Unitizations of generalized pseudo effect algebras and their ideals

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Abstract

A generalized pseudo effect algebra (GPEA) is a partially ordered partial algebraic structure with a smallest element 0, but not necessarily with a unit (i.e, a largest element). If a GPEA admits a so-called unitizing automorphism, then it can be embedded as an order ideal in its so-called unitization, which does have a unit. We study unitizations of GPEAs with respect to a unitizing automorphism, paying special attention to the behavior of congruences and ideals in this setting.

1 Introduction

An effect algebra (EA) is a bounded partially ordered structure equipped with a partially defined commutative and associative binary operation called the orthosummation, often denoted by \oplus [13]. The smallest element in an EA, called the zero or neutral element, is usually denoted by 0, and the largest element, called the unit, is often denoted by 1. For each element a in an EA, there is a unique element a^{\perp} , called the orthosupplement of a, such that $a \oplus a^{\perp} = 1$.

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The second and third authors were supported by Research and Development Support Agency under the contract No. APVV-0178-11 and grant VEGA 2/0059/12.

Effect algebras were originally introduced to represent possibly fuzzy or unsharp propositions arising either classically [17] or in quantum measurement theory [1].

The class of EAs includes the class of orthogoleans [14] and the class of orthogolean posets [19, p. 27]. The subclass consisting of lattice-ordered EAs includes the class of MV-algebras [2], the class of orthogolean lattices [19], and the class of boolean algebras.

In spite of their considerable generality, EAs have been further generalized in two ways: (1) By dropping the assumption that the orthosummation is commutative. (2) By dropping the assumption that there is a unit. Noncommutative versions of EAs called *pseudo effect algebras* (PEAs), were introduced by A. Dvurečenskij and T. Vetterlein in [4, 5] and further studied in [6, 7, 9, 10, 11, 12, 18, 26]. The investigation of *generalized effect algebras* (GEAs), i.e., versions of EAs having no unit, was pioneered by Z. Riečanová in [21, 22, 23] and further studied in [15, 20]. Finally, by dropping both the assumption of commutativity and the existence of a unit, one arrives at the notion of a *generalized pseudo effect algebra* (GPEA) [7, 8, 16, 25, 26].

It is well known that every GEA E can be embedded as a maximal proper ideal in an EA \widehat{E} called its unitization in such a way that (1) E and $\widehat{E} \setminus E$ are order-anti-isomorphic under the restrictions of the partial order on \widehat{E} ; (2) for all $a \in \widehat{E}$, either $a \in E$ or else its orthosupplement $a^{\perp} \in E$; and (3) for $x, y \in \widehat{E} \setminus E$ the orthosum of x and y is not defined [3, Theorem 1.2.6]. This construction of a unitization was extended to so-called weakly commutative GPEAs in [25]. Recently, in [16], it was shown that a GPEA P can be embedded as a maximal proper PEA-ideal in a PEA U if and only if P admits a so-called unitizing GPEA-automorphism. Indeed, the unitization of a weakly commutative GPEA is a special case in which the unitizing GPEA-automorphism is the identity mapping.

In this article, which is a continuation of [16], we focus on properties of congruences and ideals in the setting of a GPEA P and its unitization U with respect to a unitizing automorphism γ (a so-called γ -unitization). We find conditions under which a congruence on P can be extended to a congruence on its γ -unitization U such that the quotient of U is the unitization of the quotient of P with a unitizing automorphism induced by γ . In particular, we extend the results obtained in [20] and [25] for unitizations of GEAs and weakly commutative GPEAs to the more general unitizations of GPEAs with respect to a unitizing automorphism.

We briefly investigate several versions, RDP, RDP₀, RDP₁, RDP₂ of the Riesz decomposition property in GPEAs and in their γ -unitizations and show that if P is a total GPEA, i.e., if the orthosum in P is defined for all pairs of its elements, then P has one of these properties if and only if its γ -unitization U has the corresponding property.

We also study relations between the existence of a smallest nontrivial Riesz ideal in a GPEA and the existence of a smallest nontrivial Riesz ideal in its γ -unitization.

For the reader's convenience, we devote Sections 2 and 3 below to a brief review of some basic definitions and facts needed in this article.

We provide several illustrating examples. In [11] it was shown how to construct a large class of PEAs by starting with the positive cone G^+ of a po-group G, a nonempty indexing set I, and two bijections $\lambda, \rho \colon I \to I$. In [12], this construction was extended by replacing G^+ by a more general GPEA E. At the end of Section 3, we review this construction and relate it to our work in this article.

2 Generalized pseudo effect algebras

We begin this section by recalling the definition of a GPEA [16, Definition 2.6] and we observe that PEAs, GEAs, and EAs are special kinds of GPEAs. Axiomatic characterizations of PEAs, GEAs, and EAs can be found in [16, §2]. Also we review some of the basic properties of these partially ordered, partial algebraic structures. We abbreviate 'if and only if' by 'iff' and the symbol := means 'equals by definition.'

- **2.1 Definition.** A generalized pseudo effect algebra (GPEA) [7, 8] is a partial algebraic structure $(P; \oplus, 0)$, where \oplus is a partial binary operation on P called the *orthosummation*, 0 is a constant in P called the *zero element*, and the following conditions hold for all $a, b, c \in P$:
- (GPEA1) $a \oplus b$ and $(a \oplus b) \oplus c$ exist iff $b \oplus c$ and $a \oplus (b \oplus c)$ exist and in this case $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity).
- (GPEA2) If $a \oplus b$ exists, then there are elements $c, d \in P$ such that $a \oplus b = c \oplus a = b \oplus d$ (conjugation).
- (GPEA3) If $a \oplus c = b \oplus c$, or $c \oplus a = c \oplus b$, then a = b (cancellation).
- (GPEA4) $a \oplus 0 = 0 \oplus a = a$ (neutral element).

(GPEA5) If $a \oplus b = 0$, then a = 0 = b (positivity).

If no confusion threatens, we often denote the GPEA $(P; \oplus, 0)$ simply by P. If we write an equation involving an orthosum of elements of P without explicitly assuming the existence thereof, we understand that its existence is implicitly assumed.

Let P be a GPEA and let $a, b, c \in P$. By (GPEA3), the elements c, d in (GPEA2) are uniquely determined. A partial order \leq is defined on the GPEA P (the *induced partial order*) by stipulating that $a \leq b$ iff there is (a necessarily unique) $c \in P$ with $a \oplus c = b$, or equivalently, iff there is (a necessarily unique) $d \in P$ with $d \oplus a = b$. For all $a \in P$, we have $0 \leq a$. Moreover, the cancellation laws (GPEA3) can be extended to \leq as follows: If $a \oplus c \leq b \oplus c$ or $c \oplus a \leq c \oplus b$, then $a \leq b$.

Partial binary operations left subtraction \nearrow and right subtraction \searrow are defined on P as follows: For $a,b\in P$, $a\nearrow b$ and $b\searrow a$ are defined iff $a\le b$, in which case (1) $a\nearrow b:=c$, where c is the unique element of P such that $a\oplus c=b$ and (2) $b\searrow a:=d$, where d is the unique element of P such that $d\oplus a=b$.

- **2.2 Definition.** Let P be a GPEA and let I and S be nonempty subsets of P. Then:
 - (1) I is an order ideal iff $a \in I$, $b \in P$, and $b \le a$ implies that $b \in I$.
 - (2) I is an *ideal* iff I is an order ideal and whenever $a, b \in I$ and $a \oplus b$ is defined, it follows that $a \oplus b \in I$.
 - (3) An ideal I in P is said to be *normal* iff whenever $a,b,c\in P$ and $a\oplus c=c\oplus b$, then $a\in I\Leftrightarrow b\in I$.
 - (4) S is a sub-GPEA of P iff, whenever two of the elements $a, b, c \in P$ belong to S and $a \oplus b = c$, then the third element also belongs to S.

Every ideal I in P is a sub-GPEA, and every sub-GPEA S in P is a GPEA in its own right under the restriction to S of the orthosummation on P.

2.3 Lemma. Let P be a GPEA, let I be a normal ideal in P, let $a, b \in P$, and suppose that $a \oplus b$ is defined. Then $b \in I$ iff $(a \oplus b) \setminus a \in I$; likewise, $a \in I$ iff $b \not (a \oplus b) \in I$.

Proof. Observe that $((a \oplus b) \setminus a) \oplus a = a \oplus b$ and $b \oplus (b \nearrow (a \oplus b)) = a \oplus b$. \square

A GPEA P is said to be total iff $a \oplus b$ is defined for all $a, b \in P$.

2.4 Example. Let G be an additively-written, not necessarily Abelian partially ordered group (po-group). Then the positive cone $G^+ := \{a \in G : 0 \le a\}$ is a total GPEA with orthosummation \oplus given by the restriction to G^+ of the group operation + on G. In this case, the induced partial order on the GPEA G^+ is the restriction to G^+ of the partial order on G.

If P and Q are GPEAs, then a mapping $\phi: P \to Q$ is a GPEA-morphism iff, for all $a,b \in P$, if $a \oplus b$ exists in P, then $\phi a \oplus \phi b$ exists in Q, and $\phi(a \oplus b) = \phi a \oplus \phi b$. A bijective GPEA-morphism $\phi: P \to Q$ is a GPEA-isomorphism of P onto Q iff $\phi^{-1}: Q \to P$ is also a GPEA-morphism. A GPEA-automorphism of P is GPEA-isomorphism $\phi: P \to P$.

It is not difficult to show that a bijective GPEA-morphism $\phi: P \to Q$ is a GPEA-isomorphism iff, whenever $a, b \in P$ and $\phi a \oplus \phi b$ exists in Q, then $a \oplus b$ exists in P.

A pseudo effect algebra (PEA) is a GPEA with a largest element, called the unit and often denoted by 1 ([16, Definition 2.3]). Let P be a PEA with unit 1. Clearly, the only element $z \in P$ such that $z \oplus 1$ (or $1 \oplus z$) exists is z = 0. Also, for $a \in P$, we have $a \leq 1$ whence we define $a^{\sim} := a / 1$ and $a^{-} := 1 \setminus a$. Thus, a^{\sim} and a^{-} , called the right supplement and the left supplement of a, respectively, are the unique elements in P such that $a \oplus a^{\sim} = a^{-} \oplus a = 1$. If it happens that $a^{\sim} = a^{-}$, then the common element is called the orthosupplement of a and is written as a^{\perp} . The PEA P is said to be symmetric iff $a^{\perp} = a^{\sim} = a^{-}$ for all $a \in P$ [10].

