# AUTOMORPHISM GROUPS OF COUNTABLY CATEGORICAL LINEAR ORDERS ARE EXTREMELY AMENABLE

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ABSTRACT. We show that the automorphism groups of countably categorical linear orders are extremely amenable. Using methods of Kechris, Pestov, and Todorcevic, we use this fact to derive a structural Ramsey theorem for certain families of finite ordered structures with finitely many partial equivalence relations with convex classes.

## 1. INTRODUCTION

A **countably categorical linear order** is a countable (possibly finite) linear order $L$ such that every countable linear order which satisfies the same first-order theory as $L$ (in the language of linear order) is isomorphic to $L$. A topological group $G$ is **extremely amenable** if every action of $G$ on a compact Hausdorff space has a fixed point. In [5], Pestov showed that the automorphism group of the linear order of the rationals $\mathbb{Q}$ — which is a countably categorical linear order — is extremely amenable. In this paper, we generalize this result to the class of all countably categorical linear orders.

**Theorem 1.1.** *The automorphism group of a countably categorical linear order is extremely amenable.*

Our proof of this theorem uses a characterization of the countably categorical linear orders due to Rosenstein [8]. Given an indexed family $\langle \tau_i \rangle_{i \in I}$ of linear order types where the index set $I$ is a linear order, the generalized sum $\sum_{i \in I} \tau_i$ is the order type of the order obtained by concatenating intervals of type $\tau_i$ along the ordering given by $I$. Two special types of generalized sums will be of interest to us:

**Finite sums:** $s(\tau_1, \ldots, \tau_n) = \sum_{i \in \{1,\ldots,n\}} \tau_i$ where $\{1, \ldots, n\}$ has the usual ordering.

**Finite shuffles:** $\sigma(\tau_1, \ldots, \tau_n) = \sum_{q \in \mathbb{Q}} \tau_q$ where each $D_i = \{q \in \mathbb{Q} : \tau_q = \tau_i\}$ is a dense subset of $\mathbb{Q}$ and $\mathbb{Q} = D_1 \cup \cdots \cup D_n$.

See [9] for more on these operations. It turns out that these two operations generate all countably categorical linear order types.

**Theorem 1.2** (Rosenstein [8]). *The class of countably categorical linear order types is the smallest class of linear order types that contains the order type $1$ and is closed under finite sums and finite shuffles.*

Therefore, every countably categorical linear order type can be described by a sequence of applications of sums $s(T_1, \ldots, T_k)$ and shuffles $\sigma(T_1, \ldots, T_k)$ of arbitrary arity — a **sum-shuffle expression**. In general, a countably categorical order type will have many sum-shuffle expressions, for example, the expressions $\sigma(1)$, $s(\sigma(1), \sigma(1))$, $\sigma(\sigma(1))$, and $\sigma(s(1, \sigma(1)))$ all represent the order type of the rational numbers. One of our main results (Theorem 4.1) shows every countably categorical linear order has a canonical sum-shuffle expression that best captures the structure of the linear order.

To prove Theorem 1.1, we will first associate to each sum-shuffle expression $T$ a linear order $\widehat{\mathbb{L}}(T)$ of type $T$ expanded with finitely many relation symbols such that $\mathrm{Aut}(\widehat{\mathbb{L}}(T))$ is easily computed and shown to be extremely amenable (Theorem 3.1). Next we will show that every countably categorical linear order has a canonical sum-shuffle expression $T$ such that $\mathrm{Aut}(\mathbb{L}(T)) = \mathrm{Aut}(\widehat{\mathbb{L}}(T))$, where $\mathbb{L}(T)$ is the underlying linear order of $\widehat{\mathbb{L}}(T)$ (Theorem 4.1).

The analysis of $\widehat{\mathbb{L}}(T)$ will also show that $\widehat{\mathbb{L}}(T)$ can be seen as the limit of a Fraïssé class $\mathcal{K}_T$ of finite ordered structures. As a consequence of general results of Kechris, Pestov and Todorcevic [3], it follows that these Fraïssé classes $\mathcal{K}_T$ all have the Ramsey property. This gives an infinite family of structural Ramsey theorems (Corollary 5.5).

Some instances of this family of structural Ramsey theorems correspond to existing results. First, the structural Ramsey theorem for the class $\mathcal{K}_{\sigma(1)}$ is nothing but a thinly disguised form of Ramsey's theorem [7]. The case $\mathcal{K}_{\sigma(\sigma(1))}$ equivalent to a partition theorem of Rado [6] (see also [3, Corollary 6.8]). More generally, the case $\mathcal{K}_{\sigma^n(1)}$ corresponds to the fact that the Fraïssé order class $\mathcal{U}_S^{c<}$ of finite convexly ordered ultrametric spaces with distances in a fixed $n$-element subset $S$ of $(0, \infty)$ has the Ramsey property, which is a result of Nguyen Van Thé [4]. Thus the classes $\mathcal{K}_T$ are combinatorially very rich and can be used to encode a variety of natural Fraïssé order classes with the Ramsey property.

## 2. Tree Presentations and Coordinatization

While sum-shuffle expression are easy to understand, we will find it more convenient to work with parse trees for such expressions. These parse trees will be represented as trees of sequences of positive integers with labels from $\{\ell, s, \sigma\}$, for **leaf**, **sum**, **shuffle**, respectively. Formally, we define **tree presentations** via the following inductive rules.

