

The Distinguishing Number and Distinguishing Chromatic Number for Posets

Karen L. Collins

Dept. of Mathematics and Computer Science

Wesleyan University

Middletown CT 06459-0128

kcollins@wesleyan.edu

Ann N. Trenk*

Department of Mathematics

Wellesley College

Wellesley MA 02481

atrenk@wellesley.edu

June 23, 2020

Abstract

In this paper we introduce the concepts of the distinguishing number and the distinguishing chromatic number of a poset. For a distributive lattice L and its set Q_L of join-irreducibles, we use classic lattice theory to show that any linear extension of Q_L generates a distinguishing 2-coloring of L . We prove general upper bounds for the distinguishing chromatic number and particular upper bounds for the Boolean lattice and for divisibility lattices. In addition, we show that the distinguishing number of any twin-free Cohen-Macaulay planar lattice is at most 2.

Keywords: distributive lattice, distinguishing number, distinguishing chromatic number, Birkhoff's theorem

1 Introduction

The distinguishing number of a graph, introduced by Albertson and Collins [1], is the smallest integer k for which the vertices can be colored using k colors so that the only automorphism of the graph that preserves colors is the identity. The distinguishing chromatic number, introduced by Collins and Trenk [9], has the additional requirement that the coloring of the vertices is proper, that is, adjacent vertices get different colors. The distinguishing number of graph G is denoted by $D(G)$ and the distinguishing chromatic number by $\chi_D(G)$. These and related topics have received considerable

*This work was supported by a grant from the Simons Foundation (#426725, Ann Trenk).

attention by many authors in recent years; see, for example, [2, 4, 10, 13, 15, 16]. In this paper, we introduce the distinguishing number and the distinguishing chromatic number of a poset.

There are several challenges in studying these parameters. A distinguishing coloring of a graph or poset does not always yield a distinguishing coloring of induced subgraphs or subposets. It is possible to have a graph H induced in graph G for which $D(H) > D(G)$, and the same holds for posets. We provide an example of this following Definition 7. In addition, the structure inherent in posets makes these parameters qualitatively different from the graph versions.

We end this section with an overview of the rest of the paper. In Section 2, we provide background material about posets, lattices, and distributive lattices. We introduce the distinguishing number of a poset in Section 3, and prove results about sums of chains, distributive lattices, and divisibility lattices. In Section 4, we study the distinguishing chromatic number, giving upper bounds for the distinguishing chromatic number of distributive lattices, divisibility lattices, and Boolean lattices. We also show that there exist posets P for which the gap between the distinguishing chromatic number of P and that of its comparability graph is arbitrarily large. We return to the distinguishing number in Section 5 and focus on ranked planar lattices (equivalently, Cohen-Macaulay) that are rank-connected. We conclude with several open questions.

2 Preliminaries

In this section we provide definitions related to posets and lattices, and present Birkhoff's classic lattice theorem, which we use as a tool in Section 3 and Section 4. For additional details and background, see [18].

2.1 General poset definitions

The posets we consider are finite and reflexive. If P is the poset $(X, \preceq)$, we call X the *ground set* of P and refer to the elements of X (which we also call the elements of P) as *points*. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. If $x \preceq y$ or $y \preceq x$, we say points x and y are *comparable*, and otherwise they are *incomparable*. We say that y *covers* x if $x \prec y$ and there is no other point v with $x \prec v \prec y$. An *automorphism* of poset $P = (X, \preceq)$ is a bijection from X to X that preserves the relation $\preceq$.

A set of pairwise comparable points in a poset is called a *chain*, and if the points are pairwise incomparable they form an *antichain*. An r -chain is a chain with r points, and such a chain has *length* $r - 1$. The *height* of a poset is the size of a maximum chain and the *width* is the size of a maximum antichain.

If a poset has a unique minimal element, we call this element $\hat{0}$ and if it has a unique maximal element we call it $\hat{1}$. We say that a poset with a $\hat{0}$ and $\hat{1}$ is *ranked* if every maximal chain from $\hat{0}$ to $\hat{1}$ has the same length. The *rank* of a point x in a

poset, denoted by $\text{rank}(x)$ or $r(x)$, is the length of a longest chain that has x as its largest element. For example, in Figure 1, each of posets L_{pq^2} , $L_{p^2q^2}$, M and L_{pqr} has a $\hat{0}$ and a $\hat{1}$, while poset S_4 has neither, and L_{pq^2} , $L_{p^2q^2}$ and L_{pqr} are ranked, while poset M is not.

A poset is *planar* if its Hasse diagram can be drawn in the plane with no edges crossing and so that the edge from a to b has strictly increasing y -coordinate when $a \prec b$. In Figure 1, L_{pq^2} , $L_{p^2q^2}$, M and S_4 are planar, even though the drawing shown of S_4 has edges crossing. We demonstrate in Section 5 that poset L_{pqr} is not planar (see Remark 34).

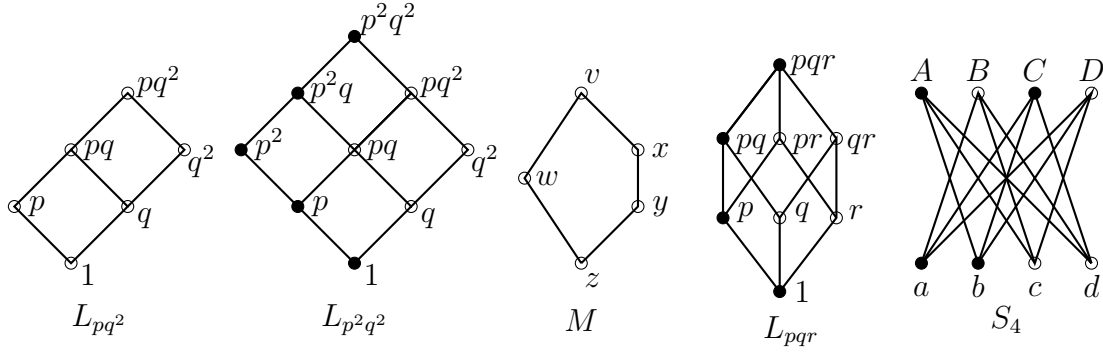


Figure 1: Examples of posets with distinguishing labelings (p , q and r are distinct primes).