2.5 Example. Let G be any (additively written and not necessarily Abelian) po-group and choose an element $0 \le u \in G$. Let $G[0,u] := \{a \in G : 0 \le a \le u\}$, and define $a \oplus b$ for $a,b \in G[0,u]$ iff $a+b \le u$, in which case $a \oplus b := a+b$. Then $(G[0,u]; \oplus, 0,u)$ is a PEA and the induced partial order coincides with the partial order on G restricted to G[0,u]. We note that, if $a,b \in G[0,u]$ and $a \le b$, then $a^{\sim} = -a+1$, $a^{-} = 1-a$, a/b = -a+b, and $b \setminus a = b-a$.

Some important basic properties of PEAs are collected in the following theorem (see [16, Theorem 2.4]).

2.6 Theorem. Let P be a PEA and let $a, b, c \in P$. Then: (i) $0^{\sim} = 0^{-} = 1$ and $1^{\sim} = 1^{-} = 0$. (iv) $a^{\sim-} = a^{-\sim} = a$. (ii) $a \oplus b = c$ iff $b^{-} = c^{-} \oplus a$. (vii)

 $a \oplus b^{\sim} = c^{\sim}$ iff $c \oplus a = b$. (iii) $a^{\sim} \oplus b = c^{\sim}$ iff $b^{-} = c \oplus a^{\sim}$ iff $b^{--} \oplus c = a$. (iv) $a \leq b$ iff $b^{\sim} \leq a^{\sim}$ iff $b^{-} \leq a^{-}$. (v) Both $a \mapsto a^{\sim}$ and $a \mapsto a^{-}$ are order-reversing bijections on P. (vi) $a \oplus b$ exists iff $b \leq a^{\sim}$ iff $a \leq b^{-}$.

In a straightforward way, one can derive formulas for left and right subtraction as illustrated by the following lemma, the proof of which we omit.

- **2.7 Lemma.** Let P be a PEA and let $a, b \in P$. Then: (i) $a \le b \Rightarrow b \setminus a = (a \oplus b^{\sim})^-$ and $a \not/ b = (b^- \oplus a)^{\sim}$. (ii) $b^{--} \le a \Rightarrow b^{--} / a = (a^{\sim} \oplus b)^-$. (iii) $b \le a^{\sim} \Rightarrow a \le b^-$, $a^{\sim} \setminus b = (b^{--} \oplus a)^{\sim}$, $b^- \setminus a = (a \oplus b)^-$, and $a^{\sim} \setminus b = (b^{--} \oplus a)^{\sim}$.
- **2.8 Lemma.** Let P be a PEA, let I be a normal ideal in P, and let $a, b \in P$. Then: (i) $a \in I$ iff $a^{--} \in I$ iff $a^{--} \in I$. (ii) $a^{-} \in I$ iff $a^{-} \in I$.

Proof. Part (i) follows from the facts that $a^{--} \oplus a^{-} = 1 = a^{-} \oplus a$ and $a \oplus a^{\sim} = 1 = a^{\sim} \oplus a^{\sim}$. Part (ii) is a consequence of $a \oplus a^{\sim} = 1 = a^{-} \oplus a$. \square

A state on a PEA P is a mapping $s: P \to [0,1] \subseteq \mathbb{R}$ such that (S1) s(1) = 1 and (S2) $s(a \oplus b) = s(a) + s(b)$ whenever $a \oplus b$ exists in P. If P is a PEA and s is a state on P, then the kernel of s, i.e., $s^{-1}(0) = \{a \in P : s(a) = 0\}$, is a normal ideal in P.

If P and Q are PEAs, then a mapping $\phi: P \to Q$ is a PEA-morphism iff it is a GPEA-morphism and $\phi 1 = 1$. The latter property is automatically satisfied if ϕ is surjective. If $\phi: P \to Q$ is a bijective PEA-morphism and ϕ^{-1} is also a PEA-morphism, then ϕ is a PEA-isomorphism of P onto Q. A PEA-automorphism of P is a PEA-isomorphism $\phi: P \to P$.

A GPEA (and in particular, a PEA) P is said to be weakly commutative iff, for all $a, b \in P$, if $a \oplus b$ is defined, then $b \oplus a$ is defined [26]. It turns out that a PEA is symmetric iff it is weakly commutative [10, 25].

Naturally, a GPEA P is said to be *commutative* iff, for all $a, b \in P$, if $a \oplus b$ is defined, then $b \oplus a$ is defined, and then $a \oplus b = b \oplus a$. A commutative GPEA is the same thing as a *generalized effect algebra* (GEA) [16, Definition 2.2] and a commutative PEA is the same thing as an *effect algebra* (EA) [13], [16, Definition 2.1].

It can be shown that a GEA is total iff it can be realized, as per Example 2.4 as the positive cone in a directed Abelian po-group.

A prototype for EAs is the system of all self-adjoint operators between zero and identity on a Hilbert space \mathfrak{H} (the system of so-called *effect operators* on \mathfrak{H}).

In many important examples, an effect algebra is an interval $G[0, u] = \{a \in G : 0 \le a \le u\}$, as per Example 2.5, in the positive cone of an Abelian po-group G. For instance, the set of all effect operators on a Hilbert space \mathfrak{H} is the interval $\mathbb{G}(\mathfrak{H})[0, I]$ in the po-group $\mathbb{G}(\mathfrak{H})$ (with the usual partial order) of all self-adjoint operators on \mathfrak{H} . For more details about EAs and GEAs, see [3].

3 Unitization of a GPEA

In this section we review the notion of a binary unitization of a GPEA and some of its properties [16].

- **3.1 Definition.** [16, Definition 3.1] If $(P; \oplus, 0)$ is a GPEA, then a PEA (U; +, 0, 1) is called a binary *unitization* of P iff the following conditions are satisfied:
 - (U1) $P \subseteq U$ and if $a, b \in P$, then $a \oplus b$ exists in P iff a + b exists in U, and then $a \oplus b = a + b$.
 - (U2) $1 \notin P$.
 - (U3) If $x, y \in U \setminus P$, then x + y is undefined.

In this paper, as in [16], we shall be considering only binary unitizations; hence, for simplicity, we usually omit the adjective 'binary' in what follows. See [16, Theorem 3.3] for some of the basic properties of a unitization U of a GPEA P; in particular, for the fact that P is a normal maximal proper ideal of U. By [16, Theorem 5.3], a PEA U is a unitization of some GPEA P iff U admits a two-valued state s, and in this case, P is the kernel of s.

3.2 Definition. If P is a GPEA, then a GPEA-automorphism $\gamma: P \to P$ is said to be *unitizing* iff, for all $a, b \in P$, $\gamma a \oplus b$ is defined iff $b \oplus a$ is defined.

Obviously, if P is a total GPEA, then every GPEA-automorphism of P is unitizing. Also, P is weakly commutative iff the identity mapping on P is a unitizing GPEA-automorphism. By [16, Lemma 2.5, Lemma 2.7], a PEA admits one and only one unitizing GPEA-automorphism, namely $a \mapsto a^{--}$.

The next theorem describes the construction of a unitization of a GPEA P having a unitizing automorphism γ .

- **3.3 Theorem.** [16, Theorem 4.2] Let $(P; \oplus, 0)$ be a GPEA, let $\gamma : P \to P$ be a unitizing GPEA-automorphism, let P^{η} be a set disjoint from P and with the same cardinality as P, and let $\eta : P \to P^{\eta}$ be a bijection. Define $U := P \cup P^{\eta}$ and let + be the partial binary operation on U defined as follows:
 - (1) If $a, b \in P$, then a + b is defined iff $a \oplus b$ is defined, in which case $a + b := a \oplus b$.
 - (2) If $a \in P$ and $x \in U \setminus P$ with $b := \eta^{-1}x$, then a + x is defined iff $a \le b$, in which case $a + x := \eta c \in P^{\eta} = U \setminus P$, where c is the unique element of P such that $c \oplus a = b$. Thus, for $a, b \in P$, $a + \eta b$ is defined iff $a \le b$, in which case $a + \eta b = \eta(b \setminus a) \in U \setminus P$.
 - (3) If $b \in P$ and $y \in U \setminus P$ with $a := \eta^{-1}y$, then y + b is defined iff $\gamma b \le a$, in which case $y + b := \eta c \in P^{\eta} = U \setminus P$ where c is the unique element of P such that $\gamma b \oplus c = a$. Thus, for $a, b \in P$, $\eta a + b$ is defined iff $\gamma b \le a$, in which case $\eta a + b = \eta(\gamma b / a) \in U \setminus P$.
 - (4) If $x, y \in P^{\eta} = U \setminus P$, then $x \oplus y$ is undefined.

Then, with $1 := \eta 0$, (U; +, 0, 1) is a PEA and it is a unitization of the GPEA $(P; \oplus, 0)$. Moreover, for $a \in P$, we have $\eta a = a^{\sim} \in U \setminus P$ and $\gamma a = a^{--} \in P$.

We shall refer to the unitization U of P as constructed in Theorem 3.3 by using the unitizing automorphism γ as the γ -unitization of P.

Suppose that V is a unitization of the GPEA P and that $\gamma \colon P \to P$ is the restriction to P of the mapping $x \mapsto x^{--}$ on V. Then by [16, Theorem 3.3 (viii)], γ is a unitizing GPEA-automorphism of P called the unitizing GPEA-automorphism of P corresponding to V. Therefore, a GPEA P admits a unitization iff it admits a unitizing GPEA-automorphism.

The proof of the following theorem, using the results of [16, Theorem 3.3], is fairly straightforward, and we omit it here.

3.4 Theorem. Let V be a unitization of the GPEA P, let $\gamma \colon P \to P$ be the corresponding unitizing GPEA-automorphism, and let U be the γ -unitization of P. Then there exists a unique PEA-isomorphism of U onto V that reduces to the identity on P, namely the mapping $\phi \colon U \to V$ defined by $\phi a = a$ for $a \in P$ and $\phi x := (x^{-U})^{\sim_V}$ for $x \in U \setminus P$.