- 1 is a tree presentation that consists only of the root $\langle \rangle$, with label $\ell$.
- If $T_1, \ldots, T_k$ are tree presentations, then so is $s(T_1, \ldots, T_k)$ whose nodes are the root $\langle \rangle$, with label $s$, and nodes of the form $\langle i \rangle {}^\frown t$ for $t \in T_i$ (including the root $t = \langle \rangle$) and $i \in \{1, \ldots, k\}$, with the same label as that of $t$ in $T_i$.
- If $T_1, \ldots, T_k$ are tree presentations, then so is $\sigma(T_1, \ldots, T_k)$ whose nodes are the root $\langle \rangle$, with label $\sigma$, and nodes of the form $\langle i \rangle {}^\frown t$ for $t \in T_i$ (including the root $t = \langle \rangle$) and $i \in \{1, \ldots, k\}$, with the same label as that of $t$ in $T_i$.

Note that only *leaf nodes* (childless nodes of the tree) have label $\ell$ and all remaining nodes have label $s$ or $\sigma$. If $T$ is a tree presentation, we say that a linear order $L$ has *type* $T$ if the order-type of $L$ is the evaluation of the sum-shuffle expression corresponding to $T$.

For each tree presentation $T$, we construct a canonical linear order $\mathbb{L}(T)$ with type $T$. To do this, we fix, once and for all, a partition $\langle \mathbb{Q}_n \rangle_{n=1}^\infty$ of $\mathbb{Q}$, each part of which is dense in $\mathbb{Q}$. This way, for each $n \geq 1$, we will have a canonical dense partition $\langle \mathbb{Q}_1, \ldots, \mathbb{Q}_n \rangle$ of the dense linear order $\mathbb{Q}_{[n]} = \mathbb{Q}_1 \cup \cdots \cup \mathbb{Q}_n$. For convenience, let us further require that $n \in \mathbb{Q}_n$ for each positive integer $n$. For each rational $q \in \mathbb{Q}$, let $\#(q)$ be the unique positive integer such that $q \in \mathbb{Q}_{\#(q)}$. Note that $\#(n) = n$ for every positive integer $n$.

The canonical linear order $\mathbb{L}(T)$ will be a lexicographically ordred set of finite sequences of rationals which is *prefix-free* (no element of the set is an initial segment of another). To determine whether a sequence $\bar{r} = \langle r_0, \ldots, r_{k-1} \rangle$ of rationals belongs to $\mathbb{L}(T)$, write $\#(\bar{r}) = \langle \#(r_0), \ldots, \#(r_{k-1}) \rangle$, then check that $\#(\bar{r})$ is a leaf-node of $T$ and, for each $i < k$, make sure that if $\#(\bar{r})\!\restriction\!i$ is a sum-node of $T$ then $r_i = \#(r_i)$. (There is nothing further to check when $\#(\bar{r})\!\restriction\!i$ is a shuffle-node.)

**Proposition 2.1.** *For each tree presentation $T$, $\mathbb{L}(T)$ is a linear order of type $T$.*

By construction, $\mathbb{L}(T)$ has some structural properties that are not always captured by the order relation alone. For each $t \in T$, let the $t$-**domain** $D_t$ be the set of all elements $\bar{r}$ of $\mathbb{L}(T)$ such that the leaf-node $\#(\bar{r})$ extends the node $t$ in $T$. In particular, $D_{\langle\rangle} = \mathbb{L}(T)$ where $\langle\rangle$ is the root of $T$.

Each element $\bar{r}$ of $D_t$ is contained in a unique $t$-**interval**: the maximal interval $I$ of $\mathbb{L}(T)$ such that $\bar{r} \in I \subseteq D_t$. In fact, it is easy to see that for each $\bar{r} \in D_t$, the $t$-interval containing $\bar{r}$ is

$$ I_t(\bar{r}) = \left\{ \bar{s} \in \mathbb{L}(T) : \bar{s}\!\restriction\!|t| = \bar{r}\!\restriction\!|t| \right\}. $$

In particular, $I_{\langle\rangle}(\bar{r}) = \mathbb{L}(T)$ for every $\bar{r} \in D_{\langle\rangle} = \mathbb{L}(T)$, and if $t$ is a leaf-node of $T$, then $I_t(\bar{r}) = \{\bar{r}\}$ for every $\bar{r} \in D_t$.

We will now enumerate a list of universal axioms for a theory that will capture the fine structure of $\mathbb{L}(T)$. In addition to the order $<$ and equality $=$ relations, we expand our language to contain one binary relation $E_t$ for each node $t$ of $T$. The intended interpretation in $\mathbb{L}(T)$ is $\bar{r} \; E_t \; \bar{s}$ if and only if $\bar{r}$ and $\bar{s}$ belong to the same $t$-interval; let $\widehat{\mathbb{L}}(T)$ denote $\mathbb{L}(T)$ expanded with these binary relations. These relations $E_t$ are *partial equivalence relations* (symmetric, transitive, but not necessarily reflexive relations) with convex equivalence classes. To talk about $t$-domains in the language of $\widehat{\mathbb{L}}(T)$, simply note that $\bar{r} \in D_t \Leftrightarrow \bar{r} \; E_t \; \bar{r}$ since $\bar{r} \in D_t$ if and only if $\bar{r}$ belongs to some $t$-interval of $\mathbb{L}(T)$.

The universal axioms characterizing $\widehat{\mathbb{L}}(T)$ are naturally divided into five groups.

**(T1)** For each node $t$ of $T$, the following are axioms:

$$ x \; E_t \; y \rightarrow y \; E_t \; x, $$
$$ x \; E_t \; y \wedge y \; E_t \; z \rightarrow x \; E_t \; z, $$
$$ x \; E_t \; y \wedge x < z \wedge z < y \rightarrow x \; E_t \; z \wedge z \; E_t \; y. $$

In other words, every $E_t$ is a partial equivalence relation with convex classes (the $t$-intervals).

