Characterizing subclasses of cover-incomparability graphs by forbidden subposets*

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Abstract. In this paper we continue investigations of cover-incomparability graphs of finite partially ordered sets (see [1,2,3,4] and [6,7]). We consider in some detail the distinction between cover-preserving subsets and isometric subsets of a partially ordered set. This is critical to understanding why forbidden subposet characterizations of certain classes of cover-incomparability graphs in [1] and [3] are not valid as presented. Here we provide examples, investigate the root of the difficulties, and formulate and prove valid revisions of these characterizations.

1 Introduction

In this paper we deal with posets and graphs associated to them. There are several ways how to associate a graph G to a given poset P. The vertex set V(G) is usually the set of points of P. Depending on the edge-set E(G), we may obtain among others the *comparability graph* of P (x and y are adjacent iff x < y or y < x), the incomparability graph of P (x and y are adjacent iff x and y are incomparable), the cover graph of P (x and y are adjacent iff x covers y or vice versa) or the cover-incomparability graph of P (x and y are adjacent iff x covers y, or y covers x, or x and y are incomparable). The incomparability graph of P is of course just the complement of its comparability graph, while the cover-incomparability graph of P is the union of the cover graph and the incomparability graph of G.

Cover graphs, comparability graphs and incomparability graphs are standard ways how to associate a graph to a given poset, while the notion of cover-incomparability graph is new. It was introduced in [1]. This notion was motivated by the theory of transit functions on posets. It turns out that the underlying graph G_P of the standard transit function T_P on the poset P is exactly the cover-incomparability graph of P (see [1] for details).

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Cover-incomparability graphs have been so far approached in two different ways. One possibility is to try to characterize graphs that are cover-incomparability graphs. In [6] it was proved that the recognition problem for cover-incomparability graphs is in general NP-complete. On the other hand there are classes of graphs (such as trees, Ptolemaic graphs, distance-hereditary graphs, block graphs, split graphs or k-trees) for which the recognition problem can be solved in linear time (see [2,3,7,8] for details and proofs).

Another approach is to study posets whose cover-incomparability graphs have certain property. Posets whose cover-incomparability graphs are chordal, Ptolemaic, distance-hereditary, claw-free or cographs were characterized in [1] and [4]. Unfortunately, there is a mistake that originated in [1] and continued in [4] and several statements from these papers do not hold as they are stated. In this paper we correct the mistake and reformulate the corresponding statements so that they hold.

Our paper is organized as follows. In Section 2 we give an overview of terminology and basic properties of cover-incomparability graphs. In Section 3 we present counterexamples to Theorem 4.1 from [3], Lemma 4.4 and 4.5 from [1] and to Proposition 5.1 from [1]. In Section 4 we show that the mistake originated in Theorem 2.4 of [1]. We reformulate this statement and give a corrected proof of it. In addition, we reformulate all the above mentioned statements so that they hold.

2 Terminology and basic properties

Let $P=(V,\leq)$ be a poset. We will use the following notation. For $u,v\in V$ we write:

- $-u < v \text{ if } u \leq v \text{ and } u \neq v.$
- $-u \prec v$ if u < v and there is no $z \in V$ such that u < z < v. We say that v covers u.
- $-u \prec \prec v \text{ if } u < v \text{ and } \neg (u \prec v).$
- $-u \parallel v$ if u and v are incomparable.

Definition 1. For a given poset $P = (V, \leq)$, let $G_P = (V, E)$ be a graph with $E = \{\{u, v\} \mid u \prec v \text{ or } v \prec u \text{ or } u \parallel v\}$. Then we say that G_P is the cover-incomparability graph of P (or the C-I graph of P for short).

Note that for any $u, v \in V(G_P)$, $u \neq v$ we have $\{u, v\} \notin E(G_P) \Leftrightarrow u \prec \prec v$ or $v \prec \prec u$.

As this is crucial for the rest of our paper let us define properly the following three concepts.

Definition 2. Let $P = (V_P, \leq_P)$ be a poset.