3.5 Example. [10, Examples 2.13 and 3.2] Let G be a directed po-group with center C(G), let $c \in G$, let $\mathbb{Z} \times G$ be the lexicographic product of the ordered group of integers \mathbb{Z} with G, and let $0 < n \in \mathbb{Z}$. Define the interval PEA $U := (\mathbb{Z} \times G)[(0,0),(n,c)]$. Then U is symmetric (weakly commutative) iff $c \in C(G)$.

Now we consider the case n=1 and $c \notin C(G)$. Then the PEA $U=(\mathbb{Z} \times G)[(0,0),(1,c)]$ is not symmetric. Define a two-valued state s on U by

(1)
$$s(0,g) = 0$$
 if $0 \le g \in G$, and (2) $s(1,g) = 1$ if $c \ge g \in G$.

Then by [16, Theorem 5.3], the set $P := s^{-1}(0) = \{(0,g) : 0 \leq g\}$ forms a normal maximal proper ideal in U, P is a GPEA isomorphic to the GPEA G^+ (Example 2.4), and U is a unitization of P. Note that P is a total GPEA, hence P is weakly commutative. To find the unitizing GPEA-automorphism γ , we observe that

$$(0,g)^- + (0,g) = (1,c) \Rightarrow (0,g)^- = (1,c-g), \text{ and}$$

 $(0,g)^{--} + (1,c-g) = (1,c) \Rightarrow (0,g)^{--} = (0,c-(c-g)).$

Thus for the element $(0, g) \in P$, $\gamma(0, g) = (0, c - (c - g))$.

We conclude this section with a presentation of an important class of PEAs, called *kite algebras*, constructed by A. Dvurečenskij in [12]. As we shall see, every kite algebra is a unitization of a GPEA. In our presentation, we shall find it convenient to make some small changes in the notation of [12]. First, for consistency with our notation above, we use P, rather than E, for the base GPEA from which a kite algebra is constructed. Second, for the bijections λ and ρ in [12], we change notation to reduce the number of occurrences of λ^{-1} and ρ^{-1} . Third, for ease of comparison with Theorem 3.3 we replace \overline{a} by ηa . Fourth, we use K rather than $K_I^{\lambda,\rho}$ for a kite algebra.

Thus, for the remainder of this section we assume that $(P; \oplus, 0)$ is a GPEA, I is a nonempty indexing set, and $\lambda, \rho: I \to I$ are bijections. We organize P^I into a GPEA $(P^I; \oplus_{P^I}, 0^I)$ where \oplus_{P^I} is the obvious coordinatewise orthosummation and 0^I is the family in P^I all the elements of which are 0. Let P^{η} be a set disjoint from P and with the same cardinality as P and let $\eta: P \to P^{\eta}$ be a bijection.

Evidently, P is total (respectively, weakly commutative) iff P^I is total (respectively, weakly commutative).

The following conditions on the bijections $\lambda, \rho \colon I \to I$ were introduced in [12] (but with λ and ρ replaced by λ^{-1} and ρ^{-1}): For all $(a_i)_{i \in I}, (b_i)_{i \in I} \in P^I$ and for all $i \in I$,

- (KCI) $a_{\rho i} \oplus b_i$ exists in P iff $b_i \oplus a_{\lambda i}$ exists in P, and
- (KCII) $a_{\lambda i} \oplus b_i$ exists in P iff $b_i \oplus a_{\rho i}$ exists in P.

Obviously, if $\lambda = \rho$, then conditions (KCI) and (KCII) are identical, and in this case it can be shown that they both hold iff P is a total GPEA. On the other hand, if P is a total GPEA, then (KCI) and(KCII) are automatically satisfied. In [12], it is assumed that the bijections $\lambda, \rho: I \to I$ satisfy both conditions (KCI) and (KCII). Later in Lemma 3.8 and Theorem 3.9, we shall assume (KCI), but not necessarily (KCII).

3.6 Definition. Noting that P^I and $(P^{\eta})^I$ are disjoint, we define

$$K := P^I \cup (P^{\eta})^I$$

and we define $0_K \in P^I$ and $1_K \in (P^{\eta})^I$ by

$$0_K := 0^I$$
 and $1_K = (\eta c_i)_{i \in I}$, where $c_i := 0$ for all $i \in I$.

We organize K into a partial algebra $(K; +, 0_K, 1_K)$, where the partial binary operation + on K is defined as follows: If $(a_i)_{i \in I}, (b_i)_{i \in I} \in P^I$, then

- (K1) $(a_i)_{i\in I} + (b_i)_{i\in I} := (a_i \oplus b_i)_{i\in I}$ iff $a_i \oplus b_i$ exists in P for all $i \in I$.
- (K2) $(a_i)_{i \in I} + (\eta b_i)_{i \in I} := (\eta(b_i \setminus a_{\lambda i}))_{i \in I} \text{ iff } a_{\lambda i} \leq b_i \text{ for all } i \in I.$
- (K3) $(\eta a_i)_{i \in I} + (b_i)_{i \in I} := (\eta(b_{\rho i} / a_i))_{i \in I} \text{ iff } b_{\rho i} \le a_i \text{ for all } i \in I.$
- (K4) $(\eta a_i)_{i \in I} + (\eta b_i)_{i \in I}$ is undefined.

If K is a PEA, it is called the kite algebra determined by P, I, λ , and ρ .

By (K1), the restriction of + to the GPEA P^I coincides with \oplus_{P^I} , whence, for all $(a_i)_{i\in I}$, we have $(a_i)_{i\in I} + 0_K = 0_K + (a_i)_{i\in I}$. Also, by (K2) and (K3), $0_K + (\eta a_i)_{i\in I} = (\eta a_i)_{i\in I} + 0_K = (\eta a_i)_{i\in I}$. Furthermore, by (K2) and (K3), $(a_i)_{i\in I} + 1_K$ is defined iff $1_K + (a_i)_{i\in I}$ is defined iff $(a_i)_{i\in I} = 0_K$, whereas by (K4), $(\eta a_i)_{i\in I} + 1_K$ and $1_K + (\eta a_i)_{i\in I}$ are undefined. If K is a kite algebra, it is obviously a (binary) unitization of P^I .

3.7 Definition. Define $\gamma \colon P^I \to P^I$ by

$$\gamma((a_i)_{i\in I}) = (a_{\rho\lambda^{-1}i})_{i\in I} \text{ for all } (a_i)_{i\in I} \in P^I.$$

3.8 Lemma. (KCI) holds iff γ is a unitizing GPEA-automorphism on P^I .

Proof. Clearly, γ is a GPEA-automorphism on P^I . Assume that (KCI) holds. Replacing i by $\lambda^{-1}i$ in (KCI), we find that, for all $(a_i)_{i\in I}, (b_i)_{i\in I} \in P^I$ and for all $i \in I$,

$$a_{\rho\lambda^{-1}i} \oplus b_{\lambda^{-1}i}$$
 exists in P iff $b_{\lambda^{-1}i} \oplus a_i$ exists in P . (1)

Now, given any $(c_i)_{i\in I} \in P^I$, we let $b_i := c_{\lambda i}$ in (1), so that $b_{\lambda^{-1}i} = c_i$, and we have, for all $(a_i)_{i\in I}, (c_i)_{i\in I} \in P^I$ and for all $i\in I$,

$$a_{o\lambda^{-1}i} \oplus c_i$$
 exists in P iff $c_i \oplus a_i$ exists in P . (2)

Therefore, γ is a unitizing GPEA-automorphism on P^I . Conversely, if (2) holds, then by first replacing i by λi , then putting $b_i := c_{\lambda i}$, we arrive back at (KCI).

Suppose that (KCI) holds, so that, by Lemma 3.8, γ is a unitizing GPEA-automorphism on P^I . Therefore, we can construct the γ -unitization U of P^I as per Theorem 3.3. To do this, we begin by putting $(P^I)^{\eta} := (P^{\eta})^I$ and we choose for the bijection from P^I to $(P^I)^{\eta}$ the mapping—also denoted by η —defined by $\eta(a_i)_{i\in I} := (\eta a_i)_{i\in I}$ for all $(a_i)_{i\in I} \in P^I$. (This dual use of η should not cause any confusion.) Then, as a set, the γ -unification $U = P^I \cup (P^I)^{\eta}$ of P^I is the same as K. However, as per Theorem 3.3 and the definition of γ , U is organized into a PEA $(U; +_U, 0_U, 1_U)$ as follows: $0_U := 0_K$, $1_U := 1_K$, and for $(a_i)_{i\in I}$, $(b_i)_{i\in I}$ in P^I :

- (U1) $(a_i)_{i\in I} +_U (b_i)_{i\in I} := (a_i \oplus b_i)_{i\in I}$ iff $a_i \oplus b_i$ is defined in P for all $i \in I$.
- (U2) $(a_i)_{i \in I} +_U (\eta b_i)_{i \in I} := (\eta(b_i \setminus a_i))_{i \in I}$ iff $a_i \leq b_i$ for all $i \in I$.
- (U3) $(\eta a_i)_{i \in I} +_U (b_i)_{i \in I} := (\eta(b_{\rho \lambda^{-1}i} / a_i))_{i \in I} \text{ iff } b_{\rho \lambda^{-1}i} \le a_i \text{ for all } i \in I.$
- (U4) $(\eta a_i)_{i \in I} +_U (\eta b_i)_{i \in I}$ is undefined.

According to (U1), the restriction of $+_U$ to P^I coincides with \oplus_{P^I} , whence it also coincides with the restriction of + to P^I .

3.9 Theorem. Suppose that (KCI) holds. Then the mapping $\phi: U \to K$ defined for all $(a_i)_{i \in I} \in P^I$ by

$$\phi(a_i)_{i\in I} := (a_i)_{i\in I} \text{ and } \phi((\eta a_i)_{i\in I}) = (\eta a_{\lambda i})_{i\in I}$$

is an isomorphism of the PEA $(U; +_U, 0_U, 1_U)$ onto the partial algebra $(K; +, 0_K, 1_K)$ such that the restriction of ϕ to P^I is the identity mapping. Therefore, $(K; +, 0_K, 1_K)$ is a PEA, K is a kite algebra, and $\phi: U \to K$ is a PEA-isomorphism. Furthermore, if $\psi: U \to K$ is any PEA-morphism such that the restriction of ψ to P^I is the identity mapping, then $\psi = \phi$.