**(T2)** The following is an axiom:

$$ x \; E_{\langle\rangle} \; y $$

In other words, there is a unique $\langle\rangle$-interval which consists of every point.

**(T3)** For every leaf $t$ of $T$, the following is an axiom:

$$ x \; E_t \; y \rightarrow x = y $$

In other words, $t$-intervals consist of only one point.

**(T4)** If $t$ is a sum or shuffle node with $k$ children in $T$, then the following are axioms:

$$x \; E_t \; x \leftrightarrow x \; E_{t^\smallfrown\langle 1 \rangle} \; x \vee \cdots \vee x \; E_{t^\smallfrown\langle k \rangle} \; x,$$
$$x \; \not{E}_{t^\smallfrown\langle i \rangle} \; x \vee x \; \not{E}_{t^\smallfrown\langle j \rangle} \; x, \quad \text{for } 1 \leq i < j \leq k.$$

In other words, the $t$-domain is the disjoint union of the $t^\smallfrown\langle i \rangle$-domains.

**(T5)** If $t$ is a sum node with $k$ children in $T$, then the following are axioms:

$$x \; E_t \; y \wedge x \; E_{t^\smallfrown\langle i \rangle} \; x \wedge y \; E_{t^\smallfrown\langle i \rangle} \; y \rightarrow x \; E_{t^\smallfrown\langle i \rangle} \; y,$$
$$x \; E_t \; y \wedge x \; E_{t^\smallfrown\langle i \rangle} \; x \wedge y \; E_{t^\smallfrown\langle j \rangle} \; y \rightarrow x < y, \quad \text{for } 1 \leq i < j \leq k.$$

In other words, each $t$-interval is a finite union of consecutive $t^\smallfrown\langle i \rangle$-intervals.

It is a simple matter to check that $\widehat{\mathbb{L}}(T)$ satisfies all of these axiom groups. More importantly, these axioms characterize the substructures of $\widehat{\mathbb{L}}(T)$.

**Proposition 2.2.** *A countable structure $L$ satisfies the axioms of* (T1–T5) *if and only if it is embeddable into $\widehat{\mathbb{L}}(T)$. In fact, if $K$ is a finite substructure of $L$, then any embedding $K \hookrightarrow \widehat{\mathbb{L}}(T)$ can be extended to an embedding $L \hookrightarrow \widehat{\mathbb{L}}(T)$.*

*Proof.* It suffices to show that if $x \in L$, $K$ is a finite substructure of $L$, and $e : K \hookrightarrow \widehat{\mathbb{L}}(T)$ is an embedding, then $e$ can be extended to an embedding $\bar{e} : K \cup \{x\} \hookrightarrow \widehat{\mathbb{L}}(T)$.

By (T4), we see that there is a unique leaf-node $t$ of $T$ such that $D_t(x)$. Let $\ell = |t|$ and define the coordinates $\bar{e}(x)(0), \ldots, \bar{e}(x)(\ell - 1)$ in order as follows.

- If $t{\restriction}i$ is a sum-node, then $\bar{e}(x)(i)$ must match the $i$-th coordinate of $t$.
- If $t{\restriction}i$ is a shuffle-node and there is a $y \in K$ such that $x \; E_{t{\restriction}(i+1)} \; y$, then $\bar{e}(x)(i)$ must match $e(y)(i)$.
- If $t{\restriction}i$ is a shuffle-node and there is no $y \in K$ such that $x \; E_{t{\restriction}(i+1)} \; y$, then we are free to choose any $\bar{e}(x)(i) \in \mathbb{Q}_{t(i)}$ such that $\bar{e}(x)(i) < e(z)(i) \Leftrightarrow x < z$ for all $z \in K$ such that $x \; E_{t{\restriction}i} \; z$.

Since $e : K \hookrightarrow \widehat{\mathbb{L}}(T)$ is an embedding, any choice of $y$ in the second case will give the same value for $\bar{e}(x)(i)$. Similarly, in the third case, the fact that $\mathbb{Q}_{t(i)}$ is dense ensures that a suitable value for $\bar{e}(x)(i)$ can always be found.

This completes the definition of the extended map $\bar{e} : K \cup \{x\} \hookrightarrow \widehat{\mathbb{L}}(T)$; we need to check that this is indeed an embedding, i.e., that $\bar{e}$ preserves the partial equivalence relations $E_{t'}$ and the order relation $<$.

*Preservation of the partial equivalence relations.* Since $e = \bar{e}{\restriction}K$ is known to be an embedding, it suffices to check that $x \; E_s \; y \Leftrightarrow \bar{e}(x) \; E_s \; \bar{e}(y)$ for every $y \in K$. First, note that if $s$ is not among $t{\restriction}0 = \langle \rangle, t{\restriction}1, \ldots, t{\restriction}\ell = t$, then $\neg D_t(x)$ and hence $\neg D_s(\bar{e}(x))$. Therefore, $x \; \not{E}_s \; y$ and $\bar{e}(x) \; \not{E}_s \; \bar{e}(y)$ for any $y \in K$. So it suffices to show that the relations $E_{t{\restriction}i}$ are preserved, for $i = 0, \ldots, \ell$.

By (T2) and (T4), we know that for every $y \in K$ there is a maximal $i \leq \ell$ such that $x \; E_{t{\restriction}i} \; y$. It suffices to show that $\bar{e}(x) \; E_{t{\restriction}i} \; \bar{e}(y)$ and, provided $i < \ell$, that $\bar{e}(x) \; \not{E}_{t{\restriction}(i+1)} \; \bar{e}(y)$.

