this Kraus decomposition to the Weyl matrices, one obtains

$$\sum_{p=1}^K M_p E_{ij} M_p^\dagger = \sum_{m,n=1}^N \Lambda_{ijmn} E_{mn} \quad (\text{III.6})$$

where Λ_{ijmn} are the components of $\Lambda[E_{ij}]$ in the Weyl basis. Taking scalar products in (III.6) one finds

$$\Lambda_{ijnm} = \sum_{p=1}^K (e_n, M_p e_i)_N (e_j, M_p^\dagger e_m)_N$$

$$= \sum_{p=1}^K (e_n, M_p e_i)_N (e_m, M_p e_j)_N^* \quad (\text{III.7a})$$

where the Parseval relation has been used. Next, define N^2 K -dimensional complex vectors by

$$f_{ij} \equiv \left((e_i, M_1 e_j)_N, \dots, (e_i, M_K e_j)_N \right) \in \mathbb{C}^K$$

Relation (III.7a) can then be rewritten as

$$\Lambda_{ijnm} = (f_{mj}, f_{ni})_K \quad (\text{III.7b})$$

Since $(\cdot, \cdot)_K$ is the standard inner product in \mathbb{C}^K , from equation (III.7b) both Hermiticity and positive semidefiniteness of the matrix $\sum_{ij=1}^N \Lambda[E_{ij}] \otimes E_{ij} \in \mathcal{M}(\mathbb{C}^{N^2})$ follows from the Hermiticity and positive definiteness of the standard scalar product $(\cdot, \cdot)_K$. In other words, it follows from the positive semidefinite character of a matrix with structure $\begin{pmatrix} S & SC^\dagger \\ CS & CSC^\dagger \end{pmatrix}$, where S is a positive definite N_1 -dimensional square matrix and C is a $N_1 \times (N^2 - N_1)$ rectangular matrix, with $N_1 \equiv \text{rg} \{f_{ij}\}$, $i, j = 1, \dots, N$. Hence (see appendix A) $\mathcal{J}_e[\Lambda] \geq 0$.

Conversely, if the linear map Λ satisfies $\mathcal{J}_e[\Lambda] \geq 0$, i.e. the matrix $\sum_{ij=1}^N \Lambda[E_{ij}] \otimes E_{ij}$ is Hermitian and positive semidefiniteness, it adopts (see appendix A) the structure $\begin{pmatrix} S & SC^\dagger \\ CS & CSC^\dagger \end{pmatrix}$ and then one can straightforwardly construct N^2 vectors $f_{ij} \in \mathbb{C}^K$, $i, j = 1, \dots, N$ such that relation (III.7b) holds for all $i, j, m, n \in \{1, \dots, N\}$.

Fix $K = N^2$. Locate those $m, n = 1, \dots, N$ such that $\Lambda_{nnmm} = 0$, say $M < K$ of them (indices set J , for brevity). Then for these indices define [23] $[f_{nm}] = C[f_{ij}]$ $i, j = 1, \dots, N_1$. The problem is then reduced to find $K_1 \equiv N^2 - M$ vectors $\{f_k\}_{k=1, \dots, N^2-M}$ such that the matrix of their scalar products is a given positive definite and Hermitian matrix. This can be accomplished using a similar method to the Gram-Schmidt orthonormalization process. An elementary proof is included in the form of a lemma in the appendix B and thus is always possible. The reader should notice that since there exists an infinity of initial orthonormal bases (cf. appendix), one can also find an infinity of vectors satisfying the required conditions. This implies that the Kraus decomposition is not unique, as it is well-known.

□

In this case of finite algebras, the use of matrices makes the application of the preceding theorem an extremely simple exercise. This is the content of the following theorem, which

is but a reformulation of theorem III.1:

Theorem III.2. *Let $\Lambda : \mathcal{M}_N(\mathbb{C}) \rightarrow \mathcal{M}_N(\mathbb{C})$ be a linear map. Then Λ is CP if, and only if, the matrix Λ_W defined below satisfies the following two properties:*

i. $\Lambda_W = \Lambda_W^\dagger$.

ii. Λ_W is positive semidefinite.