2.2 Lattice definitions

A point z in poset P is called the *meet* of x and y in P , and denoted by $x \wedge y$, if it is the unique largest element in P such that $z \preceq x$ and $z \preceq y$. Thus, if $x \wedge y$ exists and $a \preceq x$ and $a \preceq y$, then $a \preceq x \wedge y$. Similarly, a point $w \in P$ is called the *join* of x and y in P , and denoted by $x \vee y$, if it is the unique smallest element $w \in P$ such that $w \succeq x$ and $w \succeq y$. Thus, if $x \vee y$ exists and $a \succeq x$ and $a \succeq y$, then $a \succeq x \vee y$.

A poset L is a *lattice* if $x \wedge y$ and $x \vee y$ both exist for all points x and y in L . Furthermore, L is a *distributive lattice* if $\wedge$ and $\vee$ satisfy the distributive laws

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$$

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

for all $x, y, z \in L$. For example, all the posets in Figure 1 are lattices except for S_4 , and all the lattices are distributive except for M . A lattice need not be ranked (M is not ranked), but a distributive lattice is ranked, as a consequence of Birkhoff's Theorem (Theorem 3 in Section 2.3).

A point x in a lattice is called *join-irreducible* if in the Hasse diagram of the lattice x has exactly one downward edge. For example, in Figure 1, the join-irreducible points

of L_{pq^2} are p, q , and q^2 , while there are no join-irreducible points in poset S_4 . As we will see in Birkhoff's Theorem, the join-irreducible points of a distributive lattice generate all the elements in the lattice by the join operation. In this way, they act like the prime numbers in the prime factorization of an integer.

2.3 Birkhoff's Theorem

In this section, we present a fundamental theorem due to Birkhoff (Theorem 3) and a corollary, both of which will be used as tools in later sections of the paper. Let poset $P = (X, \preceq)$. The *downset* of a point $a \in X$ is defined as $down(a) = \{x \in X : x \preceq a\}$ and the downset of a subset $A \subseteq X$ is defined as $down(A) = \{x \in X : x \preceq a \text{ for some } a \in A\}$.

Definition 1. Let $P = (X, \preceq)$ be a poset. The *downset lattice* $J(P)$ has ground set $\{down(S) : S \subseteq X\}$ and the relation is $\subseteq$.

Observe that if P is a poset, then $J(P)$ is a distributive lattice, in which the meet of elements S and T is $S \cap T$ and the join of these elements is $S \cup T$.

Example 2. In Figure 2, the join-irreducible elements of L_{150} are $a = 2, b = 3, c = 5$, and $d = 25$. When these are ordered using the ordering induced by L_{150} they produce the poset labeled Q_L also shown in Figure 2. There are 16 subsets of elements of Q_L , producing 12 distinct downsets, which are given in the following table. When these 12 downsets are ordered by set inclusion, we obtain the downset lattice $J(Q_L)$ which is isomorphic to the original lattice L_{150} .

S	$\emptyset$	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
$down(S)$	$\emptyset$	a	b	c	cd	ab	ac	acd	bc	bcd	cd	abc	$abcd$	acd	bcd	$abcd$

This example illustrates the following classic theorem due to Birkhoff [5] and called the *Fundamental Theorem of Distributive Lattices* in [18].

Theorem 3. If L is a distributive lattice and Q_L is the poset induced by the join-irreducible points of L , then $J(Q_L)$ is isomorphic to L . Indeed, the function $f : L \rightarrow J(Q_L)$ defined by $f(w) = \{y \in Q_L : y \preceq w\}$ is an isomorphism.

The following notation will be helpful as we use Theorem 3 repeatedly.

Definition 4. For a distributive lattice L , denote by Q_L the induced poset of all join-irreducible points of L .

Birkhoff's theorem is fundamental in several ways. First it provides a method for checking whether a poset L is a distributive lattice without having to verify that every pair of points has a meet and a join, namely, construct the induced poset Q_L and check whether the mapping f from Theorem 3 is an isomorphism. Additionally, any

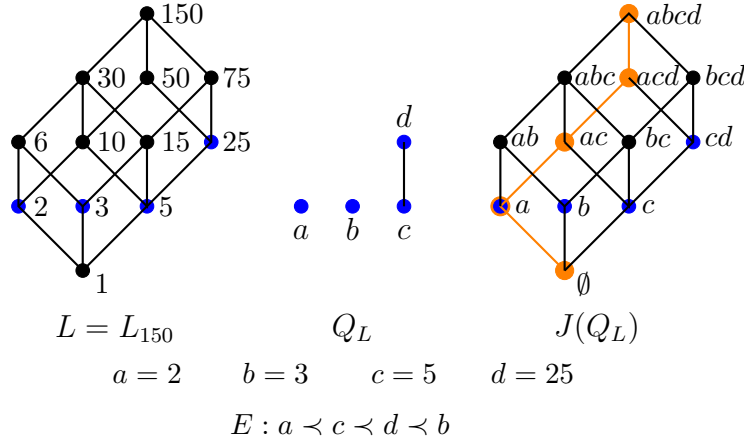


Figure 2: The lattice L_{150} , its poset Q_L of join-irreducibles, a linear extension E of Q_L , and the downset lattice $J(Q_L)$, together with a distinguishing coloring of $J(Q_L)$.

distributive lattice can be generated by starting with a poset P and constructing $J(P)$. We utilize Theorem 3 in proving that all distributive lattices have distinguishing number at most two (Theorem 14) and in characterizing those that have distinguishing number one (Theorem 13). A proof of Theorem 3 appears in [18].

Observe that in lattice L_{150} shown in Figure 2, the point 75 can be written as the join of all the join-irreducibles less or equal to it, namely $75 = 3 \vee 5 \vee 25$. Equivalently, in $J(Q_L)$, $\{b, c, d\} = \{b\} \cup \{c\} \cup \{c, d\}$. The next corollary shows this holds in general, that is, every point $w \in L$ can be written as the join of a unique subset of join-irreducibles of L . It is a well-known consequence of Birkhoff's Theorem and we provide a proof for completeness.