Proof. Obviously, the restriction of ϕ to P^I is the identity mapping, $\phi 0_U = 0_K$, $\phi 1_U = 1_K$, and $\phi \colon U \to K$ is a bijection. In fact, for all $(a_i)_{i \in I} \in P^I$, $\phi^{-1}(a_i)_{i \in I} := (a_i)_{i \in I}$ and $\phi^{-1}((\eta a_i)_{i \in I}) = (\eta a_{\lambda^{-1}i})_{i \in I}$. To complete the proof that $\phi \colon U \to K$ is a PEA-isomorphism, we have to prove that

- (i) If $p, q \in U$ and $p +_U q$ is defined in U, then $\phi p + \phi q$ is defined in K and $\phi(p +_U q) = \phi p + \phi q$ and
- (ii) If $s, t \in K$ and s + t is defined in K, then $\phi^{-1}s +_U \phi^{-1}t$ is defined in U and $\phi^{-1}(s+t) = \phi^{-1}s +_U \phi^{-1}t$.

To prove condition (i), assume that $p, q \in U$. There are only two non-trivial cases to consider.

Case 1:
$$p = (a_i)_{i \in I} \in P^I$$
 and $q = (\eta b_i)_{i \in I} \in (P^{\eta})^I$.
Case 2: $p = (\eta a_i)_{i \in I} \in (P^{\eta})^I$ and $q = (b_i)_{i \in I} \in P^I$.

In Case 1, suppose that $p +_U q$ is defined in U. Then by (U2), $a_i \leq b_i$, whence $a_{\lambda i} \leq b_{\lambda i}$ for all $i \in I$, and

$$p +_{U} q = (\eta(b_i \setminus a_i))_{i \in I}$$
 for all $i \in I$, so

$$\phi p = (a_i)_{i \in I}, \ \phi q = (\eta b_{\lambda i})_{i \in I}, \ \phi(p +_U q) = (\eta(b_{\lambda i} \setminus a_{\lambda i}))_{i \in I},$$

and by (K2),

$$\phi p + \phi q = (a_i)_{i \in I} + (\eta b_{\lambda i})_{i \in I} := (\eta (b_{\lambda i} \setminus a_{\lambda i})_{i \in I} = \phi (p +_U q).$$

In Case 2, suppose that $p +_U q$ is defined in U. Then by (U3), $b_{\rho\lambda^{-1}i} \leq a_i$, whence $a_{\rho i} \leq b_{\lambda i}$ for all $i \in I$, and

$$p +_{U} q = (\eta(b_{\rho\lambda^{-1}i}/a_i)_{i\in I} \text{ for all } i \in I, \text{ so}$$

$$\phi p = (\eta a_{\lambda i})_{i \in I}, \ \phi q = (b_i)_{i \in I}, \ \phi(p +_U q) = (\eta(b_{\rho i} / a_{\lambda i}))_{i \in I},$$

and by (K3),

$$\phi p + \phi q = (\eta a_{\lambda i})_{i \in I} + (b_i)_{i \in I} = (\eta (b_{\rho i} / a_{\lambda i})_{i \in I} = \phi (p +_U q),$$

completing the proof of (i).

The proof of (ii) is a straightforward computation similar to the proof of (i), and is therefore omitted. Thus, we may conclude that ϕ is an isomorphism of U onto K that reduces to the identity on P^I ; hence K is a PEA.

Finally, suppose that $\psi \colon U \to K$ is any PEA-morphism such that the restriction of ψ to P^I is the identity mapping. According to (U4) with $b_i = a_i$ for all $i \in I$, for all $(a_i)_{i \in I} \in P^I$,

$$(a_i)_{i \in I} +_U (\eta a_i)_{i \in I} = 1_U$$
, whence $(a_i)_{i \in I} + \psi(\eta a_i)_{i \in I} = 1_K$.

But by (K2), we also have $(a_i)_{i\in I} + (\eta a_{\lambda i})_{i\in I} = 1_K$, and it follows from cancellation that

$$\psi(\eta a_i)_{i \in I} = (\eta a_{\lambda i})_{i \in I} = \phi(\eta a_i)_{i \in I}.$$

Therefore,
$$\psi = \phi$$
.

The proof of the following corollary is now straightforward.

3.10 Corollary. Suppose that (KCI) holds. Then, for all $(a_i)_{i \in I}$, the left and right negations on the kite algebra K are as follows:

(LN)
$$((a_i)_{i\in I})^- = (\eta a_{\rho i})_{i\in I}$$
 and $((\eta a_i)_{i\in I})^- = (a_{\lambda^{-1}i})_{i\in I}$,

(RN)
$$((a_i)_{i\in I})^{\sim} = (\eta a_{\lambda i})_{i\in I} \text{ and } ((\eta a_i)_{i\in I})^{\sim} = (a_{\rho^{-1}i})_{i\in I},$$

and from (LN), we have

$$((a_i)_{i \in I})^{--} = (a_{\rho \lambda^{-1} i})_{i \in I} = \gamma(a_i)_{i \in I}.$$

Therefore, $(K; +, 0_K, 1_K)$ is a (binary) unitization of P^I with γ as its unitizing PEA-automorphism.

If G is an additively written po-group and if we put $P := G^+$ (Example 2.4), then P is total and the PEA K is what A. Dvurečenskij originally called a kite PEA in [11].

4 Congruences and ideals in GPEAs and in their unitizations

In what follows, we will need some facts about congruences and ideals in GPEAs.

- **4.1 Definition.** [25, Definition 2.1] A binary relation \sim on a GPEA P is called a *weak congruence* iff it satisfies the following conditions:
- (C1) \sim is an equivalence relation.
- (C2) If $a \oplus b$ and $a_1 \oplus b_1$ both exist, $a \sim a_1$, and $b \sim b_1$, then $a \oplus b \sim a_1 \oplus b_1$.

A weak congruence is a *congruence* iff it satisfies the following condition:

(C3) If $a \oplus b$ exists, then for any $a_1 \sim a$ there is a $b_1 \sim b$ such that $a_1 \oplus b_1$ exists, and for any $b_2 \sim b$ there is an $a_2 \sim a$ such that $a_2 \oplus b_2$ exists.

A congruence \sim is called a *c-congruence* iff it satisfies the following condition:

(C4) If $a \sim b$ and either $a \oplus a_1 \sim b \oplus b_1$ or $a_1 \oplus a \sim b_1 \oplus b$, then $a_1 \sim b_1$.

A congruence \sim is called a *p-congruence* iff it satisfies the following condition:

(C5) $a \oplus b \sim 0$ implies $a \sim b \sim 0$.

Let \sim be a weak congruence. Denote by [a] the congruence class containing $a \in P$, and let P/\sim denote the set of all congruence classes (the quotient of P with respect to \sim). We shall say that $[a] \oplus [b]$ exists iff there are $a_1, b_1 \in P$ with $a_1 \sim a$, $b_1 \sim b$, such that $a_1 \oplus b_1$ exists, in which case, $[a] \oplus [b] := [a_1 \oplus b_1]$.

4.2 Theorem. [25, Corollary 2.1] For a congruence \sim on a GPEA P, the quotient $(P/\sim; \oplus, [0])$ is a GPEA iff \sim is both a c-congruence and a p-congruence.

If P is a PEA, then every congruence on P is a c-congruence and p-congruence, as can be seen from the following theorem.

- **4.3 Theorem.** [18] A weak congruence \sim on a PEA P is a congruence iff the following conditions hold:
- (C4') If $a \sim b$, then $a^{\sim} \sim b^{\sim}$ and $a^{-} \sim b^{-}$.
- (C5') $a \sim b \oplus c$ implies that there are $a_1, a_2 \in P$, such that $a_1 \sim b$, $a_2 \sim c$ and $a = a_1 \oplus a_2$.

Moreover, (C4') is equivalent to (C4), and (C5') implies (C5).

Of course, if P is a PEA and \sim is a congruence on P, then P/\sim is also a PEA.

- **4.4 Definition.** An ideal I in a GPEA P is called an R1-ideal iff the following condition holds:
- (R1) If $i \in I$, $a, b \in P$, $a \oplus b$ exists, and $i \leq a \oplus b$, then there are $j, k \in I$ such that $j \leq a, k \leq b$ and $i \leq j \oplus k$.

An R1-ideal I is called a Riesz ideal iff the following condition holds:

- (R2) If $i \in I$, $a, b \in P$, and $i \le a$, then (i) if $(a \setminus i) \oplus b$ exists, then there is $j \in I$ such that $j \le b$ and $a \oplus (j \nearrow b)$ exists, and (ii) if $b \oplus (i \nearrow a)$ exists, then there is $k \in I$ such that $k \le b$ and $(b \setminus k) \oplus a$ exists.
- **4.5 Definition.** For an ideal I in a GPEA P, we define $a \sim_I b$ $(a, b \in P)$ iff there exist $i, j \in I$, $i \leq a, j \leq b$ such that $a \setminus i = b \setminus j$.

Notice that if I is a normal ideal in the GPEA P, then the condition in Definition 4.5 that $a \setminus i = b \setminus j$ with $i, j \in I$ is equivalent to the condition that i/a = j/b (see [25, Remark 2.4]). Also, if $i \in I$ and $i \leq a \in P$, then $(a \setminus i) \setminus 0 = a \setminus i$, so $a \setminus i \sim_I a$.

- **4.6 Theorem.** [25, Theorem 2.3] If I is a normal R1-ideal in a GPEA P, then \sim_I satisfies (C1), (C2) and (C5'). Moreover, $a \sim_I 0$ iff $a \in I$.
- **4.7 Theorem.** [25, Corollary 2.2] If I is a normal Riesz ideal in a GPEA P, then \sim_I is a c-congruence and p-congruence, hence P/\sim_I is a GPEA.

Recall that a poset V is upward directed iff for any $a, b \in V$ there is $c \in V$ with $a, b \leq c$. Similarly V is downward directed iff for any $a, b \in V$ there is $d \in V$ with $d \leq a, b$.

- **4.8 Theorem.** [25, Proposition 2.2] In an upward directed GPEA P, an ideal I is a Riesz ideal iff I is an R1-ideal.
- **4.9 Definition.** We say that a congruence \sim on a GPEA P is a *Riesz congruence* iff it satisfies (C4), (C5') and the following condition:
- (CR) If $a \sim b$, then there are $c, d \in P$, such that $c \leq a \leq d$, $c \leq b \leq d$, $a \setminus c \sim b \setminus c \sim 0$, and $a \setminus a \sim d \setminus b \sim 0$.

