# Unitizations of generalized pseudo effect algebras and their ideals 

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#### Abstract

A generalized pseudo effect algebra (GPEA) is a partially ordered partial algebraic structure with a smallest element 0 , but not necessarily with a unit (i.e, a largest element). If a GPEA admits a so-called unitizing automorphism, then it can be embedded as an order ideal in its so-called unitization, which does have a unit. We study unitizations of GPEAs with respect to a unitizing automorphism, paying special attention to the behavior of congruences and ideals in this setting.


## 1 Introduction

An effect algebra (EA) is a bounded partially ordered structure equipped with a partially defined commutative and associative binary operation called the orthosummation, often denoted by $\oplus$ [13]. The smallest element in an EA, called the zero or neutral element, is usually denoted by 0 , and the largest element, called the unit, is often denoted by 1 . For each element $a$ in an EA, there is a unique element $a^{\perp}$, called the orthosupplement of $a$, such that $a \oplus a^{\perp}=1$.

[^0]Effect algebras were originally introduced to represent possibly fuzzy or unsharp propositions arising either classically [17] or in quantum measurement theory [1].

The class of EAs includes the class of orthoalgebras [14] and the class of orthomodular posets [19, p. 27]. The subclass consisting of lattice-ordered EAs includes the class of MV-algebras [2], the class of orthomodular lattices [19], and the class of boolean algebras.

In spite of their considerable generality, EAs have been further generalized in two ways: (1) By dropping the assumption that the orthosummation is commutative. (2) By dropping the assumption that there is a unit. Noncommutative versions of EAs called pseudo effect algebras (PEAs), were introduced by A. Dvurečenskij and T. Vetterlein in [4, 5] and further studied in [6, 7, 9, 10, 11, 12, 18, 26]. The investigation of generalized effect algebras (GEAs), i.e., versions of EAs having no unit, was pioneered by Z. Riečanová in [21, 22, 23] and further studied in [15, 20]. Finally, by dropping both the assumption of commutativity and the existence of a unit, one arrives at the notion of a generalized pseudo effect algebra (GPEA) [7, 8, 16, 25, 26].

It is well known that every GEA $E$ can be embedded as a maximal proper ideal in an EA $\widehat{E}$ called its unitization in such a way that (1) $E$ and $\widehat{E} \backslash E$ are order-anti-isomorphic under the restrictions of the partial order on $\widehat{E}$; (2) for all $a \in \widehat{E}$, either $a \in E$ or else its orthosupplement $a^{\perp} \in E$; and (3) for $x, y \in \widehat{E} \backslash E$ the orthosum of $x$ and $y$ is not defined [3, Theorem 1.2.6]. This construction of a unitization was extended to so-called weakly commutative GPEAs in [25]. Recently, in [16], it was shown that a GPEA $P$ can be embedded as a maximal proper PEA-ideal in a PEA $U$ if and only if $P$ admits a so-called unitizing GPEA-automorphism. Indeed, the unitization of a weakly commutative GPEA is a special case in which the unitizing GPEA-automorphism is the identity mapping.

In this article, which is a continuation of [16], we focus on properties of congruences and ideals in the setting of a GPEA $P$ and its unitization $U$ with respect to a unitizing automorphism $\gamma$ (a so-called $\gamma$-unitization). We find conditions under which a congruence on $P$ can be extended to a congruence on its $\gamma$-unitization $U$ such that the quotient of $U$ is the unitization of the quotient of $P$ with a unitizing automorphism induced by $\gamma$. In particular, we extend the results obtained in [20] and [25] for unitizations of GEAs and weakly commutative GPEAs to the more general unitizations of GPEAs with respect to a unitizing automorphism.

We briefly investigate several versions, $\mathrm{RDP}, \mathrm{RDP}_{0}, \mathrm{RDP}_{1}, \mathrm{RDP}_{2}$ of the Riesz decomposition property in GPEAs and in their $\gamma$-unitizations and show that if $P$ is a total GPEA, i.e., if the orthosum in $P$ is defined for all pairs of its elements, then $P$ has one of these properties if and only if its $\gamma$-unitization $U$ has the corresponding property.

We also study relations between the existence of a smallest nontrivial Riesz ideal in a GPEA and the existence of a smallest nontrivial Riesz ideal in its $\gamma$-unitization.

For the reader's convenience, we devote Sections 2 and 3 below to a brief review of some basic definitions and facts needed in this article.

We provide several illustrating examples. In [11] it was shown how to construct a large class of PEAs by starting with the positive cone $G^{+}$of a po-group $G$, a nonempty indexing set $I$, and two bijections $\lambda, \rho: I \rightarrow I$. In [12], this construction was extended by replacing $G^{+}$by a more general GPEA $E$. At the end of Section 3, we review this construction and relate it to our work in this article.

## 2 Generalized pseudo effect algebras

We begin this section by recalling the definition of a GPEA [16, Definition $2.6]$ and we observe that PEAs, GEAs, and EAs are special kinds of GPEAs. Axiomatic characterizations of PEAs, GEAs, and EAs can be found in 16, §2]. Also we review some of the basic properties of these partially ordered, partial algebraic structures. We abbreviate 'if and only if' by 'iff' and the symbol $:=$ means 'equals by definition.'
2.1 Definition. A generalized pseudo effect algebra (GPEA) [7, 8] is a partial algebraic structure $(P ; \oplus, 0)$, where $\oplus$ is a partial binary operation on $P$ called the orthosummation, 0 is a constant in $P$ called the zero element, and the following conditions hold for all $a, b, c \in P$ :
(GPEA1) $a \oplus b$ and $(a \oplus b) \oplus c$ exist iff $b \oplus c$ and $a \oplus(b \oplus c)$ exist and in this case $(a \oplus b) \oplus c=a \oplus(b \oplus c)$ (associativity).
(GPEA2) If $a \oplus b$ exists, then there are elements $c, d \in P$ such that $a \oplus b=$ $c \oplus a=b \oplus d$ (conjugation).
(GPEA3) If $a \oplus c=b \oplus c$, or $c \oplus a=c \oplus b$, then $a=b$ (cancellation).
(GPEA4) $a \oplus 0=0 \oplus a=a$ (neutral element).
(GPEA5) If $a \oplus b=0$, then $a=0=b$ (positivity).
If no confusion threatens, we often denote the GPEA $(P ; \oplus, 0)$ simply by $P$. If we write an equation involving an orthosum of elements of $P$ without explicitly assuming the existence thereof, we understand that its existence is implicitly assumed.

Let $P$ be a GPEA and let $a, b, c \in P$. By (GPEA3), the elements $c, d$ in (GPEA2) are uniquely determined. A partial order $\leq$ is defined on the GPEA $P$ (the induced partial order) by stipulating that $a \leq b$ iff there is (a necessarily unique) $c \in P$ with $a \oplus c=b$, or equivalently, iff there is (a necessarily unique) $d \in P$ with $d \oplus a=b$. For all $a \in P$, we have $0 \leq a$. Moreover, the cancellation laws (GPEA3) can be extended to $\leq$ as follows: If $a \oplus c \leq b \oplus c$ or $c \oplus a \leq c \oplus b$, then $a \leq b$.

Partial binary operations left subtraction / and right subtraction \are defined on $P$ as follows: For $a, b \in P, a / b$ and $b \backslash a$ are defined iff $a \leq b$, in which case (1) $a / b:=c$, where $c$ is the unique element of $P$ such that $a \oplus c=b$ and (2) $b \backslash a:=d$, where $d$ is the unique element of $P$ such that $d \oplus a=b$.
2.2 Definition. Let $P$ be a GPEA and let $I$ and $S$ be nonempty subsets of $P$. Then:
(1) $I$ is an order ideal iff $a \in I, b \in P$, and $b \leq a$ implies that $b \in I$.
(2) $I$ is an ideal iff $I$ is an order ideal and whenever $a, b \in I$ and $a \oplus b$ is defined, it follows that $a \oplus b \in I$.
(3) An ideal $I$ in $P$ is said to be normal iff whenever $a, b, c \in P$ and $a \oplus c=c \oplus b$, then $a \in I \Leftrightarrow b \in I$.
(4) $S$ is a sub-GPEA of $P$ iff, whenever two of the elements $a, b, c \in P$ belong to $S$ and $a \oplus b=c$, then the third element also belongs to $S$.

Every ideal $I$ in $P$ is a sub-GPEA, and every sub-GPEA $S$ in $P$ is a GPEA in its own right under the restriction to $S$ of the orthosummation on $P$.
2.3 Lemma. Let $P$ be a GPEA, let $I$ be a normal ideal in $P$, let $a, b \in P$, and suppose that $a \oplus b$ is defined. Then $b \in I$ iff $(a \oplus b) \backslash a \in I$; likewise, $a \in I$ iff $b /(a \oplus b) \in I$.

Proof. Observe that $((a \oplus b) \backslash a) \oplus a=a \oplus b$ and $b \oplus(b /(a \oplus b))=a \oplus b$.
A GPEA $P$ is said to be total iff $a \oplus b$ is defined for all $a, b \in P$.
2.4 Example. Let $G$ be an additively-written, not necessarily Abelian partially ordered group (po-group). Then the positive cone $G^{+}:=\{a \in G: 0 \leq$ a\} is a total GPEA with orthosummation $\oplus$ given by the restriction to $G^{+}$ of the group operation + on $G$. In this case, the induced partial order on the GPEA $G^{+}$is the restriction to $G^{+}$of the partial order on $G$.

If $P$ and $Q$ are GPEAs, then a mapping $\phi: P \rightarrow Q$ is a GPEA-morphism iff, for all $a, b \in P$, if $a \oplus b$ exists in $P$, then $\phi a \oplus \phi b$ exists in $Q$, and $\phi(a \oplus b)=\phi a \oplus \phi b$. A bijective GPEA-morphism $\phi: P \rightarrow Q$ is a GPEAisomorphism of $P$ onto $Q$ iff $\phi^{-1}: Q \rightarrow P$ is also a GPEA-morphism. A GPEA-automorphism of $P$ is GPEA-isomorphism $\phi: P \rightarrow P$.

It is not difficult to show that a bijective GPEA-morphism $\phi: P \rightarrow Q$ is a GPEA-isomorphism iff, whenever $a, b \in P$ and $\phi a \oplus \phi b$ exists in $Q$, then $a \oplus b$ exists in $P$.

A pseudo effect algebra (PEA) is a GPEA with a largest element, called the unit and often denoted by 1 ([16, Definition 2.3]). Let $P$ be a PEA with unit 1. Clearly, the only element $z \in P$ such that $z \oplus 1$ (or $1 \oplus z$ ) exists is $z=0$. Also, for $a \in P$, we have $a \leq 1$ whence we define $a^{\sim}:=a / 1$ and $a^{-}:=1 \backslash a$. Thus, $a^{\sim}$ and $a^{-}$, called the right supplement and the left supplement of $a$, respectively, are the unique elements in $P$ such that $a \oplus a^{\sim}=a^{-} \oplus a=1$. If it happens that $a^{\sim}=a^{-}$, then the common element is called the orthosupplement of $a$ and is written as $a^{\perp}$. The PEA $P$ is said to be symmetric iff $a^{\perp}=a^{\sim}=a^{-}$for all $a \in P[10]$.
2.5 Example. Let $G$ be any (additively written and not necessarily Abelian) po-group and choose an element $0 \leq u \in G$. Let $G[0, u]:=\{a \in G: 0 \leq a \leq$ $u\}$, and define $a \oplus b$ for $a, b \in G[0, u]$ iff $a+b \leq u$, in which case $a \oplus b:=a+b$. Then $(G[0, u] ; \oplus, 0, u)$ is a PEA and the induced partial order coincides with the partial order on $G$ restricted to $G[0, u]$. We note that, if $a, b \in G[0, u]$ and $a \leq b$, then $a^{\sim}=-a+1, a^{-}=1-a, a / b=-a+b$, and $b \backslash a=b-a$.

