On the non-existence of an R-labeling*

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Abstract

We present a family of Eulerian posets which does not have any R-labeling. The result uses a structure theorem for R-labelings of the butterfly poset.

1 Introduction

For a graded poset the property of having an R-labeling is a precursor to face enumerative results. The slightly stronger condition of an EL-labeling gives the topological condition of shellability of the order complex of the poset. If a graded poset has an R-labeling, this implies that every entry of its flag h-vector is non-negative. See [4, 10] for details. Hence the most straightforward way to show that a poset lacks an R-labeling is to demonstrate a negative entry in its flag h-vector. If the poset has a non-negative flag h-vector, the problem is more difficult.

In this paper we construct a family of posets where each member has a positive flag h-vector but has no R-labeling. Moreover, half of the examples have the added attribute that they are Eulerian posets, that is, each nontrivial interval satisfies the Euler-Poincaré relation. It is noteworthy that these Eulerian posets also have negative coefficients in their **cd**-indexes. It is premature for us to assert that the lack of an R-labeling is related to these negative coefficients. Further research regarding these types of issues is necessary.

We begin by reviewing the definition of an R-labeling, a notion that has been extended since it was first discovered by Björner and Stanley. We reformulate this notion to a triple assignment. We then study R-labelings of the butterfly poset, that is, the unique poset which has two elements of each rank and every element covers all of the elements of one lower rank. Using triple assignments we give a structure theorem for R-labelings on the butterfly poset. We construct a family of examples by gluing two butterfly posets together. The structure theorem is used to show that these examples cannot have an R-labeling. We end the paper with a number of open questions.

2 Graded posets and R-labelings

We recall some basic properties of partially ordered sets (posets), including their flag f- and flag h-vectors. We refer the reader to Chapter 3 of Stanley's book [10] for a more complete introduction.

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A poset P is *graded* if has a minimal element $\hat{0}$, maximal element $\hat{1}$ and a rank function ρ such that $\rho(\hat{0}) = 0$. We say that a graded poset P is of rank n if $\rho(\hat{1}) = n$. For a poset of rank n and a subset $S = \{s_1 < s_2 < \cdots < s_k\}$ of $\{1, 2, \ldots, n-1\}$, define f_S to be the number of chains through the ranks of S, that is,

$$f_S = |\{\{\hat{0} < x_1 < x_2 < \cdots < x_k < \hat{1}\} \;:\; \rho(x_i) = s_i\}|.$$

These 2^{n-1} values constitute the *flag f-vector* of the poset. An equivalent notion is the *flag h-vector* defined by the invertible relation

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T.$$

For certain classes of posets the entries in the flag h-vector are non-negative. This is not at all apparent from the alternating sum defining the flag h-vector. One explanation of this non-negativity is given by R-labelings. Let $E(P)$ the set of all cover relations of the poset P, that is, $E(P) = \{(x, y) \in P^2 \;:\; x \prec y\}$.

Definition 2.1 *An R-labeling of a poset P is a labeling set Λ with a relation $\sim$ on its elements and a function $\lambda : E(P) \longrightarrow \Lambda$ such that in every non-trivial interval $[x, y]$ in the poset P there is a unique maximal chain $x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ such that $\lambda(x_0, x_1) \sim \lambda(x_1, x_2) \sim \cdots \sim \lambda(x_{k-1}, x_k)$. This unique chain is called* rising.

In the original definition by Björner and Stanley [4] the set Λ is a totally ordered set. This was later extended to a partially ordered set Λ by Björner and Wachs [5]. However, since none of the poset axioms are used from the poset Λ, the most general definition so far is the one given above.

The next result presents the connection between R-labelings and the flag h-vector. For a maximal chain $c = \{\hat{0} = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n = \hat{1}\}$, we define its *descent set* to be

$$D(c) = \{i \in \{1, \ldots, n-1\} \;:\; \lambda(x_{i-1}, x_i) \not\sim \lambda(x_i, x_{i+1})\}.$$

Theorem 2.2 (Björner and Stanley) *Let P be a graded poset with an R-labeling. The number of maximal chains with descent set S is given by the flag h-vector entry h_S.*

Although we extended the original notion of R-labelings, the proof in [4] still applies.