4.10 Theorem. [25, Lemma 2.4] If I is a normal Riesz ideal in an upward directed GPEA P, then \sim_I is a Riesz congruence.

4.11 Theorem. Let P be a GPEA. Then:

- (i) A congruence \sim on P satisfying (C4), (C5') is a Riesz congruence iff every equivalence class is both downward and upward directed.
- (ii) If \sim is a Riesz congruence on P, then $I := \{i \in P : i \sim 0\}$ is a normal Riesz ideal in P. Moreover, $\sim = \sim_I$.
- *Proof.* (i) If \sim is a Riesz congruence, then every equivalence class is upward and downward directed by [25, Proposition 2.8 (1)]. Conversely, assume that \sim is a congruence such that every equivalence class is downward and upward directed and suppose that $a \sim b$. Then there exist c, d such that $c \leq a \leq d$, $c \leq b \leq d$, and $c \sim a \sim b \sim d$. By (C4) we then obtain $a \setminus c \sim b \setminus c \sim 0$, and $a \setminus a \sim d \setminus b \sim 0$. Hence \sim satisfies (CR).
- (ii) If $x \in I$ and $y \le x$, then $(x \setminus y) \oplus y \sim 0$. Since \sim is a p-congruence, $y \sim 0$ and $y \in I$. If $i \sim 0, j \sim 0$ and $i \oplus j$ exists, then $i \oplus j \sim 0$. Thus I is an ideal. Assume $i \in I$, $a, j \in P$ and $a \oplus i = j \oplus a$. Then $0 \oplus a \sim a \oplus i = j \oplus a$, and by (C4) we have $0 \sim j$, whence $j \in I$. Thus I is a normal ideal.

The proof of the R1 property is the same as in the proof of [25, Lemma 2.3].

To prove R2, assume that $a, b \in P$, $i \in I$, and $(a \setminus i) \oplus b$ exists. Thus $a \setminus i \sim a$ and by (C3), there is $b_1 \sim b$ such that $a \oplus b_1$ exists. Then there is $b_2 \leq b, b_1$ with $b_2 \sim b$. From this we get $b_2 = j/b$ and $a \oplus b_2$ exists. Finally, from $b_2 = j/b \sim b$ we get by (C4) that $j \sim 0$. To prove the remaining statement, assume first that $a \setminus i = b \setminus j$ with $i, j \in I$. Then by (C4), $a \setminus i \oplus i \sim b \setminus j \oplus j$, hence $a \sim_I b$ implies $a \sim b$. Conversely, assume that $a \sim b$. Then by (CR), there is $c \in P$ with $a \setminus c \sim b \setminus c \sim 0$, and with $i = a \setminus c \in I$, $j = b \setminus c \in I$ we have i/a = c = j/b, whence $a \sim_I b$.

From here on in this paper, we shall be considering the situation in which P is a GPEA, $\gamma \colon P \to P$ is a unitizing GPEA-automorphism on P, and the PEA U is the γ -unitization of P as per Theorem 3.3. Thus, P^{η} is disjoint from P, $U = P \cup P^{\eta}$, and $\eta \colon P \to P^{\eta}$ is a bijection. Moreover, for all $a \in P$, $\eta a = a^{\sim}$, $\gamma a = a^{--}$, and $\gamma^{-1}a = a^{\sim}$. In what follows, we use a, b, c, d, and e, with or without subscripts, to denote elements of P.

4.12 Theorem. Let P be a GPEA, and let U be its γ -unitization. Then P is a normal Riesz ideal in U iff P is upward directed.

Proof. As mentioned earlier, by Theorem [16, Theorem 3.3 (vii)] P is a maximal proper normal ideal in U. Let P be a Riesz ideal in U and let $a, b \in P$. Then $b \leq 1_U = a + \eta a$ implies that there are $j, k \in P$ such that $j \leq a, k \leq \eta a$ and $b \leq j \oplus k$. But then $a \oplus k$ is an upper bound of both a and b, hence P is upward directed.

Conversely, assume that P is upward directed. By Theorem 4.8, it suffices to check condition R1. If $i, a, b \in P$, $i \leq a \oplus b$, this is obvious. Let $i \leq a + \eta b = \eta(b \setminus a)$. Then $(b \setminus a) \oplus i$ exists in P, and we choose $d \in P$ such that $(b \setminus a) \oplus i \leq d$, $b \leq d$. Then $i \leq (b \setminus a) / d = a \oplus (b / d)$, where $b / d \leq \eta b$.

Now let $i \leq \eta a + b$. We have $\eta a + b = b + \eta a_1$ for some $a_1 \in P$. From the previous part of this proof we have that $i \leq b \oplus c$ where $c \leq \eta a_1$. Let $c + u = \eta a_1$. Then $\eta a + b = b + \eta a_1 = b + c + u = c_1 + b + u = u_1 + c_1 + b$. From this, $u_1 + c_1 = \eta a$, hence $c_1 \leq \eta a$, and $i \leq b \oplus c = c_1 \oplus b$.

- **4.13 Definition.** Let P be a GPEA with a unitizing automorphism γ . An ideal I in P is called γ -closed (or simply a γ -ideal) iff for all $a \in P$, $a \in I$ $\Leftrightarrow \gamma a \in I$, or equivalently iff $I = \gamma I = \{\gamma i : i \in I\}$. A congruence \sim on P is called a γ -congruence iff $a \sim b \Leftrightarrow \gamma a \sim \gamma b$.
- **4.14 Theorem.** Let P be a GPEA, γ a unitizing automorphism of P, and U the γ -unitization of P. If I is a normal Riesz ideal in U, then its restriction $I \cap P$ to P is a γ -closed normal Riesz ideal in P.

Proof. Taking into account Lemma 2.8 (i) and the fact that for any $a \in P$, $\gamma a = a^{--} \in P$, the proof is straightforward.

Now we shall consider the question of extending a congruence on the GPEA P to a congruence on its γ -unitization U.

4.15 Definition. Let U be the γ -unitization of a GPEA P and let \sim be a weak congruence on P. Since $U = P \cup P^{\eta}$, $P^{\eta} = U \setminus P$, and $\eta: P \to P^{\eta}$ is a bijection, we can, and do, define a binary relation \sim^* on U by

 $a \sim^* b$ iff $a \sim b$ and $\eta a \sim^* \eta b$ iff $a \sim b$ for all $a, b \in P$.

If $a \in P$ and $x \in P^{\eta} = U \setminus P$, we understand that $a \not\sim^* x$.

4.16 Lemma. Let U be the γ -unitization of the GPEA P, let \sim be a γ -congruence on P, and let $a,b\in P$. Then: (i) $a^-\sim^*b^-$ iff $a\sim b$. (ii) $a\sim b^{--}$ iff $a^{\sim}\sim b$ iff $a^{\sim}\sim^*b^-$.

Proof. Assume the hypotheses of the lemma. (i) $a^- \sim^* b^-$ iff $a^{---} \sim^* b^{---}$ iff $(\gamma a)^- \sim^* (\gamma b)^-$ iff $(\gamma a) \sim (\gamma b)$ iff $a \sim b$.

- (ii) $a \sim b^{--}$ iff $a^{\sim \sim --} \sim b^{--}$ iff $\gamma(a^{\sim \sim}) \sim \gamma b$ iff $a^{\sim \sim} \sim b$. Also, by (i), $a^{\sim \sim} \sim b$ iff $a^{\sim} = a^{\sim \sim -} \sim^* b^-$.
- **4.17 Theorem.** Let U be the γ -unitization of the GPEA P and let \sim be a congruence on P. Then \sim^* is a congruence on U iff \sim is a γ -congruence that satisfies (C4) and (C5').

Proof. Assume that \sim is a γ -congruence on P that satisfies (C4) and (C5'). Evidently, the restriction of \sim^* to P as well as to P^{η} is an equivalence relation; hence \sim^* is an equivalence relation on $U = P \cup P^{\eta}$, and we have (C1).

To prove (C2), we concentrate only on the two nontrivial cases corresponding to parts (2) and (3) of Theorem 3.3. Thus let $a \sim^* a_1$, $b^{\sim} \sim^* b_1^{\sim}$ and suppose that $a + b^{\sim}$ and $a_1 + b_1^{\sim}$ exist. Then $a \leq b$ and $a_1 \leq b_1$, so we have $(b \setminus a) \oplus a = b \sim b_1 = (b_1 \setminus a_1) \oplus a_1$. Therefore, by (C4), $b \setminus a \sim b_1 \setminus a_1$, whence $a + b^{\sim} = (b \setminus a)^{\sim} \sim^* (b_1 \setminus a_1)^{\sim} = a_1 + b_1^{\sim}$ by Theorem 3.3 (2).

For the remaining case, let $a^{\sim} \sim^* a_1^{\sim}$, $b \sim^* b_1$ and suppose that $a^{\sim} + b$ and $a_1^{\sim} + b_1$ exist. Then $\gamma b = b^{--} \leq a$, so we have $\gamma b \oplus (\gamma b / a) = a \sim a_1 = \gamma b_1 \oplus (\gamma b_1 / a_1)$. Since $b \sim b_1$, it follows that $\gamma b \sim \gamma b_1$ and (C4) implies that $\gamma b / a \sim \gamma b_1 / a_1$. But then, $(\gamma b / a)^{\sim} \sim^* (\gamma b_1 / a_1)^{\sim}$, and we infer from Theorem 3.3 (3) that $a^{\sim} + b \sim^* a_1^{\sim} \oplus b_1$. Thus \sim^* satisfies (C2).

To prove (C3), we again concentrate only on the nontrivial cases corresponding to parts (2) and (3) of Theorem 3.3. Thus, on the one hand, assume that $a+b^{\sim}$ exists and let $a_1 \in P$ with $a_1 \sim a$. Then $a \leq b$ and $b = (b \setminus a) \oplus a$. Therefore by (C3) (in P) there exists $c \in P$ such that $c \sim b \setminus a$ and $c \oplus a_1$ exists. By (C2) (in P), $b_1 := c \oplus a_1 \sim b$, whence $b_1^{\sim} \sim^* b^{\sim}$ and, since $a_1 \leq b_1$, it follows that $a_1 + b_1^{\sim}$ exists.