By (T4), we know that $x \; E_{t{\restriction}j} \; y$ for every $j < i$. By the above construction, we see that at every stage $j < i$ where $t{\restriction}j$ is a sum-node, we picked $\bar{e}(x)(j) = t(j) = \bar{e}(y)(j)$. Also, at every stage $j < i$ where $t{\restriction}j$ is a shuffle-node, we explicitly picked $\bar{e}(x)(j) = \bar{e}(y)(j)$. Therefore, $\bar{e}(x)(j) = e(y)(j)$ for every $j < i$, which implies that $\bar{e}(x) \; E_{t{\restriction}i} \; \bar{e}(y)$.

Suppose now that $i < \ell$. We want to show that $\bar{e}(x) \; \not{E}_{t\restriction(i+1)} \; \bar{e}(y)$. We consider two cases.

- If $t\restriction i$ is a sum-node, then the first axiom of group (T5) implies that $\neg D_{t\restriction(i+1)}(y)$. Therefore $\neg D_{t\restriction(i+1)}(\bar{e}(y))$, which implies that $\bar{e}(x) \; \not{E}_{t\restriction(i+1)} \; \bar{e}(y)$.
- If $t\restriction i$ is a shuffle node, then either $x \; E_{t\restriction(i+1)} \; z$ for some $z \in K$, and $\bar{e}(x) \; E_{t\restriction(i+1)} \; \bar{e}(z)$ by definition of $\bar{e}(x)$. However, we must then have $y \; \not{E}_{t\restriction(i+1)} \; z$ and hence $\bar{e}(y) \; \not{E}_{t\restriction(i+1)} \; \bar{e}(z)$. Otherwise, $\bar{e}(x)(i)$ was chosen to be either strictly smaller or strictly bigger than $\bar{e}(z)(i)$ for every $z \in K$ such that $z \; E_{t\restriction i} \; x$. In particular, $\bar{e}(x)(i) \neq \bar{e}(y)(i)$ which means that $\bar{e}(x) \; \not{E}_{t\restriction(i+1)} \; \bar{e}(y)$.

*Preservation of the order relation.* Since $e = \bar{e}\restriction K$ is known to be an embedding and the order relation on $\widehat{\mathbb{L}}(T)$ is total, it suffices to check that $x = y \Leftrightarrow \bar{e}(x) = \bar{e}(y)$ and $x < y \Leftrightarrow \bar{e}(x) < \bar{e}(y)$ for every $y \in K$. By (T2), (T3), and (T4) it is true that $x = y \Leftrightarrow x \; E_{t\restriction i} \; y$ for every $i = 0, \dots, \ell$. Since $\bar{e}$ is known to preserve the partial equivalence relations, for every $y \in K$ we have $x = y \Leftrightarrow \bar{e}(x) = \bar{e}(y)$ and $x \neq y$ if and only if the maximal $i$ such that $x \; E_{t\restriction i} \; y$ satisfies $i < \ell$.

Suppose $x < y$. Let $i$ be the maximal number less than $\ell$ such that $x \; E_{t\restriction i} \; y$ and let $(t\restriction i)^\smallfrown\langle 1 \rangle, \dots, (t\restriction i)^\smallfrown\langle k \rangle$ be all the children of $t\restriction i$. Thus by (T4) we can let $(t\restriction i)^\smallfrown\langle m \rangle$ and $(t\restriction i)^\smallfrown\langle n \rangle$ be the two distinct children of $t\restriction i$ such that $D_{(t\restriction i)^\smallfrown\langle m \rangle}(x)$ (i.e., $t\restriction(i+1) = (t\restriction i)^\smallfrown\langle m \rangle$) and $D_{(t\restriction i)^\smallfrown\langle n \rangle}(y)$.

If $t\restriction i$ is a sum-node then $m < n$ by (T5). Since $\bar{e}$ is known to preserve the partial equivalence relations, $\bar{e}(x) \; E_{t\restriction i} \; \bar{e}(y) \wedge D_{(t\restriction i)^\smallfrown\langle m \rangle}(\bar{e}(x)) \wedge D_{(t\restriction i)^\smallfrown\langle n \rangle}(\bar{e}(y))$. Thus $\bar{e}(x) < \bar{e}(y)$ by (T5).

If $t\restriction i$ is a shuffle-node then by our construction $\bar{e}(x)(j) = \bar{e}(y)(j)$ for all $j = 0, \dots, i-1$ but $\bar{e}(x)(i) < \bar{e}(y)(i)$. Therefore $\bar{e}(x) < \bar{e}(y)$ because the ordering is lexicographic. □

## 3. Extreme Amenability of $\mathrm{Aut}(\widehat{\mathbb{L}}(T))$

In this section, we will establish the first step in the proof of Theorem 1.1.

**Theorem 3.1.** *For every tree presentation $T$, the automorphism group of $\widehat{\mathbb{L}}(T)$ is extremely amenable.*

We proceed by induction on the structure of $T$. The result is trivial for $T = 1$ since the automorphism group of $\widehat{\mathbb{L}}(1)$ is the trivial group, which is clearly extremely amenable. To complete the induction, it suffices to show that if $\mathrm{Aut}(\widehat{\mathbb{L}}(T_1)), \dots, \mathrm{Aut}(\widehat{\mathbb{L}}(T_k))$ are extremely amenable, then so are $\mathrm{Aut}(\widehat{\mathbb{L}}(s(T_1, \dots, T_k)))$ and $\mathrm{Aut}(\widehat{\mathbb{L}}(\sigma(T_1, \dots, T_k)))$.