Corollary 5. *Let L be a distributive lattice and $f : L \rightarrow J(Q_L)$ be the isomorphism from Theorem 3. If $w \in L$, then $w = \bigvee_{z \in f(w)} z$.*

Proof. Fix $w \in L$ and let $f(w) = \{y_1, y_2, \dots, y_t\}$. For each $i : 1 \leq i \leq t$, the reflexive property implies that $y_i \in f(y_i)$, and the fact that $f(w)$ is a downset implies that $f(y_i) \subseteq f(w)$. Therefore, $f(w) = f(y_1) \cup f(y_2) \cup \dots \cup f(y_t)$. However, f is an isomorphism, so applying f^{-1} to both sides yields $w = y_1 \vee y_2 \vee \dots \vee y_t$, as desired. $\square$

For a distributive lattice L , the points of $J(Q_L)$ are downsets, and the rank of each point is the cardinality of its downset. We record this below.

Remark 6. *Every distributive lattice L is ranked and the rank of a point w is $|\{z \in Q_L : z \preceq w\}|$.*

3 Distinguishing numbers

We begin with the definition of the distinguishing number of a poset and some examples.

Definition 7. A coloring of the points of poset P is *distinguishing* if the only automorphism of P that preserves colors is the identity. The *distinguishing number* of P , denoted $D(P)$ is the least integer k so that P has a distinguishing coloring using k colors.

Distinguishing colorings are shown in Figure 1, and the distinguishing numbers are the following: $D(L_{pq^2}) = 1$, $D(L_{p^2q^2}) = 2$, $D(M) = 1$, $D(L_{pqr}) = 2$, $D(S_4) = 2$. Note that while $D(M) = 1$, if we remove point x from M , the resulting induced subposet $M - x$ has $D(M - x) = 2$; thus an induced subposet can have a larger distinguishing number than that of the original. Observe that antichains are the only posets for which each point must receive a different color in a distinguishing coloring.

The *comparability graph* of poset P is the graph $G_P = (V, E)$ where V is the ground set of P and $xy \in E$ if and only if x and y are comparable in P . Any automorphism of a poset P is also an automorphism of its comparability graph, G_P . This justifies the following remark.

Remark 8. $D(P) \leq D(G_P)$.

Some automorphisms of the graph G_P are not automorphisms of the poset P because they do not preserve the ordering of points in P . The following example shows that $D(P)$ can differ significantly from $D(G_P)$ and $D(\overline{G_P})$. If P is an n -chain then $D(P) = 1$. However, the comparability graph of P is the complete graph K_n , and the incomparability graph of P is its complement $\overline{K_n}$, and each of these has distinguishing number n .

3.1 Sums of chains

In the next two results, we find the distinguishing number for posets consisting of the sum of chains.

Proposition 9. *Let P be the poset consisting of the sum of t chains, each consisting of r points and let k be the positive integer for which $(k - 1)^r < t \leq k^r$. Then $D(P) = k$.*

Proof. First we find a distinguishing coloring of P using k colors. There are k^r different ways to color the elements of an r -chain when k colors are available. Coloring the elements of each r -chain differently is a distinguishing coloring since any automorphism of P maps an r -chain to an r -chain. Thus $D(P) \leq k$. We next show $D(P) > k - 1$. For a contradiction, suppose there is a distinguishing coloring of P using $k - 1$ colors. There are $(k - 1)^r$ ways to color each chain and since $t > (k - 1)^r$,

two chains have the same coloring. The automorphism that swaps those two chains is non-trivial, a contradiction. $\square$

Combining Proposition 9 with the following proposition, allows us to compute the distinguishing number of any poset that consists of the sum of chains.

Proposition 10. *Let P be the sum of chains and partition P as $P_1 + P_2 + \cdots + P_m$ where P_i consists of t_i chains, each consisting of r_i points, where $r_1, r_2, \dots, r_m$ are distinct. Then $D(P) = \max\{D(P_i) : 1 \leq i \leq m\}$.*

Proof. The result follows immediately from the fact that any automorphism of P will map P_i to itself for each i . $\square$

3.2 Distributive lattices

We find the distinguishing number of any distributive lattice in Theorems 13 and 14.

In showing that a coloring is distinguishing it can be helpful to analyze the points individually or in groups using the following concept of *pinning*.

Definition 11. Let P be a poset with a color assigned to each point. We say that a point x is *pinned* if every automorphism of P that preserves colors maps x to itself.

Note that a coloring of the ground set of a poset P is distinguishing precisely when every point is pinned.

Proposition 12. *Let ϕ be any coloring of a distributive lattice L . If ϕ restricted to Q_L pins every point of Q_L , then ϕ pins every point of L .*

Proof. Let ϕ be a coloring of L so that ϕ restricted to Q_L pins every point of Q_L . By Corollary 5, every element of L is the join of a unique set of elements of Q_L . Since joins are preserved by isomorphism, it follows that every point of L is pinned. $\square$

We now have the tools to determine the distinguishing number of any distributive lattice. Theorem 14 shows that any distributive lattice has distinguishing number at most two and Theorem 13 characterizes those distributive lattices whose distinguishing number is one. The proof of Theorem 14 is illustrated in Examples 15 and 16.

Theorem 13. *If L is a distributive lattice, then $D(L) = 1$ if and only if $D(Q_L) = 1$.*

Proof. By definition, any poset has distinguishing number equal to 1 if and only if it has no non-trivial automorphisms. Let ϕ be a coloring of L in which every vertex is colored the same. If Q_L has no non-trivial automorphisms, then every point in Q_L is pinned by ϕ and by Proposition 12, every point in L is pinned. Conversely, if Q_L has a non-trivial automorphism σ , then by Corollary 5, σ can be extended to a non-trivial automorphism of L , contradicting $D(L) = 1$. $\square$

Theorem 14. *If $L = (X, \preceq)$ is a distributive lattice, then $D(L) \leq 2$ and $D(L) = 2$ if and only if $D(Q_L) > 1$.*

Proof. Let $L = (X, \preceq)$ be a distributive lattice and $Q_L = (Y, \prec)$ where $Y = \{y_1, y_2, \dots, y_t\}$. By Theorem 3, L is isomorphic to $J(Q_L)$. We will provide a distinguishing coloring of $J(Q_L)$ using two colors, showing $D(L) \leq 2$. The remainder of the theorem follows from Theorem 13.

Let $f : L \rightarrow J(Q_L)$ be the isomorphism defined in Theorem 3, and let $f(Y) = \{f(y_1), f(y_2), \dots, f(y_t)\}$. The property of being join-irreducible is preserved under isomorphism, thus $f(Y)$ is the set of join-irreducible points of $J(Q_L)$.