Continuing to assume that $a + b^{\sim}$ exists, we let $b_2 \in P$ with $b_2^{\sim} \sim^* b^{\sim}$, whence $b_2 \sim b = (b \setminus a) \oplus a$. But then, by (C5'), there exist $c, a_2 \in P$ such that $b_2 = c \oplus a_2$ where $c \sim b \setminus a$ and $a_2 \sim a$. Since $a_2 \leq b_2$, it follows that $a_2 + b_2^{\sim}$ exists.

On the other hand, assume that $a^{\sim} + b$ exists and let $a_1 \in P$ with $a_1^{\sim} \sim^* a^{\sim}$. Then $\gamma b \leq a$ and $a_1 \sim a = \gamma b \oplus (\gamma b / a)$. But then, by (C5'), there exist $c, d \in P$ such that $a_1 = c \oplus d$ where $c \sim \gamma b = b^{--}$ and $d \sim \gamma b / a$. But then, by Lemma 4.16 (ii), $b_1 := c^{\sim} \sim b$; moreover, since $c \leq a_1$, we have $a_1^{\sim} \leq c^{\sim}$, so $a_1^{\sim} + c^{\sim} = a_1^{\sim} + b_1$ exists.

Continuing to assume that $a^{\sim} + b$ exists, we let $b_2 \in P$ with $b_2 \sim^* b$, i.e., $b_2 \sim b$. Then $\gamma b_2 \sim \gamma b$, and since $\gamma b \oplus (\gamma b / a) = a$ exists, (C3) implies that

there exists $c \in P$ such that $c \sim \gamma b / a$ and $\gamma b_2 \oplus c$ exists. But then, by (C2), $a_2 := \gamma b_2 \oplus c \sim \gamma b \oplus (\gamma b / a) = a$, so $a_2^{\sim} \sim^* a^{\sim}$. Since $b_2^{--} = \gamma b_2 \leq a_2$, we have $a_2^{\sim} \leq b_2^{--} = b_2^{-}$, whence $\gamma b_2 = b_2^{--} \leq a_2^{\sim} = a$, and it follows that $a_2^{\sim} + b_2$ exists. Thus \sim^* satisfies (C3), so it is a congruence on U.

Conversely, if \sim^* is a congruence on the PEA U, then by Theorem 4.3, \sim^* satisfies (C4'), (C4), and (C5'), and thus, of course, \sim satisfies (C4) and (C5') on P. Also, since \sim^* satisfies (C4'), it follows that \sim is a γ -congruence on P.

4.18 Proposition. Let I be a normal (R1)- γ -ideal in the GPEA P, put $\sim := \sim_I$, and let U be the γ -unitization of P. Then the following conditions are mutually equivalent: (i) \sim^* satisfies (C3) on U. (ii) I is a normal Riesz γ -ideal in P. (iii) \sim is a γ -congruence on P that satisfies (C4) and (C5'). (iv) \sim^* is a congruence on U that satisfies (C4), (C4'), (C5'), and (C5).

Proof. By Theorem 4.6, $\sim = \sim_I$ is a weak congruence on P satisfying (C5').

- (i) \Rightarrow (ii). Suppose that \sim^* satisfies (C3) on U. Then also \sim satisfies (C3) on P and so it is a congruence on P. Assume that $i \in I$ with $i \leq a \in P$ and that $(a \setminus i) \oplus b$ exists for some $b \in P$. Then we have $a \setminus i \sim a$ and so by (C3) there is $b_1 \in P$ such that $a \oplus b_1$ exists, whence $b_1 \leq a^{\sim}$, and $b \sim b_1$. Since I is normal and $b \sim b_1$, there exist $j, k \in I$ with $j/b = k/b_1$. Thus we have $j/b = k/b_1 \leq k \oplus (k/b_1) = b_1 \leq a^{\sim}$, and it follows that $a \oplus j/b$ exists. This proves part (i) of condition (R2) in Definition 4.4, a similar argument proves part (ii), whence I is also an (R2)-ideal, and we have (ii).
- (ii) \Rightarrow (iii). Suppose that I is a normal Riesz γ -ideal in P. Then by Theorems 4.6 and 4.7, $\sim = \sim_I$ is a γ -congruence on P satisfying (C4) and (C5').
- (iii) \Rightarrow (iv). If \sim is a γ -congruence on P satisfying (C4) and (C5'), then by Theorem 4.17, \sim * is a congruence on U. By Theorem 4.3, \sim * satisfies (C4'), (C4), (C5'), and (C5).
- (iv) \Rightarrow (i). A congruence \sim^* on U satisfies (C3) on U.
- **4.19 Theorem.** Let I be a normal Riesz γ -ideal in the GPEA P, put $\sim := \sim_I$, and let U be the γ -unitization of P. Then \sim^* is a Riesz congruence on U iff \sim satisfies the following condition on P:
 - (GCR) $a \sim b \Rightarrow \exists i, j \in I : i \oplus a = j \oplus b$, or equivalently $\exists k, \ell \in I : a \oplus k = b \oplus \ell$.

Proof. Assume the hypotheses of the theorem. By Proposition 4.18, \sim is a γ -congruence on P, \sim^* is a congruence on U, and both satisfy (C4) and (C5'). Also, by Theorem 4.6, $I = \{a \in P : a \sim 0\}$. Thus, if $a \in P$, $i \in I$, and $i \oplus a$ exists, then by (C2), $a = 0 \oplus a \sim i \oplus a$, and therefore $a^{\sim} \sim^* (i \oplus a)^{\sim}$.

Assume that \sim satisfies condition (GCR). Consider the following condition on \sim^* : For all $u, v \in U$,

(CR') $u \sim^* v \Rightarrow \exists s, t \in U : u+s \sim^* v+s \sim^* 1$, and $t+u \sim^* t+v \sim^* 1$.

According to [25, Proposition 3.1], on the PEA U, condition (CR') is equivalent to condition (CR). We shall show that \sim^* satisfies condition (CR').

Let $a, b \in P$. On the one hand, suppose that $a \sim^* b$. Then $a \sim b$, whence by (GCR), there exist $i, j, k, \ell \in I$ with $i \oplus a = j \oplus b$ and $a \oplus k = b \oplus \ell$. Put $s := (i \oplus a)^{\sim} = (j \oplus b)^{\sim}$ and $t := (a \oplus k)^{-} = (b \oplus \ell)^{-}$. Thus, $t = (\gamma(a \oplus k))^{\sim} = (\gamma(b \oplus \ell))^{\sim}$. Since $i \oplus a \sim a$, we have $s \sim^* a^{\sim} \sim^* b^{\sim}$ and similarly, $t \sim^* a^{-} \sim^* b^{-}$. Also, $a \leq i \oplus a$, and it follows that a + s exists and $a + s \sim^* a + a^{\sim} = 1$. Similar computations show that $b + s \sim^* 1$, $b \sim^* 1$.

On the other hand, suppose that $a^{\sim} \sim^* b^{\sim}$. Then $a \sim_I b$, whence there exist $i, j \in I$ with $a \setminus i = b \setminus j$. Put $s := (a \setminus i)^{\sim} = (b \setminus j)^{\sim}$ and $t := a \setminus i = b \setminus j$. We have $t = a \setminus i \sim a$, whence $s \sim a^{\sim}$. Also $\gamma s = \gamma((a \setminus i)^{\sim}) = a \setminus i \leq a$, and it follows that $a^{\sim} + s$ exists and $a^{\sim} + s \sim^* a^{\sim} + a^{\sim} = 1$. Similar computations show that $b^{\sim} + s \sim^* 1$, $t + a^{\sim} \sim^* 1$, and $t + b^{\sim} \sim^* 1$. Thus, \sim^* satisfies (CR'), whence it satisfies (CR), and consequently it is a Riesz congruence on U.

Conversely, let \sim^* be a Riesz congruence on U and suppose that $a, b \in P$ with $a \sim b$. Then by (CR), there exists $d \in U$ with $d \setminus a \sim^* d \setminus b \sim^* 0$. Thus $(d \setminus a) + a = d$, and since 0 + a = a with $0 \sim^* d \setminus a$ and $a \sim^* a$, it follows that $d \sim^* a \in P$; hence $d \in P$, $d \sim a \sim b$, and $d \setminus a \sim_I d \setminus b$ in P. Therefore, there exist $i, j \in I$ such that $d \setminus (i \oplus a) = (d \setminus a) \setminus i = (d \setminus b) \setminus j = d \setminus (j \oplus b)$. Consequently, $i \oplus a = j \oplus b$ and (GCR) holds.

4.20 Theorem. If I is a normal Riesz γ -ideal in the GPEA P, then I is a normal Riesz ideal in the γ -unitization U of P iff \sim_I satisfies the (GCR) condition on P.

Proof. If I is a normal Riesz γ -ideal in P and \sim_I satisfies the (GCR) condition, then by Theorem 4.19 \sim_I^* is a Riesz congruence on U. Since I is the zero class of this congruence, it is a normal Riesz ideal in U (Theorem 4.11 (ii)).

Conversely, suppose that I is a normal Riesz ideal in U and let $\sim_{U,I}$ be the relation induced on U by I as per Definition 4.5. Clearly, the restriction of $\sim_{U,I}$ to P coincides with \sim_I . Also, since U is upward directed, $\sim_{U,I}$ is a Riesz congruence on U by Theorem 4.10. Thus, by Theorem 4.3, $\sim_{U,I}$ satisfies (C4) and (C5') on U, whence by Theorem 4.11 (i), all equivalence classes in U modulo $\sim_{U,I}$ are upward directed. To prove that \sim_I satisfies GCR, assume that $a,b \in P$ with $a \sim_I b$. Then $a \sim_{U,I} b$, so there exists $r \in U$ such that $a,b \leq r$ and $a \sim_{U,I} b \sim_{U,I} r$. As $a,b \leq r$, there exist $p,q \in U$ with a+p=b+q=r. Moreover, as $a \sim_{U,I} r$, there exist $i,j \in I$ such that $i \leq a, j \leq r = a+p$, and $a \setminus i = r \setminus j = (a+p) \setminus j$. Therefore, $(a \setminus i) + j = a + p = (a \setminus i) + i + p$, whence j = i + p. But then, $p \leq j \in I$, so $p \in I$. A similar argument shows that $q \in I$, and therefore GCR holds. \square

We remark that, in the proof of Theorem 4.20, $\sim_{U,I} = \sim_I^*$. To prove this, it will be sufficient to show that, for all $a, b \in P$, $a \sim_I b \Leftrightarrow a^{\sim} \sim_{U,I} b^{\sim}$ and that $a \sim_{U,I} b^{\sim}$ cannot hold. But by Theorem 4.3, $a \sim_I b \Rightarrow a \sim_{U,I} b \Rightarrow a^{\sim} \sim_{U,I} b^{\sim}$. Conversely, by the same theorem, $a^{\sim} \sim_{U,I} b^{\sim} \Rightarrow a = a^{\sim} \sim_{U,I} b^{\sim} = b \Rightarrow a \sim_I b$. Moreover, if $a \sim_{U,I} b^{\sim}$, then there exist $i, j \in I$ such that $a \setminus i = b^{\sim} \setminus j$, which yields $(a \setminus i) \oplus j = a \setminus i + j = b^{\sim} \notin P$, contradicting the definition of U.