To handle sums, we make the following simple observation.

**Lemma 3.2.** *If $T = s(T_1, \dots, T_k)$ then*

$$\mathrm{Aut}(\widehat{\mathbb{L}}(T)) \cong \mathrm{Aut}(\widehat{\mathbb{L}}(T_1)) \times \cdots \times \mathrm{Aut}(\widehat{\mathbb{L}}(T_k)).$$

Since extremely amenable groups are closed under products [3, Lemma 6.7], it follows that if $\mathrm{Aut}(\widehat{\mathbb{L}}(T_1)), \dots, \mathrm{Aut}(\widehat{\mathbb{L}}(T_k))$ are extremely amenable then so is $\mathrm{Aut}(\widehat{\mathbb{L}}(s(T_1, \dots, T_k)))$.

Shuffles require a more subtle argument. We begin with this observation, which the main part of the proof of [3, Lemma 8.4].

**Lemma 3.3.** *For every positive integer $k$, the group $H_k$ is extremely amenable, where $H_k$ consists of all order automorphisms of $\mathbb{Q}_{[k]} = \mathbb{Q}_1 \cup \cdots \cup \mathbb{Q}_k$ that preserve each $\mathbb{Q}_i$ setwise.*

The heart of the proof is the following key fact.

**Lemma 3.4.** *If $T = \sigma(T_1, \ldots, T_k)$ then $\mathrm{Aut}(\widehat{\mathbb{L}}(T)) \cong H_k \ltimes G$ where*

$$G = \prod_{q \in \mathbb{Q}_{[k]}} \mathrm{Aut}(\widehat{\mathbb{L}}(T_{\#(q)})) \cong \left( \mathrm{Aut}(\widehat{\mathbb{L}}(T_1)) \times \cdots \times \mathrm{Aut}(\widehat{\mathbb{L}}(T_k)) \right)^\omega$$

*and $H_k$ acts on $G$ by permuting the index set $\mathbb{Q}_{[k]} = \mathbb{Q}_1 \cup \cdots \cup \mathbb{Q}_k$.*

*Proof.* For each $q \in \mathbb{Q}_{[k]}$, let $I_q$ be the interval of $\widehat{\mathbb{L}}(T)$ consisting of elements with first coordinate $q$. Note that deleting the first coordinate gives a natural isomorphism $I_q \cong \widehat{\mathbb{L}}(T_{\#(q)})$.

Since every automorphism of $\widehat{\mathbb{L}}(T)$ maps each interval $I_q$ onto a similar interval $I_{q'}$, we have a natural homomorphism $h : \mathrm{Aut}(\widehat{\mathbb{L}}(T)) \to H_k$ defined by the relation

$$h(\alpha)(x(0)) = \alpha(x)(0)$$

for all $x \in \widehat{\mathbb{L}}(T)$. Moreover, $h$ has a right inverse $s : H_k \to \mathrm{Aut}(\widehat{\mathbb{L}}(T))$ where, for each $\pi \in H_k$, $s(\pi)$ acts on the first coordinate according to $\pi$ but leaves all other coordinates unchanged.

The kernel of $h$ is the set

$$K = \{\alpha \in \mathrm{Aut}(\widehat{\mathbb{L}}(T)) : \alpha(x)(0) = x(0) \text{ for every } x \in \widehat{\mathbb{L}}(T)\}.$$

Thus the restriction of an element $\alpha \in K$ to $I_q$ is an automorphism of $I_q$. Pasting these restrictions together and piping them through the natural isomorphisms $I_q \cong \widehat{\mathbb{L}}(T_{\#(q)})$ yields isomorphisms

$$\ker(h) \cong \prod_{q \in \mathbb{Q}_{[k]}} \mathrm{Aut}(I_q) \cong \prod_{q \in \mathbb{Q}_{[k]}} \mathrm{Aut}(\widehat{\mathbb{L}}(T_{\#(q)})) = G.$$

It follows at once that $\mathrm{Aut}(\widehat{\mathbb{L}}(T)) \cong H_k \ltimes G$, as described in the statement of the lemma.  $\square$

To finish the proof of Theorem 3.1, we appeal again to [3, Lemma 6.7] where it is shown that extremely amenable groups are closed under arbitrary products and semidirect products. It follows that if $\mathrm{Aut}(\widehat{\mathbb{L}}(T_1)), \ldots, \mathrm{Aut}(\widehat{\mathbb{L}}(T_k))$ are extremely amenable, then so is $\mathrm{Aut}(\widehat{\mathbb{L}}(\sigma(T_1, \ldots, T_k)))$.

## 4. Canonical Tree Presentations

In this section, we will establish the final step in the proof of Theorem 1.1. A tree presentation $T$ is said to be **canonical** if every automorphism of $\mathbb{L}(T)$ is also an automorphism of $\widehat{\mathbb{L}}(T)$, hence $\mathrm{Aut}(\mathbb{L}(T)) = \mathrm{Aut}(\widehat{\mathbb{L}}(T))$. Since $\mathrm{Aut}(\widehat{\mathbb{L}}(T))$ is known to be extremely amenable, it follows that $\mathrm{Aut}(\mathbb{L}(T))$ is extremely amenable too.

