# Diagonal Orbits in a Type A Double Flag Variety of Complexity One 

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#### Abstract

We continue our study of the inclusion posets of diagonal $S L(n)$-orbit closures in a product of two partial flag varieties. We prove that, if the diagonal action is of complexity one, then the poset is isomorphic to one of the 28 posets that we determine explicitly. Furthermore, our computations show that the number of diagonal $S L(n)$ orbits in any of these posets is at most 10 for any positive integer $n$. This is in contrast with the complexity 0 case, where, in some cases, the resulting posets attain arbitrary heights.


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## 1 Introduction

Let $G$ be a connected reductive complex algebraic group, and let $B$ be a Borel subgroup in $G$. Let $X$ be an irreducible complex algebraic $G$-variety. We denote the action of $G$ on $X$ by $G: X$. A typical example for such a variety is the homogeneous space $G / H$, where $H$ is a closed subgroup of $G$, and the action of $G$ on $G / H$ is given by the multiplication action of $G$ on the left cosets of $H$ in $G$. The complexity of $G: X$, denoted by $c_{G}(X)$, is defined as the codimension of a general $B$-orbit in $X$. This notion plays an important role in the theory of equivariant embeddings of homogeneous spaces, see [19]. As it is demonstrated by the seminal paper of Panyushev [14], among all homogeneous spaces of $G$, the ones with complexity at most one form the most remarkable subclass.

An enduring problem in representation theory is to decompose the tensor products of irreducible representations of $G$. Let $\lambda_{i}(1 \leq i \leq 2)$ be two dominant weights corresponding
to the irreducible representations $V_{i}(1 \leq i \leq 2)$ of $G$, and let $P_{i}(1 \leq i \leq 2)$ denote the corresponding parabolic subgroups that arise as the stabilizer subgroups of highest weight vectors $v_{i} \in V_{i}(1 \leq i \leq 2)$. There is a close relationship between the decomposition of $V_{1} \otimes V_{2}$ as a $G$-module and the polynomial invariants of the diagonal action of $G$ on the double flag variety $X:=G / P_{1} \times G / P_{2}$. By using the coordinate ring of the affine cone over the double flag variety, in [11], Littelmann obtained precise description of the decompositions of the tensor products of two fundamental representations of simple groups. This progress motivated the works [12, 13], and [16]. In the last reference, Stembridge classified all multiplicityfree tensor products of irreducible representations of semisimple complex Lie groups. This classification amounts to the classification of the parabolic subgroups $P_{i}(1 \leq i \leq 2)$ such that $c_{G}\left(G / P_{1} \times G / P_{2}\right)=0$. Finally, in [15], Ponomareva classified all double flag varieties of complexity one. In the same paper, Ponomareva showed by examples how one could use the results of Brion [4] and Timashev [18] for decomposing the spaces of global sections of the line bundles on a double flag variety of complexity $\leq 1$. In the present article, we focus on the double flag varieties of complexity one. Our purpose here is to give a complete description of the inclusion order on the closures of the $G$-orbits in $G / P_{1} \times G / P_{2}$ when $c_{G}\left(G / P_{1} \times G / P_{2}\right)=1$.

To further motivate our discussion, let us mention another setup where the diagonal orbits are of crucial importance. In [8], Deligne and Lusztig constructed the complex linear representations of finite groups of Lie type by using the $\ell$-adic cohomology with compact support on certain varieties. Let $\mathbf{F}_{q}$ denote the finite field with $q$ elements, let $G$ be a reductive group defined over an algebraic closure of $\mathbf{F}_{q}$, and let $F$ denote a Frobenius map on $G$. Let $w$ be an element of the Weyl group of $G$. The Deligne-Lusztig variety associated with $w$, denoted by $X(w)$, consists of all Borel subgroups $B$ of $G$ such that $B$ and $F(B)$ are in relative position $w$. In other words, $X(w)$ is the intersection of the $G$-orbit corresponding to $w$ in $G / B \times G / B^{-}$, where $B^{-}$is the unique opposite Borel subgroup to $B$, with the graph of the Frobenius map. More recently, Digne and Michel extended this theory to the setting of partial flag varieties, see [9]. In essence, the poset that we study in our paper is about the natural hierarchy between the parabolic Deligne-Lusztig characters, namely, the characters of the representations of $G^{F}$ on $H_{c}^{*}\left(X(w), \overline{\mathbb{Q}_{\ell}}\right)$, where $X(w)$ is a Deligne-Lusztig variety in $G / P \times G / P^{-}$. Here, $P^{-}$is a parabolic subgroup such that $P \cap P^{-}$is a common Levi subgroup of both of $P$ and $P^{-}$. Finally, let us mention that the same partial order arises rather naturally in the study of the nilpotent variety of the dual canonical monoids, see [17] and [6].

Let $X$ be a normal $G$-variety, and let $B$ denote a Borel subgroup of $G$. In many ways the geometry of $X$, as a $G$-variety, depends on how $G$ - and $B$-orbits in $X$ fit together. For example, if $X$ has finitely many $G$-orbits, then the rational Chow group of $X$ has a decomposition with respect to $G$-orbits, see [1]. With this fact in mind, in our earlier work [5], for $G=S L(n)$, we showed that if $c_{G}(X)=0$, then the inclusion poset of $G$-orbit closures in $X$ is a particular kind of graded lattice; it is either a chain, or it is what we called a 'ladder poset.' In higher complexity, these posets can be very complicated; they are not necessarily graded. However, they always have a unique minimal and a unique maximal
element. In the case of complexity one, as we show, most of them turn out to be lattices, and not all of them are graded. Our main theorem is the following statement.

Theorem 1.1. Let $G$ denote $S L(n)$ and let $X$ be a double flag variety $G / P_{1} \times G / P_{2}$. If $c_{G}(X)=1$, then the inclusion poset of $G$-orbit closures in $X$ is one of the 28 posets whose Hasse diagrams are as in Figure 1.

For us, the most surprising outcome of our computation is the number of $G$-orbits in $G / P_{1} \times G / P_{2}$. Although there are infinitely many complexity one double flag varieties, in each case, the number of $G$-orbits turns out to be bounded by 10 ; this is in contrast with the complexity zero case, where there are infinitely many non-isomorphic $G$-orbit containment posets, and they can be of arbitrary height.

Next, we give a brief description of our paper. In Section 2, we present some background material regarding our posets. Section 3 forms the main body of our paper; we depict the Hasse diagrams of our posets in Figure 1. The subsequent Section 4 is the concluding section for the proof of our main theorem. Finally, in Section 5, we mention an alternative method for proving our theorem.

## 2 Preliminaries

## 2.1

Let $G$ be a complex semisimple algebraic group, let $B$ be a Borel subgroup in $G$, and let $T$ be a maximal torus of $G$ that is contained in $B$. We denote by $\Phi$ the root system corresponding to the pair $(G, T)$, and we denote by $\Delta$ the set of simple roots determined by $B$. A parabolic subgroup $P$ of $G$ is said to be standard with respect to $B$ if $B \subseteq P$. In this case, $P$ is uniquely determined by a subset $I \subseteq \Delta$ such that $|I|=\operatorname{dim} P / B$.