In particular, if there is a zero element in the diagonal, then their corresponding row and column must be also zero. The matrix Λ_W is defined by:

$$\Lambda_W = \begin{pmatrix} \Lambda_{1111} & \dots & \Lambda_{111N} & \dots & \Lambda_{N111} & \dots & \Lambda_{1N1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_{11N1} & \dots & \Lambda_{11NN} & \dots & \Lambda_{1NN1} & \dots & \Lambda_{1NNN} \\ & & \vdots & & \vdots & & \\ \Lambda_{N111} & \dots & \Lambda_{N11N} & \dots & \Lambda_{NN11} & \dots & \Lambda_{NN1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_{N1N1} & \dots & \Lambda_{N1NN} & \dots & \Lambda_{NNN1} & \dots & \Lambda_{NNNN} \end{pmatrix}$$

Proof. It is the explicit expression of $\sum_{ij=1}^N \Lambda[E_{ij}] \otimes E_{ij}$ in the tensor product basis.

However for completeness' sake we include the implication stated in theorem III.2. If a matrix is Hermitian and positive semidefinite, then there exists a diagonal matrix D and a unitary matrix P such that $M = P^\dagger D P$. The elements of D are the eigenvalues of M and the columns of P the corresponding eigenvectors. Let J denote the subset of indexes such that $d_{kk} = 0$, $k \in J$. Let us also denote $\bar{J} = I - J$, where $I = \{1, \dots, N\}$, $N = \dim M$. Then suppose $M_{jj} = 0$, which implies

$$M_{jj} = \sum_{m \in I} |p_{mj}|^2 d_{mm} = 0 \Rightarrow p_{mj} = 0 \quad \forall j \in \bar{J} \quad (\text{III.8})$$

We immediately conclude

$$M_{jk} = \sum_{m \in J} p_{mj}^* p_{mk} d_{mm} + \sum_{m \in \bar{J}} p_{mj}^* p_{mk} d_{mm} = 0 \quad \forall k \in I \quad (\text{III.9a})$$

$$M_{kj} = \sum_{m \in J} p_{mk}^* p_{mj} d_{mm} + \sum_{m \in \bar{J}} p_{mk}^* p_{mj} d_{mm} = 0 \quad \forall k \in I \quad (\text{III.9b})$$

Note that the converse is not always true. □

This proof, in conjunction with lemma B, allows to extract the following corollaries:

Corollary III.1. *Let Λ be a CP map. Then $\mathcal{J}_e[\Lambda] = QQ^\dagger$ and the entries of the k th column of Q^* are the components of the Kraus matrix M_k in the Weyl basis E_{ij} .*

Corollary III.2. *Let Q be as in corollary III.1. Then the entries of the k th column of $\tilde{Q}^* \equiv Q^*U$, where U is a unitary N^2 -dimensional matrix are the components of the Kraus matrix \tilde{M}_k in the Weyl basis E_{ij} .*

These two corollaries can be summarized in words in a very simple way: any set of Kraus matrices of a CP map is given by the columns of a square root of $\mathcal{J}_e[\Lambda]$. Since there are infinite square roots, there will be infinite sets of Kraus representations, as already known. Finally, the number of Kraus matrices in such representations can also be readily obtained.

Corollary III.3. *The number of matrices in a minimal Kraus representation of a CP linear map equals the number of positive eigenvalues of $\mathcal{J}_e[\Lambda]$.*

Proof. The number of matrices in a minimal Kraus representation equals the range of the set of Kraus matrices in any representation [14]. Since Kraus matrices are given by the columns of Q^* and since eigenvectors are linearly independent, the number of non-null Kraus matrices will coincide with the number of eigenvector whose eigenvalue is non-null. \square

These tools will be illustrated with common well-known examples.