Some important basic properties of PEAs are collected in the following theorem (see [16, Theorem 2.4]).
2.6 Theorem. Let $P$ be a PEA and let $a, b, c \in P$. Then: (i) $0^{\sim}=0^{-}=1$ and $1^{\sim}=1^{-}=0$. (iv) $a^{\sim-}=a^{-\sim}=a$. (ii) $a \oplus b=c$ iff $b^{-}=c^{-} \oplus a$. (vii)
$a \oplus b^{\sim}=c^{\sim}$ iff $c \oplus a=b$. (iii) $a^{\sim} \oplus b=c^{\sim}$ iff $b^{-}=c \oplus a^{\sim}$ iff $b^{--} \oplus c=a$. (iv) $a \leq b$ iff $b^{\sim} \leq a^{\sim}$ iff $b^{-} \leq a^{-}$. (v) Both $a \mapsto a^{\sim}$ and $a \mapsto a^{-}$are order-reversing bijections on $P$. (vi) $a \oplus b$ exists iff $b \leq a^{\sim}$ iff $a \leq b^{-}$.

In a straightforward way, one can derive formulas for left and right subtraction as illustrated by the following lemma, the proof of which we omit.
2.7 Lemma. Let $P$ be $a P E A$ and let $a, b \in P$. Then: (i) $a \leq b \Rightarrow b \backslash a=$ $\left(a \oplus b^{\sim}\right)^{-}$and $a / b=\left(b^{-} \oplus a\right)^{\sim}$. (ii) $b^{--} \leq a \Rightarrow b^{--} / a=\left(a^{\sim} \oplus b\right)^{-}$. (iii) $b \leq a^{\sim} \Rightarrow a \leq b^{-}, a^{\sim} \backslash b=\left(b^{--} \oplus a\right)^{\sim}, b^{-} \backslash a=(a \oplus b)^{-}$, and $a^{\sim} \backslash b=$ $\left(b^{--} \oplus a\right)^{\sim}$.
2.8 Lemma. Let $P$ be a PEA, let $I$ be a normal ideal in $P$, and let $a, b \in P$. Then: (i) $a \in I$ iff $a^{--} \in I$ iff $a^{\sim \sim} \in I$. (ii) $a^{-} \in I$ iff $a^{\sim} \in I$.

Proof. Part (i) follows from the facts that $a^{--} \oplus a^{-}=1=a^{-} \oplus a$ and $a \oplus a^{\sim}=1=a^{\sim} \oplus a^{\sim \sim}$. Part (ii) is a consequence of $a \oplus a^{\sim}=1=a^{-} \oplus a$.

A state on a PEA $P$ is a mapping $s: P \rightarrow[0,1] \subseteq \mathbb{R}$ such that (S1) $s(1)=1$ and $(\mathrm{S} 2) s(a \oplus b)=s(a)+s(b)$ whenever $a \oplus b$ exists in $P$. If $P$ is a PEA and $s$ is a state on $P$, then the kernel of $s$, i.e., $s^{-1}(0)=\{a \in P$ : $s(a)=0\}$, is a normal ideal in $P$.

If $P$ and $Q$ are PEAs, then a mapping $\phi: P \rightarrow Q$ is a $P E A$-morphism iff it is a GPEA-morphism and $\phi 1=1$. The latter property is automatically satisfied if $\phi$ is surjective. If $\phi: P \rightarrow Q$ is a bijective PEA-morphism and $\phi^{-1}$ is also a PEA-morphism, then $\phi$ is a PEA-isomorphism of $P$ onto $Q$. A PEA-automorphism of $P$ is a PEA-isomorphism $\phi: P \rightarrow P$.

A GPEA (and in particular, a PEA) $P$ is said to be weakly commutative iff, for all $a, b \in P$, if $a \oplus b$ is defined, then $b \oplus a$ is defined [26]. It turns out that a PEA is symmetric iff it is weakly commutative [10, 25].

Naturally, a GPEA $P$ is said to be commutative iff, for all $a, b \in P$, if $a \oplus b$ is defined, then $b \oplus a$ is defined, and then $a \oplus b=b \oplus a$. A commutative GPEA is the same thing as a generalized effect algebra (GEA) [16, Definition 2.2 ] and a commutative PEA is the same thing as an effect algebra (EA) [13], [16, Definition 2.1].

It can be shown that a GEA is total iff it can be realized, as per Example 2.4 as the positive cone in a directed Abelian po-group.

A prototype for EAs is the system of all self-adjoint operators between zero and identity on a Hilbert space $\mathfrak{H}$ (the system of so-called effect operators on $\mathfrak{H}$ ).

In many important examples, an effect algebra is an interval $G[0, u]=$ $\{a \in G: 0 \leq a \leq u\}$, as per Example 2.5, in the positive cone of an Abelian po-group $G$. For instance, the set of all effect operators on a Hilbert space $\mathfrak{H}$ is the interval $\mathbb{G}(\mathfrak{H})[0, I]$ in the po-group $\mathbb{G}(\mathfrak{H})$ (with the usual partial order) of all self-adjoint operators on $\mathfrak{H}$. For more details about EAs and GEAs, see [3].

## 3 Unitization of a GPEA

In this section we review the notion of a binary unitization of a GPEA and some of its properties [16].
3.1 Definition. [16, Definition 3.1] If $(P ; \oplus, 0)$ is a GPEA, then a PEA ( $U ;+, 0,1$ ) is called a binary unitization of $P$ iff the following conditions are satisfied:
(U1) $P \subseteq U$ and if $a, b \in P$, then $a \oplus b$ exists in $P$ iff $a+b$ exists in $U$, and then $a \oplus b=a+b$.
(U2) $1 \notin P$.
(U3) If $x, y \in U \backslash P$, then $x+y$ is undefined.
In this paper, as in [16], we shall be considering only binary unitizations; hence, for simplicity, we usually omit the adjective 'binary' in what follows. See [16, Theorem 3.3] for some of the basic properties of a unitization $U$ of a GPEA $P$; in particular, for the fact that $P$ is a normal maximal proper ideal of $U$. By [16, Theorem 5.3], a PEA $U$ is a unitization of some GPEA P iff $U$ admits a two-valued state $s$, and in this case, $P$ is the kernel of $s$.
3.2 Definition. If $P$ is a GPEA, then a GPEA-automorphism $\gamma: P \rightarrow P$ is said to be unitizing iff, for all $a, b \in P, \gamma a \oplus b$ is defined iff $b \oplus a$ is defined.

Obviously, if $P$ is a total GPEA, then every GPEA-automorphism of $P$ is unitizing. Also, $P$ is weakly commutative iff the identity mapping on $P$ is a unitizing GPEA-automorphism. By [16, Lemma 2.5, Lemma 2.7], a PEA admits one and only one unitizing GPEA-automorphism, namely $a \mapsto a^{--}$.

The next theorem describes the construction of a unitization of a GPEA $P$ having a unitizing automorphism $\gamma$.
3.3 Theorem. [16, Theorem 4.2] Let $(P ; \oplus, 0)$ be a GPEA, let $\gamma: P \rightarrow P$ be a unitizing GPEA-automorphism, let $P^{\eta}$ be a set disjoint from $P$ and with the same cardinality as $P$, and let $\eta: P \rightarrow P^{\eta}$ be a bijection. Define $U:=P \cup P^{\eta}$ and let + be the partial binary operation on $U$ defined as follows:
(1) If $a, b \in P$, then $a+b$ is defined iff $a \oplus b$ is defined, in which case $a+b:=a \oplus b$.
(2) If $a \in P$ and $x \in U \backslash P$ with $b:=\eta^{-1} x$, then $a+x$ is defined iff $a \leq b$, in which case $a+x:=\eta c \in P^{\eta}=U \backslash P$, where $c$ is the unique element of $P$ such that $c \oplus a=b$. Thus, for $a, b \in P, a+\eta b$ is defined iff $a \leq b$, in which case $a+\eta b=\eta(b \backslash a) \in U \backslash P$.
(3) If $b \in P$ and $y \in U \backslash P$ with $a:=\eta^{-1} y$, then $y+b$ is defined iff $\gamma b \leq a$, in which case $y+b:=\eta c \in P^{\eta}=U \backslash P$ where $c$ is the unique element of $P$ such that $\gamma b \oplus c=a$. Thus, for $a, b \in P, \eta a+b$ is defined iff $\gamma b \leq a$, in which case $\eta a+b=\eta(\gamma b / a) \in U \backslash P$.
(4) If $x, y \in P^{\eta}=U \backslash P$, then $x \oplus y$ is undefined.

Then, with $1:=\eta 0,(U ;+, 0,1)$ is a PEA and it is a unitization of the GPEA $(P ; \oplus, 0)$. Moreover, for $a \in P$, we have $\eta a=a^{\sim} \in U \backslash P$ and $\gamma a=a^{--} \in P$.

We shall refer to the unitization $U$ of $P$ as constructed in Theorem 3.3 by using the unitizing automorphism $\gamma$ as the $\gamma$-unitization of $P$.

Suppose that $V$ is a unitization of the GPEA $P$ and that $\gamma: P \rightarrow P$ is the restriction to $P$ of the mapping $x \mapsto x^{--}$on $V$. Then by [16, Theorem 3.3 (viii)], $\gamma$ is a unitizing GPEA-automorphism of $P$ called the unitizing GPEAautomorphism of $P$ corresponding to $V$. Therefore, a GPEA $P$ admits a unitization iff it admits a unitizing GPEA-automorphism.

The proof of the following theorem, using the results of [16, Theorem 3.3], is fairly straightforward, and we omit it here.
3.4 Theorem. Let $V$ be a unitization of the GPEA $P$, let $\gamma: P \rightarrow P$ be the corresponding unitizing GPEA-automorphism, and let $U$ be the $\gamma$-unitization of $P$. Then there exists a unique PEA-isomorphism of $U$ onto $V$ that reduces to the identity on $P$, namely the mapping $\phi: U \rightarrow V$ defined by $\phi a=a$ for $a \in P$ and $\phi x:=\left(x^{-U}\right)^{\sim V}$ for $x \in U \backslash P$.
3.5 Example. [10, Examples 2.13 and 3.2] Let $G$ be a directed po-group with center $C(G)$, let $c \in G$, let $\mathbb{Z} \overrightarrow{\times} G$ be the lexicographic product of the ordered group of integers $\mathbb{Z}$ with $G$, and let $0<n \in \mathbb{Z}$. Define the interval PEA $U:=(\mathbb{Z} \overrightarrow{\times} G)[(0,0),(n, c)]$. Then $U$ is symmetric (weakly commutative) iff $c \in C(G)$.

Now we consider the case $n=1$ and $c \notin C(G)$. Then the PEA $U=$ $(\mathbb{Z} \times \overrightarrow{\times})[(0,0),(1, c)]$ is not symmetric. Define a two-valued state $s$ on $U$ by
(1) $s(0, g)=0$ if $0 \leq g \in G$, and (2) $s(1, g)=1$ if $c \geq g \in G$.