Let $N_{G}(T)$ denote the normalizer subgroup of $T$ in $G$. The Weyl group $W:=N_{G}(T) / T$ of $G$ is a Coxeter group, and we denote its Coxeter generating system corresponding to $\Delta$ by

$$
R(\Delta):=\left\{s_{\alpha} \in W: \alpha \in \Delta\right\}
$$

The elements of $R(\Delta)$ are called the simple reflections relative to $B$. If the set of simple roots we are using is fixed, then we will denote $R(\Delta)$ by $R$ to ease our notation.

We will interchangeably use the letters $I$ and $J$ to denote subsets of $\Delta$ and the corresponding subsets of simple reflections in $R(\Delta)$. The length of an element $w \in W$, denoted by $\ell(w)$, is the minimal number of simple reflections $s_{\alpha_{i}} \in R(\Delta)$ that is needed for the equality $w=s_{\alpha_{1}} \cdots s_{\alpha_{k}}$ hold true. In this case, the product $s_{\alpha_{1}} \cdots s_{\alpha_{k}}$ is called a reduced expression for $w$.

The Bruhat-Chevalley order on $W$ is be defined by declaring $v \leq w(w, v \in W)$ if a reduced expression of $v$ is obtained from a reduced expression $s_{\alpha_{1}} \cdots s_{\alpha_{k}}=w$ by deleting some of the simple reflections $s_{\alpha_{i}}$ in $w$. More geometrically, the Bruhat-Chevalley order is given by $v \leq w \Longleftrightarrow B \dot{v} B / B \subseteq \overline{B \dot{w} B / B}$. Here, $\dot{v}$ and $\dot{w}$ are any representatives of $v$ and $w$ in $N_{G}(T)$, respectively. The sets $B \dot{v} B / B, B \dot{w} B / B$ denote the $B$-orbits of $\dot{v}, \dot{w}$ in $G / B$,
and the bar on $B \dot{w} B / B$ indicates the Zariski closure. In this notation, $\ell(w)$ is equal to the dimension of the orbit $B \dot{w} B / B$.

Let $G$ be a classical semisimple matrix group with entries in $\mathbb{C}$, and let $B$ denote its Borel subgroup consisting of upper triangular matrices. The parabolic subgroups of $G$ containing $B$ have block-triangular structure, and they are determined by the sizes of the diagonal blocks. Following Ponomareva's notation from [15], if $P$ is a parabolic subgroup containing $B$, then we will denote by $B l(P)$ the sequence $\left(p_{1}, \ldots, p_{r}\right)$, where $p_{i}$ denotes the size of the $i$-th block in $P_{I}$. For example, if $P$ is the Borel subgroup of upper triangular matrices in $S L(n)$, then each diagonal block of $P$ is a $1 \times 1$ matrix, therefore, $B l\left(P_{I}\right)$ is the sequence $(1,1, \ldots, 1)$ with $n$ entries.

Our primary example is the matrix group $G=S L(n)$. We take $B$ as the Borel subgroup of upper triangular matrices, and we take $T$ as the maximal torus of diagonal matrices in $B$. The Weyl group $W$ of $S L(n)$ is denoted by $S_{n}$, which is isomorphic to the symmetric group of permutations of $\{1, \ldots, n\}$. The set of simple roots relative to $B$, that is $\Delta_{n-1}:=$ $\left\{\alpha_{1}, \ldots, \alpha_{n-1}\right\}$, is ordered so that the $i$-th simple reflection $s_{\alpha_{i}}(1 \leq i \leq n-1)$ is the simple transposition $s_{i} \in S_{n}$ that interchanges $i$ and $i+1$. Thus we set

$$
R_{n-1}:=R\left(\Delta_{n-1}\right)=\left\{s_{1}, \ldots, s_{n-1}\right\}
$$

If a permutation $w$ in $S_{n}$ is given in one-line notation $w=w_{1} \ldots w_{n}$, then its length is equal to the cardinality of the following set: $\left\{1 \leq i<j \leq n: w_{i}>w_{j}\right\}$.

An important fact that we repeatedly use in our paper is that $S L(n)$ is the stabilizer subgroup in $S L(n+1)$ of the standard basis vector $e_{n+1}$ of $\mathbb{C}^{n+1}$, where $S L(n+1)$ acts by its defining representation. In particular, by using this identification of $S L(n)$ as a subgroup of $S L(n+1)$, we will use the following containments without further mentioning in the sequel:

$$
\Delta_{n-1} \hookrightarrow \Delta_{n}, R_{n-1} \hookrightarrow R_{n}, \quad \text { and } S_{n} \hookrightarrow S_{n+1} \text { (as a subgroup). }
$$

## 2.2

Let $X_{1}$ and $X_{2}$ be two $G$-varieties. Let $x_{i} \in X_{i}(1 \leq i \leq 2)$ be two points in general positions. If $G_{i} \subset G$ denotes the stabilizer subgroup of $x_{i}$ in $G$, then $\operatorname{Stab}_{G}\left(x_{1} \times x_{2}\right)$ coincides with the stabilizer in $G_{1}$ of a point in general position from $G / G_{2}$ (or, equivalently, with the stabilizer in $G_{2}$ of a point in general position from $\left.G / G_{1}\right)$, see [14]. As a special case, we consider the $G$-variety $X:=G / P_{1} \times G / P_{2}$. The proof of the following lemma is not difficult, see [5, Lemma 2.1].

Lemma 2.1. The poset of $G$-orbit closures in $X$ is isomorphic to the poset of $P_{2}$-orbit closures in $G / P_{1}$.

From now on we assume that $P_{1}$ and $P_{2}$ are standard parabolic subgroups with respect to $B$. If $I$ and $J$ are the subsets of $R:=R(\Delta)$ (or, of $\Delta$ ) that determine $P_{1}$ and $P_{2}$, respectively, then we will write $P_{I}$ (resp. $P_{J}$ ) in place of $P_{1}$ (resp. $P_{2}$ ). The Weyl groups of $P_{I}$ and $P_{J}$ are denoted by $W_{I}$ and $W_{J}$, respectively. The set of $\left(W_{I}, W_{J}\right)$-double cosets in $W$ is denoted by $W_{I} \backslash W / W_{J}$.

## 2.3

It follows from Bruhat-Chevalley decomposition that the set of $B$-orbits in $G / P_{J}$ are in a bijection with the set of minimal length left coset representatives for $W / W_{J}$, which we denote by $W^{J}$. The set of minimal length right coset representatives for $W_{I} \backslash W$ is denoted by ${ }^{I} W$. In a similar way, $W_{I} \backslash W / W_{J}$ is in a bijection with the set of $P_{I^{-}}$-orbits in $G / P_{J}$, see [3, Section 21.16]. Let $w$ be an element from $W$, and let $[w]$ denote the double coset $W_{I} w W_{J}$. Let

$$
\pi: W \rightarrow W_{I} \backslash W / W_{J}
$$

denote the canonical projection onto the set of $\left(W_{I}, W_{J}\right)$-double cosets. Then the preimage in $W$ of every double coset in $W_{I} \backslash W / W_{J}$ is an interval with respect to Bruhat-Chevalley order. Therefore, there is a unique maximal and a unique minimal element, see [7]. Moreover, if $[w]$ and $\left[w^{\prime}\right]$ are two elements from $W_{I} \backslash W / W_{J}$, and $\bar{w}$ and $\bar{w}^{\prime}$ are the maximal elements in the cosets $[w]$ and $\left[w^{\prime}\right]$, respectively, then $w \leq w^{\prime}$ if and only if $\bar{w} \leq \bar{w}^{\prime}$. (This can be seen directly by a geometric argument, but see [10] also.) Therefore, the set of ( $W_{I}, W_{J}$ )-cosets has a natural combinatorial partial order defined by

$$
[w] \leq\left[w^{\prime}\right] \Longleftrightarrow w \leq w^{\prime} \Longleftrightarrow \bar{w} \leq \bar{w}^{\prime}
$$

where $[w],\left[w^{\prime}\right] \in W_{I} \backslash W / W_{J}$. There is a geometric interpretation of this partial order: If $O_{1}$ and $O_{2}$ are two $P_{I^{-}}$-orbits in $G / P_{J}$ with the corresponding double cosets $[w]$ and $\left[w^{\prime}\right]$, respectively, then $O_{1} \subseteq \overline{O_{2}}$ if and only if $w \leq w^{\prime}$. The bar on $O_{2}$ stands for the Zariski closure in $G / P_{J}$.