- We say that $Q = (V_Q, \leq_Q)$ is a subposet of $P = (V_P, \leq_P)$ if 1. $V_Q \subseteq V_P$ and
 - 2. for any $u, v \in V_Q$ we have $u \leq_Q v \Leftrightarrow u \leq_P v$.

- We say that $R = (V_R, \leq_R)$ is an isometric subposet of $P = (V_P, \leq_P)$ if
 - 1. $V_R \subseteq V_P$ and
 - 2. for any $u, v \in V_R$ we have $u \leq_R v \Leftrightarrow u \leq_P v$ and
 - 3. for any $u, v \in V_R$ such that $u \leq_R v$ there exists a chain of a shortest length between u and v in P is also in R.
- We say that $S = (V_S, \leq_S)$ is a \prec -preserving subposet of $P = (V_P, \leq_P)$ if
 - 1. $V_S \subseteq V_P$ and
 - 2. for any $u, v \in V_S$ we have $u \leq_S v \Leftrightarrow u \leq_P v$ and
 - 3. for any $u, v \in V_S$ we have $u \prec_S v \Leftrightarrow u \prec_P v$.

Note that an isometric subposet is always \prec -preserving but there are \prec -preserving subposets that are not isometric. For example, the poset P' depicted in Fig. 1 is a nonisometric \prec -preserving subposet of P in Fig. 1.



Fig. 1: A nonisometric \prec -preserving subposet.

Let us also mention a few easy observations about C-I graphs. They follow immediately from the definition.

Lemma 1. Let $P = (V, \leq)$ be a poset and $G_P = (V, E)$ its C-I graph. Then the following holds.

- (i) G_P is connected.
- (ii) If $U \subseteq V$ is an antichain in P then U induces a complete subgraph in G_P .
- (iii) If $I \subseteq V$ is an independent set in G_P then all points of I lie on a common chain in P.
- (iv) There are at most 2 vertices of degree 1 in G_P .
- (v) If $P^* = (V, \leq^*)$ is the dual poset to P (i.e. $u \leq v$ in $P \Leftrightarrow v \leq^* u$ in P^*), then $G(P^*) = G_P$.
- (vi) If the vertices x, y, z form a triangle in G_P then at least two of them are incomparable.
- (vii) Let x, y, z be vertices of G_P such that $xy \in E$, $xz \notin E$, $yz \notin E$. Then $(x \prec \prec z \text{ and } y \prec \prec z)$ or $(z \prec \prec x \text{ and } z \prec \prec y)$.

3 Counterexamples

In this section we present counterexamples to several statements from [1] and [3]. Let us start with the easiest case, with Proposition 5.1 from [1].

3.1 A counterexample to Proposition 5.1 from [1]

First we cite the statement of this proposition in the original text:

Proposition (Proposition 5.1 [1]). Let P be a poset. Then G_P contains an induced claw if and only if P contains one of S_1 , S_2 or S_3 as an isometric subposet, see Fig. 2.

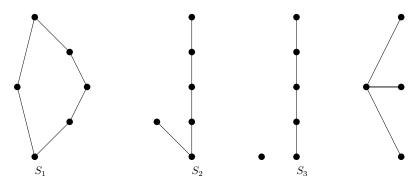


Fig. 2: Subposets S_1 , S_2 and S_3 and the claw.

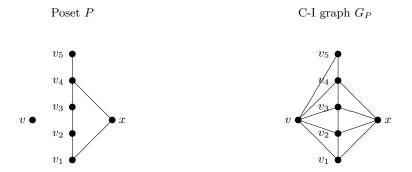


Fig. 3: A counterexample to Proposition 5.1.

This statement does not hold. Let P be the poset depicted in Fig. 3. Clearly, neither S_1 nor S_2 are subposets of P. S_3 is a subposet of P but it is not an

isometric subposet of P. This is because there is a chain of length two between u and v in P while there is no chain of length two between u and v in S_3 . Thus P does not contain any of S_1 , S_2 and S_3 as an isometric subposet. But G_P contains an induced claw on vertices v, v_1, v_3, v_5 , a contradiction.