4.21 Corollary. In an upward directed GPEA P, a normal Riesz γ -ideal I in P is also a normal Riesz ideal in the γ -unitization U of P.

Proof. By the previous theorem, it is sufficient to show that \sim_I satisfies (GCR). So assume that $a \sim_I b$. Since P is upward directed, there exists $c \in P$ with $a, b \leq c$, and by (C4), $c \setminus a \sim_I c \setminus b$. Thus there are $i, j \in I$ such that $c \setminus (i \oplus a) = (c \setminus a) \setminus i = (c \setminus b) \setminus j = c \setminus (j \oplus b)$; hence $i \oplus a = j \oplus b$. \square

4.22 Theorem. Let \sim be a γ -congruence on a GPEA P which satisfies conditions (C4) and (C5'). Let \sim^* be the extension of \sim in the γ -unitization U of P. Define $\widetilde{\gamma}: P/\sim \to P/\sim$ by $\widetilde{\gamma}[a] = [\gamma a]$. Then $\widetilde{\gamma}$ is a unitizing automorphism in P/\sim , and U/\sim^* is the $\widetilde{\gamma}$ -unitization of P/\sim . In particular, if I is a normal Riesz γ -ideal on P such that \sim_I satisfies (GCR), then the $\widetilde{\gamma}$ -unitization of P/I is U/I.

Proof. By Theorem 4.17, \sim^* is a congruence on U and so U/\sim^* is a PEA. Clearly, if $s, t \in U$ and $s \in [t]$ (i.e. $s \sim^* t$), then either $s, t \in P$ or else $s, t \in U \setminus P =: \eta P$; hence $[a] \cap [\eta b] = \emptyset$ for all $a, b \in P$. Consequently,

 $U/\sim^*=P/\sim^*\cup\eta P/\sim^*$, where the sets on the right are disjoint and obviously of the same cardinality. Thus we may put $\eta[a]:=[\eta a]$.

We have $[a] \oplus [b] \in P/\sim$ for all $a,b \in P$ and for $a,b \in \eta P$, $[a] \oplus [b]$ does not exist. Also $[1] = [a]^- \oplus [a] = [a] \oplus [a]^\sim$ for any $a \in P$ and thus $[1] \not\in P/\sim$. Therefore U/\sim^* satisfies (U1)-(U3) and it is a unitization of P/\sim according to Definition 3.1.

As \sim is a γ -congruence, $\widetilde{\gamma}[a] = [\gamma a], a \in P$, is well defined. If $[a] \oplus [b]$ is defined, there are $a_1 \sim a, b_1 \sim b$ such that $[a] \oplus [b] = [a_1 \oplus b_1]$. Then $\widetilde{\gamma}([a] \oplus [b]) = [\gamma(a_1 \oplus b_1)] = [\gamma a_1] \oplus [\gamma b_1] = \widetilde{\gamma}[a] \oplus \widetilde{\gamma}[b]$, hence $\widetilde{\gamma}$ is additive. Moreover, $\widetilde{\gamma}[a] = \widetilde{\gamma}[b]$ implies $\gamma a \sim \gamma b$, which in turn implies $a \sim b$, so that [a] = [b]. This shows that $\widetilde{\gamma}$ is injective. To prove that $\widetilde{\gamma}$ is surjective, observe that for all $a \in P$, $a = \gamma(\gamma^{-1}a)$, hence $[a] = \widetilde{\gamma}[\gamma^{-1}a]$. This proves that $\widetilde{\gamma}$ is an automorphism of P/\sim . In addition, if $[a] \oplus [b]$ exists, then there are $a_1 \sim a$ and $b_1 \sim b$ such that $a_1 \oplus b_1$ is defined in P. Now $a_1 \oplus b_1$ is defined iff $\gamma b_1 \oplus a_1$ is defined in P, whence $[a] \oplus [b]$ exists iff $\widetilde{\gamma}[b] \oplus [a]$ exists. This proves that $\widetilde{\gamma}$ is a unitizing automorphism of P/\sim .

In U/\sim^* we have $[a] + \eta[b]$ is defined iff there are $a_1, b_1, a_1 \sim a, b_1 \sim b$ such that $a_1 + \eta b_1 = \eta(b_1 \setminus a_1)$ is defined and then $[a] + \eta[b] = \eta[b_1 \setminus a_1] = \eta([b] \setminus [a])$. Similarly, $\eta[a] + [b]$ is defined iff there are $a_1 \sim a, b_1 \sim b$ such that $\eta a_1 + b_1 = \eta(\gamma b_1 / a_1)$ is defined, and then $\eta[a] + [b] = \eta[\gamma b_1 / a_1] = \eta(\widetilde{\gamma}[b] / [a])$. This shows that U/\sim^* is a $\widetilde{\gamma}$ -unitization of P/\sim .

4.23 Example. [10, Example 2.3] Let \mathbb{Z} be the group of integers and $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. For any two elements of G we define:

$$(a,b,c) + (x,y,z) = \begin{cases} (a+x,b+y,c+z) & : & x \text{ is even} \\ (a+x,c+y,b+z) & : & x \text{ is odd} \end{cases}$$

and define $(a, b, c) \leq (x, y, z)$ iff a < x or $a = x, b \leq y, c \leq z$. Then $(G, +, \leq)$ is a lattice-ordered group with strong unit u = (1, 0, 0) and the interval E := G[(0, 0, 0), (1, 0, 0)] is a PEA.

The elements between (0,0,0) and (1,0,0) are of two kinds—(0,a,b) and (1,c,d)—where $a,b \ge 0$ and $c,d \le 0$. The two sets are of the same cardinality and $P := \{(0,a,b) \in G : a,b \ge 0\}$ is a normal Riesz ideal of E. We also have $(0,b,c)^{\sim} = (1,-c,-b)$ and $(0,b,c)^{-} = (1,-b,-c)$, thus $\gamma(0,a,b) = (0,a,b)^{--} = (0,b,a)$. With this γ on P, which is a weakly commutative GPEA, we have a GPEA with a unitizing automorphism and its γ -unitization is exactly the G[(0,0,0),(1,0,0)] that we started with.

We may also consider the identity as a unitizing automorphism on P and then we obtain a unitization G[(0,0,0),(1,0,0)] but with the + operation on G defined by

$$(a, b, c) + (x, y, z) = (a + x, b + y, c + z)$$

for all $a, b, c, x, y, z \in \mathbb{Z}$.

5 The Riesz decomposition property

- **5.1 Definition.** We say, that a GPEA P satisfies
 - (i) the Riesz decomposition property (RDP for short), iff for all $a, b, c, d \in P$ such that $a \oplus b = c \oplus d$, there are four elements $e_{11}, e_{12}, e_{21}, e_{22} \in P$ such that $a = e_{11} \oplus e_{12}, b = e_{21} \oplus e_{22}, c = e_{11} \oplus e_{21}$ and $d = e_{12} \oplus e_{22}$. We shall denote these four decompositions by the following table:

$$\begin{array}{c|cc} & c & d \\ \hline a & e_{11} & e_{12} \\ b & e_{21} & e_{22} \end{array}$$

- (ii) RDP₁, if for the decomposition in (i) it moreover holds: if $f, g \in P$ are such that $f \leq e_{12}$, $g \leq e_{21}$, then f and g commute $(f \oplus g = g \oplus f)$.
- (iii) RDP₂, if for the decomposition in (i) it moreover holds that $e_{12} \wedge e_{21} = 0$.
- (iv) RDP₀, if for $a, b, c \in P$ such that $a \leq b \oplus c$, there exists $b_1, c_1 \in P$ such that $b_1 \leq b, c_1 \leq c$ and $a = b_1 \oplus c_1$.

From the next example we see that, even in the commutative case, if P has RDP, its unitization U need not have it.

5.2 Example. [22, Example 4.1] Consider the GEA $P = \{0, a, b, c, a \oplus c, b \oplus c\}$ (see Fig. 1 below) and let the unitizing automorphism γ be the identity (thus the unitization U of P is the "classical" unitization \widehat{P}). Then P has RDP, but U does not. Indeed, for example, in U, $(b \oplus c)^{\perp} = b^{\perp} \ominus c$ cannot be decomposed except for $b^{\perp} \ominus c = (b^{\perp} \ominus c) \oplus 0$. But $(b^{\perp} \ominus c) \oplus b = c^{\perp} = (a^{\perp} \ominus c) \oplus a$ and therefore, for these elements, there is no decomposition in the sense of RDP.

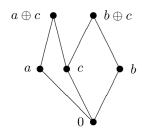


Fig. 1

In the following theorem, we show that if a GPEA P has a total operation \oplus , then P has any of the RDP properties iff its γ -unitization (with respect to any unitizing automorphism γ) has the corresponding RDP property. We do not know whether the above condition is also necessary.

5.3 Theorem. Let P be a total GPEA and let U be the unitization of P by a unitizing GPEA-automorphism γ . Then P has RDP (RDP_0 , RDP_1 , RDP_2 , respectively) iff U has RDP (RDP_0 , RDP_1 , RDP_2 , respectively).

Proof. Notice first, that, since $a \oplus b$ exists for all $a, b \in P$, then of course, P is upward directed. Also, since $a + b = a \oplus b$ exists, then $b \leq a^{\sim} = \eta a$ for all $a, b \in P$.