Then by [16, Theorem 5.3], the set $P:=s^{-1}(0)=\{(0, g): 0 \leq g\}$ forms a normal maximal proper ideal in $U, P$ is a GPEA isomorphic to the GPEA $G^{+}$(Example 2.4), and $U$ is a unitization of $P$. Note that $P$ is a total GPEA, hence $P$ is weakly commutative. To find the unitizing GPEA-automorphism $\gamma$, we observe that

$$
\begin{gathered}
(0, g)^{-}+(0, g)=(1, c) \Rightarrow(0, g)^{-}=(1, c-g), \text { and } \\
(0, g)^{--}+(1, c-g)=(1, c) \Rightarrow(0, g)^{--}=(0, c-(c-g)) .
\end{gathered}
$$

Thus for the element $(0, g) \in P, \gamma(0, g)=(0, c-(c-g))$.
We conclude this section with a presentation of an important class of PEAs, called kite algebras, constructed by A. Dvurečenskij in [12]. As we shall see, every kite algebra is a unitization of a GPEA. In our presentation, we shall find it convenient to make some small changes in the notation of [12]. First, for consistency with our notation above, we use $P$, rather than $E$, for the base GPEA from which a kite algebra is constructed. Second, for the bijections $\lambda$ and $\rho$ in [12], we change notation to reduce the number of occurrences of $\lambda^{-1}$ and $\rho^{-1}$. Third, for ease of comparison with Theorem 3.3 we replace $\bar{a}$ by $\eta a$. Fourth, we use $K$ rather than $K_{I}^{\lambda, \rho}$ for a kite algebra.

Thus, for the remainder of this section we assume that $(P ; \oplus, 0)$ is a GPEA, $I$ is a nonempty indexing set, and $\lambda, \rho: I \rightarrow I$ are bijections. We organize $P^{I}$ into a GPEA $\left(P^{I} ; \oplus_{P^{I}}, 0^{I}\right)$ where $\oplus_{P^{I}}$ is the obvious coordinatewise orthosummation and $0^{I}$ is the family in $P^{I}$ all the elements of which are 0 . Let $P^{\eta}$ be a set disjoint from $P$ and with the same cardinality as $P$ and let $\eta: P \rightarrow P^{\eta}$ be a bijection.

Evidently, $P$ is total (respectively, weakly commutative) iff $P^{I}$ is total (respectively, weakly commutative).

The following conditions on the bijections $\lambda, \rho: I \rightarrow I$ were introduced in [12] (but with $\lambda$ and $\rho$ replaced by $\lambda^{-1}$ and $\rho^{-1}$ ): For all $\left(a_{i}\right)_{i \in I},\left(b_{i}\right)_{i \in I} \in P^{I}$ and for all $i \in I$,
(KCI) $a_{\rho i} \oplus b_{i}$ exists in $P$ iff $b_{i} \oplus a_{\lambda i}$ exists in $P$, and
(KCII) $a_{\lambda i} \oplus b_{i}$ exists in $P$ iff $b_{i} \oplus a_{\rho i}$ exists in $P$.
Obviously, if $\lambda=\rho$, then conditions (KCI) and (KCII) are identical, and in this case it can be shown that they both hold iff $P$ is a total GPEA. On the other hand, if $P$ is a total GPEA, then (KCI) and(KCII) are automatically satisfied. In [12], it is assumed that the bijections $\lambda, \rho: I \rightarrow I$ satisfy both conditions (KCI) and (KCII). Later in Lemma 3.8 and Theorem 3.9, we shall assume (KCI), but not necessarily (KCII).
3.6 Definition. Noting that $P^{I}$ and $\left(P^{\eta}\right)^{I}$ are disjoint, we define

$$
K:=P^{I} \cup\left(P^{\eta}\right)^{I}
$$

and we define $0_{K} \in P^{I}$ and $1_{K} \in\left(P^{\eta}\right)^{I}$ by

$$
0_{K}:=0^{I} \text { and } 1_{K}=\left(\eta c_{i}\right)_{i \in I}, \text { where } c_{i}:=0 \text { for all } i \in I
$$

We organize $K$ into a partial algebra $\left(K ;+, 0_{K}, 1_{K}\right)$, where the partial binary operation + on $K$ is defined as follows: If $\left(a_{i}\right)_{i \in I},\left(b_{i}\right)_{i \in I} \in P^{I}$, then
(K1) $\left(a_{i}\right)_{i \in I}+\left(b_{i}\right)_{i \in I}:=\left(a_{i} \oplus b_{i}\right)_{i \in I}$ iff $a_{i} \oplus b_{i}$ exists in $P$ for all $i \in I$.
(K2) $\left(a_{i}\right)_{i \in I}+\left(\eta b_{i}\right)_{i \in I}:=\left(\eta\left(b_{i} \backslash a_{\lambda i}\right)\right)_{i \in I}$ iff $a_{\lambda i} \leq b_{i}$ for all $i \in I$.
(K3) $\left(\eta a_{i}\right)_{i \in I}+\left(b_{i}\right)_{i \in I}:=\left(\eta\left(b_{\rho i} / a_{i}\right)\right)_{i \in I}$ iff $b_{\rho i} \leq a_{i}$ for all $i \in I$.
(K4) $\left(\eta a_{i}\right)_{i \in I}+\left(\eta b_{i}\right)_{i \in I}$ is undefined.
If $K$ is a PEA, it is called the kite algebra determined by $P, I, \lambda$, and $\rho$.
By (K1), the restriction of + to the GPEA $P^{I}$ coincides with $\oplus_{P^{I}}$, whence, for all $\left(a_{i}\right)_{i \in I}$, we have $\left(a_{i}\right)_{i \in I}+0_{K}=0_{K}+\left(a_{i}\right)_{i \in I}$. Also, by (K2) and (K3), $0_{K}+\left(\eta a_{i}\right)_{i \in I}=\left(\eta a_{i}\right)_{i \in I}+0_{K}=\left(\eta a_{i}\right)_{i \in I}$. Furthermore, by (K2) and (K3), $\left(a_{i}\right)_{i \in I}+1_{K}$ is defined iff $1_{K}+\left(a_{i}\right)_{i \in I}$ is defined iff $\left(a_{i}\right)_{i \in I}=0_{K}$, whereas by (K4), $\left(\eta a_{i}\right)_{i \in I}+1_{K}$ and $1_{K}+\left(\eta a_{i}\right)_{i \in I}$ are undefined. If $K$ is a kite algebra, it is obviously a (binary) unitization of $P^{I}$.
3.7 Definition. Define $\gamma: P^{I} \rightarrow P^{I}$ by

$$
\gamma\left(\left(a_{i}\right)_{i \in I}\right)=\left(a_{\rho \lambda^{-1} i}\right)_{i \in I} \text { for all }\left(a_{i}\right)_{i \in I} \in P^{I} .
$$

3.8 Lemma. (KCI) holds iff $\gamma$ is a unitizing GPEA-automorphism on $P^{I}$.

Proof. Clearly, $\gamma$ is a GPEA-automorphism on $P^{I}$. Assume that (KCI) holds. Replacing $i$ by $\lambda^{-1} i$ in (KCI), we find that, for all $\left(a_{i}\right)_{i \in I},\left(b_{i}\right)_{i \in I} \in P^{I}$ and for all $i \in I$,

$$
\begin{equation*}
a_{\rho \lambda^{-1} i} \oplus b_{\lambda^{-1} i} \text { exists in } P \text { iff } b_{\lambda^{-1} i} \oplus a_{i} \text { exists in } P . \tag{1}
\end{equation*}
$$

Now, given any $\left(c_{i}\right)_{i \in I} \in P^{I}$, we let $b_{i}:=c_{\lambda i}$ in (11), so that $b_{\lambda^{-1}}=c_{i}$, and we have, for all $\left(a_{i}\right)_{i \in I},\left(c_{i}\right)_{i \in I} \in P^{I}$ and for all $i \in I$,

$$
\begin{equation*}
a_{\rho \lambda^{-1} i} \oplus c_{i} \text { exists in } P \text { iff } c_{i} \oplus a_{i} \text { exists in } P . \tag{2}
\end{equation*}
$$

Therefore, $\gamma$ is a unitizing GPEA-automorphism on $P^{I}$. Conversely, if (2) holds, then by first replacing $i$ by $\lambda i$, then putting $b_{i}:=c_{\lambda i}$, we arrive back at (KCI).

Suppose that (KCI) holds, so that, by Lemma 3.8, $\gamma$ is a unitizing GPEAautomorphism on $P^{I}$. Therefore, we can construct the $\gamma$-unitization $U$ of $P^{I}$ as per Theorem 3.3. To do this, we begin by putting $\left(P^{I}\right)^{\eta}:=\left(P^{\eta}\right)^{I}$ and we choose for the bijection from $P^{I}$ to $\left(P^{I}\right)^{\eta}$ the mapping-also denoted by $\eta$ defined by $\eta\left(a_{i}\right)_{i \in I}:=\left(\eta a_{i}\right)_{i \in I}$ for all $\left(a_{i}\right)_{i \in I} \in P^{I}$. (This dual use of $\eta$ should not cause any confusion.) Then, as a set, the $\gamma$-unification $U=P^{I} \cup\left(P^{I}\right)^{\eta}$ of $P^{I}$ is the same as $K$. However, as per Theorem 3.3 and the definition of $\gamma, U$ is organized into a PEA $\left(U ;+_{U}, 0_{U}, 1_{U}\right)$ as follows: $0_{U}:=0_{K}, 1_{U}:=1_{K}$, and for $\left(a_{i}\right)_{i \in I},\left(b_{i}\right)_{i \in I}$ in $P^{I}$ :
(U1) $\left(a_{i}\right)_{i \in I}+_{U}\left(b_{i}\right)_{i \in I}:=\left(a_{i} \oplus b_{i}\right)_{i \in I}$ iff $a_{i} \oplus b_{i}$ is defined in $P$ for all $i \in I$.
(U2) $\left(a_{i}\right)_{i \in I}+_{U}\left(\eta b_{i}\right)_{i \in I}:=\left(\eta\left(b_{i} \backslash a_{i}\right)\right)_{i \in I}$ iff $a_{i} \leq b_{i}$ for all $i \in I$.
(U3) $\left(\eta a_{i}\right)_{i \in I}+_{U}\left(b_{i}\right)_{i \in I}:=\left(\eta\left(b_{\rho \lambda^{-1} i} / a_{i}\right)\right)_{i \in I}$ iff $b_{\rho \lambda^{-1} i} \leq a_{i}$ for all $i \in I$.
(U4) $\left(\eta a_{i}\right)_{i \in I}+_{U}\left(\eta b_{i}\right)_{i \in I}$ is undefined.
According to (U1), the restriction of $+_{U}$ to $P^{I}$ coincides with $\oplus_{P^{I}}$, whence it also coincides with the restriction of + to $P^{I}$.
3.9 Theorem. Suppose that (KCI) holds. Then the mapping $\phi: U \rightarrow K$ defined for all $\left(a_{i}\right)_{i \in I} \in P^{I}$ by

$$
\phi\left(a_{i}\right)_{i \in I}:=\left(a_{i}\right)_{i \in I} \text { and } \phi\left(\left(\eta a_{i}\right)_{i \in I}\right)=\left(\eta a_{\lambda i}\right)_{i \in I}
$$

is an isomorphism of the PEA $\left(U ;+_{U}, 0_{U}, 1_{U}\right)$ onto the partial algebra ( $K$; $\left.+, 0_{K}, 1_{K}\right)$ such that the restriction of $\phi$ to $P^{I}$ is the identity mapping. Therefore, $\left(K ;+, 0_{K}, 1_{K}\right)$ is a PEA, $K$ is a kite algebra, and $\phi: U \rightarrow K$ is a PEAisomorphism. Furthermore, if $\psi: U \rightarrow K$ is any PEA-morphism such that the restriction of $\psi$ to $P^{I}$ is the identity mapping, then $\psi=\phi$.