Let $[w](w \in W)$ be an element from $W_{I} \backslash W / W_{J}$ such that $\ell(w) \leq \ell(v)$ for all $v \in[w]$. Such minimal length double coset representatives are parametrized by the set ${ }^{I} W \cap W^{J}$. From now on, we denote ${ }^{I} W \cap W^{J}$ by $U_{I, J}^{-}$. Set $H=I \cap w J w^{-1}$. Then $u w \in W^{J}$ for $u \in W_{I}$ if and only if $u$ is a minimal length coset representative for $W_{I} / W_{H}$. In particular, every element of $W_{I} w W_{J}$ has a unique expression of the form $u w v$ with $u \in W_{I}$ is a minimal length coset representative of $W_{I} / W_{H}, v \in W_{J}$ and

$$
\begin{equation*}
\ell(u w v)=\ell(u)+\ell(w)+\ell(v) \tag{1}
\end{equation*}
$$

For $i \in\{1, \ldots, n-1\}$, let $s_{i}$ denote the $i$-th simple transposition. Let $w$ be a permutation in $S_{n}$, and let $w_{1} \ldots w_{n}$ be the one-line notation for $w$. The number $i$ is called a right descent of $w$ if $w_{i}>w_{i+1}$. Equivalently, $i$ is a right descent if $\ell\left(w s_{i}\right)<\ell(w)$. The set of all right descents of $w$, denoted by $\operatorname{Des}_{R}(w)$, is called the right descent set of $w$. In a similar way, the integer $i$ is said to be a right ascent of $w$ if $w_{i}<w_{i+1}$, or, equivalently, $\ell\left(w s_{i}\right)>w$. The right ascent set of $w$, denoted by $\operatorname{Des}_{R}(w)$, is the set of all right ascents of $w$. In this notation, the following characterization of $U_{I, J}^{-}$will be useful for our purposes:

$$
\begin{aligned}
U_{I, J}^{-} & =\left\{w \in W: I \subseteq \operatorname{Asc}_{L}(w) \text { and } J \subseteq \operatorname{Asc}_{R}(w)\right\} \\
& =\left\{w \in W: I^{c} \supseteq \operatorname{Des}_{R}\left(w^{-1}\right) \text { and } J^{c} \supseteq \operatorname{Des}_{R}(w)\right\}
\end{aligned}
$$

Remark 2.2. Let $\theta$ denote the involution of the set $R_{n-1}$ that is defined by $s_{i} \mapsto s_{n-i}$ for $i \in\{1, \ldots, n-1\}$. Then $U_{I, J}^{-}$and $U_{\theta(I), \theta(J)}^{-}$are isomorphic as posets.

## 3 Computations

As we mentioned before, Ponomareva [15] has determined the parabolic subgroups $P_{I}$ and $P_{J}$ in a semisimple complex algebraic group $G$ such that the complexity of the diagonal action of $G$ on $G / P_{I} \times G / P_{J}$ is one. For $G=S L(n)$, the possible $P_{I}$ and $P_{J}$ 's, according to their block sizes, are listed in Table 1. There are in total eight major cases.

|  | Number of blocks | $B l\left(P_{I}\right)$ | $B l\left(P_{J}\right)$ |
| :---: | :---: | :---: | :---: |
| 1. | 2,3 | $\left(3, p_{2}\right), p_{2} \geq 3$ | $\left(q_{1}, q_{2}, q_{3}\right), q_{1}, q_{2}, q_{3} \geq 2$ |
| 2. | 2,3 | $\left(p_{1}, p_{2}\right), p_{1}, p_{2} \geq 3$ | $\left(2,2, q_{3}\right), q_{3} \geq 2$ |
| 3. | 2,3 | $\left(p_{1}, p_{2}\right), p_{1}, p_{2} \geq 3$ | $\left(2, q_{2}, 2\right), q_{2} \geq 2$ |
| 4. | 2,4 | $\left(2, p_{2}\right), p_{2} \geq 3$ | $\left(q_{1}, q_{2}, q_{3}, q_{4}\right)$ |
| 5. | 2,4 | $\left(p_{1}, p_{2}\right), p_{1}, p_{2} \geq 2$ | $\left(1,1,1, q_{4}\right)$ |
| 6. | 2,4 | $\left(p_{1}, p_{2}\right), p_{1}, p_{2} \geq 2$ | $\left(1,1, q_{3}, 1\right), q_{3} \geq 2$ |
| 7. | 3,3 | $\left(1, p_{2}, 1\right), p_{2} \geq 2$ | $\left(q_{1}, q_{2}, q_{3}\right)$ |
| 8. | 3,3 | $\left(1,1, p_{3}\right), p_{3} \geq 2$ | $\left(q_{1}, q_{2}, q_{3}\right)$ |

Table 1: The list of all complexity 1 double flag varieties for $G=S L(n)$.
In the rest of this section, we will describe the structure of the poset $U_{I, J}^{-}$for each pair of parabolic subgroups $\left(P_{I}, P_{J}\right)$ from Ponomareva's list. For $i \in\{1, \ldots, 8\}$, the $i$-th row of Table 1 will be analyzed in Subsection 3.i.

Notation 3.1. If $n$ is a positive integer, then we will use the shorthand $[n]$ to denote the set $\{1, \ldots, n\}$.

## 3.1 $B l\left(P_{I}\right)=\left(3, p_{2}\right), p_{2} \geq 3$ and $B l\left(P_{J}\right)=\left(q_{1}, q_{2}, q_{3}\right), q_{1}, q_{2}, q_{3} \geq 2$.

Let $n$ denote $3+p_{2}$, which is equal to $q_{1}+q_{2}+q_{3}$. Clearly, $n \geq 6$ and $p_{2}>q_{3}$. Since $I^{c}=\left\{s_{3}\right\}$, and $J^{c}=\left\{s_{q_{1}}, s_{q_{1}+q_{2}}\right\}$, we see that if $w=w_{1} \ldots w_{n} \in U_{I, J}^{-}$, then
(i) for $i \in[n-1] \backslash\{3\}, i$ comes before $i+1$ in $w$;
(ii) $w_{1}<\cdots<w_{q_{1}}, w_{q_{1}+1}<\cdots<w_{q_{1}+q_{2}}, w_{q_{1}+q_{2}+1}<\cdots<w_{n}$.