Counterexamples to other statements can be derived in a similar way:

3.2 A counterexample to Lemma 4.4 from [1]

Proposition (Proposition 4.4 in [1]). Let P be a poset. Then G_P contains an induced house if and only if P contains one of R_1 , R_2 , R_3 , R_4 or R_5 as an isometric subposet, see Figure 4.

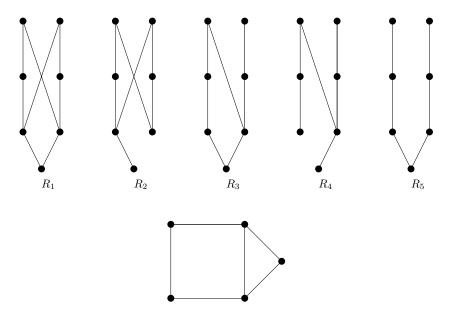


Fig. 4: Subposets R_i , i = 1, ... 5 and the house.

Let P be the poset depicted in Fig. 5. It is easy to see that it is a counterexample to Lemma 4.4 [1]. Indeed, P does not contain any of the posets R_1 , R_2 , R_3 , R_4 or R_5 as an **isometric subposet**. But G_P contains an induced house on vertices v_1, v_2, v_4, v_5, v_7 , a contradiction.

3.3 A counterexample to Lemma 4.5 from [1]

Proposition (Proposition 4.5 in [1]). Let P be a poset. Then G_P contains an induced domino if and only if P contains one of D_1 , D_2 , D_3 , D_4 , D_5 , D_6 or D_7 as an isometric subposet, see Fig. 6.

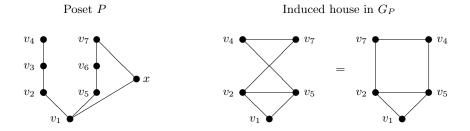


Fig. 5: A counterexample to Lemma 4.4

Let P be the poset depicted in Figure 7. Clearly that it is a counterexample to Lemma 4.5. [1]. Indeed, P does not contain any of the posets D_1 , D_2 , D_3 , D_4 , D_5 , D_6 or D_7 as an *isometric subposet*. But G_P contains an induced domino on vertices v_1 , v_2 , v_4 , v_5 , v_7 , a contradiction.

3.4 A counterexample to Theorem 4.1 from [3]

Theorem (Theorem 4.1 [3]). Let P be a poset. Then G_P is a cograph if and only if P contains neither any of Q_1, Q_2, \ldots, Q_7 nor duals of Q_2 or Q_5 as an isometric subposet, see Fig. 8.

Let P be the poset depicted in Fig. 7. It is easy to see that it is a counterexample to Theorem 4.1 [3]. Indeed, P contains neither any of the posets Q_1 , Q_2 , ..., Q_7 nor the duals of Q_2 or Q_5 as an isometric subposet. But G_P contains an induced path on four vertices v_1, v_2, v_3, v_4 . Thus, G_P is not a cograph, a contradiction.

4 Restatements and proofs

The mistake originated in Theorem 2.4 [1].

Theorem (Theorem 2.4 [1]). Let \mathcal{G} be a class of graphs with a forbidden induced subgraphs characterization. Let $\mathcal{P} = \{P \mid P \text{ is a poset with } G_P \in \mathcal{G}\}$. Then \mathcal{P} has a forbidden isometric characterization.

If we go carefully through the proof of this theorem in [1] we notice that it is not proved that the poset P contains one of the constructed posets $\{P_i\}_{i\in I}$ as an *isometric* subposet. The condition of isometry is too strong and it has to be replaced by the weaker concept of \prec -preserving subposet. See Section 2 for the definition.

Theorem 1 (see Theorem 2.4 [1]). Let \mathcal{G} be a class of graphs with a characterization by forbidden induced subgraphs. Let $\mathcal{P} = \{P \mid P \text{ is a poset with } G_P \in \mathcal{G}\}$. Then \mathcal{P} has a characterization by forbidden \prec -preserving subposets.