**Theorem 4.1.** *Every countably categorical linear order has a canonical tree presentation.*

Here is an outline of the proof. Given a countably categorical linear order $L$ we will inductively construct a sequence $D_1, \ldots, D_k$ of dense linear orders (possibly with endpoints and possibly trivial). Each $D_i$ will be equipped with a labeling that assigns to each point $q \in D_i$ a tree presentation $T_q$. At each stage, we will have an isomorphism $h_i : L \cong \sum_{q \in D_i} \mathbb{L}(T_q)$. The final linear order $D_k$ will be trivial, so $h_k$ will be an isomorphism from $L$ onto $\mathbb{L}(T_\star)$, where $\star$ is the unique element of $D_k$.

To ensure that $T_\star$ is a canonical tree presentation for $L$, we will show that at each stage $h_i$ induces an isomorphism

$$\hat{h}_i : \mathrm{Aut}(L) \cong G_i \ltimes \prod_{q \in D_i} \mathrm{Aut}(\widehat{\mathbb{L}}(T_q)),$$

where $G_i$ is the group of automorphisms of $D_i$ that preserve the labeling $q \in D_i \mapsto T_q$ and $G_i$ acts on $\prod_{q \in D_i} \mathrm{Aut}(\widehat{\mathbb{L}}(T_q))$ by permuting the indices. At the last stage, $G_k$ is trivial and hence $h_k : L \cong \mathbb{L}(T_\star)$ induces an isomorphism $\mathrm{Aut}(L) \cong \mathrm{Aut}(\widehat{\mathbb{L}}(T_\star))$, which will show that $T_\star$ is a canonical tree presentation of $L$.

The method for constructing the dense linear orders $D_1, \ldots, D_k$ was developed by Rosenstein. We will appeal to the proof of [9, Theorem 8.40] for some useful facts about the construction, but we need to recall the main steps of the the construction in some detail in order to establish the relevant facts about automorphism groups. Our notation will diverge from Rosenstein's, but the translation will always be clear.

The first dense linear order $D_1$ is the finite condensation of $L$, i.e., $D_1$ is the collection of all maximal finite intervals of $L$ with the induced ordering. As observed by Rosenstein, every element of $L$ is contained in a maximal finite interval of $L$, so we have a unique isomorphism $h_1 : L \cong \sum_{q \in D_1} \mathbb{L}(T_q)$. The labeling $q \in D_1 \mapsto T_q$ simply assigns to each maximal finite interval $q$ the tree presentation of the finite linear order of length $|q|$. An automorphism of $L$ must map each maximal finite interval to a maximal finite interval of the same length and thus corresponds to a unique element of $G_1$. Conversely, any $\alpha \in G_1$ has a unique expansion to an automorphism of $L$ by mapping each element of the finite interval $q$ to the corresponding element of $\alpha(q)$. Since $\mathrm{Aut}(\widehat{\mathbb{L}}(T_q))$ is trivial for each $q \in D_1$, this correspondence gives an isomorphism

$$\hat{h}_1 : \mathrm{Aut}(L) \cong G_1 \cong G_1 \ltimes \prod_{q \in D_1} \mathrm{Aut}(\widehat{\mathbb{L}}(T_q)).$$

The next dense linear orders are obtained by a two-stage process. We first perform the label condensation of $D_i$ with respect to the labeling $q \in D_i \mapsto T_q$ to obtain a linear order $E_i$. We say that an interval $I \subseteq D_i$ is **homogeneous** if it has no endpoints and $\{T_q : q \in I\} = \{T_q : r < q < s\}$ holds for all $r, s \in I$ with $r < s$. The linear order $E_i$ consists of the collection of all maximal homogeneous intervals of $D_i$ together with all singleton intervals for elements of $D_i$ that are not contained in any homogeneous interval of $D_i$.

To each $I \in E_i$, we assign a tree presentation $S_I$. Rosenstein shows that the set $\{T_q : q \in D_i\}$ is always finite, so fix, once and for all, an enumeration $T_1, \ldots, T_n$ of this set. If $I$ is a singleton, say $I = \{q\}$, we simply define $S_I = T_q$. Otherwise, we assign $S_I = \sigma(T_{i(1)}, \ldots, T_{i(k)})$ where $i(1) < \cdots < i(k)$ are such that $T_{i(1)}, \ldots, T_{i(k)}$ enumerates $\{T_q : q \in I\}$. Note that if $I$ and $J$ are two maximal homogeneous intervals such that $\{T_q : q \in I\} = \{T_r : r \in J\}$ then $S_I = S_J$.

It is easy to see that if $I \in E_i$, then $\sum_{q \in I} \mathbb{L}(T_q) \cong \mathbb{L}(S_I)$. Therefore,

$$L \cong \sum_{q \in D_i} \mathbb{L}(T_q) \cong \sum_{I \in E_i} \sum_{q \in I} \mathbb{L}(T_q) \cong \sum_{I \in E_i} \mathbb{L}(S_I).$$

Let $k_i : L \cong \sum_{I \in E_i} \mathbb{L}(S_I)$ be the isomorphism just described. There is more than one choice for $k_i$ but any choice which respects the above decompositions will do. In particular, $k_i$ must be compatible with $h_i$, which realizes the first of these decompositions, in the sense that $h_i(x) = (q, \bar{r})$ and $k_i(x) = (I, \bar{s})$, then $q \in I$, if $I$ is a singleton then $\bar{s} = \bar{r}$, and if $I$ is a maximal homogeneous interval then $\bar{s} = \langle s_0 \rangle ^\frown \bar{r}$.