It is clear, that if U has RDP (RDP₁, RDP₂), then P has RDP (RDP₁, RDP₂), because if the decomposition of $a \oplus b = c \oplus d$ is:

$$\begin{array}{c|cc} & c & d \\ a & e_{11} & e_{12} \\ b & e_{21} & e_{22} \end{array}$$

then $e_{11}, e_{12} \leq a$ and $e_{21}, e_{22} \leq b$ and so if $a, b, c, d \in P$, then $e_{11}, e_{12}, e_{21}, e_{22} \in P$, because P is an ideal in U. Similarly, if U has RDP₀, so does P. Now assume that RDP holds in P. Then we have to consider three cases:

(a) $\eta a + b = \eta c + d$ (where $a, b, c, d \in P$). Then $\eta(\gamma b/a) = \eta(\gamma d/c)$, so $\gamma b/a = \gamma d/c$. Since P is upward directed, there exists $e \in P$ with e > a, c and therefore

$$e\diagdown(\gamma b\diagup a)=e\diagdown(\gamma d\diagup c), \text{ i.e., } (e\diagdown a)\oplus\gamma b=(e\diagdown c)\oplus\gamma d$$

where all these elements are in P, so by RDP we have a decomposition:

$$\begin{array}{c|cc} & e \backslash c & \gamma d \\ \hline e \backslash a & e_{11} & e_{12} \\ \gamma b & e_{21} & e_{22} \end{array}$$

Also, as $(e \setminus a) + a$ exists, $e \setminus a \leq a^-$, whence $\gamma^{-1}(e \setminus a) \leq \eta a$, and likewise $\gamma^{-1}(e \setminus c) \leq \eta c$. Therefore

$$\exists x, y \in U, \ x + \gamma^{-1}(e \setminus a) = \eta a \text{ and } y + \gamma^{-1}(e \setminus c) = \eta c.$$

But by assumption $\eta a + b = \eta c + d$, and therefore

$$x + \gamma^{-1}(e \setminus a) + b = \eta a + b = \eta c + d = y + \gamma^{-1}(e \setminus c) + d.$$

Moreover, from $(e \setminus a) \oplus \gamma b = (e \setminus c) \oplus \gamma d$, we deduce that

$$\gamma^{-1}(e \setminus a) + b = \gamma^{-1}(e \setminus a) \oplus b = \gamma^{-1}(e \setminus c) \oplus d = \gamma^{-1}(e \setminus c) + d$$

and it follows that x = y. Thus, $\eta a = x + \gamma^{-1}(e \setminus a)$, $\eta c = x + \gamma^{-1}(e \setminus c)$, and we have

$$\begin{array}{c|cccc}
 & \eta c & d \\
\hline
 & \eta a & x + \gamma^{-1} e_{11} & \gamma^{-1} e_{12} \\
 & b & \gamma^{-1} e_{21} & \gamma^{-1} e_{22}
\end{array}$$

(b) $a + \eta b = c + \eta d$. Then $\eta(b \setminus a) = \eta(d \setminus c)$, so $b \setminus a = d \setminus c$. Again, there exists $e \in P$ with $e \ge b, d$, so that

$$(b \setminus a)/e = (d \setminus c)/e$$
, i.e., $a \oplus (b/e) = c \oplus (d/e)$.

Thus, since RDP holds in P, we get

$$\begin{array}{c|cc}
 & c & d/e \\
\hline
 & a & e_{11} & e_{12} \\
 & b/e & e_{21} & e_{22} \\
\end{array}$$

As $b \oplus (b/e)$ exists, we have $b/e \le b^{\sim} = \eta b$ and thus $\eta b = (b/e) + x$ for some $x \in U$. Similarly, $\eta d = (d/e) + y$ for some $y \in U$. Then

$$a + (b/e) + x = a + \eta b = c + \eta d = c + (d/e) + y$$

and, since a + (b/e) = c + (d/e), it follows that x = y. Consequently, we obtain the decomposition:

$$\begin{array}{c|cccc}
c & (d/e) + x \\
\hline
a & e_{11} & e_{12} \\
(b/e) + x & e_{21} & e_{22} + x
\end{array}$$

where $(b/e) + x = \eta b$ and $(d/e) + x = \eta d$.

(c) $a \oplus \eta b = \eta c \oplus d$. As $d \leq b^{\sim} = \eta b$ there exists $x \in U$ with $x + d = \eta b$. Then we have $a + x + d = a + \eta b = \eta c + d$, and therefore $a + x = \eta c$. Thus the required decomposition is:

$$\begin{array}{c|ccc}
 & \eta c & d \\
\hline
 a & a & 0 \\
 \eta b & x & d
\end{array}$$

From the decompositions obtained in (a), (b), and (c), it is clear that, if P satisfies RDP_1 or RDP_2 , then so does U.

Finally we prove that P has RDP_0 iff U has RDP_0 . Clearly, if U has RDP_0 , then P has RDP_0 .

Assume that P has RDP_0 . Then there are four cases to check:

- (i) If $a \leq \eta b \oplus c$, then as $a \leq b^{\sim} = \eta b$, it follows that $a = a \oplus 0$ provides the desired decomposition.
- (ii) If $a \leq b \oplus \eta c$, then similarly, $a = 0 \oplus a$ provides the desired decomposition.
- (iii) If $\eta a \leq \eta b \oplus c$, then we have $c \leq \eta a$ and so $\eta a = x \oplus c \leq \eta b \oplus c$ for some $x \in U$ and that implies $x \leq \eta b$. So $\eta a = x + c$ provides the desired decomposition.
- (iv) If $\eta a \leq b \oplus \eta c$, then we make use of $b \leq \eta a$ and so there is again an $x \in U$ such that $\eta a = b + x \leq b \oplus \eta c$, which implies $x \leq \eta c$, whence $\eta a = b + x$ provides the desired decomposition.

We note that only in cases (c) and (iv) do we use the hypothesis that P is total.

We recall that a pseudo effect algebra E is a subdirect product of a system of pseudo effect algebras $(E_t:t\in T)$ if there is an injective homomorphism $h:E\to\prod_{t\in T}E_t$ such that $\pi_t(h(E))=E_t$ for each $t\in T$, where π_t is the t-th projection of $\prod_{t\in T}E_t$ onto E_t . In addition, E is subdirectly irreducible if whenever E is a subdirect product of $(E_t:t\in T)$, there is $t_0\in T$ such that $\pi_{t_0}\circ h$ is an isomorphism of pseudo effect algebras.

In the theory of total algebras, it is well known that an algebra is subdirectly irreducible iff it has a smallest nontrivial ideal. As pseudo effect algebras are only partial algebraic structures, some results of universal algebra need not hold for arbitrary pseudo effect algebras. For example, while for total algebras, there is one-to-one correspondence between congruences and ideals, in pseudo effect algebras the relations between ideals and congruences are more complicated. Therefore, in what follows, instead of studying irreducibility, we study relations between the smallest nontrivial (i.e. different from $\{0\}$) normal Riesz ideals in GPEAs and in their unitizations.

5.4 Theorem. Let P be an upward directed GPEA and let U be a γ -unitization of P. Then there is a smallest nontrivial normal Riesz ideal in U iff there is a smallest normal Riesz γ -ideal in P.

Proof. Let I be a smallest nontrivial normal Riesz ideal in U. Then $I \cap P$ is a nontrivial normal Riesz γ -ideal in P (Theorem 4.14). Let $\{0\} \neq I_0 \subset I \cap P$ be a normal Riesz γ -ideal in P. By Corollary 4.21, I_0 is also normal Riesz ideal in U, therefore we must have $I_0 = I = I \cap P$. It follows that I is also the smallest nontrivial Riesz γ -ideal in P.

Conversely, let I be a smallest nontrivial normal Riesz γ -ideal in P. By Corollary 4.21, I is also a normal Riesz ideal in U. If there is a normal Riesz ideal $\{0\} \neq J$ in U such that $J \subseteq I$, then $\{0\} \neq J \cap P$ is a normal Riesz γ -ideal in P, and hence $J \cap P = I$. It follows that $I \subseteq J$, hence I is the smallest nontrivial Riesz ideal in U.

Notice that in an upward directed GPEA P with RDP, every ideal is Riesz ideal. Similarly as for GEAs [9, Lemma 3.2] (see also [12, Lemma 4.3]), it can be proved that a (nontrivial) upward directed GPEA P with RDP₁ is subdirectly irreducible iff P possesses a smallest non-trivial normal ideal. This yields the following corollary.

- **5.5 Corollary.** Let P be a GPEA with total operation \oplus satisfying RDP_1 and let U be its γ -unitization. Then U is subdirectly irreducible iff P has a smallest nontrivial normal γ -ideal. If γ is the identity, then U is subdirectly irreducible iff P is subdirectly irreducible.
- **5.6 Remark.** Assume that $(P; \oplus, 0)$ is an upward directed GPEA, I is a nonempty indexing set, $\lambda, \rho: I \to I$ are bijections, condition (KCI) holds, and $(K; +, 0_K, 1_K)$ is the resulting kite algebra (Theorem 3.9). By Corollary 3.10, K is the γ -unitization of P^I with the unitizing automorphism $\gamma(a_i)_{i \in I} = (a_{\rho \circ \lambda^{-1}(i)})_{i \in I}$. We shall say that $i, j \in I$ are connected iff $(\rho \circ \lambda^{-1})^m(i) = j$, or $(\rho \circ \lambda^{-1})^m(j) = i$ for some integer $m \geq 0$, otherwise i and j are disconnected.

Let $i_0, j_0 \in I$ be disconnected, and let I_0 and I_1 be maximal subsets of mutually connected elements in I containing i_0 and j_0 , respectively. Then clearly, every element in I_0 is disconnected with every element in I_1 . Let $H_0 := \{(f_i)_{i \in I} : f_i = 0 \ \forall i \notin I_0\}, \ H_1 := \{(f_j)_{j \in I} : f_j = 0 \ \forall j \notin I_1\}.$ It is easy to see that H_0 and H_1 are normal γ -ideals in P^I . Assume that K has RDP₁. Then also P^I has RDP₁, and H_0 and H_1 are normal Riesz γ -ideals in P^I . By Corollary 4.21, they are also normal Riesz ideals in K. Clearly, $H_0 \cap H_1 = \{0\}$. We can deduce that, under the above suppositions, if K has a smallest nontrivial normal Riesz ideal, then the set I is connected. (Compare with [12, Theorem 4.6].)

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