For a poset P let $W(P)$ denote the set of triplets of elements that cover each other, that is,

$$W(P) = \{(x, y, z) \in P^3 \;:\; x \prec y \prec z\}.$$

Definition 2.3 *A* triple assignment *of a poset is a function $\tau : W(P) \longrightarrow \{\mathbf{a}, \mathbf{b}\}$ such that for every non-trivial interval $[x, y]$ in the poset there is a unique maximal chain $x = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_k = y$ such that $\tau(x_i, x_{i+1}, x_{i+2}) = \mathbf{a}$ for $0 \le i \le k-2$.*

Proposition 2.4 *The two notions of R-labelings and triple assignments are equivalent.*

Proof: Given a labeling λ of the poset P, define the function τ by $\tau(x, y, z) = \mathbf{a}$ if and only if $\lambda(x, y) \sim \lambda(y, z)$. If λ is an R-labeling then directly we have that τ is a triple assignment.

Conversely, let $\tau : W(P) \longrightarrow \{\mathbf{a}, \mathbf{b}\}$ be a triple assignment function. We define a labeling as follows. Let the label set Λ be the set of all cover relations, that is, $\Lambda = E(P)$ and the labeling λ is given by $\lambda(x, y) = (x, y)$. Define the relation $\sim$ on Λ by $(x, y) \sim (y, z)$ if and only if $\tau(x, y, z) = \mathbf{a}$. It follows now that if τ is a triple assignment then the labeling λ is an R-labeling. $\square$

The reason the two element set $\{\mathbf{a}, \mathbf{b}\}$ is used as the range of a triple function stems from the notion of the **ab**-index of a poset. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-commutative variables of degree 1. For S a subset of the set $\{1, 2, \ldots, n-1\}$ define the monomial $u_S = u_1 u_2 \cdots u_{n-1}$ by letting $u_i = \mathbf{b}$ if $i \in S$ and otherwise $u_i = \mathbf{a}$. The **ab**-*index* is the non-commutative polynomial

$$\Psi(P) = \sum_S h_S \cdot u_S.$$

The **ab**-index is an equivalent encoding of the flag h-vector of a poset and it has degree one less than the rank of the poset.

For a maximal chain $c = \{\hat{0} = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n = \hat{1}\}$, define its *weight* $\mathrm{wt}(c)$ by the product $\mathrm{wt}(c) = \tau(x_0, x_1, x_2) \cdot \tau(x_1, x_2, x_3) \cdots \tau(x_{n-2}, x_{n-1}, x_n)$. The **ab**-index of a poset P having triple assignment τ is then given by $\Psi(P) = \sum_c \mathrm{wt}(c)$, where the sum is over all maximal chains c in P.

Recall a poset is *Eulerian* if every non-trivial interval satisfies the Euler-Poincaré relation, that is, it has the same number of elements of odd rank as even rank. For Eulerian posets Bayer and Klapper [3] proved that the **ab**-index can be written in terms of the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This invariant is called the **cd**-*index*. It offers an efficient encoding of the entries of the flag h-vector of an Eulerian poset. That a poset has a **cd**-index is equivalent to that the flag f-vector of the poset satisfies the generalized Dehn-Somerville relations; see [1].

3 The butterfly poset

The butterfly poset T_n is the unique graded poset of rank n such that there are two elements of rank i for $1 \leq i \leq n-1$ and every element different from $\hat{0}$ covers all of the elements of one rank below. Note that every interval in the butterfly poset is a butterfly poset of smaller rank and that the butterfly poset is an Eulerian poset. We will denote the elements of T_n by $\{\hat{0}, x_1, \overline{x_1}, \ldots, x_{n-1}, \overline{x_{n-1}}, \hat{1}\}$, where $\rho(x_i) = \rho(\overline{x_i}) = i$. For an element x in the butterfly poset different from the minimal and maximal elements, let $\overline{x}$ denote the unique element different from x but of the same rank as x. Furthermore, let $\overline{\cdot}$ also denote the natural involution on the 2-element set $\{\mathbf{a}, \mathbf{b}\}$, that is, $\overline{\mathbf{a}} = \mathbf{b}$ and $\overline{\mathbf{b}} = \mathbf{a}$.