This implies that $1 \in\left\{w_{1}, w_{q_{1}+1}, w_{q_{1}+q_{2}+1}\right\}$, and that $n \in\left\{w_{q_{1}+q_{2}}, w_{n}\right\}$.
We start with the assumption that $q_{3} \geq 4$. By Remark 2.2 , we know that $U_{I, J}^{-}$is isomorphic to $U_{\theta(I), \theta(J)}^{-}$. Therefore, to prove that we can reduce to $q_{3} \leq 3$, we are going to work with the isomorphic poset $U_{\theta(I), \theta(J)}^{-}$, which is given by $B l\left(P_{\theta(I)}\right)=\left(p_{2}, 3\right), p_{2} \geq 3$ and $B l\left(P_{\theta(J)}\right)=\left(q_{3}, q_{2}, q_{1}\right), q_{1}, q_{2}, q_{3} \geq 2$. Note that $p_{2}=n-3$. Since $\theta(I)^{c}=\left\{s_{p_{2}}\right\}$, and $\theta(J)^{c}=\left\{s_{q_{3}}, s_{q_{3}+q_{2}}\right\}$, we see that if $w=w_{1} \ldots w_{n} \in U_{\theta(I), \theta(J)}^{-}$, then

1. for $i \in[n-1] \backslash\{n-3\}, i$ comes before $i+1$ in $w$;
2. $w_{1}<\cdots<w_{q_{3}}$, $w_{q_{3}+1}<\cdots<w_{q_{3}+q_{2}}, w_{q_{3}+q_{2}+1}<\cdots<w_{n}$.

This implies that $1 \in\left\{w_{1}, w_{q_{3}+1}, w_{q_{3}+q_{2}+1}\right\}$. If 1 appears as $w_{q_{3}+1}$ or $w_{q_{3}+q_{2}+1}$, then we cannot fit $2,3, \ldots, n-3$ in $w$ since they come after 1 in $w$. Therefore, we have $w_{1}=1$. Then we remove 1 from all permutations in $U_{\theta(I), \theta(J)}^{-}$and we reduce each remaining number by 1 . This operation gives us a poset $U_{\theta(I)^{\prime}, \theta(J)^{\prime}}^{\prime-}$, isomorphic to $U_{\theta(I), \theta(J)}^{-}$, where $B l\left(P_{\theta(I)^{\prime}}\right)=\left(p_{2}-1,3\right)$, $p_{2}-1 \geq 3$ and $B l\left(P_{\theta(J)^{\prime}}\right)=\left(q_{3}-1, q_{2}, q_{1}\right), q_{1}, q_{2}, q_{3}-1 \geq 2$. Therefore, we can assume that $q_{3} \leq 3$.

Let us proceed with the assumption that $q_{1} \geq 4$, and let $w=w_{1} \ldots w_{n}$ be an element from $U_{I, J}^{-}$. By condition (i), we know that 5 appears either in the first segment $w_{1} \ldots w_{q_{1}}$, or in the second segment $w_{q_{1}+1} \ldots w_{q_{1}+q_{2}}$. If it appears in the first segment, then 4 has to precede 5 otherwise it creates a descent which gives a contradiction. If 5 appears in the second segment $w_{q_{1}+1} \ldots w_{q_{1}+q_{2}}$, then we must have $w_{5}=5$ by conditions (i) and (ii), and by our assumption that $q_{1}+1 \geq 5$. In this case, condition (ii) shows that 4 has to be equal to $w_{4}$. These arguments show that if $q_{1} \geq 4$, then 4 precedes 5 in every element $w \in U_{I, J}^{-}$. Therefore, removing 4 from $w$ and reducing every number bigger than 4 by 1 give us a new poset $U_{I^{\prime}, J^{\prime}}^{-}$, isomorphic to $U_{I, J}^{-}$, where $\operatorname{Bl}\left(P_{I^{\prime}}\right)=\left(3, p_{2}-1\right), p_{2}-1 \geq 3$ and $\operatorname{Bl}\left(P_{J^{\prime}}\right)=\left(q_{1}-1, q_{2}, q_{3}\right)$, $q_{1}-1, q_{2}, q_{3} \geq 2$.

Now we assume that $q_{2} \geq 4$ along with $2 \leq q_{1}, q_{3} \leq 3$. We will look for where in $w=w_{1} \ldots w_{n} \in U_{I, J}^{-}$the numbers $n-q_{3}$ and $n-q_{3}+1$ appear. Since $q_{2} \geq 4$, we see from conditions (i) and (ii) that $n-q_{3}$ appears in the segment $w_{q_{1}+1}<\cdots<w_{q_{1}+q_{2}}$. We claim that if $w_{k}=w_{n-q_{3}}$ for some $k \in\left\{q_{1}+1, \ldots, q_{1}+q_{2}\right\}$, then $w_{k+1}=w_{n-q_{3}+1}$. This is clearly true if $n-q_{3}$ appears in the same segment $w_{q_{1}+1} \ldots w_{q_{1}+q_{2}}$ since there is no descents within this segment. On the other hand, if $n-q_{3}+1$ appears in the segment $w_{q_{1}+q_{2}+1}<\cdots<w_{n}$, then we must have $w_{q_{1}+q_{2}+1}=w_{n-q_{3}+1}=n-q_{3}+1$. But in this case, $w_{q_{1}+q_{2}+i}=n-q_{3}+i$, therefore, $w_{q_{1}+q_{2}}<w_{q_{1}+q_{2}+1}$. This implies that $n-q_{3}$ appears as the last entry $w_{q_{1}+q_{2}}$ of the segment $w_{q_{1}+1} \ldots w_{q_{1}+q_{2}}$, hence the proof of our claim follows. Now we know that $n-q_{3}$ and $n-q_{3}+1$ appear in any $w \in U_{I, J}^{-}$consecutively. Therefore, the removal of $n-q_{3}$ from $w$, and the reduction of all entries bigger than $n-q_{3}$ in $w$ by 1 gives a permutation in $S_{n-1}$. Furthermore, this operation preserves the relative ordering (in Bruhat-Chevalley order) of the elements of $U_{I, J}^{-}$. In other words, we obtain a new poset $U_{I^{\prime}, J^{\prime}}^{\prime}$, isomorphic to $U_{I, J}^{-}$, where $B l\left(P_{I^{\prime}}\right)=\left(3, p_{2}-1\right), p_{2}-1 \geq 3$ and $\operatorname{Bl}\left(P_{J^{\prime}}\right)=\left(q_{1}, q_{2}-1, q_{3}\right), q_{1}, q_{2}-1, q_{3} \geq 2$. These reduction arguments show that it suffices to consider the following eight cases only:

1. $B l\left(P_{I}\right)=(3,3), B l\left(P_{J}\right)=(2,2,2)$;
2. $B l\left(P_{I}\right)=(3,4), B l\left(P_{J}\right)=(2,2,3)$;
3. $B l\left(P_{I}\right)=(3,4), B l\left(P_{J}\right)=(3,2,2)$;
4. $B l\left(P_{I}\right)=(3,4), B l\left(P_{J}\right)=(2,3,2)$;
5. $B l\left(P_{I}\right)=(3,5), B l\left(P_{J}\right)=(2,3,3)$;
6. $B l\left(P_{I}\right)=(3,5), B l\left(P_{J}\right)=(3,2,3)$;
7. $B l\left(P_{I}\right)=(3,5), B l\left(P_{J}\right)=(3,3,2)$;
8. $B l\left(P_{I}\right)=(3,6), B l\left(P_{J}\right)=(3,3,3)$.