Proof. Obviously, the restriction of $\phi$ to $P^{I}$ is the identity mapping, $\phi 0_{U}=$ $0_{K}, \phi 1_{U}=1_{K}$, and $\phi: U \rightarrow K$ is a bijection. In fact, for all $\left(a_{i}\right)_{i \in I} \in P^{I}$, $\phi^{-1}\left(a_{i}\right)_{i \in I}:=\left(a_{i}\right)_{i \in I}$ and $\phi^{-1}\left(\left(\eta a_{i}\right)_{i \in I}\right)=\left(\eta a_{\lambda^{-1}}\right)_{i \in I}$. To complete the proof that $\phi: U \rightarrow K$ is a PEA-isomorphism, we have to prove that
(i) If $p, q \in U$ and $p+{ }_{U} q$ is defined in $U$, then $\phi p+\phi q$ is defined in $K$ and $\phi\left(p+_{U} q\right)=\phi p+\phi q$ and
(ii) If $s, t \in K$ and $s+t$ is defined in $K$, then $\phi^{-1} s+_{U} \phi^{-1} t$ is defined in $U$ and $\phi^{-1}(s+t)=\phi^{-1} s+_{U} \phi^{-1} t$.

To prove condition (i), assume that $p, q \in U$. There are only two nontrivial cases to consider.
Case 1: $p=\left(a_{i}\right)_{i \in I} \in P^{I}$ and $q=\left(\eta b_{i}\right)_{i \in I} \in\left(P^{\eta}\right)^{I}$.
Case 2: $p=\left(\eta a_{i}\right)_{i \in I} \in\left(P^{\eta}\right)^{I}$ and $q=\left(b_{i}\right)_{i \in I} \in P^{I}$.
In Case 1, suppose that $p+_{U} q$ is defined in $U$. Then by (U2), $a_{i} \leq b_{i}$, whence $a_{\lambda i} \leq b_{\lambda i}$ for all $i \in I$, and

$$
\begin{gathered}
p+_{U} q=\left(\eta\left(b_{i} \backslash a_{i}\right)\right)_{i \in I} \text { for all } i \in I \text {, so } \\
\phi p=\left(a_{i}\right)_{i \in I}, \phi q=\left(\eta b_{\lambda i}\right)_{i \in I}, \phi\left(p+_{U} q\right)=\left(\eta\left(b_{\lambda i} \backslash a_{\lambda i}\right)\right)_{i \in I}
\end{gathered}
$$

and by (K2),

$$
\phi p+\phi q=\left(a_{i}\right)_{i \in I}+\left(\eta b_{\lambda i}\right)_{i \in I}:=\left(\eta\left(b_{\lambda i} \backslash a_{\lambda i}\right)_{i \in I}=\phi\left(p+_{U} q\right)\right.
$$

In Case 2, suppose that $p+_{U} q$ is defined in $U$. Then by (U3), $b_{\rho \lambda^{-1} i} \leq a_{i}$, whence $a_{\rho i} \leq b_{\lambda i}$ for all $i \in I$, and

$$
\begin{gathered}
p+_{U} q=\left(\eta\left(b_{\rho \lambda^{-1} i} / a_{i}\right)_{i \in I} \text { for all } i \in I,\right. \text { so } \\
\phi p=\left(\eta a_{\lambda i}\right)_{i \in I}, \phi q=\left(b_{i}\right)_{i \in I}, \phi\left(p+_{U} q\right)=\left(\eta\left(b_{\rho i} / a_{\lambda i}\right)\right)_{i \in I}
\end{gathered}
$$

and by (K3),

$$
\phi p+\phi q=\left(\eta a_{\lambda i}\right)_{i \in I}+\left(b_{i}\right)_{i \in I}=\left(\eta\left(b_{\rho i} / a_{\lambda i}\right)_{i \in I}=\phi\left(p+_{U} q\right),\right.
$$

completing the proof of (i).
The proof of (ii) is a straightforward computation similar to the proof of (i), and is therefore omitted. Thus, we may conclude that $\phi$ is an isomorphism of $U$ onto $K$ that reduces to the identity on $P^{I}$; hence $K$ is a PEA.

Finally, suppose that $\psi: U \rightarrow K$ is any PEA-morphism such that the restriction of $\psi$ to $P^{I}$ is the identity mapping. According to (U4) with $b_{i}=a_{i}$ for all $i \in I$, for all $\left(a_{i}\right)_{i \in I} \in P^{I}$,

$$
\left(a_{i}\right)_{i \in I}+_{U}\left(\eta a_{i}\right)_{i \in I}=1_{U}, \text { whence }\left(a_{i}\right)_{i \in I}+\psi\left(\eta a_{i}\right)_{i \in I}=1_{K} .
$$

But by (K2), we also have $\left(a_{i}\right)_{i \in I}+\left(\eta a_{\lambda i}\right)_{i \in I}=1_{K}$, and it follows from cancellation that

$$
\psi\left(\eta a_{i}\right)_{i \in I}=\left(\eta a_{\lambda i}\right)_{i \in I}=\phi\left(\eta a_{i}\right)_{i \in I}
$$

Therefore, $\psi=\phi$.
The proof of the following corollary is now straightforward.
3.10 Corollary. Suppose that (KCI) holds. Then, for all $\left(a_{i}\right)_{i \in I}$, the left and right negations on the kite algebra $K$ are as follows:
(LN) $\left(\left(a_{i}\right)_{i \in I}\right)^{-}=\left(\eta a_{\rho i}\right)_{i \in I}$ and $\left(\left(\eta a_{i}\right)_{i \in I}\right)^{-}=\left(a_{\lambda^{-1} i}\right)_{i \in I}$,
$(\mathrm{RN})\left(\left(a_{i}\right)_{i \in I}\right)^{\sim}=\left(\eta a_{\lambda i}\right)_{i \in I}$ and $\left(\left(\eta a_{i}\right)_{i \in I}\right)^{\sim}=\left(a_{\rho^{-1} i}\right)_{i \in I}$,
and from (LN), we have

$$
\left(\left(a_{i}\right)_{i \in I}\right)^{--}=\left(a_{\rho \lambda^{-1}}\right)_{i \in I}=\gamma\left(a_{i}\right)_{i \in I} .
$$

Therefore, $\left(K ;+, 0_{K}, 1_{K}\right)$ is a (binary) unitization of $P^{I}$ with $\gamma$ as its unitizing PEA-automorphism.

If $G$ is an additively written po-group and if we put $P:=G^{+}$(Example (2.4), then $P$ is total and the PEA $K$ is what A. Dvurečenskij originally called a kite PEA in [11].

## 4 Congruences and ideals in GPEAs and in their unitizations

In what follows, we will need some facts about congruences and ideals in GPEAs.
4.1 Definition. [25, Definition 2.1] A binary relation $\sim$ on a GPEA $P$ is called a weak congruence iff it satisfies the following conditions:
(C1) $\sim$ is an equivalence relation.
(C2) If $a \oplus b$ and $a_{1} \oplus b_{1}$ both exist, $a \sim a_{1}$, and $b \sim b_{1}$, then $a \oplus b \sim a_{1} \oplus b_{1}$.
A weak congruence is a congruence iff it satisfies the following condition:
(C3) If $a \oplus b$ exists, then for any $a_{1} \sim a$ there is a $b_{1} \sim b$ such that $a_{1} \oplus b_{1}$ exists, and for any $b_{2} \sim b$ there is an $a_{2} \sim a$ such that $a_{2} \oplus b_{2}$ exists.

A congruence $\sim$ is called a c-congruence iff it satisfies the following condition:
(C4) If $a \sim b$ and either $a \oplus a_{1} \sim b \oplus b_{1}$ or $a_{1} \oplus a \sim b_{1} \oplus b$, then $a_{1} \sim b_{1}$.
A congruence $\sim$ is called a p-congruence iff it satisfies the following condition:
(C5) $a \oplus b \sim 0$ implies $a \sim b \sim 0$.
Let $\sim$ be a weak congruence. Denote by $[a]$ the congruence class containing $a \in P$, and let $P / \sim$ denote the set of all congruence classes (the quotient of $P$ with respect to $\sim$ ). We shall say that $[a] \oplus[b]$ exists iff there are $a_{1}, b_{1} \in P$ with $a_{1} \sim a, b_{1} \sim b$, such that $a_{1} \oplus b_{1}$ exists, in which case, $[a] \oplus[b]:=\left[a_{1} \oplus b_{1}\right]$.
4.2 Theorem. [25, Corollary 2.1] For a congruence ~on a GPEA $P$, the quotient $(P / \sim ; \oplus,[0])$ is a GPEA iff $\sim$ is both a c-congruence and a pcongruence.

If $P$ is a PEA, then every congruence on $P$ is a c-congruence and pcongruence, as can be seen from the following theorem.
4.3 Theorem. [18] A weak congruence $\sim$ on a PEA $P$ is a congruence iff the following conditions hold:
(C4') If $a \sim b$, then $a^{\sim} \sim b^{\sim}$ and $a^{-} \sim b^{-}$.
$\left(\mathrm{C} 5^{\prime}\right) a \sim b \oplus c$ implies that there are $a_{1}, a_{2} \in P$, such that $a_{1} \sim b, a_{2} \sim c$ and $a=a_{1} \oplus a_{2}$.

Moreover, ( $\mathrm{C} 4{ }^{\prime}$ ) is equivalent to ( C 4 ), and ( C 5 ) implies (C5).

Of course, if $P$ is a PEA and $\sim$ is a congruence on $P$, then $P / \sim$ is also a PEA.
4.4 Definition. An ideal $I$ in a GPEA $P$ is called an $R 1$-ideal iff the following condition holds:
(R1) If $i \in I, a, b \in P, a \oplus b$ exists, and $i \leq a \oplus b$, then there are $j, k \in I$ such that $j \leq a, k \leq b$ and $i \leq j \oplus k$.

An R1-ideal $I$ is called a Riesz ideal iff the following condition holds:
(R2) If $i \in I, a, b \in P$, and $i \leq a$, then (i) if ( $a \backslash i$ ) $\oplus b$ exists, then there is $j \in I$ such that $j \leq b$ and $a \oplus(j / b)$ exists, and (ii) if $b \oplus(i / a)$ exists, then there is $k \in I$ such that $k \leq b$ and $(b \backslash k) \oplus a$ exists.
4.5 Definition. For an ideal $I$ in a GPEA $P$, we define $a \sim_{I} b(a, b \in P)$ iff there exist $i, j \in I, i \leq a, j \leq b$ such that $a \backslash i=b \backslash j$.

Notice that if $I$ is a normal ideal in the GPEA $P$, then the condition in Definition 4.5 that $a \backslash i=b \backslash j$ with $i, j \in I$ is equivalent to the condition that $i / a=j / b$ (see [25, Remark 2.4]). Also, if $i \in I$ and $i \leq a \in P$, then $(a \backslash i) \backslash 0=a \backslash i$, so $a \backslash i \sim_{I} a$.
4.6 Theorem. [25, Theorem 2.3] If I is a normal R1-ideal in a GPEA P, then $\sim_{I}$ satisfies (C1), (C2) and (C5'). Moreover, $a \sim_{I} 0$ iff $a \in I$.
4.7 Theorem. [25, Corollary 2.2] If $I$ is a normal Riesz ideal in a GPEA $P$, then $\sim_{I}$ is a c-congruence and p-congruence, hence $P / \sim_{I}$ is a GPEA.

Recall that a poset $V$ is upward directed iff for any $a, b \in V$ there is $c \in V$ with $a, b \leq c$. Similarly $V$ is downward directed iff for any $a, b \in V$ there is $d \in V$ with $d \leq a, b$.
4.8 Theorem. [25, Proposition 2.2] In an upward directed GPEA P, an ideal I is a Riesz ideal iff $I$ is an R1-ideal.
4.9 Definition. We say that a congruence $\sim$ on a GPEA $P$ is a Riesz congruence iff it satisfies (C4), (C5') and the following condition:
(CR) If $a \sim b$, then there are $c, d \in P$, such that $c \leq a \leq d, c \leq b \leq d$, $a \backslash c \sim b \backslash c \sim 0$, and $d \backslash a \sim d \backslash b \sim 0$.
4.10 Theorem. [25, Lemma 2.4] If I is a normal Riesz ideal in an upward directed GPEA $P$, then $\sim_{I}$ is a Riesz congruence.
4.11 Theorem. Let $P$ be a GPEA. Then:
(i) A congruence $\sim$ on $P$ satisfying (C4), (C5') is a Riesz congruence iff every equivalence class is both downward and upward directed.
(ii) If $\sim$ is a Riesz congruence on $P$, then $I:=\{i \in P: i \sim 0\}$ is a normal Riesz ideal in $P$. Moreover, $\sim=\sim_{I}$.