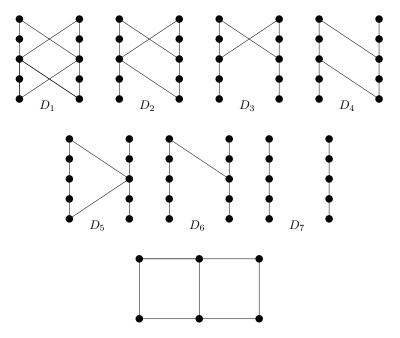


Fig. 6: Subposets D_i , i = 1, ... 7, and the domino graph.

For the proof of this theorem we need a slightly stronger version of Lemma 2.3 [1].

Lemma 2 (see Lemma 2.3 [1]). Let Q be a \triangleleft -preserving subposet of a poset P. Then G_Q is isomorphic to a subgraph of G_P induced by the points of Q.

Proof. Let H be the subgraph of G_P induced by the points of Q. Let u and v be arbitrary points in Q. We show that

$$\{u, v\} \in E(H) \Leftrightarrow \{u, v\} \in E(G_Q).$$

First suppose that $\{u, v\} \in E(H)$. This happens if and only if either $u \prec_P v$, or $v \prec_P u$, or $u \parallel_P v$. As Q is a \lhd -preserving subposet of P we have

$$u \prec_P v \Rightarrow u \prec_Q v \Rightarrow \{u, v\} \in E(G_Q),$$

$$v \prec_P u \Rightarrow v \prec_Q u \Rightarrow \{u, v\} \in E(G_Q),$$

$$u \parallel_P v \Rightarrow u \parallel_Q v \Rightarrow \{u, v\} \in E(G_Q).$$

Thus if $\{u, v\} \in E(H)$ then also $\{u, v\} \in E(G_Q)$.

Now suppose that $\{u,v\} \notin E(H)$. Then $u \prec \prec_P v$ or $v \prec \prec_P u$. As Q is a \lhd -preserving subposet of P it follows that $u \prec \prec_Q v$ or $v \prec \prec_Q u$, and thus $\{u,v\} \notin E(G_Q)$.

We conclude that H and G_Q are isomorphic graphs as stated.

Now we are ready to prove Theorem 1.



Fig. 7: A counterexample to Lemma 4.5.

Proof ((of Theorem 1)). Let G_{forb} be one of the forbidden induced subgraphs for the class \mathcal{G} . Let $P \in \mathcal{P}$ be any poset in the class \mathcal{P} . By the definition of \mathcal{P} , G_P does not contain G_{forb} as an induced subgraph. By Lemma 2, P does not contain any \prec -preserving subposet Q such that G_Q is isomorphic to G_{forb} . Hence any subposet Q s.t. G_Q is isomorphic to G_{forb} is forbidden for \mathcal{P} . Repeating this for all the forbidden induced subgraphs for \mathcal{G} we find a list of forbidden \prec -preserving subposets $\{Q_i\}_{i\in I}$.

We will show that the class \mathcal{P} is characterized by forbidden \prec -preserving subposets $\{Q_i\}_{i\in I}$.

First, let $P \in \mathcal{P}$. Then P clearly contains no Q_i as a \prec -preserving subposet. Otherwise (by Lemma 2) the graph G_P would contain a forbidden induced subgraph for \mathcal{G} .

Conversely, suppose that P contains no Q_i as a \prec -preserving subposet. Then (by the construction of $\{Q_i\}_{i\in I}$) G_P contains no forbidden subgraph for \mathcal{G} . Thus $G_P \in \mathcal{G}$, and hence $P \in \mathcal{P}$.

The previous theorem can be applied for various graph classes that admit a characterization by forbidden induced subgraphs, such as chordal graphs, clawfree graphs, distance-hereditary graphs, Ptolemaic graphs etc.

Theorem 2 (corrected Lemma 5.1 [1]). Let P be a poset. Then G_P contains an induced claw if and only if P contains one of S_1 , S_2 , S_3 or S_2^* (the dual of S_2) as a \prec -preserving subposet, see Fig. 2.