The Hasse diagrams of the posets corresponding to these eight cases are given by the diagrams $P .1-P .8$ in Figure 1.

## 3.2 $B l\left(P_{I}\right)=\left(p_{1}, p_{2}\right), p_{1}, p_{2} \geq 3$ and $\operatorname{Bl}\left(P_{J}\right)=\left(2,2, q_{3}\right), q_{3} \geq 2$.

First, we assume that $p_{2} \geq 5$, and we apply $\theta$ to $I$ and $J$. Then $\theta(I)^{c}=\left\{s_{p_{2}}\right\}$, and $\theta(J)^{c}=\left\{s_{n-4}, s_{n-2}\right\}$, we see that if $w=w_{1} \ldots w_{n} \in U_{\theta(I), \theta(J)}^{-}$, then
(i) for $i \in[n-1] \backslash\left\{p_{2}\right\}, i$ comes before $i+1$ in $w$;
(ii) $w_{1}<\cdots<w_{n-4}, w_{n-3}<w_{n-2}, w_{n-1}<w_{n}$.

This means that 1 is contained in $\left\{w_{1}, w_{n-3}, w_{n-1}\right\}$. Recall that $p_{2} \geq 5$. Thus, we cannot place the sequence $1,2, \ldots, p_{2}$ in $w$ as an increasing substring unless $w_{1}=1$. So, $w$ starts with 1 . Since this is true for all elements of $U_{\theta(I), \theta(J)}^{-}$, by first removing $w_{1}=1$ from all $w \in U_{\theta(I), \theta(J)}^{-}$, and then reducing the remaining entries by 1 , we obtain an isomorphic poset $U_{\theta(I)^{\prime}, \theta(J)^{\prime}}^{-}$in $S_{n-1}$, where $\theta(I)^{\prime c}=\left\{s_{p_{2}-1}\right\}$ and $\theta(J)^{\prime c}=\left\{s_{n-4}, s_{n-2}\right\}$. Therefore, we see that we can assume $p_{2} \leq 4$.

We now proceed with the assumption that $p_{1} \geq 5$ and that $p_{2} \leq 4$. If $w=w_{1} \ldots w_{n} \in$ $U_{I, J}^{-}$, then

1. for $i \in[n-1] \backslash\left\{p_{1}\right\}, i$ comes before $i+1$ in $w$;
2. $w_{1}<w_{2}, w_{3}<w_{4}, w_{5}<\cdots<w_{n}$.

We will look for where in $w=w_{1} \ldots w_{n}$ the numbers $p_{1}-1$ and $p_{1}$ appear. Since $p_{1} \geq 5$, we see from conditions 1 and 2 that $p_{1}$ appears in the segment $w_{5}<w_{6}<\cdots<w_{n}$. If $w_{k}=p_{1}$ and $k>5$, then clearly $w_{k-1}=p_{1}-1$ otherwise we must have a descent in the segment $w_{5} w_{6} \ldots w_{n}$, which would contradict with Condition 2. On the other hand, if $w_{5}=p_{1}$, then we see that $5=p_{1}$, hence $w_{4}=p_{1}-1$. In both of these cases, we see that if $w_{k}=p_{1}$, then $w_{k-1}=p_{1}-1$. Now, by removing $p_{1}$ from $w \in U_{I, J}^{-}$and reducing by 1 all entries $w_{j}$ with $w_{j}>p_{1}$, we obtain a poset $U_{I^{\prime}, J^{\prime}}^{-}$, isomorphic to $U_{I, J}^{-}$, in $S_{n-1}$. Furthermore, $B l\left(P_{I^{\prime}}\right)=\left(p_{1}-1, p_{2}\right), p_{1}-1, p_{2} \geq 3$ and $B l\left(P_{J^{\prime}}\right)=\left(2,2, q_{3}-1\right), q_{3}-1 \geq 2$. In other words, we can assume that $p_{1} \leq 4$.

These two reduction arguments show that it suffices to consider the following four cases only:

1. $B l\left(P_{I}\right)=(3,3), B l\left(P_{J}\right)=(2,2,2)$;
2. $B l\left(P_{I}\right)=(3,4), B l\left(P_{J}\right)=(2,2,3)$;
3. $B l\left(P_{I}\right)=(4,3), B l\left(P_{J}\right)=(2,2,3)$;
4. $B l\left(P_{I}\right)=(4,4), B l\left(P_{J}\right)=(2,2,4)$.

The Hasse diagrams of the posets corresponding to these four cases are given by the diagrams P.1, P.2, P.3, and P. 6 of Figure 1.

## 3.3 $B l\left(P_{I}\right)=\left(p_{1}, p_{2}\right), p_{1}, p_{2} \geq 3$ and $B l\left(P_{J}\right)=\left(2, q_{2}, 2\right), q_{2} \geq 2$.

First, we assume that $p_{1} \geq 5$. Since $I^{c}=\left\{s_{p_{1}}\right\}, J^{c}=\left\{s_{2}, s_{n-2}\right\}$ in $R_{n-1}$, we see that if $w=w_{1} \ldots w_{n} \in U_{I, J}^{-}$, then
(i) for $i \in[n-1] \backslash\left\{p_{1}\right\}, i$ comes before $i+1$ in $w$;
(ii) $w_{1}<w_{2}, w_{3}<\cdots<w_{n-2}, w_{n-1}<w_{n}$.

We look for the positions of $p_{1}-3$ and $p_{1}-2$. Since $p_{1} \geq 5$, we see from condition (i) that $p_{1}-2$ appears in the segment $w_{3} w_{4} \ldots w_{n-2}$. If $w_{k}=p_{1}-2$ for some $k>3$, then we see that $p_{1}-3$ must also be in the same segment, hence, we must have that $w_{k-1}=p_{1}-3$. If $w_{3}=p_{1}-2$, then, by conditions (i) and (ii), we have only one choice that $p_{1}=5$, and $p_{1}-3=2=w_{2}$. In both of these two cases we see that $p_{1}-3$ must come immediately before $p_{1}-2$ in every $w \in U_{I, J}^{-}$. Therefore, by removing $p_{1}-2$ from $w$ and reducing every entry which is greater than $p_{1}-2$ by 1 , we do not change the structure of the underlying poset; we obtain a poset $U_{I^{\prime}, J^{\prime}}^{-}$in $S_{n-1}$ such that $B l\left(P_{I^{\prime}}\right)=\left(p_{1}-1, p_{2}\right), p_{1}-1, p_{2} \geq 3$ and $B l\left(P_{J^{\prime}}\right)=\left(2, q_{2}-1,2\right), q_{2}-1 \geq 2$. In other words, we can assume that $p_{1} \leq 4$.

For $p_{2} \geq 5$, we repeat the same arguments after applying $\theta$ to $I$ and $J$. Therefore, without loss of generality we can assume that $3 \leq p_{1}, p_{2} \leq 4$. This reduction argument shows that our poset is isomorphic to one of the following three cases:

1. $B l\left(P_{I}\right)=(3,3), B l\left(P_{J}\right)=(2,2,2)$;
2. $B l\left(P_{I}\right)=(3,4), B l\left(P_{J}\right)=(2,3,2)$;
3. $B l\left(P_{I}\right)=(4,4), B l\left(P_{J}\right)=(2,4,2)$.

The Hasse diagrams of the posets corresponding to these three cases are given by the diagrams P.1, P. 4 and P. 9 in Figure 1.