It is easy to verify that the flag f- and flag h-vectors of the butterfly poset T_n are given by

$$f_S = 2^{|S|} \quad \text{and} \quad h_S = 1$$

for S a subset of the set $\{1, 2, \ldots, n-1\}$. Hence the **ab**-index of the butterfly poset is given by $\Psi(T_n) = (\mathbf{a} + \mathbf{b})^{n-1} = \mathbf{c}^{n-1}$.

Assume τ is a function $\tau : W(T_n) \longrightarrow \{\mathbf{a}, \mathbf{b}\}$ such that every interval of length 2 has a unique rising chain. Since every length 2 interval $[x, z]$ is a diamond, we conclude that $\tau(x, \overline{y}, z) = \overline{\tau(x, y, z)}$ where y and $\overline{y}$ are the two unique atoms (and coatoms!) in the interval $[x, z]$.

For a function $\tau : W(T_n) \longrightarrow \{\mathbf{a}, \mathbf{b}\}$ call an element y a *breakpoint* if $\hat{0} < y < \hat{1}$ and the value of $\tau(x, y, z)$ does not depend on x and z. Note that if y is a breakpoint then so is $\overline{y}$.

Theorem 3.1 *Let n be a positive integer greater than or equal to 2 and let τ be a function $\tau : W(T_n) \longrightarrow \{\mathbf{a}, \mathbf{b}\}$ such that in every interval of rank 3 or less there is a unique rising chain. Then the following two statements hold:*

(i) There is a breakpoint y in the poset T_n.

(ii) The function τ is a triple assignment.

Proof: First we show the existence of a breakpoint by induction on the rank. For the case $n = 2$ the statement is straightforward to verify. Next consider the case $n = 3$. Assuming that $\overline{x_1}$ is not a breakpoint, we have that $\tau(\hat{0}, \overline{x_1}, \overline{x_2}) = \overline{\tau(\hat{0}, \overline{x_1}, x_2)} = \tau(\hat{0}, x_1, x_2)$. Similarly, assuming that $\overline{x_2}$ is not a breakpoint we have $\tau(\overline{x_1}, \overline{x_2}, \hat{1}) = \overline{\tau(x_1, \overline{x_2}, \hat{1})} = \tau(x_1, x_2, \hat{1})$. Hence the two chains $\{\hat{0} < x_1 < x_2 < \hat{1}\}$ and $\{\hat{0} < \overline{x_1} < \overline{x_2} < \hat{1}\}$ have the same weight. This is a contradiction since every entry in the flag h-vector of T_3 is at most 1. Thus at least one assumption is wrong and we conclude that there is a breakpoint.

For the induction step, assume that $n \geq 4$. Consider the three intervals $[x_1, \hat{1}]$, $[\overline{x_1}, \hat{1}]$ and $[\hat{0}, x_3]$. All are butterfly posets of rank less than n and the induction hypothesis holds for them. Hence the interval $[x_1, \hat{1}]$ contains a breakpoint x_i, for some $2 \leq i \leq n - 1$. If $i \geq 3$ this is a breakpoint for the whole poset and we are done. Hence we assume that $i = 2$ and we have that $\tau(x_1, x_2, x_3) = \tau(x_1, x_2, \overline{x_3})$. Similarly, the interval $[\overline{x_1}, \hat{1}]$ has a breakpoint. Avoiding a breakpoint of rank 3 or higher in $[\overline{x_1}, \hat{1}]$, yields $\tau(\overline{x_1}, x_2, x_3) = \tau(\overline{x_1}, x_2, \overline{x_3})$.

Finally, the interval $[\hat{0}, x_3]$ contains a breakpoint. If it is x_1 then it is a breakpoint for the entire poset. If it is x_2 we have that $\tau(x_1, x_2, x_3) = \tau(\overline{x_1}, x_2, x_3)$. By concatenating these three equalities we obtain that x_2 is a breakpoint for the poset T_n, completing the induction.

It remains to show that τ is a triple assignment. Let $[x, y]$ be an interval in T_n. Since the interval $[x, y]$ is isomorphic to a butterfly poset there is a breakpoint z in this interval. Also note that $\overline{z}$ is also a breakpoint. Without loss of generality we may assume that the value of the function τ at z is $\mathbf{a}$, that is, $\tau(z', z, z'') = \mathbf{a}$ for all z' and z''. Now concatenate the two unique rising chains in the intervals $[x, z]$ and $[z, y]$. The result is a rising chain. Furthermore, it is the only possible rising chain in the interval $[x, y]$. This proves that τ is a triple assignment. $\square$

4 A class of posets without an R-labeling

Let P_n consist of two copies of the butterfly poset T_n where we have identified the minimal elements and the maximal elements. See Figure 1 for the two posets P_3 and P_5.