Let $E : y_1 \prec y_2 \prec y_3 \prec \dots \prec y_t$ be a linear extension of Q_L . Color the following chain of elements of $J(Q_L)$ using the color red:

$\{f(y_1)\}, \{f(y_1), f(y_2)\}, \{f(y_1), f(y_2), f(y_3)\}, \dots, \{f(y_1), f(y_2), f(y_3), \dots, f(y_{t-1})\}$. Color the remaining elements green. We show this is a distinguishing coloring of $J(Q_L)$ by showing that every nontrivial automorphism of $J(Q_L)$ preserves colors.

Since poset automorphisms preserve rank and there is at most one red vertex at each rank of $J(Q_L)$, we know the red vertices are pinned. Next we show all green points in $f(Y)$ are pinned. For $2 \leq i \leq t-1$, each $f(y_i) \in f(Y)$ is less than a unique lowest red point in the chain of red vertices of $J(Q_L)$. In particular, $\{f(y_i)\} \preceq \{f(y_1), f(y_2), f(y_3), \dots, f(y_i)\}$ but $\{f(y_i)\}$ is incomparable to all lower ranked red points. Hence each is pinned. Also, $f(y_t)$ is the only point in $f(Y)$ that is not less than any red vertex, hence it is pinned. Thus the green points in $f(Y)$ are pinned. By Corollary 5, every point of $J(Q_L)$ that is not in $f(Y)$ is the join of a unique set of elements of $f(Y)$, and hence is pinned. Thus all points are pinned and the coloring is distinguishing. $\square$

The next two examples illustrate the proof of Theorem 14.

Example 15. For the distributive lattice $L = L_{150}$ in Figure 2, the set of join-irreducible points is $Y = \{a, b, c, d\}$, where $a = 2$, $b = 3$, $c = 5$ and $d = 25$. For the linear extension $E : a \prec c \prec d \prec b$ of Q_L , the chain of points in $J(Q_L)$ colored red in the proof of Theorem 14 is $a \prec ac \prec acd$ and the remaining vertices are green. Observe that each join-irreducible point in $J(Q_L)$ except b is indeed less than or equal to a unique lowest red point: $a \preceq a$ (rank 1), $c \preceq ac$ (rank 2), $cd \preceq acd$ (rank 3).

Example 16. For the distributive lattice in Figure 3, the set of join-irreducible points is $Y = \{a, b, c, d\}$, where $a = w_1$, $b = w_4$, $c = w_5$, and $d = w_2$. For the linear extension $E : d \prec a \prec b \prec c$ of Q_L , the chain of points in $J(Q_L)$ colored red in the proof of Theorem 14 is $d \prec ad \prec abd$ and the remaining vertices are green. Observe that each join-irreducible point of $J(Q_L)$ except c is indeed less than or equal to a unique lowest red point: $a \preceq ad$ (rank 2), $ab \preceq abd$ (rank 3), $d \preceq d$ (rank 1). Each point of $J(Q_L)$ is the join of a unique set of join-irreducible points of $J(Q_L)$, for example, point acd is the join of a, d, ac .

Our proof of Theorem 14 provides a distinguishing coloring of L for each linear extension of Q_L . We record this in Corollary 17.

Corollary 17. *For any distributive lattice L , each linear extension of Q_L leads to a distinguishing coloring of L using two colors, one of which appears on exactly $|Q_L| - 1$ points.*

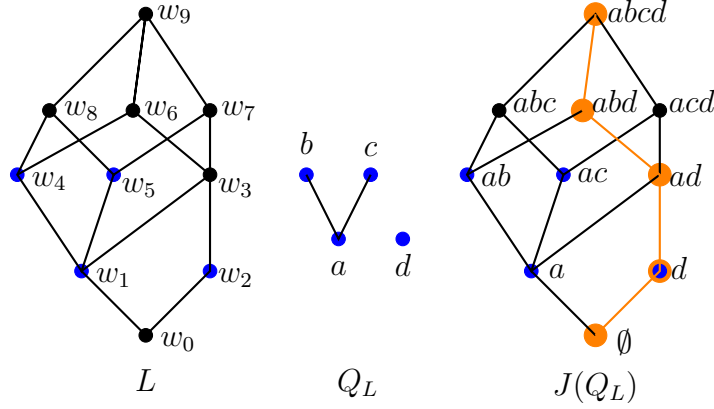


Figure 3: A lattice L , its poset Q_L of join-irreducibles, and the downset lattice $J(Q_L)$, together with a distinguishing coloring of $J(Q_L)$.

3.3 Divisibility lattices

Divisibility lattices form a subset of the set of distributive lattices. The meet of two integers is their greatest common divisor and their join is their least common multiple. For positive integer n , the *divisibility lattice* is the poset L_n consisting of the positive integer divisors of n ordered by divisibility. Figure 1 shows the poset L_n for $n = pq^2$ and $n = p^2q^2$ when p and q are distinct primes. As illustrated by this figure, the structure of L_n is determined by the prime factorization of n .

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where the p_i are distinct primes and each a_i is a positive integer. It is straightforward to check that if f is an automorphism of divisibility lattice L_n and $f(p_i) = p_j$ then $a_i = a_j$. The join-irreducible elements of L_n are the factors of n of the form $p_i^{b_i}$. When Theorem 14 is translated to divisibility lattices, we can say precisely when $D(L) = 1$ and when $D(L) = 2$.

Theorem 18. *Let n be an integer greater than 1 and write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where p_i are distinct primes and $a_i \geq 1$ for each i . The divisibility lattice L_n has $D(L_n) = 1$ if the a_i are distinct and $D(L_n) = 2$ otherwise.*

When $D(L_n) = 2$, the coloring in the proof of Theorem 14 does not always use a minimum number of red vertices, as seen in the following example.

Example 19. Consider the divisibility lattice L_n where $n = 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. The join-irreducibles of L_{2310} are the primes 2, 3, 5, 7 and 11. Each of these points has rank 1. Thus, in the proof of Theorem 18, the points colored red are the four in the chain $2 \prec 2 \cdot 3 \prec 2 \cdot 3 \cdot 5 \prec 2 \cdot 3 \cdot 5 \cdot 7$, while the remaining vertices are colored green.