Let $H_i$ be the group of automorphisms of $E_i$ that preserve the labeling $I \in E_i \mapsto S_I$. Each $\alpha \in G_i$ must map a maximal homogeneous interval $I$ of $D_i$ to another maximal homogeneous interval $\alpha(I)$ in such a way that $S_I = S_{\alpha(I)}$. Similarly, if $q$ is not contained in any homogeneous interval of $D_i$ then $\alpha(q)$ has the same property and $S_{\{q\}} = T_q = T_{\alpha(q)} = S_{\{\alpha(q)\}}$. Therefore, we have a group homomorphism $g_i : G_i \to H_i$. Note that this homomorphism $g_i$ has a section $s_i : H_i \to G_i$ where each $\eta \in H_i$ is expanded to $s_i(\eta) \in G_i$ using $h_i$ to select a canonical isomorphism between $I$ and $\eta(I)$.

The kernel of $g_i$ is the subgroup $K_i = \{\alpha \in G_i : (\forall I \in E_i)(\alpha(I) = I)\}$. Observe that $G_i \cong H_i \ltimes K_i$ and that

$$K_i \ltimes \prod_{q \in D_i} \mathrm{Aut}(\widehat{\mathbb{L}}(T_q)) \cong \prod_{I \in E_i} \mathrm{Aut}(\widehat{\mathbb{L}}(S_I)),$$

where $K_i$ acts on $\prod_{q \in D_i} \mathrm{Aut}(\widehat{\mathbb{L}}(T_q))$ by permuting the indices. By the induction hypothesis, the isomorphism $h_i$ induces an isomorphism

$$\hat{h}_i : \mathrm{Aut}(L) \cong G_i \ltimes \prod_{q \in D_i} \mathrm{Aut}(\widehat{\mathbb{L}}(T_q)).$$

It follows from the above computations that the isomorphism $k_i$ similarly induces an isomorphism

$$\hat{k}_i : \mathrm{Aut}(L) \cong H_i \ltimes \prod_{I \in E_i} \mathrm{Aut}(\widehat{\mathbb{L}}(S_I)).$$

Finally, to obtain $D_{i+1}$, we perform the finite condensation of $E_i$. Rosenstein shows that every element of $E_i$ is contained in a maximal finite interval of $E_i$, so $D_{i+1}$ is a well-defined dense linear order. To each $q \in D_{i+1}$ we define $T_q = s(S_{I_1}, \ldots, S_{I_k})$, where $I_1 < \cdots < I_k$ is the increasing enumeration of $q$. Any $\alpha \in H_i$ must map a maximal finite interval $q$ of $E_i$ to a maximal finite interval $\alpha(q)$ of $E_i$ in such a way that $T_q = T_{\alpha(q)}$ and thus $\alpha$ corresponds to a unique element of $G_{i+1}$. Conversely, any $\alpha \in G_{i+1}$ has a unique expansion to an element of $H_i$ by mapping each element of the finite interval $q$ to the corresponding element of $\alpha(q)$. Therefore, $G_{i+1} \cong H_i$ and since

$$\mathrm{Aut}(\widehat{\mathbb{L}}(T_q)) = \mathrm{Aut}(\widehat{\mathbb{L}}(S_{I_1})) \times \cdots \times \mathrm{Aut}(\widehat{\mathbb{L}}(S_{I_k})),$$

where $I_1 < \cdots < I_k$ is the increasing enumeration of $q \in D_{i+1}$, we have

$$\prod_{q \in D_{i+1}} \mathrm{Aut}(\widehat{\mathbb{L}}(T_q)) \cong \prod_{I \in E_i} \mathrm{Aut}(\widehat{\mathbb{L}}(S_I)).$$

It follows immediately that

$$\hat{h}_{i+1} : \mathrm{Aut}(L) \cong G_{i+1} \ltimes \prod_{q \in D_{i+1}} \mathrm{Aut}(\widehat{\mathbb{L}}(T_q)).$$

The only detail that remains is to show that this process must eventually terminate by reaching a step $k$ where $D_k$ is trivial — this termination argument is also given by Rosenstein.

## 5. The Fraïssé Order Class $\mathcal{K}_T$

In Section 2, we have expanded the linear order $\mathbb{L}(T)$ by adding certain binary relations to form the structure $\widehat{\mathbb{L}}(T)$. In this section, we will show that this new structure $\widehat{\mathbb{L}}(T)$ can be regarded as a Fraïssé limit of a Fraïssé order class $\mathcal{K}_T$. Before we prove this, let us first briefly review the relevant parts of Fraïssé theory; a detailed discussion can be found in [2], for example.

Let $L$ be a first-order language with finitely many relation symbols and no function symbols. A class $\mathcal{K}$ of finite $L$-structures is a **Fraïssé class** if it satisfies the following three properties.

**Hereditary Property:** If $A \in \mathcal{K}$ and $B \hookrightarrow A$ then $B \in \mathcal{K}$.

**Joint Embedding Property:** If $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ such that $A \hookrightarrow C$ and $B \hookrightarrow C$.

**Amalgamation Property:** If $A, B, C \in \mathcal{K}$ are such that $f : A \hookrightarrow B$, $g : A \hookrightarrow C$, there are a $D \in \mathcal{K}$ and $f' : B \hookrightarrow D$, $g' : C \hookrightarrow D$ such that $f' \circ f = g' \circ g$.

If, moreover, there is a distinguished binary relation symbol $<$ in $L$ which is interpreted as a linear order in every element of $\mathcal{K}$, then we say that $\mathcal{K}$ is a **Fraïssé order class**.