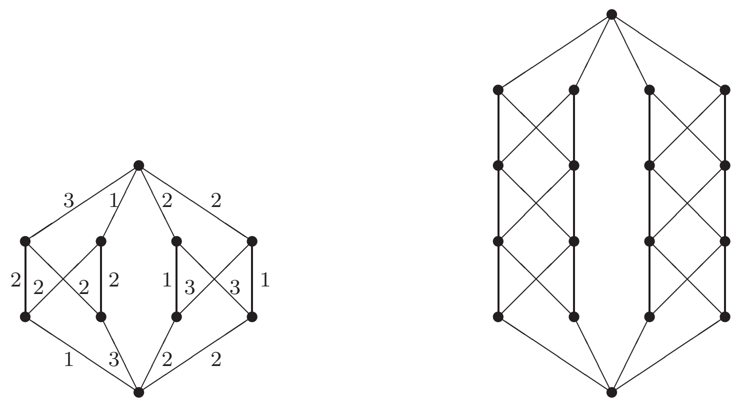

Figure 1: The two Eulerian posets P_3 and P_5. The poset P_3 has an R-labeling, whereas P_5 does not.

The flag f-vector of the poset P_n is given by $f_S = 2 \cdot 2^{|S|}$ for S non-empty and $f_\emptyset = 1$. Hence its flag h-vector is non-negative and is given by

$$h_S(P_n) = 2 - (-1)^{|S|} = \begin{cases} 1 & \text{if } |S| \text{ is even,} \\ 3 & \text{if } |S| \text{ is odd.} \end{cases}$$

Another way to observe this is to compute the **ab**-index of this poset. It is $\Psi(P_n) = 2 \cdot \Psi(T_n) - (\mathbf{a} - \mathbf{b})^{n-1} = 2 \cdot \mathbf{c}^{n-1} - (\mathbf{a} - \mathbf{b})^{n-1}$; see for instance [7, Section 11].

When n is odd the poset P_n is Eulerian. In fact, its **cd**-index is given by $\Psi(P_n) = 2 \cdot \mathbf{c}^{2k} - (\mathbf{c}^2 - 2 \cdot \mathbf{d})^k$ for $n = 2k + 1$. For $k \geq 2$ note that every **cd**-monomial having an even number of **d**'s and different from the monomial $\mathbf{c}^{n-1}$ has a negative coefficient.

Theorem 4.1 *The poset P_n for $n \geq 4$ does not have an R-labeling.*

Proof: Let P and Q be the two subposets of P_n such that they are both isomorphic to T_n, their union is P_n and they intersect in $\{\hat{0}, \hat{1}\}$. Assume that P_n has a triple assignment τ. Consider τ restricted to the subposet P. Since every interval of length 3 or less in P is an interval in P_n, the poset P with the function τ satisfies the condition of Theorem 3.1. Hence τ is a triple assignment for the poset P. Hence there is a rising chain in the poset P.

By the exact same reasoning, there is a rising chain in the poset Q, yielding the contradiction that P_n has two rising chains. $\square$

5 Concluding remarks

In the literature there are examples of non-shellable simplicial complexes whose geometric realization are 3-dimensional balls and spheres. For instance, see [6, 8, 9, 11] and the references therein. The difficulty in each of these papers is not to find a non-shellable complex, but to find a non-shellable object having a natural geometric realization. Thus we sharpen the question of this paper to: Is there a poset which lacks an R-labeling having a positive flag h-vector such that

(i) it is also a lattice?
(ii) its chain complex has the geometric realization of a sphere?

Furthermore, can one find a poset having R-labelings but where none of the labelings is an EL-labeling? The similar question concerning whether there are posets which are shellable but not EL-shellable has been answered independently in two papers [12, 13].

Observe that the poset P_n for n odd is obtained by doubling a half-Eulerian poset. This notion was introduced by Bayer and Hetyei [2]. Their paper gives a plethora of examples. What is known about labelings for these posets in general?

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