Instead, we could color the three points $5 \cdot 7$, $7 \cdot 11$, $3 \cdot 5 \cdot 7$ red and the remaining points green. Each of the rank 1 points is pinned as follows: 2 is pinned since it is the only rank one point not below any red point, 3 is pinned since it is below a rank 3 red point and no others, 5 is pinned since it is below one rank 2 red point and one rank 3 red point, 7 is pinned since it is below all three red points, and 11 is pinned since it is below one rank 2 red point and no others.

4 Distinguishing Chromatic Number

The distinguishing chromatic number $\chi_D(G)$ of a graph G was introduced in [9] and studied further by other authors, see for example [3, 4, 8, 12]. It is defined as the minimum number of colors needed to properly color the vertices of G so that the only automorphism that preserves colors is the identity. We next define an analogous parameter for posets.

Definition 20. A coloring of the points of poset P is *proper* if comparable points are assigned different colors, that is, each color class induces an antichain. The *distinguishing chromatic number* of poset P , denoted $\chi_D(P)$, is the least integer k for which there is a coloring of P that is both proper and distinguishing.

For example, for the poset M in Figure 1, $\chi_D(M) = 4$ and one proper distinguishing coloring is the following: color 1 for z , color 2 for y and w , color 3 for x , and color 4 for v .

The fact that any automorphism of a poset P is also an automorphism of its comparability graph G_P justifies the following remark.

Remark 21. $\chi_D(P) \leq \chi_D(G_P)$.

However, some automorphisms of the graph G_P are not automorphisms of the poset P because they do not preserve the ordering of points in P . The following result shows that $\chi_D(P)$ can differ significantly from $\chi_D(G_P)$.

Proposition 22. *There exist posets P for which the gap between $\chi_D(P)$ and $\chi_D(G_P)$ is arbitrarily large.*

Proof. Let $j \geq 3$ be a positive integer and P be the poset consisting of $\binom{2j}{j}$ disjoint j -chains. The graph G_P consists of $\binom{2j}{j}$ copies of the graph K_j , and thus $\chi_D(G_P) = 2j$. However, $\chi_D(P) = t$, where t is the smallest integer such that $(t)_j = t(t-1)(t-2)\dots(t-j+1) \geq \binom{2j}{j}$. We will show that $\chi_D(P) \leq j+1$, demonstrating that the gap between $\chi_D(G_P)$ and $\chi_D(P)$ can be made arbitrarily large.

It remains to show $t \leq j + 1$. For the initial value $j = 3$, we have $(j + 1)_j = 4 \cdot 3 \cdot 2 = 24 > 20 = \binom{6}{3} = \binom{2j}{j}$, so $t \leq j + 1$. Observe that

$$\binom{2j+2}{j+1} = \frac{2(j+1)(2j+1)}{(j+1)^2} \binom{2j}{j} < 4 \binom{2j}{j}$$

whereas $(t)_{j+1} = (j+1)(t)_j$, so $(t)_j$ grows at a faster rate than $\binom{2j}{j}$ for $j \geq 3$. Thus $t \leq j + 1$ for $j \geq 3$ as desired. $\square$

Application: The definition of $\chi_D(P)$ is related to the following problem of designing a student's course schedule. Form a poset P in which the points of P are the courses a student plans to take to complete a major, and $x \prec y$ if course x is a prerequisite for course y . In a proper coloring, if two courses receive the same color, neither is a prerequisite of the other and they can be taken in the same semester. The minimum number of colors needed for a proper coloring of P is the minimum number of semesters needed to complete the major. If the coloring is also distinguishing, then P together with its coloring will uniquely identify the courses as well as specify which ones are taken in which semester.

The next result can be used to determine the distinguishing chromatic number of posets consisting of the sum of chains. We denote the falling factorial as $(k)_r = k(k-1)(k-2)\cdots(k-r+1)$.

Proposition 23. (i) If P is the poset consisting of the sum of t chains in which each chain contains r elements, and k is the positive integer for which $(k-1)_r < t \leq (k)_r$, then $\chi_D(P) = k$.

(ii) Let P be the sum of chains and partition P as $P_1 + P_2 + \cdots + P_m$ where P_i consists of t_i chains, each consisting of r_i points, where $r_1, r_2, \dots, r_m$ are distinct. Then $\chi_D(P) = \max\{\chi_D(P_i) : 1 \leq i \leq m\}$.

Proof. Use the arguments given in the proofs of Propositions 9 and 10, except here the vertices of a chain must get different colors, so there are $(k)_r$ ways to color a chain of r points if there are k colors available. $\square$

An alternate way to properly color the points of a poset is to color two points distinctly if they are *incomparable*, or equivalently, so that each color class induces a chain. We call this a *chain-proper* coloring. A coloring that is both chain-proper and distinguishing is related to the following problem of assigning rooms to a set of scheduled events.

Application: Represent a set of events as a poset P in which the events are the points of P and $x \prec y$ if event x ends before event y begins. In a chain-proper coloring, each color class is a set of events that can be assigned to the same room, and thus the minimum number of color classes is the number of rooms needed to schedule all of the events. If the coloring is distinguishing as well as proper, then the poset

together with its coloring will uniquely identify the events as well as specifying which room each would occupy.

As an example, the poset M in Figure 1, requires two colors for a chain-proper coloring that is distinguishing: color x , y , and z red, and color v and w blue. The next proposition is a lovely consequence of Dilworth's theorem.

Proposition 24. *For any poset P , the minimum number of colors needed for a chain-proper coloring that is also distinguishing is the width of P .*

Proof. Let k be the width of P and let A be an antichain of P with $|A| = k$. Coloring the points of A properly requires k colors, hence at least k colors are required. To show that k colors suffice, use Dilworth's theorem to partition the points of P into k sets, each of which induces a chain in P . Color all points on chain i using color i for $i = 1, 2, 3, \dots, k$. By definition, this coloring is proper. Observe that all chain-proper colorings are distinguishing because each point on chain i has a unique height on that chain, and height is preserved by automorphisms. $\square$

4.1 Bounds for distributive lattices

In our next result, we again use Birkhoff's Theorem, this time to relate the distinguishing chromatic number of a distributive lattice to that of its poset of join-irreducibles. Note that Lemma 25 is tight for L_{pq} when p and q are distinct primes.