IV. EXAMPLES

A very illustrative example of the immediate application of these results can be included by considering the transposition map $\Lambda[X] = X^T$. It is well-known that this is a non CP map, and to prove it one had traditionally to resort to consider maps $\Lambda \otimes I_M$ over the tensor product algebra $\mathcal{M}_N(\mathbb{C}) \otimes \mathcal{M}_M(\mathbb{C})$ and study its positivity (cf. e.g. [3]). With the preceding results, the task of deciding whether this is a CP map or not is elementary. Consider $\mathcal{M}_2(\mathbb{C})$ (nothing but computational complexity is gained in considering algebras of higher dimension). The matrix Λ_W is easily obtained:

$$\Lambda_W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is clear that Λ_W fails to satisfy the second condition in theorem III.2, thus the transposition is not a CP map.

Observe that once a linear map is proven to be CP, then we can also easily find one of its Kraus representations (indeed, we can find as many as we want). As an example of this technique, let us study in $\mathcal{M}_2(\mathbb{C})$ the linear map $\Lambda[X] = \frac{\lambda}{2}\text{tr}X I_2 + \mu X$ with, in principle, $\lambda, \mu \in \mathbb{C}$. This map is especially relevant in quantum information theory [22] and is known, under certain restrictions[24], as the depolarizing channel of one qubit. The matrix Λ_W is

$$\Lambda_W = \begin{pmatrix} \frac{\lambda}{2} + \mu & 0 & 0 & \mu \\ 0 & \frac{\lambda}{2} & 0 & 0 \\ 0 & 0 & \frac{\lambda}{2} & 0 \\ \mu & 0 & 0 & \frac{\lambda}{2} + \mu \end{pmatrix}$$

If Λ_W is to be Hermitian, then both λ and μ must be real numbers. If moreover it must also be positive semidefinite, then [25] $\lambda \geq 0$ and $\frac{\lambda}{2} + 2\mu \geq 0$. The cases in which one of them is zero are trivial, so we shall focus on $\lambda, \mu \neq 0$. Since none of the eigenvalues is zero, we will need 4 Kraus matrices. Following the preceding proof and the appendix, we can decompose $\Lambda_W = QQ^\dagger = PDP^\dagger$, with P unitary and $D \geq 0$ diagonal, so that the columns of the matrix $Q^* \equiv P^*D^{1/2}$ contain the elements of each Kraus matrix:

$$M_1 = \sqrt{\frac{\lambda + 4\mu}{2}} \mathbb{I}_2 \tag{IV.10}$$

$$M_2 = \sqrt{\frac{\lambda}{4}} E_{12} \tag{IV.11}$$

$$M_3 = \sqrt{\frac{\lambda}{4}} E_{21} \tag{IV.12}$$

$$M_4 = \sqrt{\frac{\lambda}{4}} \sigma_z \tag{IV.13}$$

where σ_z is the Z Pauli matrix. Notice that this is not the only set of Kraus matrices. We

can make use of corollary III.2 to find another set. In particular, choosing

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{IV.14})$$

we obtain another matrix \tilde{Q} , which produces the widely used set

$$\tilde{M}_1 = \sqrt{\frac{\lambda + 4\mu}{2}} \mathbb{I}_2 \quad (\text{IV.15})$$

$$\tilde{M}_2 = \sqrt{\frac{\lambda}{4}} \sigma_x \quad (\text{IV.16})$$

$$\tilde{M}_3 = \sqrt{\frac{\lambda}{4}} \sigma_y \quad (\text{IV.17})$$

$$\tilde{M}_4 = \sqrt{\frac{\lambda}{4}} \sigma_z \quad (\text{IV.18})$$

This technique shows the advantage of being possible to be systematically applied to any linear map in any dimension. Everything is reduced to a matter of computation.

V. CONCLUSIONS

The main conclusion to be drawn is that to check the CP character of a linear map of matrix algebras, one only needs to check the Hermiticity and positive semidefiniteness of a properly constructed matrix. Besides, once complete positivity is assured, we also have a systematic method to find *any* set of Kraus matrices of its Kraus representation.