Proof. (i) If $\sim$ is a Riesz congruence, then every equivalence class is upward and downward directed by [25, Proposition 2.8 (1)]. Conversely, assume that $\sim$ is a congruence such that every equivalence class is downward and upward directed and suppose that $a \sim b$. Then there exist $c, d$ such that $c \leq a \leq d$, $c \leq b \leq d$, and $c \sim a \sim b \sim d$. By (C4) we then obtain $a \backslash c \sim b \backslash c \sim 0$, and $d \backslash a \sim d \backslash b \sim 0$. Hence $\sim$ satisfies (CR).
(ii) If $x \in I$ and $y \leq x$, then $(x \backslash y) \oplus y \sim 0$. Since $\sim$ is a p-congruence, $y \sim 0$ and $y \in I$. If $i \sim 0, j \sim 0$ and $i \oplus j$ exists, then $i \oplus j \sim 0$. Thus $I$ is an ideal. Assume $i \in I, a, j \in P$ and $a \oplus i=j \oplus a$. Then $0 \oplus a \sim a \oplus i=j \oplus a$, and by (C4) we have $0 \sim j$, whence $j \in I$. Thus $I$ is a normal ideal.

The proof of the R1 property is the same as in the proof of [25, Lemma 2.3].

To prove R2, assume that $a, b \in P, i \in I$, and $(a \backslash i) \oplus b$ exists. Thus $a \backslash i \sim a$ and by (C3), there is $b_{1} \sim b$ such that $a \oplus b_{1}$ exists. Then there is $b_{2} \leq b, b_{1}$ with $b_{2} \sim b$. From this we get $b_{2}=j / b$ and $a \oplus b_{2}$ exists. Finally, from $b_{2}=j / b \sim b$ we get by (C4) that $j \sim 0$. To prove the remaining statement, assume first that $a \backslash i=b \backslash j$ with $i, j \in I$. Then by (C4), $a \backslash i \oplus i \sim b \backslash j \oplus j$, hence $a \sim_{I} b$ implies $a \sim b$. Conversely, assume that $a \sim b$. Then by (CR), there is $c \in P$ with $a \backslash c \sim b \backslash c \sim 0$, and with $i=a \backslash c \in I, j=b \backslash c \in I$ we have $i / a=c=j / b$, whence $a \sim_{I} b$.

From here on in this paper, we shall be considering the situation in which $P$ is a GPEA, $\gamma: P \rightarrow P$ is a unitizing GPEA-automorphism on $P$, and the PEA $U$ is the $\gamma$-unitization of $P$ as per Theorem 3.3. Thus, $P^{\eta}$ is disjoint from $P, U=P \cup P^{\eta}$, and $\eta: P \rightarrow P^{\eta}$ is a bijection. Moreover, for all $a \in P$, $\eta a=a^{\sim}$, $\gamma a=a^{--}$, and $\gamma^{-1} a=a^{\sim \sim}$. In what follows, we use $a, b, c, d$, and $e$, with or without subscripts, to denote elements of $P$.
4.12 Theorem. Let $P$ be a GPEA, and let $U$ be its $\gamma$-unitization. Then $P$ is a normal Riesz ideal in $U$ iff $P$ is upward directed.

Proof. As mentioned earlier, by Theorem [16, Theorem 3.3 (vii)] $P$ is a maximal proper normal ideal in $U$. Let $P$ be a Riesz ideal in $U$ and let $a, b \in P$. Then $b \leq 1_{U}=a+\eta a$ implies that there are $j, k \in P$ such that $j \leq a, k \leq \eta a$ and $b \leq j \oplus k$. But then $a \oplus k$ is an upper bound of both $a$ and $b$, hence $P$ is upward directed.

Conversely, assume that $P$ is upward directed. By Theorem 4.8, it suffices to check condition R1. If $i, a, b \in P, i \leq a \oplus b$, this is obvious. Let $i \leq$ $a+\eta b=\eta(b \backslash a)$. Then $(b \backslash a) \oplus i$ exists in $P$, and we choose $d \in P$ such that $(b \backslash a) \oplus i \leq d, b \leq d$. Then $i \leq(b \backslash a) / d=a \oplus(b / d)$, where $b / d \leq \eta b$.

Now let $i \leq \eta a+b$. We have $\eta a+b=b+\eta a_{1}$ for some $a_{1} \in P$. From the previous part of this proof we have that $i \leq b \oplus c$ where $c \leq \eta a_{1}$. Let $c+u=\eta a_{1}$. Then $\eta a+b=b+\eta a_{1}=b+c+u=c_{1}+b+u=u_{1}+c_{1}+b$. From this, $u_{1}+c_{1}=\eta a$, hence $c_{1} \leq \eta a$, and $i \leq b \oplus c=c_{1} \oplus b$.
4.13 Definition. Let $P$ be a GPEA with a unitizing automorphism $\gamma$. An ideal $I$ in $P$ is called $\gamma$-closed (or simply a $\gamma$-ideal) iff for all $a \in P, a \in I$ $\Leftrightarrow \gamma a \in I$, or equivalently iff $I=\gamma I=\{\gamma i: i \in I\}$. A congruence $\sim$ on $P$ is called a $\gamma$-congruence iff $a \sim b \Leftrightarrow \gamma a \sim \gamma b$.
4.14 Theorem. Let $P$ be a GPEA, $\gamma$ a unitizing automorphism of $P$, and $U$ the $\gamma$-unitization of $P$. If I is a normal Riesz ideal in $U$, then its restriction $I \cap P$ to $P$ is a $\gamma$-closed normal Riesz ideal in $P$.

Proof. Taking into account Lemma 2.8 (i) and the fact that for any $a \in P$, $\gamma a=a^{--} \in P$, the proof is straightforward.

Now we shall consider the question of extending a congruence on the GPEA $P$ to a congruence on its $\gamma$-unitization $U$.
4.15 Definition. Let $U$ be the $\gamma$-unitization of a GPEA $P$ and let $\sim$ be a weak congruence on $P$. Since $U=P \cup P^{\eta}, P^{\eta}=U \backslash P$, and $\eta: P \rightarrow P^{\eta}$ is a bijection, we can, and do, define a binary relation $\sim^{*}$ on $U$ by

$$
a \sim^{*} b \text { iff } a \sim b \text { and } \eta a \sim^{*} \eta b \text { iff } a \sim b \text { for all } a, b \in P .
$$

If $a \in P$ and $x \in P^{\eta}=U \backslash P$, we understand that $a \not \chi^{*} x$.
4.16 Lemma. Let $U$ be the $\gamma$-unitization of the GPEA $P$, let $\sim$ be a $\gamma$ congruence on $P$, and let $a, b \in P$. Then: (i) $a^{-} \sim^{*} b^{-}$iff $a \sim b$. (ii) $a \sim b^{--}$iff $a^{\sim \sim} \sim b$ iff $a^{\sim} \sim^{*} b^{-}$.

Proof. Assume the hypotheses of the lemma. (i) $a^{-} \sim^{*} b^{-}$iff $a^{--\sim} \sim^{*} b^{--\sim}$ iff $(\gamma a)^{\sim} \sim^{*}(\gamma b)^{\sim}$ iff $(\gamma a) \sim(\gamma b)$ iff $a \sim b$.
(ii) $a \sim b^{--}$iff $a^{\sim \sim--} \sim b^{--}$iff $\gamma\left(a^{\sim \sim}\right) \sim \gamma b$ iff $a^{\sim \sim} \sim b$. Also, by (i), $a^{\sim \sim} \sim b$ iff $a^{\sim}=a^{\sim \sim-} \sim^{*} b^{-}$.
4.17 Theorem. Let $U$ be the $\gamma$-unitization of the GPEA $P$ and let $\sim$ be a congruence on $P$. Then $\sim^{*}$ is a congruence on $U$ iff $\sim$ is a $\gamma$-congruence that satisfies ( C 4 ) and ( C 5 ) .

Proof. Assume that $\sim$ is a $\gamma$-congruence on $P$ that satisfies (C4) and (C5'). Evidently, the restriction of $\sim^{*}$ to $P$ as well as to $P^{\eta}$ is an equivalence relation; hence $\sim^{*}$ is an equivalence relation on $U=P \cup P^{\eta}$, and we have ( C 1 ).

To prove (C2), we concentrate only on the two nontrivial cases corresponding to parts (2) and (3) of Theorem 3.3. Thus let $a \sim^{*} a_{1}, b^{\sim} \sim^{*} b_{1}^{\sim}$ and suppose that $a+b^{\sim}$ and $a_{1}+b_{1}^{\sim}$ exist. Then $a \leq b$ and $a_{1} \leq b_{1}$, so we have $(b \backslash a) \oplus a=b \sim b_{1}=\left(b_{1} \backslash a_{1}\right) \oplus a_{1}$. Therefore, by (C4), $b \backslash a \sim b_{1} \backslash a_{1}$, whence $a+b^{\sim}=(b \backslash a)^{\sim} \sim^{*}\left(b_{1} \backslash a_{1}\right)^{\sim}=a_{1}+b_{1}^{\sim}$ by Theorem 3.3 (2).

For the remaining case, let $a^{\sim} \sim^{*} a_{1}^{\sim}, b \sim^{*} b_{1}$ and suppose that $a^{\sim}+b$ and $a_{1}^{\sim}+b_{1}$ exist. Then $\gamma b=b^{--} \leq a$, so we have $\gamma b \oplus(\gamma b / a)=a \sim$ $a_{1}=\gamma b_{1} \oplus\left(\gamma b_{1} / a_{1}\right)$. Since $b \sim b_{1}$, it follows that $\gamma b \sim \gamma b_{1}$ and (C4) implies that $\gamma b / a \sim \gamma b_{1} / a_{1}$. But then, $(\gamma b / a)^{\sim} \sim^{*}\left(\gamma b_{1} / a_{1}\right)^{\sim}$, and we infer from Theorem 3.3 (3) that $a^{\sim}+b \sim^{*} a_{1}^{\sim} \oplus b_{1}$. Thus $\sim^{*}$ satisfies (C2).

To prove (C3), we again concentrate only on the nontrivial cases corresponding to parts (2) and (3) of Theorem 3.3. Thus, on the one hand, assume that $a+b^{\sim}$ exists and let $a_{1} \in P$ with $a_{1} \sim a$. Then $a \leq b$ and $b=(b \backslash a) \oplus a$. Therefore by (C3) (in $P$ ) there exists $c \in P$ such that $c \sim b \backslash a$ and $c \oplus a_{1}$ exists. By (C2) (in $P), b_{1}:=c \oplus a_{1} \sim b$, whence $b_{1}^{\sim} \sim^{*} b^{\sim}$ and, since $a_{1} \leq b_{1}$, it follows that $a_{1}+b_{1}^{\sim}$ exists.

Continuing to assume that $a+b^{\sim}$ exists, we let $b_{2} \in P$ with $b_{2}^{\sim} \sim^{*} b^{\sim}$, whence $b_{2} \sim b=(b \backslash a) \oplus a$. But then, by (C5'), there exist $c, a_{2} \in P$ such that $b_{2}=c \oplus a_{2}$ where $c \sim b \backslash a$ and $a_{2} \sim a$. Since $a_{2} \leq b_{2}$, it follows that $a_{2}+b_{2}^{\sim}$ exists.