Lemma 25. *If L is a distributive lattice, then $\chi_D(L) \leq \chi_D(Q_L) + |Q_L|$.*

Proof. First color each point of L at rank j using color j for $0 \leq j \leq |Q_L|$. This provides a proper coloring of L and by Remark 6, it uses $|Q_L| + 1$ colors. Next, recolor the points of Q_L using a proper and distinguishing coloring with $\chi_D(Q_L)$ new colors. The resulting coloring of L is still proper. It is also distinguishing since by Corollary 5 any point of L can be written uniquely as the join of elements of Q_L , and automorphisms preserve joins. All rank 1 points of L are in Q_L , so the color 1 is never used in the final coloring, thus we have a proper and distinguishing coloring of L using $\chi_D(Q_L) + |Q_L|$ colors. $\square$

The lattice L in Figure 3 has $\chi_D(Q_L) = 3$ and $|Q_L| = 4$. The proof of Lemma 25 provides a proper and distinguishing coloring of L using 7 colors. We can show that $\chi_D(L) \geq 6$ as follows. At least 5 colors are needed for a proper coloring, and any proper coloring using 5 colors assigns the same color to w_4 and w_5 , and thus is not distinguishing. The reverse inequality, $\chi_D(L) \leq 6$, follows from our next theorem, and thus lattice L is an example that shows the bound in Theorem 26 is tight.

Theorem 26. *If L is a distributive lattice and $\chi_D(Q_L) \geq 3$, then $\chi_D(L) \leq |Q_L| + \chi_D(Q_L) - 1$.*

Proof. Let $d = \chi_D(Q_L)$ and let ϕ be a proper and distinguishing coloring of Q_L , using the colors in the set $A = \{a_1, a_2, \dots, a_d\}$. Let A_i be the set of points in Q_L with color a_i for $1 \leq i \leq d$, that is $A_i = \{x \in Q_L : \phi(x) = a_i\}$. We use a different set of colors, the *rank colors*, for the remaining points of L . For each point x in L , let $r(x)$ be its rank in L . As before, each uncolored point x in L has $r(x)$ in the set $R = \{0, 2, 3, 4, \dots, |Q_L|\}$, and we color point x using color $r(x)$. As before, this is a proper and distinguishing coloring of L using $|Q_L| + d$ colors. We will construct a new coloring that uses one fewer color by eliminating the color 2.

We define three subsets of points of Q_L as follows.

$$S_1 = \{z \in A_1 : r(z) \geq 3\}$$

$$S_2 = \{z \in A_2 : r(z) \geq 3 \text{ and } \exists y \in A_1 \text{ with } r(y) = 1 \text{ and } y \prec z\}$$

$$S_3 = \{z \in A_3 : r(z) \geq 3 \text{ and } \exists y \in A_1 \text{ and } \exists w \in A_2 \text{ with } r(y) = r(w) = 1, \text{ and } y, w \prec z\}$$

Define a new coloring ψ on points x of Q_L as follows.

$$\psi(x) = \begin{cases} r(x) & \text{if } x \in S_1 \cup S_2 \cup S_3 \\ \phi(x) & \text{otherwise.} \end{cases}$$

The coloring ψ is proper on the points of Q_L because ϕ gives a proper coloring of the points not in $S_1 \cup S_2 \cup S_3$, the rank function gives a proper coloring of the points in $S_1 \cup S_2 \cup S_3$, and the colors in R are different from the colors in A . We next show that ψ is a distinguishing coloring of Q_L . Let h be an automorphism of Q_L that preserves the coloring ψ .

First we show that $h(S_1) = S_1$, $h(S_2) = S_2$ and $h(S_3) = S_3$, that is, h preserves membership in each of the sets S_1 , S_2 , S_3 . For $z \in S_1$ we have $\psi(z) = r(z)$, so $h(z)$ has a color in the set R and thus $h(z) \in S_1 \cup S_2 \cup S_3$. By the definition of S_1 , we know $\phi(z) = a_1$, and because ϕ is a proper coloring, the point z is incomparable to all other points of color a_1 . Each point in $S_2 \cup S_3$ is comparable to a rank 1 point with color a_1 . Since h preserves coloring ψ , we know $h(z) \notin S_2 \cup S_3$. Hence $h(z) \in S_1$, and $h(S_1) = S_1$. For $z \in S_2$, we have $\psi(z) = r(z)$, so $h(z)$ also has a color in the set R , and thus $h(z) \in S_1 \cup S_2 \cup S_3$. However, h is an automorphism and $h(S_1) = S_1$, hence $h(z) \in S_2 \cup S_3$. Each point in S_3 is comparable to a rank 1 point with color a_2 , and since h preserves ψ , we know $h(z) \notin S_3$. Hence $h(z) \in S_2$ and $h(S_2) = S_2$. Similarly, each point in S_3 has a color in R and thus $h(S_3) = S_3$ as well.

Since h preserves ψ and $h(S_i) = S_i$, then $h(A_i - S_i) = A_i - S_i$ and $h(A_i) = A_i$ for $1 \leq i \leq 3$. We know $h(A_i) = A_i$ for $i \geq 4$ because h preserves ψ . Thus, ψ preserves ϕ . Since ϕ is a distinguishing coloring of Q_L , we conclude that h is the identity automorphism. Thus we have shown that ψ is a distinguishing coloring of Q_L .

Now we extend ψ to L . For each pair of distinct rank 1 points x, y of Q_L , define

$$g(x, y) = \begin{cases} a_1 & \text{if } a_1 \notin \{\phi(x), \phi(y)\} \\ a_2 & \text{if } a_1 \in \{\phi(x), \phi(y)\} \text{ and } a_2 \notin \{\phi(x), \phi(y)\} \\ a_3 & \{\phi(x), \phi(y)\} = \{a_1, a_2\} \end{cases}$$

We now extend ψ to the elements of L that are not in Q_L . By Remark 6, each rank 2 point in L that is not in Q_L covers exactly two rank 1 points of L . This allows us to define $\psi(z)$ when $r(z) = 2$ and $z \notin Q_L$ as follows.

$$\psi(z) = \begin{cases} r(z) & z \notin Q_L \text{ and } r(z) > 2 \text{ or } r(z) = 0 \\ g(x, y) & z \notin Q_L, r(z) = 2 \text{ and } z \text{ covers } x, y. \end{cases}$$