## 3.4 $B l\left(P_{I}\right)=\left(2, p_{2}\right), p_{2} \geq 3$ and $\operatorname{Bl}\left(P_{J}\right)=\left(q_{1}, q_{2}, q_{3}, q_{4}\right)$.

Let us first assume that $q_{4} \geq 3$. Since $I^{c}=\left\{s_{2}\right\}, J^{c}=\left\{s_{q_{1}}, s_{q_{1}+q_{2}}, s_{q_{1}+q_{2}+q_{3}}\right\}$ in $R_{n-1}$, we see that if $w=w_{1} \ldots w_{n} \in U_{I, J}^{-}$, then
(i) for $i \in[n-1] \backslash\{2\}, i$ comes before $i+1$ in $w$;
(ii) $w_{1}<\cdots<w_{q_{1}}, w_{q_{1}+1}<\cdots<w_{q_{1}+q_{2}}, w_{q_{1}+q_{2}+1}<\cdots<w_{q_{1}+q_{2}+q_{3}}$, and $w_{q_{1}+q_{2}+q_{3}+1}<$ $\cdots<w_{n}$.

This implies that $n \in\left\{w_{q_{1}}, w_{q_{1}+q_{2}}, w_{q_{1}+q_{2}+q_{3}}, w_{n}\right\}$. By (i) we know that $n$ is preceded by $3, \ldots, n-1$, which prevents the possibilities $n \in\left\{w_{q_{1}}, w_{q_{1}+q_{2}}, w_{q_{1}+q_{2}+q_{3}}\right\}$. Therefore, $w_{n}=n$. Thus, by removing $n$ from $w \in U_{I, J}^{-}$, we do not change the structure of the underlying poset; we obtain a poset $U_{I^{\prime}, J^{\prime}}^{-}$in $S_{n-1}$, which is isomorphic to $U_{I, J}^{-}$, such that $B l\left(P_{I^{\prime}}\right)=\left(2, p_{2}-1\right)$, $p_{2}-1 \geq 3$ and $\operatorname{Bl}\left(P_{J^{\prime}}^{\prime}\right)=\left(q_{1}, q_{2}, q_{3}, q_{4}-1\right)$. In other words, we can assume without loss of generality that $1 \leq q_{4} \leq 2$.

We proceed with the assumption that $q_{3} \geq 3$. Then we look at the relative positions of the numbers $m:=q_{1}+q_{2}+q_{3}$ and $m+1$ in $w$. Since we assumed that $1 \leq q_{4} \leq 2$, we have $n \in\left\{w_{m+1}, w_{n}\right\}$. If $n=w_{m+1}$, then the following implication is obvious:

$$
w_{k}=m \Longrightarrow w_{k+1}=m+1
$$

On the other hand, if $n=w_{n}$, then since $q_{3} \geq 3$, we know that $m+1$ has to appear in the following segment of $w: w_{q_{1}+q_{2}+1} \ldots w_{q_{1}+q_{2}+q_{3}}$. In particular, we have one of the following cases:

$$
w_{q_{1}+q_{2}+q_{3}-i}=m \quad \text { and } w_{q_{1}+q_{2}+q_{3}-i+1}=m+1
$$

for $i=0,1$. Therefore, $m$ and $m+1$ appear as consecutive terms in $w$, furthermore, $m$ appears in $w_{q_{1}+q_{2}+1} \ldots w_{q_{1}+q_{2}+q_{3}}$. In this case, by removing $m$ from $w$ and reducing every number greater than $m$ by 1 , we obtain a poset $U_{I^{\prime}, J^{\prime}}^{-}$in $S_{n-1}$, which is isomorphic to $U_{I, J}^{-}$, such that $B l\left(P_{I^{\prime}}\right)=\left(2, p_{2}-1\right), p_{2}-1 \geq 3$ and $B l\left(P_{J^{\prime}}\right)=\left(q_{1}, q_{2}, q_{3}-1, q_{4}\right)$. In other words, we can assume without loss of generality that $1 \leq q_{3} \leq 2$ as well.

Next, we proceed with the assumptions that $q_{2} \geq 3$ and $1 \leq q_{3}, q_{4} \leq 2$. In this case, after applying the involution $\theta$ to $I$ and $J$, we assume that $B l\left(P_{I}\right)=\left(p_{2}, 2\right), p_{2} \geq 3$ and $B l\left(P_{J}\right)=\left(q_{4}, q_{3}, q_{2}, q_{1}\right)$, where $q_{2} \geq 3$ and $1 \leq q_{3}, q_{4} \leq 2$. In other words, we have one of the following four possibilities for the first few terms of $J$ :

1. $s_{1}, s_{3}, s_{5}, s_{6} \in J$ and $s_{2}, s_{4} \notin J$, or
2. $s_{1}, s_{4}, s_{5} \in J$ and $s_{2}, s_{3} \notin J$, or
3. $s_{2}, s_{4}, s_{5} \in J$ and $s_{1}, s_{3} \notin J$, or
4. $s_{3}, s_{4} \in J$ and $s_{1}, s_{2} \notin J$.

In the first case, we have that

$$
w_{k}=4 \Longrightarrow w_{k+1}=5
$$

for some $k \geq 1$. In the second case, we have

$$
w_{k}=3 \Longrightarrow w_{k+1}=4
$$

for some $k \geq 1$. In the third case, we have

$$
w_{k}=3 \Longrightarrow w_{k+1}=4
$$

for some $k \geq 1$. Finally, in the fourth case, we have

$$
w_{k}=2 \Longrightarrow w_{k+1}=3
$$

for some $k \geq 1$. In all of these cases, removing $w_{k+1}$ from $w$ and reducing every number that is greater than $w_{k+1}$ by 1 give a poset $U_{I^{\prime}, J^{\prime}}^{-}$in $S_{n-1}$, which is isomorphic to $U_{I, J}^{-}$, such that $B l\left(P_{I^{\prime}}\right)=\left(p_{2}-1,2\right), p_{2}-1 \geq 3$ and $\operatorname{Bl}\left(P_{J^{\prime}}\right)=\left(q_{4}, q_{3}, q_{2}-1, q_{1}\right)$. In other words, we can assume without loss of generality that $1 \leq q_{2} \leq 2$.

Finally, we assume that $q_{1} \geq 3$ and $1 \leq q_{2}, q_{3}, q_{4} \leq 2$. The proof of this case develops similar to the previous case; we apply $\theta$ to $I$ and $J$; we assume that $B l\left(P_{I}\right)=\left(p_{2}, 2\right), p_{2} \geq 3$ and $B l\left(P_{J}\right)=\left(q_{4}, q_{3}, q_{2}, q_{1}\right)$, where $q_{1} \geq 3$ and $1 \leq q_{2}, q_{3}, q_{4} \leq 2$. This time we have 8 possibilities, instead of 4 as in the previous case. In each of these eight cases, we consider the simple reflection $s_{j}$ with smallest index $j$ among the elements of $J$ associated to its block of size $q_{1}$. Then, as in the previous case,

$$
w_{k}=j-1 \Longrightarrow w_{k+1}=j
$$

for some $k \geq 1$. Therefore, removing $j$ from $w$ and reducing every number that is greater than $j$ by 1 give a poset $U_{I^{\prime}, J^{\prime}}^{-}$in $S_{n-1}$, isomorphic to $U_{I, J}^{-}$, such that $B l\left(P_{I^{\prime}}\right)=\left(p_{2}-1,2\right), p_{2}-1 \geq 3$ and $B l\left(P_{J^{\prime}}\right)=\left(q_{4}, q_{3}, q_{2}, q_{1}-1\right)$. In other words, we can assume without loss of generality that $1 \leq q_{1} \leq 2$. Now we know that if $B l\left(P_{I}\right)=\left(2, p_{2}\right), p_{2} \geq 3$ and $B l\left(P_{J}\right)=\left(q_{1}, q_{2}, q_{3}, q_{4}\right)$, then $U_{I, J}^{-}$is isomorphic to one of the following 15 cases:

1. $B l\left(P_{I}\right)=(2,3)$ and $B l\left(P_{J}\right)=(1,1,1,2)$;
2. $B l\left(P_{I}\right)=(2,3)$ and $B l\left(P_{J}\right)=(1,1,2,1)$;
3. $\operatorname{Bl}\left(P_{I}\right)=(2,3)$ and $B l\left(P_{J}\right)=(1,2,1,1)$;
4. $B l\left(P_{I}\right)=(2,3)$ and $B l\left(P_{J}\right)=(2,1,1,1)$;
5. $B l\left(P_{I}\right)=(2,4)$ and $B l\left(P_{J}\right)=(1,1,2,2)$;
6. $B l\left(P_{I}\right)=(2,4)$ and $B l\left(P_{J}\right)=(1,2,1,2)$;
7. $\operatorname{Bl}\left(P_{I}\right)=(2,4)$ and $B l\left(P_{J}\right)=(2,1,1,2)$;
8. $B l\left(P_{I}\right)=(2,4)$ and $B l\left(P_{J}\right)=(1,2,2,1)$;
9. $B l\left(P_{I}\right)=(2,4)$ and $B l\left(P_{J}\right)=(2,1,2,1)$;
10. $B l\left(P_{I}\right)=(2,4)$ and $B l\left(P_{J}\right)=(2,2,1,1)$;
11. $B l\left(P_{I}\right)=(2,5)$ and $B l\left(P_{J}\right)=(1,2,2,2)$;
12. $B l\left(P_{I}\right)=(2,5)$ and $B l\left(P_{J}\right)=(2,1,2,2)$;
13. $B l\left(P_{I}\right)=(2,5)$ and $B l\left(P_{J}\right)=(2,2,1,2)$;
14. $B l\left(P_{I}\right)=(2,5)$ and $B l\left(P_{J}\right)=(2,2,2,1)$;
15. $B l\left(P_{I}\right)=(2,6)$ and $B l\left(P_{J}\right)=(2,2,2,2)$.

The Hasse diagrams of the posets corresponding to these 15 cases are given by the diagrams P.10, P.11, P.12, P.13, P.14, P.4, P.15, P.4, P.15, P.14, P.5, P.16, P.17, P.7, P. 8 in Figure 1. Note that several Hasse diagrams appear multiple times in this list.

## 3.5 $B l\left(P_{I}\right)=\left(p_{1}, p_{2}\right), p_{1}, p_{2} \geq 2$ and $B l\left(P_{J}\right)=\left(1,1,1, q_{4}\right)$.

We consider this situation in two different cases:
(a) $B l\left(P_{I}\right)=(2,2)$ and $B l\left(P_{J}\right)=(1,1,1,1)$;
(b) $B l\left(P_{I}\right)=\left(p_{1}, p_{2}\right), p_{1}, p_{2} \geq 2$ and $B l\left(P_{J}\right)=\left(1,1,1, q_{4}\right), q_{4} \geq 2$.

We explain the reduction argument for (b); we claim that we can assume $2 \leq p_{1}, p_{2} \leq 3$.
First, we assume that $p_{2} \geq 4$. Since $I^{c}=\left\{s_{p_{1}}\right\}, J^{c}=\left\{s_{1}, s_{2}, s_{3}\right\}$ in $R_{n-1}$, we see that if $w=w_{1} \ldots w_{n} \in U_{I, J}^{-}$, then
(i) for $i \in[n-1] \backslash\left\{p_{1}\right\}$, $i$ comes before $i+1$ in $w$;
(ii) $w_{4}<\cdots<w_{n}$.

Therefore, $n \in\left\{w_{1}, w_{2}, w_{3}, w_{n}\right\}$. But there are at least $p_{2}-1 \geq 3$ numbers before $n$ in $w$, therefore, $n$ cannot appear in $\left\{w_{1}, w_{2}, w_{3}\right\}$. This means that $n$ is equal to $w_{n}$. Now we see that removing $n$ from $w$, for all $w \in U_{I, J}^{-}$gives us an isomorphic poset $U_{I^{\prime}, J^{\prime}}^{-}$, where $B l\left(P_{I^{\prime}}\right)=\left(p_{1}, p_{2}-1\right), p_{2}-1, p_{1} \geq 2$ and $B l\left(P_{J^{\prime}}\right)=\left(1,1,1, q_{4}-1\right), q_{4}-1 \geq 2$.

We now proceed with the assumption that $p_{1} \geq 4$. In this case, we look at the relative positions of numbers 3 and 4 . If 3 appears in the segment $w_{4} w_{5} \ldots w_{n}$, then 3 is immediately followed by 4 since there are no descents in this portion of $w$. On the other hand, if 3 does not appear in the segment $w_{4} w_{5} \ldots w_{n}$, then it can only appear at $w_{3}$ since in this case it has to be preceded by 1 and 2 by condition (i). But then, 4 has to appear as $w_{4}$, otherwise, there would be a descent in $w_{4} w_{5} \ldots w_{n}$. This argument shows that the numbers 3 and 4 appear in $w$ consecutively. Hence, if we remove 4 from $w$, and reduce every number greater than 4 by 1 , then we do not change the Bruhat-Chevalley order. In other words, we obtain a poset $U_{I^{\prime}, J^{\prime}}^{-}$, isomorphic to $U_{I, J}^{-}$, where $B l\left(P_{I^{\prime}}\right)=\left(p_{1}-1, p_{2}\right), p_{1}-1, p_{2} \geq 2$ and $B l\left(P_{J^{\prime}}\right)=\left(1,1,1, q_{4}-1\right)$, $q_{4}-1 \geq 2$. Therefore, we can assume that $p_{1} \leq 3$. Consequently, we see in this case that there are only the following four possibilities:

1. $\operatorname{Bl}\left(P_{I}\right)=(2,2)$ and $B l\left(P_{J}\right)=(1,1,1,1)$;
2. $B l\left(P_{I}\right)=(2,3)$ and $B l\left(P_{J}\right)=(1,1,1,2)$;
3. $B l\left(P_{I}\right)=(3,2)$ and $B l\left(P_{J}\right)=(1,1,1,2)$;
4. $B l\left(P_{I}\right)=(3,3)$ and $B l\left(P_{J}\right)=(1,1,1,3)$.

The Hasse diagrams of the corresponding posets of these four cases are given by the diagrams $P .18, P .10, P .13$, and $P .19$ in Figure 1.

## 3.6 $\operatorname{Bl}\left(P_{I}\right)=\left(p_{1}, p_{2}\right), p_{1}, p_{2} \geq 2$ and $B l\left(P_{J}\right)=\left(1,1, q_{3}, 1\right), q_{3} \geq 2$.