On the other hand, assume that $a^{\sim}+b$ exists and let $a_{1} \in P$ with $a_{1}^{\sim} \sim^{*}$ $a^{\sim}$. Then $\gamma b \leq a$ and $a_{1} \sim a=\gamma b \oplus(\gamma b / a)$. But then, by (C5'), there exist $c, d \in P$ such that $a_{1}=c \oplus d$ where $c \sim \gamma b=b^{--}$and $d \sim \gamma b / a$. But then, by Lemma 4.16 (ii), $b_{1}:=c^{\sim \sim} \sim b$; moreover, since $c \leq a_{1}$, we have $a_{1}^{\sim} \leq c^{\sim}$, so $a_{1}^{\sim}+c^{\sim \sim}=a_{1}^{\sim}+b_{1}$ exists.

Continuing to assume that $a^{\sim}+b$ exists, we let $b_{2} \in P$ with $b_{2} \sim^{*} b$, i.e., $b_{2} \sim b$. Then $\gamma b_{2} \sim \gamma b$, and since $\gamma b \oplus(\gamma b / a)=a$ exists, (C3) implies that
there exists $c \in P$ such that $c \sim \gamma b / a$ and $\gamma b_{2} \oplus c$ exists. But then, by (C2), $a_{2}:=\gamma b_{2} \oplus c \sim \gamma b \oplus(\gamma b / a)=a$, so $a_{2}^{\sim} \sim^{*} a^{\sim}$. Since $b_{2}^{--}=\gamma b_{2} \leq a_{2}$, we have $a_{2}^{\sim} \leq b_{2}^{--\sim}=b_{2}^{-}$, whence $\gamma b_{2}=b_{2}^{--} \leq a_{2}^{\sim-}=a$, and it follows that $a_{2}^{\sim}+b_{2}$ exists. Thus $\sim^{*}$ satisfies (C3), so it is a congruence on $U$.

Conversely, if $\sim^{*}$ is a congruence on the PEA $U$, then by Theorem 4.3, $\sim^{*}$ satisfies $\left(\mathrm{C} 4^{\prime}\right),(\mathrm{C} 4)$, and ( $\mathrm{C} 5^{\prime}$ ), and thus, of course, $\sim$ satisfies $(\mathrm{C} 4)$ and (C5 $5^{\prime}$ ) on $P$. Also, since $\sim^{*}$ satisfies (C4'), it follows that $\sim$ is a $\gamma$-congruence on $P$.
4.18 Proposition. Let I be a normal (R1)- $\gamma$-ideal in the GPEA $P$, put $\sim:=\sim_{I}$, and let $U$ be the $\gamma$-unitization of $P$. Then the following conditions are mutually equivalent: (i) $\sim^{*}$ satisfies (C3) on $U$. (ii) I is a normal Riesz $\gamma$-ideal in $P$. (iii) $\sim$ is a $\gamma$-congruence on $P$ that satisfies $(\mathrm{C} 4)$ and $\left(\mathrm{C} 5^{\prime}\right)$. (iv) $\sim^{*}$ is a congruence on $U$ that satisfies ( C 4$),\left(\mathrm{C} 4^{\prime}\right),\left(\mathrm{C} 5^{\prime}\right)$, and ( C 5 ).

Proof. By Theorem 4.6, $\sim=\sim_{I}$ is a weak congruence on $P$ satisfying (C5'). (i) $\Rightarrow$ (ii). Suppose that $\sim^{*}$ satisfies (C3) on $U$. Then also $\sim$ satisfies (C3) on $P$ and so it is a congruence on $P$. Assume that $i \in I$ with $i \leq a \in P$ and that $(a \backslash i) \oplus b$ exists for some $b \in P$. Then we have $a \backslash i \sim a$ and so by (C3) there is $b_{1} \in P$ such that $a \oplus b_{1}$ exists, whence $b_{1} \leq a^{\sim}$, and $b \sim b_{1}$. Since $I$ is normal and $b \sim b_{1}$, there exist $j, k \in I$ with $j / b=k / b_{1}$. Thus we have $j / b=k / b_{1} \leq k \oplus\left(k / b_{1}\right)=b_{1} \leq a^{\sim}$, and it follows that $a \oplus j / b$ exists. This proves part (i) of condition (R2) in Definition 4.4, a similar argument proves part (ii), whence $I$ is also an (R2)-ideal, and we have (ii).
(ii) $\Rightarrow$ (iii). Suppose that $I$ is a normal Riesz $\gamma$-ideal in $P$. Then by Theorems 4.6 and 4.7, $\sim=\sim_{I}$ is a $\gamma$-congruence on $P$ satisfying (C4) and (C5').
(iii) $\Rightarrow$ (iv). If $\sim$ is a $\gamma$-congruence on $P$ satisfying (C4) and (C5'), then by Theorem 4.17, $\sim^{*}$ is a congruence on $U$. By Theorem 4.3, $\sim^{*}$ satisfies $\left(\mathrm{C} 4^{\prime}\right)$, (C4), (C5'), and (C5).
(iv) $\Rightarrow$ (i). A congruence $\sim^{*}$ on $U$ satisfies (C3) on $U$.
4.19 Theorem. Let I be a normal Riesz $\gamma$-ideal in the GPEA $P$, put $\sim:=\sim_{I}$, and let $U$ be the $\gamma$-unitization of $P$. Then $\sim^{*}$ is a Riesz congruence on $U$ iff $\sim$ satisfies the following condition on $P$ :
(GCR) $a \sim b \Rightarrow \exists i, j \in I: i \oplus a=j \oplus b$, or equivalently $\exists k, \ell \in I$ : $a \oplus k=b \oplus \ell$.

Proof. Assume the hypotheses of the theorem. By Proposition 4.18, $\sim$ is a $\gamma$-congruence on $P, \sim^{*}$ is a congruence on $U$, and both satisfy (C4) and (C5'). Also, by Theorem4.6, $I=\{a \in P: a \sim 0\}$. Thus, if $a \in P, i \in I$, and $i \oplus a$ exists, then by (C2), $a=0 \oplus a \sim i \oplus a$, and therefore $a^{\sim} \sim^{*}(i \oplus a)^{\sim}$.

Assume that $\sim$ satisfies condition (GCR). Consider the following condition on $\sim^{*}$ : For all $u, v \in U$,
$(\mathrm{CR}) \quad u \sim^{*} v \Rightarrow \exists s, t \in U: u+s \sim^{*} v+s \sim^{*} 1$, and $t+u \sim^{*} t+v \sim^{*} 1$.
According to [25, Proposition 3.1], on the PEA $U$, condition ( $\mathrm{CR}^{\prime}$ ) is equivalent to condition (CR). We shall show that $\sim^{*}$ satisfies condition $\left(\mathrm{CR}^{\prime}\right)$.

Let $a, b \in P$. On the one hand, suppose that $a \sim^{*} b$. Then $a \sim b$, whence by (GCR), there exist $i, j, k, \ell \in I$ with $i \oplus a=j \oplus b$ and $a \oplus k=b \oplus \ell$. Put $s:=(i \oplus a)^{\sim}=(j \oplus b)^{\sim}$ and $t:=(a \oplus k)^{-}=(b \oplus \ell)^{-}$. Thus, $t=$ $(\gamma(a \oplus k))^{\sim}=(\gamma(b \oplus \ell))^{\sim}$. Since $i \oplus a \sim a$, we have $s \sim^{*} a^{\sim} \sim^{*} b^{\sim}$ and similarly, $t \sim^{*} a^{-} \sim^{*} b^{-}$. Also, $a \leq i \oplus a$, and it follows that $a+s$ exists and $a+s \sim^{*} a+a^{\sim}=1$. Similar computations show that $b+s \sim^{*} 1, t+a \sim^{*} 1$, and $t+b \sim^{*} 1$.

On the other hand, suppose that $a^{\sim} \sim^{*} b^{\sim}$. Then $a \sim_{I} b$, whence there exist $i, j \in I$ with $a \backslash i=b \backslash j$. Put $s:=(a \backslash i)^{\sim \sim}=(b \backslash j)^{\sim \sim}$ and $t:=a \backslash i=$ $b \backslash j$. We have $t=a \backslash i \sim a$, whence $s \sim a^{\sim \sim}$. Also $\gamma s=\gamma\left((a \backslash i)^{\sim \sim}\right)=$ $a \backslash i \leq a$, and it follows that $a^{\sim}+s$ exists and $a^{\sim}+s \sim^{*} a^{\sim}+a^{\sim \sim}=1$. Similar computations show that $b^{\sim}+s \sim^{*} 1, t+a^{\sim} \sim^{*} 1$, and $t+b^{\sim} \sim^{*} 1$. Thus, $\sim^{*}$ satisfies ( $\mathrm{CR}^{\prime}$ ), whence it satisfies (CR), and consequently it is a Riesz congruence on $U$.

Conversely, let $\sim^{*}$ be a Riesz congruence on $U$ and suppose that $a, b \in P$ with $a \sim b$. Then by (CR), there exists $d \in U$ with $d \backslash a \sim^{*} d \backslash b \sim^{*} 0$. Thus $(d \backslash a)+a=d$, and since $0+a=a$ with $0 \sim^{*} d \backslash a$ and $a \sim^{*} a$, it follows that $d \sim^{*} a \in P$; hence $d \in P, d \sim a \sim b$, and $d \backslash a \sim_{I} d \backslash b$ in $P$. Therefore, there exist $i, j \in I$ such that $d \backslash(i \oplus a)=(d \backslash a) \backslash i=(d \backslash b) \backslash j=d \backslash(j \oplus b)$. Consequently, $i \oplus a=j \oplus b$ and (GCR) holds.
4.20 Theorem. If $I$ is a normal Riesz $\gamma$-ideal in the GPEA P, then I is a normal Riesz ideal in the $\gamma$-unitization $U$ of $P$ iff $\sim_{I}$ satisfies the (GCR) condition on $P$.

Proof. If $I$ is a normal Riesz $\gamma$-ideal in $P$ and $\sim_{I}$ satisfies the (GCR) condition, then by Theorem 4.19 $\sim_{I}^{*}$ is a Riesz congruence on $U$. Since $I$ is the zero class of this congruence, it is a normal Riesz ideal in $U$ (Theorem 4.11 (ii)).

Conversely, suppose that $I$ is a normal Riesz ideal in $U$ and let $\sim_{U, I}$ be the relation induced on $U$ by $I$ as per Definition 4.5. Clearly, the restriction of $\sim_{U, I}$ to $P$ coincides with $\sim_{I}$. Also, since $U$ is upward directed, $\sim_{U, I}$ is a Riesz congruence on $U$ by Theorem 4.10. Thus, by Theorem 4.3, $\sim_{U, I}$ satisfies (C4) and ( C 5 ) on $U$, whence by Theorem 4.11 (i), all equivalence classes in $U$ modulo $\sim_{U, I}$ are upward directed. To prove that $\sim_{I}$ satisfies GCR, assume that $a, b \in P$ with $a \sim_{I} b$. Then $a \sim_{U, I} b$, so there exists $r \in U$ such that $a, b \leq r$ and $a \sim_{U, I} b \sim_{U, I} r$. As $a, b \leq r$, there exist $p, q \in U$ with $a+p=b+q=r$. Moreover, as $a \sim_{U, I} r$, there exist $i, j \in I$ such that $i \leq a, j \leq r=a+p$, and $a \backslash i=r \backslash j=(a+p) \backslash j$. Therefore, $(a \backslash i)+j=a+p=(a \backslash i)+i+p$, whence $j=i+p$. But then, $p \leq j \in I$, so $p \in I$. A similar argument shows that $q \in I$, and therefore GCR holds.