If $D$ is any $L$-structure, the **age** of $D$ is the class $\mathrm{Age}(D)$ of finite $L$-structure of structures that can be embedded into $D$. It is clear that $\mathrm{Age}(D)$ satisfies the hereditary and joint embedding properties. We say that $D$ is **ultrahomogeneous** every isomorphism between finite substructures of $D$ can be extended to an automorphism of $D$; this condition guarantees that $\mathrm{Age}(D)$ also satisfies the amalgamation property. Hence, $\mathrm{Age}(D)$ is a Fraïssé class whenever $D$ is an ultrahomogeneous $L$-structure. The converse of this fact is the basis of Fraïssé theory.

**Theorem 5.1** (Fraïssé [1])**.** *If $\mathcal{K}$ is a Fraïssé class of finite $L$-structures, then there is a countable ultrahomogeneous $L$-structure $D$, unique up to isomorphism, such that $\mathcal{K}$ is the age of $D$.*

The unique countable structure $D$ of Theorem 5.1 is called the **Fraïssé limit** of the class $\mathcal{K}$.

A standard back-and-forth argument using Proposition 2.2 shows that:

**Proposition 5.2.** *For every tree presentation $T$, the expanded structure $\widehat{\mathbb{L}}(T)$ is ultrahomogeneous.*

Denote by $\mathcal{K}_T$ the age of $\widehat{\mathbb{L}}(T)$. It follows from Theorem 5.2 that $\mathcal{K}_T$ is a Fraïssé order class. Thus, by Proposition 2.2, the class $\mathcal{K}_T$ is precisely the class of finite ordered structures that satisfy the axioms (T1–T5) of Section 2.

**Theorem 5.3.** *For every tree presentation $T$, the class $\mathcal{K}_T$ is a Fraïssé order class and $\widehat{\mathbb{L}}(T)$ is its Fraïssé limit.*

We will now turn to structural Ramsey theory. Let $L$ be a first-order language with finitely many relation symbols and no function symbols. If $A, B$ are finite $L$-structures, we denote by $\binom{B}{A}$ the set of all substructures of $B$ which are isomorphic to $A$. If $C$ is another finite $L$-structure $k$ is a positive integer, we write

$$C \to (B)^A_k$$

if for every coloring $\chi : \binom{C}{A} \to \{1, 2, ..., k\}$ there exists $B' \in \binom{C}{B}$ such that the $\chi$ is constant on $\binom{B'}{A}$. We say that the class $\mathcal{H}$ of finite $L$-structures satisfies the **Ramsey property** if for any two structures $A, B \in \mathcal{H}$ and every positive integer $k$, there exists $C \in \mathcal{H}$ such that $C \to (B)^A_k$. This property was considered by Kechris, Pestov, and Todorcevic, who characterized which Fraïssé order classes satisfy the Ramsey property as follows.

**Theorem 5.4** (Kechris–Pestov–Todorcevic [3])**.** *Let $\mathcal{K}$ be a Fraïssé order class with Fraïssé limit $D$. Then $\mathrm{Aut}(D)$ is extremely amenable if and only if $\mathcal{K}$ has the Ramsey property.*

Since $\widehat{\mathbb{L}}(T)$ is extremely amenable by Theorem 3.1 and $\widehat{\mathbb{L}}(T)$ is the Fraïssé limit of the Fraïssé order class $\mathcal{K}_T$, we can apply the above theorem to obtain that $\mathcal{K}_T$ has the Ramsey property.

**Corollary 5.5.** *For each tree presentation $T$, the Fraïssé order class $\mathcal{K}_T$ has the Ramsey property.*

As announced in the introduction, structures $\mathcal{K}_T$ can be seen as convexly ordered ultrametric spaces whose open balls coincide with the various $t$-classes of the structure. In fact, there is a precise biinterpretation between $\mathcal{K}_{\sigma^n(1)}$ and the Fraïssé order class $\mathcal{U}_S^{c<}$ of finite convexly ordered ultrametric spaces with distances in a fixed $n$-element set $S \subseteq (0, \infty)$, as previously considered by Nguyen Van The [4].

To see how this correspondence works, suppose $S = \{s_1, \ldots, s_n\}$ where $s_0 = 0 < s_1 < \cdots < s_n$. Similarly, suppose $t_0, \ldots, t_n$ enumerates the tree presentation $\sigma^n(1)$ from the leaf $t_0 = \langle 1, \ldots, 1 \rangle$ to the root $t_n = \langle \rangle$. A finite convexly ordered ultrametric space $(A, d, <)$ with distances in $S$ can be viewed as an element of $\mathcal{K}_{\sigma^n(1)}$ by definining the relations $x\ E_{t_i}\ y \Leftrightarrow d(x,y) \le s_i$. Conversely, a structure $(A, E_{t_0}, E_{t_0}, \ldots, E_{t_n}, <)$ in $\mathcal{K}_{\sigma^n(1)}$ can be made into a convexly ordered ultrametric space with distances in $S$ by defining $d(x,y) = s_i$ where $i$ is least such that $x\ E_{t_i}\ y$.

This back and forth translation gives an equivalence between the Fraïssé order classes $\mathcal{U}_S^{c<}$ and $\mathcal{K}_{\sigma^n(1)}$. Therefore, Corollary 5.5 gives the following.

**Corollary 5.6** (Nguyen Van Thé [4])**.** *Let $S$ be a finite set of positive real numbers. The Fraïssé order class $\mathcal{U}_S^{c<}$ has the Ramsey property.*

Nguyen Van Thé further shows that $\mathcal{U}_S^{c<}$ has the Ramsey property even when $S$ is an infinite subset of $(0, \infty)$. This general case does not correspond to a special case of Corollary 5.5, but one can derive this more general result from the case where $S$ is finite.

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