Observe that ψ uses colors from the set $\{0, 3, 4, \dots, |Q_L|\} \cup A$, for a total of $d + |Q_L| - 1$ colors. Since ψ is distinguishing on Q_L , it is distinguishing on L . We need additionally to show that ψ is proper on L . The color j , for $j = 0, 3, 4, \dots, |Q_L|$, is used only on points of rank j , so the use of the rank colors is proper. We need to show that the set of points of L colored a_1 by ψ form an antichain, and similarly for the points colored a_2 and the points colored a_3 . We partition the set of $z \in L$ with $r(z) = 2$ as $T_1 \cup T_2 \cup T_3$ as follows, where the rank 1 points covered by z are denoted x and y :

- (i) $z \in T_1$ if $a_1 \notin \{\phi(x), \phi(y)\}$
- (ii) $z \in T_2$ if $a_1 \in \{\phi(x), \phi(y)\}$ and $a_2 \notin \{\phi(x), \phi(y)\}$
- (iii) $z \in T_3$ if $a_1, a_2 \in \{\phi(x), \phi(y)\}$

Then the a_i -color class of ψ is $T_i \cup (A_i - S_i)$, for $1 \leq i \leq 3$. Since ϕ is proper, A_i is an antichain, hence $A_i - S_i$ is an antichain. Since every point in T_i has rank 2, T_i is an antichain.

Suppose that $z \in T_i$ and $w \in A_i - S_i$ are comparable. Since $w \in Q_L$, $r(w) \neq 0$. Since z covers only x, y and neither is in A_i , then $r(w) \neq 1$. Since $r(z) = 2$, and points of the same rank are not comparable, then $r(w) \neq 2$. Hence $r(w) \geq 3$ and $w \succ z$.

Case 1: Let $i = 1$. Then $\psi(z) = a_1$ and $w \in A_1 - S_1$, so $\psi(w) \neq a_1$ by the definition of S_1 . Therefore, $\psi(z) \neq \psi(w)$.

Case 2: Let $i = 2$. Then $\psi(z) = a_2$ and $w \in A_2 - S_2$. Thus w is not above any rank 1 point colored a_1 , but $z \in T_2$ and $z \succ x$ and $\psi(x) = a_1$. By transitivity of the order relation, this contradicts the assumption that w and z are comparable.

Case 3: Let $i = 3$. Then $\psi(z) = a_3$ and $w \in A_3 - S_3$. Thus, w cannot be above both a rank 1 point colored a_1 and a rank 1 point colored a_2 , but $z \in T_3$ and $z \succ x$, $\psi(x) = a_1$ and $z \succ y$, $\psi(y) = a_2$. By transitivity of the order relation, this contradicts the assumption that w and z are comparable.

Thus sets of the points colored a_1, a_2, a_3 , respectively are antichains, and ψ is proper. Since we have shown it is distinguishing, $\chi_D(L) \leq d + |Q_L| - 1$, as desired. $\square$

Theorem 26 is tight, as we will see in Section 4.3.

4.2 Bounds for divisibility lattices

In the next theorem, we provide an alternative bound in Theorem 26 in the instance when the distributive lattice is a divisibility lattice. We begin by coloring each point by its rank, and then recolor the join-irreducible points using the method in Proposition 23. Not all join-irreducible points need to receive new colors, so we can use fewer colors than the number needed in Proposition 23. Recall the falling factorial function is $(n)_k = n(n-1)(n-2)\cdots(n-k+1)$ where $(n)_0 = 1$.

Theorem 27. *Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where the p_i are distinct primes and each a_i is a positive integer and let $Q = Q_{L_n}$. Partition Q as $Q_1 + Q_2 + \cdots + Q_j$ where Q_i consists of t_i chains, each consisting of r_i points, where $r_1, r_2, \dots, r_j$ are distinct. For $1 \leq i \leq k$, let m_i be the smallest integer such that $t_i \leq \sum_{\ell=0}^{\min\{r_i, m_i\}} \binom{r_i}{\ell} (m_i)_\ell$. Let $m = \max\{m_i : 1 \leq i \leq j\}$. Then $\chi_D(L_n) \leq m + |Q| + 1$.*

Proof. We begin by coloring each point in L_n with its rank. There are $|Q|+1$ ranks in L_n . We then re-color some points in Q with m new colors as follows. We can choose ℓ points on each r_i -chain to recolor in $\binom{r_i}{\ell}$ ways, and the number of ways to recolor these points with m_i colors is $(m_i)_\ell$. Thus the total number of ways to recolor ℓ points of an r_i -chain is $\binom{r_i}{\ell} (m_i)_\ell$. Since two chains with a different number of points recolored will not have the same coloring, the total number of ways to recolor the points of Q_i using m_i colors is $\sum_{\ell=0}^{\min\{r_i, m_i\}} \binom{r_i}{\ell} (m_i)_\ell$. By our choice of m_i , we can color the t_i chains, each containing r_i points, differently. Similarly, by our choice of m , there are enough colors for each value of i . Chains of different lengths in Q can not map to one another under any automorphism, hence every point in Q is pinned by this coloring. By Proposition 12, every point in L_n is pinned and our coloring is distinguishing. $\square$

As seen from the proof of the theorem, chains of different lengths may be considered independently. Let Q_L have t chains, each of length r , and let m be the smallest integer such that $t \leq \sum_{\ell=0}^{\min\{r, m\}} \binom{r}{\ell} (m)_\ell$. Then the upper bound in Theorem 26 for $\chi_D(L)$ is $\chi_D(Q_L) + |Q_L| - 1$, and since an r -chain needs at least r colors, the smallest value of $\chi_D(Q_L)$ is r . The coloring in Theorem 27 allows us to use fewer than r new colors, and thus can be a better bound when t is not too large. For example, let $r = 5$ and $t = 31$, then Theorem 27 allows us to use only two new colors, whereas the proof in Theorem 26 uses at least five. Thus, Theorem 26 gives an upper bound of $5 + 5 \cdot 31 - 1 = 159$, whereas Theorem 27 gives an upper bound of $2 + 5 \cdot 31 + 1 = 158$. The formula $\sum_{\ell=0}^{\min\{r, m\}} \binom{r}{\ell} (m)_\ell$ is a well-known formula for the number of ways of placing ℓ rooks on an $m \times r$ chessboard [14].

4.3 Bounds for Boolean lattices

The Boolean lattice B_n is the lattice of subsets of $\{1, 2, 3, \dots, n\}$, ordered by inclusion. It is a distributive lattice and its join-irreducibles are the singletons $\{\{k\} \mid 1 \leq k \leq n\}$. The number of points in any longest chain of B_n is $n + 1$.