We remark that, in the proof of Theorem 4.20, $\sim_{U, I}=\sim_{I}^{*}$. To prove this, it will be sufficient to show that, for all $a, b \in P, a \sim_{I} b \Leftrightarrow a^{\sim} \sim_{U, I} b^{\sim}$ and that $a \sim_{U, I} b^{\sim}$ cannot hold. But by Theorem 4.3, $a \sim_{I} b \Rightarrow a \sim_{U, I} b \Rightarrow$ $a^{\sim} \sim_{U, I} b^{\sim}$. Conversely, by the same theorem, $a^{\sim} \sim_{U, I} b^{\sim} \Rightarrow a=a^{\sim-} \sim_{U, I}$ $b^{\sim-}=b \Rightarrow a \sim_{I} b$. Moreover, if $a \sim_{U, I} b^{\sim}$, then there exist $i, j \in I$ such that $a \backslash i=b^{\sim} \backslash j$, which yields $(a \backslash i) \oplus j=a \backslash i+j=b^{\sim} \notin P$, contradicting the definition of $U$.
4.21 Corollary. In an upward directed GPEA P, a normal Riesz $\gamma$-ideal I in $P$ is also a normal Riesz ideal in the $\gamma$-unitization $U$ of $P$.

Proof. By the previous theorem, it is sufficient to show that $\sim_{I}$ satisfies (GCR). So assume that $a \sim_{I} b$. Since $P$ is upward directed, there exists $c \in P$ with $a, b \leq c$, and by (C4), $c \backslash a \sim_{I} c \backslash b$. Thus there are $i, j \in I$ such that $c \backslash(i \oplus a)=(c \backslash a) \backslash i=(c \backslash b) \backslash j=c \backslash(j \oplus b)$; hence $i \oplus a=j \oplus b$.
4.22 Theorem. Let $\sim$ be a $\gamma$-congruence on a GPEA $P$ which satisfies conditions ( C 4$)$ and $\left(\mathrm{C} 5\right.$ ') . Let $\sim^{*}$ be the extension of $\sim$ in the $\gamma$-unitization $U$ of $P$. Define $\widetilde{\gamma}: P / \sim \rightarrow P / \sim$ by $\widetilde{\gamma}[a]=[\gamma a]$. Then $\widetilde{\gamma}$ is a unitizing automorphism in $P / \sim$, and $U / \sim^{*}$ is the $\widetilde{\gamma}$-unitization of $P / \sim$. In particular, if I is a normal Riesz $\gamma$-ideal on $P$ such that $\sim_{I}$ satisfies (GCR), then the $\widetilde{\gamma}$-unitization of $P / I$ is $U / I$.

Proof. By Theorem 4.17, $\sim^{*}$ is a congruence on $U$ and so $U / \sim^{*}$ is a PEA. Clearly, if $s, t \in U$ and $s \in[t]$ (i.e. $s \sim^{*} t$ ), then either $s, t \in P$ or else $s, t \in U \backslash P=: \eta P$; hence $[a] \cap[\eta b]=\emptyset$ for all $a, b \in P$. Consequently,
$U / \sim^{*}=P / \sim^{*} \cup \eta P / \sim^{*}$, where the sets on the right are disjoint and obviously of the same cardinality. Thus we may put $\eta[a]:=[\eta a]$.

We have $[a] \oplus[b] \in P / \sim$ for all $a, b \in P$ and for $a, b \in \eta P,[a] \oplus[b]$ does not exist. Also $[1]=[a]^{-} \oplus[a]=[a] \oplus[a]^{\sim}$ for any $a \in P$ and thus $[1] \notin P / \sim$. Therefore $U / \sim^{*}$ satisfies (U1)-(U3) and it is a unitization of $P / \sim$ according to Definition 3.1.

As $\sim$ is a $\gamma$-congruence, $\widetilde{\gamma}[a]=[\gamma a], a \in P$, is well defined. If $[a] \oplus[b]$ is defined, there are $a_{1} \sim a, b_{1} \sim b$ such that $[a] \oplus[b]=\left[a_{1} \oplus b_{1}\right]$. Then $\widetilde{\gamma}([a] \oplus[b])=\left[\gamma\left(a_{1} \oplus b_{1}\right)\right]=\left[\gamma a_{1}\right] \oplus\left[\gamma b_{1}\right]=\widetilde{\gamma}[a] \oplus \widetilde{\gamma}[b]$, hence $\widetilde{\gamma}$ is additive. Moreover, $\widetilde{\gamma}[a]=\widetilde{\gamma}[b]$ implies $\gamma a \sim \gamma b$, which in turn implies $a \sim b$, so that $[a]=[b]$. This shows that $\widetilde{\gamma}$ is injective. To prove that $\widetilde{\gamma}$ is surjective, observe that for all $a \in P, a=\gamma\left(\gamma^{-1} a\right)$, hence $[a]=\widetilde{\gamma}\left[\gamma^{-1} a\right]$. This proves that $\widetilde{\gamma}$ is an automorphism of $P / \sim$. In addition, if $[a] \oplus[b]$ exists, then there are $a_{1} \sim a$ and $b_{1} \sim b$ such that $a_{1} \oplus b_{1}$ is defined in $P$. Now $a_{1} \oplus b_{1}$ is defined iff $\gamma b_{1} \oplus a_{1}$ is defined in $P$, whence $[a] \oplus[b]$ exists iff $\widetilde{\gamma}[b] \oplus[a]$ exists. This proves that $\widetilde{\gamma}$ is a unitizing automorphism of $P / \sim$.

In $U / \sim^{*}$ we have $[a]+\eta[b]$ is defined iff there are $a_{1}, b_{1}, a_{1} \sim a, b_{1} \sim b$ such that $a_{1}+\eta b_{1}=\eta\left(b_{1} \backslash a_{1}\right)$ is defined and then $[a]+\eta[b]=\eta\left[b_{1} \backslash a_{1}\right]=$ $\eta([b] \backslash[a])$. Similarly, $\eta[a]+[b]$ is defined iff there are $a_{1} \sim a, b_{1} \sim b$ such that $\eta a_{1}+b_{1}=\eta\left(\gamma b_{1} / a_{1}\right)$ is defined, and then $\eta[a]+[b]=\eta\left[\gamma b_{1} / a_{1}\right]=$ $\eta(\widetilde{\gamma}[b] /[a])$. This shows that $U / \sim^{*}$ is a $\widetilde{\gamma}$-unitization of $P / \sim$.
4.23 Example. [10, Example 2.3] Let $\mathbb{Z}$ be the group of integers and $G=$ $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. For any two elements of $G$ we define:

$$
(a, b, c)+(x, y, z)= \begin{cases}(a+x, b+y, c+z) & : x \text { is even } \\ (a+x, c+y, b+z) & : \quad x \text { is odd }\end{cases}
$$

and define $(a, b, c) \leq(x, y, z)$ iff $a<x$ or $a=x, b \leq y, c \leq z$. Then $(G,+, \leq)$ is a lattice-ordered group with strong unit $u=(1,0,0)$ and the interval $E:=G[(0,0,0),(1,0,0)]$ is a PEA.

The elements between $(0,0,0)$ and $(1,0,0)$ are of two kinds- $(0, a, b)$ and $(1, c, d)$-where $a, b \geq 0$ and $c, d \leq 0$. The two sets are of the same cardinality and $P:=\{(0, a, b) \in G: a, b \geq 0\}$ is a normal Riesz ideal of $E$. We also have $(0, b, c)^{\sim}=(1,-c,-b)$ and $(0, b, c)^{-}=(1,-b,-c)$, thus $\gamma(0, a, b)=$ $(0, a, b)^{--}=(0, b, a)$. With this $\gamma$ on $P$, which is a weakly commutative GPEA, we have a GPEA with a unitizing automorphism and its $\gamma$-unitization is exactly the $G[(0,0,0),(1,0,0)]$ that we started with.

We may also consider the identity as a unitizing automorphism on $P$ and then we obtain a unitization $G[(0,0,0),(1,0,0)]$ but with the + operation on $G$ defined by

$$
(a, b, c)+(x, y, z)=(a+x, b+y, c+z)
$$

for all $a, b, c, x, y, z \in \mathbb{Z}$.

## 5 The Riesz decomposition property

5.1 Definition. We say, that a GPEA $P$ satisfies
(i) the Riesz decomposition property ( $R D P$ for short), iff for all $a, b, c, d \in$ $P$ such that $a \oplus b=c \oplus d$, there are four elements $e_{11}, e_{12}, e_{21}, e_{22} \in P$ such that $a=e_{11} \oplus e_{12}, b=e_{21} \oplus e_{22}, c=e_{11} \oplus e_{21}$ and $d=e_{12} \oplus e_{22}$. We shall denote these four decompositions by the following table:

|  | c | d |
| :---: | :---: | :---: |
| a | $e_{11}$ | $e_{12}$ |
| b | $e_{21}$ | $e_{22}$ |

(ii) $\mathrm{RDP}_{1}$, if for the decomposition in (i) it moreover holds: if $f, g \in P$ are such that $f \leq e_{12}, g \leq e_{21}$, then $f$ and $g$ commute $(f \oplus g=g \oplus f)$.
(iii) $\mathrm{RDP}_{2}$, if for the decomposition in (i) it moreover holds that $e_{12} \wedge e_{21}=0$.
(iv) $\mathrm{RDP}_{0}$, if for $a, b, c \in P$ such that $a \leq b \oplus c$, there exists $b_{1}, c_{1} \in P$ such that $b_{1} \leq b, c_{1} \leq c$ and $a=b_{1} \oplus c_{1}$.

From the next example we see that, even in the commutative case, if $P$ has RDP, its unitization $U$ need not have it.
5.2 Example. [22, Example 4.1] Consider the GEA $P=\{0, a, b, c, a \oplus c$, $b \oplus c\}$ (see Fig. 1 below) and let the unitizing automorphism $\gamma$ be the identity (thus the unitization $U$ of $P$ is the "classical" unitization $\widehat{P}$ ). Then $P$ has $R D P$, but $U$ does not. Indeed, for example, in $U,(b \oplus c)^{\perp}=b^{\perp} \ominus c$ cannot be decomposed except for $b^{\perp} \ominus c=\left(b^{\perp} \ominus c\right) \oplus 0$. But $\left(b^{\perp} \ominus c\right) \oplus b=c^{\perp}=\left(a^{\perp} \ominus c\right) \oplus a$ and therefore, for these elements, there is no decomposition in the sense of $R D P$.


Fig. 1

In the following theorem, we show that if a GPEA $P$ has a total operation $\oplus$, then $P$ has any of the RDP properties iff its $\gamma$-unitization (with respect to any unitizing automorphism $\gamma$ ) has the corresponding RDP property. We do not know whether the above condition is also necessary.
5.3 Theorem. Let $P$ be a total GPEA and let $U$ be the unitization of $P$ by a unitizing GPEA-automorphism $\gamma$. Then $P$ has $R D P\left(R D P_{0}, R D P_{1}, R D P_{2}\right.$, respectively) iff $U$ has $R D P\left(R D P_{0}, R D P_{1}, R D P_{2}\right.$, respectively $)$.

Proof. Notice first, that, since $a \oplus b$ exists for all $a, b \in P$, then of course, $P$ is upward directed. Also, since $a+b=a \oplus b$ exists, then $b \leq a^{\sim}=\eta a$ for all $a, b \in P$.