Proof. If P contains one of the posets S_1 , S_2 , S_3 or S_2^* as a \prec -preserving subposet then clearly G_P contains an induced claw.

Conversely, suppose that G_P contains an induced claw. We want to find S_1 , S_2 , S_3 or S_2^* as a \prec -preserving subposet of P. Let us denote by x the middle vertex and by u, v, w the other vertices of the claw. By Lemma 1(iii), as u, v, w form an independent set in G_P they lie on a common chain in P. Without loss of generality we may suppose that $u \prec \prec v \prec \prec w$.

Note that $x \prec v$ is not possible, otherwise $x \prec \prec w$ and hence $\{x, w\} \notin E(G_P)$, a contradiction. Similarly, it is not possible that $x \prec u$, $v \prec x$ or $w \prec x$. Thus there are only five cases to distinguish:

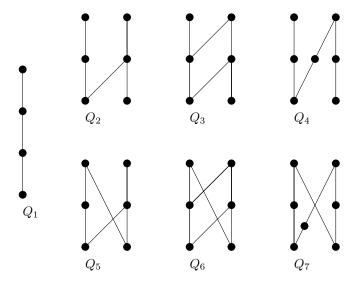


Fig. 8: Subposets Q_i , i = 1, ... 7.



Fig. 9: A counterexample to Theorem 4.1

- Case 1 $x \parallel u$, $x \parallel v$, $x \parallel w$. Then P obviously contains S_3 as a \prec -preserving subposet.
- Case 2 $u \prec x$, $x \parallel v$, $x \parallel w$. Then P obviously contains S_2 as a \prec -preserving subposet.
- Case 3 $x \prec w, x || v, x || u$. Then P obviously contains S_2^* as a \prec -preserving subposet.
- Case $4u \prec x, x \prec w, x || v$ and the length of the shortest chain in P between u and w is equal to 4. Then P obviously contains S_3 as a \prec -preserving subposet.
- Case 5 $u \prec x, x \prec w, x || v$ and the length of the shortest chain in P between u and w is greater than 4. Then P obviously contains S_2 as a \prec -preserving subposet.

Now let us restate the corresponding statements from [1] and [3]. We skip their proofs as they are the same as the ones presented in [1] and [3], the only mistake was claiming that the forbidden subposets must be isometric subposets of P.

Theorem 3 (corrected Lemma 4.4 [1]). Let P be a poset. Then G_P contains an induced house if and only if P contains one of R_1 , R_2 , R_3 , R_4 , R_5 or its duals as a \prec -preserving subposet, see Fig. 4.

Theorem 4 (corrected Lemma 4.5 [1]). Let P be a poset. Then G_P contains an induced domino if and only if P contains one of D_1 , D_2 , D_3 , D_4 , D_5 , D_6 , D_7 or its duals as a \prec -preserving subposet, see Fig. 6.

Let us remark that for P_1 , P_2 , and P_3 the notion of isometric subposet and \prec -preserving subposet coincide. More precisely, a poset P contains P_1 , P_2 , or P_3 as an isometric subposet if and only if P contains P_1 , P_2 , or P_3 as a \prec -preserving subposet. This is because the length of the longest chain in P_1 , P_2 , and P_3 is only two. Hence, Theorem 3.1 [3] holds as it was stated in [3].

Theorem 5 (corrected Theorem 4.1 [3]). Let P be a poset. Then G_P is a cograph if and only if P contains none of Q_1, Q_2, \ldots, Q_7 and neither of the duals of Q_2 and Q_5 as a \prec -preserving subposet, see Fig. 8.

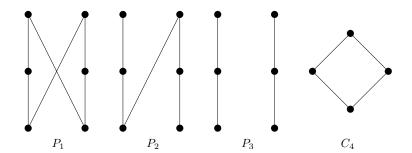


Fig. 10: Subposets P_1 , P_2 , P_3 and C_4

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