By arguing as in the previous cases, we see that each subcase reduces to the one of the following three subcases:

1. $B l\left(P_{I}\right)=(2,3)$ and $B l\left(P_{J}\right)=(1,1,2,1)$;
2. $B l\left(P_{I}\right)=(3,2)$ and $B l\left(P_{J}\right)=(1,1,2,1)$;
3. $B l\left(P_{I}\right)=(3,3)$ and $B l\left(P_{J}\right)=(1,1,3,1)$.

The Hasse diagrams of the corresponding posets of these three cases are given by P.11, P. 12 and $P .20$ in Figure 1.

## 3.7 $B l\left(P_{I}\right)=\left(1, p_{2}, 1\right), p_{2} \geq 2$ and $\operatorname{Bl}\left(P_{J}\right)=\left(q_{1}, q_{2}, q_{3}\right)$.

In this case, by the appropriate reduction arguments as in the previous cases we may assume that $q_{1}, q_{2}, q_{3} \leq 2$. Then we obtain the following five cases:

1. $B l\left(P_{I}\right)=(1,2,1)$ and $B l\left(P_{J}\right)=(1,1,2)$;
2. $B l\left(P_{I}\right)=(1,2,1)$ and $B l\left(P_{J}\right)=(1,2,1)$;
3. $\operatorname{Bl}\left(P_{I}\right)=(1,3,1)$ and $B l\left(P_{J}\right)=(1,2,2)$;
4. $B l\left(P_{I}\right)=(1,3,1)$ and $B l\left(P_{J}\right)=(2,1,2)$;
5. $B l\left(P_{I}\right)=(1,4,1)$ and $B l\left(P_{J}\right)=(2,2,2)$.

The Hasse diagrams of the corresponding posets of these five cases are given by P.21, P.1, P.4, P.22, and P. 9 in Figure 1.
3.8 $B l\left(P_{I}\right)=\left(1,1, p_{3}\right), p_{3} \geq 2$ and $B l\left(P_{J}\right)=\left(q_{1}, q_{2}, q_{3}\right)$.

Once again, by the appropriate reduction arguments as we did in the previous cases, we may assume that $q_{1}, q_{2}, q_{3} \leq 2$. Then we obtain the following five cases:

1. $B l\left(P_{I}\right)=(1,1,2)$ and $B l\left(P_{J}\right)=(1,1,2)$;
2. $B l\left(P_{I}\right)=(1,1,2)$ and $B l\left(P_{J}\right)=(1,2,1)$;
3. $B l\left(P_{I}\right)=(1,1,2)$ and $B l\left(P_{J}\right)=(2,1,1)$;
4. $B l\left(P_{I}\right)=(1,1,3)$ and $B l\left(P_{J}\right)=(1,2,2)$;
5. $B l\left(P_{I}\right)=(1,1,3)$ and $B l\left(P_{J}\right)=(2,1,2)$;
6. $\operatorname{Bl}\left(P_{I}\right)=(1,1,3)$ and $\operatorname{Bl}\left(P_{J}\right)=(2,2,1)$;

P. 1




P. 6

$P .7$
P. 5







P. 8
$P .9$
P. 10
$P .11$
$P .12$
P. 13
$P .14$

P. 15

$P .16$



P. 19
$P .20$

$P .21$

P.22

P. 23

$P .24$

$P .25$

P. 26

P.27


Figure 1: The inclusion posets of $S L(n)$ orbit closures in complexity 1.
7. $\operatorname{Bl}\left(P_{I}\right)=(1,1,4)$ and $\operatorname{Bl}\left(P_{J}\right)=(2,2,2)$;

The Hasse diagrams of the corresponding posets of these seven cases are given by P.23, P.21, $P .24, P .25, P .26, P .27$, and $P .28$ in Figure 1.

## 4 Proof of Theorem 1.1

Let $P_{I}$ and $P_{J}$ be two standard parabolic subgroups of $S L(n)$ such that the diagonal action $S L(n): S L(n) / P_{I} \times S L(n) / P_{J}$ has complexity 1. Then the block sizes of $P_{I}$ and $P_{J}$ 's are listed in Table 1. Let $P$ denote the corresponding inclusion poset of the $S L(n)$-orbit closures. The computations that we performed in the previous section show that the Hasse diagram of $P$ is one of the 28 non-isomorphic Hasse diagrams which are depicted in Figure 1. Using this figure, it is easy to verify the following assertions:

- the posets P.i $(i \in\{1, \ldots, 28\})$ have at most 10 elements;
- P.21, P.22, P.25, P.27, P. 28 are non-graded posets. The poset $P .21$ appears in both of the cases of the 7 -th and the 8 -th rows of Table 1. The poset $P .22$ appears only in the case of the 7 -th row of Table 1. The posets $P .25, P .27$, and $P .28$ appear only in the case of the 8 -th row of Table 1 .
- The posets $P .1-P .20$ are lattices, and $P .21-P .28$ are non-lattices. In particular, all posets of the 8 -th row of Table 1 are non-lattices, and two of the five posets of the 7 -th row, namely $P .21$ and $P .22$, are non-lattices.

This finishes the proof of Theorem 1.1.
Remark 4.1. The height of a finite poset is the maximum of the lengths of its saturated chains. The maximum of the set of the heights of P.i's $(i \in\{1, \ldots, 28\})$ is 6 .

## 5 Final Remarks

There is an alternative approach, which is attributed to Bongartz [2], for studying the order relations between the closures of the diagonal $G$-orbits in a double flag variety $G / P_{I} \times G / P_{J}$. For completeness and for the convenience to the reader, next, we will summarize this approach, as described by Magyar, Weyman, and Zelevinsky in [12, Example 4.7]; the exact statement is somewhat difficult to see from Bongartz's original paper.

Let $P_{I}$ and $P_{J}$ denote the corresponding standard parabolic subgroups in $G$. As before, let us denote by $B l\left(P_{I}\right)=\left(p_{1}, \ldots, p_{r}\right)$ and $B l\left(P_{J}\right)=\left(q_{1}, \ldots, q_{s}\right)$ the sizes of the blocks of $P_{I}$ and $P_{J}$, respectively. For every $G$-orbit in $G / P_{I} \times G / P_{J}$, there is a nonnegative integer matrix $M=\left(m_{i j}\right)$ with row sums $p_{1}, \ldots, p_{r}$, and with column sums $q_{1}, \ldots, q_{s}$. By [12, Proposition 4.5] and by the general results of Bongartz, if $M=\left(m_{i j}\right)$ and $M^{\prime}=\left(m_{i j}^{\prime}\right)$ are two such matrices corresponding to the $G$-orbits $O_{1}$ and $O_{2}$ in $G / P_{I} \times G / P_{J}$, then

$$
\begin{equation*}
O_{1} \subseteq \overline{O_{2}} \Longleftrightarrow \sum_{k=1}^{i} \sum_{l=1}^{j} m_{k l} \geq \sum_{k=1}^{i} \sum_{l=1}^{j} m_{k l}^{\prime} \text { for all } i \text { and } j \tag{2}
\end{equation*}
$$

This ordering is helpful if the data of two $G$-orbits are provided. However, to obtain the full Hasse diagram from (2), one needs to generate all possible matrices, and then compare them
by using the double-summations as in (2). We checked our Figure 1 by following these steps as well. Our conclusion is that the amount of work that is required for the creation of the Hasse diagrams in our method, which uses the minimal double coset representatives, and the method of Bongartz, which uses matrices, do not significantly differ from each other.

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