The generalization to infinite algebras is currently under study. In this case, though the criterion is still valid, we must investigate the tractability of the method in these infinite cases.

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APPENDIX A

We will prove the following

Lemma A.1. *A Hermitian N^2 -dimensional matrix M is positive semidefinite if, and only if, M has the structure $\begin{pmatrix} S & SC^\dagger \\ CS & CSC^\dagger \end{pmatrix}$, where S is a Hermitian positive definite N_1 -dimensional square matrix ($1 \leq N_1 \leq N^2$) and C is a $N_1 \times (N^2 - N_1)$ rectangular matrix.*

Proof. (\Rightarrow). Any Hermitian N^2 -dimensional positive semidefinite matrix can be written as

$$M = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^\dagger \quad (\text{A.1})$$

where D is a positive definite N_1 -dimensional matrix, X and T are unitary matrices, with dimensions N_1 and $N^2 - N_1$, respectively, and Y and Z are null matrices. Then M adopts the preceding structure after recognizing $S = XDX^\dagger$ and $C = ZX^\dagger (= 0)$.

(\Leftarrow). Let us suppose that M shows the preceding structure. We will prove that then it can be written as $M = Q^\dagger Q$, thus it is positive semidefinite. Since S is positive definite, one can find an N_1 -dimensional matrix ξ such that $S = \xi^\dagger \xi$. A valid Q can then be $\frac{1}{\sqrt{2}} \begin{pmatrix} \xi & \xi C^\dagger \\ V\xi & V\xi C^\dagger \end{pmatrix}$, where V is an arbitrary $[(N^2 - N_1) \times N_1]$ -dimensional matrix such that $V^\dagger V = \mathbb{I}_{N_1}$. Note that this matrix, in the given conditions, always exists, since $N^2 - N_1 \geq N_1$ and then one can always find a matrix $V = \begin{pmatrix} \sqrt{1-\lambda} \mathbb{I}_{N_1} \\ b \end{pmatrix}$ such that $b^\dagger b = \lambda \mathbb{I}_{N_1}$, where b is an $(N^2 - 2N_1) \times N_1$ -dimensional matrix and $\lambda \in (0, 1)$. The Hermiticity is evident. \square

APPENDIX B

The problem is, given a positive definite Hermitian matrix $S \in \mathcal{M}_N(\mathbb{C})$, to find N vectors $\{f_j\}_{j=1, \dots, N}$ in \mathbb{C}^N such that $(f_i, f_j)_N = S_{ij}$. This is always possible.

Lemma B.1. *Given a positive definite Hermitian matrix $[S_{ij}]$, $i, j = 1, \dots, N$, there exists a set of vectors $\{f_k\}_{k=1, \dots, N}$ such that $(f_i, f_j)_N = S_{ij}$.*

Proof. Consider an arbitrary orthonormal basis $\{b_k\}_{k=1, \dots, N}$ (which can always be found with the orthonormalization Gram-Schmidt method). Since the matrix S is positive definite

and Hermitian, one can always find a unitary matrix P and a diagonal positive matrix D such $S = PDP^\dagger$. Define the matrix $Q \equiv PD^{1/2}$. Then one can elementary check that the set of vectors

$$f_i = \sum_{j=1}^N Q_{ij}^* b_j \quad i = 1, \dots, N \quad (\text{B.1})$$

satisfies the desired properties. □

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- [22] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- [23] $[f_{ij}]$ denotes a vector whose components are the vectors f_{ij} themselves.
- [24] Namely, the domain is not all $\mathcal{M}_2(\mathbb{C})$, but only the set of quantum states in which $X = X^\dagger$, $\text{tr}X = 1$ and $X \geq 0$; also Λ must be trace-preserving, i.e. $\lambda + \mu = 1$ and $\lambda, \mu \in [0, 1]$.
- [25] The eigenvalues of Λ_W are $\frac{\lambda}{2}$ (triple) and $\frac{\lambda}{2} + 2\mu$.