It is clear, that if $U$ has $\mathrm{RDP}\left(\mathrm{RDP}_{1}, \mathrm{RDP}_{2}\right)$, then $P$ has $\mathrm{RDP}\left(\mathrm{RDP}_{1}\right.$, $\mathrm{RDP}_{2}$ ), because if the decomposition of $a \oplus b=c \oplus d$ is:

|  | c | d |
| :---: | :---: | :---: |
| a | $e_{11}$ | $e_{12}$ |
| b | $e_{21}$ | $e_{22}$ |

then $e_{11}, e_{12} \leq a$ and $e_{21}, e_{22} \leq b$ and so if $a, b, c, d \in P$, then $e_{11}, e_{12}, e_{21}$, $e_{22} \in P$, because $P$ is an ideal in $U$. Similarly, if $U$ has $\operatorname{RDP}_{0}$, so does $P$.

Now assume that RDP holds in $P$. Then we have to consider three cases:
(a) $\eta a+b=\eta c+d$ (where $a, b, c, d \in P)$. Then $\eta(\gamma b / a)=\eta(\gamma d / c)$, so $\gamma b / a=\gamma d / c$. Since $P$ is upward directed, there exists $e \in P$ with $e \geq a, c$ and therefore

$$
e \backslash(\gamma b / a)=e \backslash(\gamma d / c) \text {, i.e., }(e \backslash a) \oplus \gamma b=(e \backslash c) \oplus \gamma d
$$

where all these elements are in $P$, so by RDP we have a decomposition:

|  | $e \backslash c$ | $\gamma d$ |
| :---: | :---: | :---: |
| $e \backslash a$ | $e_{11}$ | $e_{12}$ |
| $\gamma b$ | $e_{21}$ | $e_{22}$ |

Also, as $(e \backslash a)+a$ exists, $e \backslash a \leq a^{-}$, whence $\gamma^{-1}(e \backslash a) \leq \eta a$, and likewise $\gamma^{-1}(e \backslash c) \leq \eta c$. Therefore

$$
\exists x, y \in U, x+\gamma^{-1}(e \backslash a)=\eta a \text { and } y+\gamma^{-1}(e \backslash c)=\eta c .
$$

But by assumption $\eta a+b=\eta c+d$, and therefore

$$
x+\gamma^{-1}(e \backslash a)+b=\eta a+b=\eta c+d=y+\gamma^{-1}(e \backslash c)+d .
$$

Moreover, from $(e \backslash a) \oplus \gamma b=(e \backslash c) \oplus \gamma d$, we deduce that

$$
\gamma^{-1}(e \backslash a)+b=\gamma^{-1}(e \backslash a) \oplus b=\gamma^{-1}(e \backslash c) \oplus d=\gamma^{-1}(e \backslash c)+d
$$

and it follows that $x=y$. Thus, $\eta a=x+\gamma^{-1}(e \backslash a), \eta c=x+\gamma^{-1}(e \backslash c)$, and we have

|  | $\eta c$ | $d$ |
| :---: | :---: | :---: |
| $\eta a$ | $x+\gamma^{-1} e_{11}$ | $\gamma^{-1} e_{12}$ |
| $b$ | $\gamma^{-1} e_{21}$ | $\gamma^{-1} e_{22}$ |

(b) $a+\eta b=c+\eta d$. Then $\eta(b \backslash a)=\eta(d \backslash c)$, so $b \backslash a=d \backslash c$. Again, there exists $e \in P$ with $e \geq b, d$, so that

$$
(b \backslash a) / e=(d \backslash c) / e \text {, i.e., } a \oplus(b / e)=c \oplus(d / e) .
$$

Thus, since RDP holds in $P$, we get

|  | $c$ | $d / e$ |
| :---: | :---: | :---: |
| $a$ | $e_{11}$ | $e_{12}$ |
| $b / e$ | $e_{21}$ | $e_{22}$ |

As $b \oplus(b / e)$ exists, we have $b / e \leq b^{\sim}=\eta b$ and thus $\eta b=(b / e)+x$ for some $x \in U$. Similarly, $\eta d=(d / e)+y$ for some $y \in U$. Then

$$
a+(b / e)+x=a+\eta b=c+\eta d=c+(d / e)+y
$$

and, since $a+(b / e)=c+(d / e)$, it follows that $x=y$. Consequently, we obtain the decomposition:

|  | $c$ | $(d / e)+x$ |
| :---: | :---: | :---: |
| $a$ | $e_{11}$ | $e_{12}$ |
| $(b / e)+x$ | $e_{21}$ | $e_{22}+x$ |

where $(b / e)+x=\eta b$ and $(d / e)+x=\eta d$.
(c) $a \oplus \eta b=\eta c \oplus d$. As $d \leq b^{\sim}=\eta b$ there exists $x \in U$ with $x+d=\eta b$. Then we have $a+x+d=a+\eta b=\eta c+d$, and therefore $a+x=\eta c$. Thus the required decomposition is:

|  | $\eta c$ | $d$ |
| :---: | :---: | :---: |
| $a$ | $a$ | 0 |
| $\eta b$ | $x$ | $d$ |

From the decompositions obtained in (a), (b), and (c), it is clear that, if $P$ satisfies $\mathrm{RDP}_{1}$ or $\mathrm{RDP}_{2}$, then so does $U$.

Finally we prove that $P$ has $\mathrm{RDP}_{0}$ iff $U$ has $\mathrm{RDP}_{0}$. Clearly, if $U$ has $\mathrm{RDP}_{0}$, then $P$ has $\mathrm{RDP}_{0}$.

Assume that $P$ has $\mathrm{RDP}_{0}$. Then there are four cases to check:
(i) If $a \leq \eta b \oplus c$, then as $a \leq b^{\sim}=\eta b$, it follows that $a=a \oplus 0$ provides the desired decomposition.
(ii) If $a \leq b \oplus \eta c$, then similarly, $a=0 \oplus a$ provides the desired decomposition.
(iii) If $\eta a \leq \eta b \oplus c$, then we have $c \leq \eta a$ and so $\eta a=x \oplus c \leq \eta b \oplus c$ for some $x \in \bar{U}$ and that implies $x \leq \eta b$. So $\eta a=x+c$ provides the desired decomposition.
(iv) If $\eta a \leq b \oplus \eta c$, then we make use of $b \leq \eta a$ and so there is again an $x \in U$ such that $\eta a=b+x \leq b \oplus \eta c$, which implies $x \leq \eta c$, whence $\eta a=b+x$ provides the desired decomposition.

We note that only in cases (c) and (iv) do we use the hypothesis that $P$ is total.

We recall that a pseudo effect algebra $E$ is a subdirect product of a system of pseudo effect algebras $\left(E_{t}: t \in T\right)$ if there is an injective homomorphism $h: E \rightarrow \prod_{t \in T} E_{t}$ such that $\pi_{t}(h(E))=E_{t}$ for each $t \in T$, where $\pi_{t}$ is the $t$-th projection of $\prod_{t \in T} E_{t}$ onto $E_{t}$. In addition, $E$ is subdirectly irreducible if whenever $E$ is a subdirect product of $\left(E_{t}: t \in T\right)$, there is $t_{0} \in T$ such that $\pi_{t_{0}} \circ h$ is an isomorphism of pseudo effect algebras.

In the theory of total algebras, it is well known that an algebra is subdirectly irreducible iff it has a smallest nontrivial ideal. As pseudo effect
algebras are only partial algebraic structures, some results of universal algebra need not hold for arbitrary pseudo effect algebras. For example, while for total algebras, there is one-to-one correspondence between congruences and ideals, in pseudo effect algebras the relations between ideals and congruences are more complicated. Therefore, in what follows, instead of studying irreducibility, we study relations between the smallest nontrivial (i.e. different from $\{0\}$ ) normal Riesz ideals in GPEAs and in their unitizations.
5.4 Theorem. Let $P$ be an upward directed GPEA and let $U$ be a $\gamma$-unitization of $P$. Then there is a smallest nontrivial normal Riesz ideal in $U$ iff there is a smallest normal Riesz $\gamma$-ideal in $P$.

Proof. Let $I$ be a smallest nontrivial normal Riesz ideal in $U$. Then $I \cap P$ is a nontrivial normal Riesz $\gamma$-ideal in $P$ (Theorem4.14). Let $\{0\} \neq I_{0} \subset I \cap P$ be a normal Riesz $\gamma$-ideal in $P$. By Corollary 4.21, $I_{0}$ is also normal Riesz ideal in $U$, therefore we must have $I_{0}=I=I \cap P$. It follows that $I$ is also the smallest nontrivial Riesz $\gamma$-ideal in $P$.

Conversely, let $I$ be a smallest nontrivial normal Riesz $\gamma$-ideal in $P$. By Corollary 4.21, $I$ is also a normal Riesz ideal in $U$. If there is a normal Riesz ideal $\{0\} \neq J$ in $U$ such that $J \subseteq I$, then $\{0\} \neq J \cap P$ is a normal Riesz $\gamma$-ideal in $P$, and hence $J \cap P=I$. It follows that $I \subseteq J$, hence $I$ is the smallest nontrivial Riesz ideal in $U$.

Notice that in an upward directed GPEA $P$ with RDP, every ideal is Riesz ideal. Similarly as for GEAs [9, Lemma 3.2] (see also [12, Lemma $4.3]$ ), it can be proved that a (nontrivial) upward directed GPEA $P$ with $\mathrm{RDP}_{1}$ is subdirectly irreducible iff $P$ possesses a smallest non-trivial normal ideal. This yields the following corollary.
5.5 Corollary. Let $P$ be a GPEA with total operation $\oplus$ satisfying $R D P_{1}$ and let $U$ be its $\gamma$-unitization. Then $U$ is subdirectly irreducible iff $P$ has a smallest nontrivial normal $\gamma$-ideal. If $\gamma$ is the identity, then $U$ is subdirectly irreducible iff $P$ is subdirectly irreducible.
5.6 Remark. Assume that $(P ; \oplus, 0)$ is an upward directed GPEA, $I$ is a nonempty indexing set, $\lambda, \rho: I \rightarrow I$ are bijections, condition (KCI) holds, and ( $K ;+, 0_{K}, 1_{K}$ ) is the resulting kite algebra (Theorem 3.9). By Corollary 3.10, $K$ is the $\gamma$-unitization of $P^{I}$ with the unitizing automorphism $\gamma\left(a_{i}\right)_{i \in I}=$ $\left(a_{\rho \circ \lambda^{-1}(i)}\right)_{i \in I}$. We shall say that $i, j \in I$ are connected iff $\left(\rho \circ \lambda^{-1}\right)^{m}(i)=j$, or $\left(\rho \circ \lambda^{-1}\right)^{m}(j)=i$ for some integer $m \geq 0$, otherwise $i$ and $j$ are disconnected.

Let $i_{0}, j_{0} \in I$ be disconnected, and let $I_{0}$ and $I_{1}$ be maximal subsets of mutually connected elements in $I$ containing $i_{0}$ and $j_{0}$, respectively. Then clearly, every element in $I_{0}$ is disconnected with every element in $I_{1}$. Let $H_{0}:=\left\{\left(f_{i}\right)_{i \in I}: f_{i}=0 \forall i \notin I_{0}\right\}, H_{1}:=\left\{\left(f_{j}\right)_{j \in I}: f_{j}=0 \forall j \notin I_{1}\right\}$. It is easy to see that $H_{0}$ and $H_{1}$ are normal $\gamma$-ideals in $P^{I}$. Assume that $K$ has $\mathrm{RDP}_{1}$. Then also $P^{I}$ has $\mathrm{RDP}_{1}$, and $H_{0}$ and $H_{1}$ are normal Riesz $\gamma$-ideals in $P^{I}$. By Corollary 4.21, they are also normal Riesz ideals in $K$. Clearly, $H_{0} \cap H_{1}=\{0\}$. We can deduce that, under the above suppositions, if $K$ has a smallest nontrivial normal Riesz ideal, then the set $I$ is connected. (Compare with [12, Theorem 4.6].)

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