By Theorem 26, $\chi_D(B_n) \leq n + n = 2n$. The next theorem has a tighter bound.

Theorem 28. *Let B_n be the Boolean lattice of $\{1, 2, 3, \dots, n\}$. Then $\chi_D(B_n) \leq n + 3$.*

Proof. Let n be odd. We initially color each element x with $r(x)$, for $0 \leq r(x) \leq n$. This coloring is proper. Next we will recolor some of the vertices to obtain a distinguishing coloring, using two new colors, a and b .

Let $S_i = \{i, i + 1, i + 2, \dots, 2i - 1\}$ for $1 \leq i \leq \frac{n+1}{2}$. For $1 \leq i < j \leq \frac{n+1}{2}$, we have $i \in S_i - S_j$ and $2j - 1 \in S_j - S_i$, thus the S_i form an antichain. We change the color of each of the S_i to a .

Similarly, let $T_i = \{n - (i - 1), n - i, n - (i + 1), \dots, n - 2(i - 1)\}$ for $1 \leq i \leq \frac{n+1}{2}$. For $1 \leq i < j \leq \frac{n+1}{2}$, we have $n - (i - 1) \in T_i - T_j$ and $n - 2(j - 1) \in T_j - T_i$, thus the T_i form an antichain. We change the color of each of the T_i to b .

For example, let $n = 7$. Then we color $\{1\}, \{2, 3\}, \{3, 4, 5\}, \{4, 5, 6, 7\}$ with a , and we color $\{7\}, \{6, 5\}, \{5, 4, 3\}, \{4, 3, 2, 1\}$ with b . An alternative description of the red elements is, for each $1 \leq i \leq \frac{n+1}{2}$, S_i contains the i smallest elements greater than or equal to i . Similarly, the blue elements are described as, for each $1 \leq j \leq \frac{n+1}{2}$, T_j contains the j largest elements less than or equal to $n - j$.

We show that the coloring is distinguishing. Each point of B_n colored a is pinned because it is the only point colored a in its rank. Similarly, each point of B_n colored b is pinned. Since Q_{B_n} is the set of rank 1 points, it is enough to show that the rank 1 points are pinned. Given i , $1 \leq i \leq \frac{n+1}{2}$, then the highest ranked point colored a that contains i is S_i . For $\frac{n+1}{2} \leq j$, the highest ranked point colored a that contains j is $S_{\frac{n+1}{2}}$. Thus $\{i\}$ is pinned for $1 \leq i \leq \frac{n+1}{2}$.

Similarly, given j , $\frac{n+1}{2} \leq j \leq n$, the lowest ranked point colored b that contains j is T_j . The lowest ranked point colored b that contains i is $T_{\frac{n+1}{2}}$, for $1 \leq i \leq \frac{n+1}{2}$. Thus $\{j\}$ is pinned for $\frac{n+3}{2} \leq j \leq n$. Now $\frac{n+1}{2}$ is the only element that is contained in both a point colored a and a point colored b at rank $\frac{n+1}{2}$. Thus, $\{\frac{n+1}{2}\}$ is pinned. Hence all the join-irreducibles of B_n are pinned, and this coloring is distinguishing.

Let n be even. Then color all the elements by their rank and then alter the coloring using the same red and blue elements as in the case for B_{n-1} . In this coloring, n will not appear in any point colored a or b , but each of $1, 2, \dots, n - 1$ will do so. Thus $\{n\}$ is pinned. The other join-irreducibles of B_n are pinned for the same reasons as in the previous argument. $\square$

The examples in the next two propositions show that the bound in Theorem 26 is tight when $\chi_D(Q_L) = 3$ or 4 .

Proposition 29. *The Boolean Lattice B_3 has $\chi_D(B_3) = 5$.*

Proof. Observe that $|Q_{B_3}| = 3$ and $\chi_D(Q_{B_3}) = 3$, so by Theorem 28, $\chi_D(B_3) \leq 5$. The following is a proper and distinguishing coloring of B_3 using five colors: $\emptyset$ is red, $\{1\}$ and $\{2, 3\}$ are blue, $\{3\}$ and $\{1, 2\}$ are green, $\{2\}$ and $\{1, 3\}$ are yellow, and $\{1, 2, 3\}$ is purple. Thus, $\chi_D(B_3) = 5$. $\square$

Proposition 30. *The Boolean lattice B_4 has $\chi_D(B_4) = 7$.*

Proof. Observe that $|Q_{B_4}| = 4$ and $\chi_D(Q_{B_4}) = 4$, so by Theorem 28, $\chi_D(B_4) \leq 7$. Thus we must show that $\chi_D(B_4) > 6$. Suppose for a contradiction that we have a proper and distinguishing coloring ϕ of B_4 using 6 colors. One color is used for $\emptyset$ and another for $\{1, 2, 3, 4\}$, so the remaining points must be colored using four colors: red, blue, yellow and green. We consider cases depending on the number of colors used on the rank 1 points.

Case 1: Four colors are used on the rank 1 points.

Without loss of generality we may assume $\{1\}$ is red, $\{2\}$ is blue, $\{3\}$ is green, and $\{4\}$ is yellow. Since ϕ is proper, we know $\{1, 2, 3\}$ must be yellow and $\{1, 2, 4\}$ must be green. Now no color is available for $\{1, 2\}$ since it is comparable to points that use all four colors.

Case 2: Three colors are used on the rank 1 points.

Without loss of generality we may assume $\{1\}$ is red, $\{2\}$ is blue, and $\{3\}$ and $\{4\}$ are yellow. Since ϕ is proper, the colors are forced on all points except $\{3, 4\}$, namely, $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are green, $\{1, 2\}$ is yellow, $\{2, 3\}$ is red, $\{1, 3\}$ is blue, $\{2, 4\}$ is red, $\{1, 3, 4\}$ and $\{2, 3, 4\}$ are green, and $\{1, 4\}$ is blue. Regardless of whether $\{3, 4\}$ is red or blue, the automorphism that swaps all instances of 3 and 4 preserves colors, contradicting our assumption that ϕ is distinguishing.

Case 3: Two colors are used on the rank 1 points.

First consider the instance that the two colors each appear on two rank 1 points. Without loss of generality we may assume $\{1\}$ and $\{2\}$ are red, $\{3\}$ and $\{4\}$ are blue, and $\{1, 3\}$ is green. Since ϕ is proper, the colors are forced on all points except $\{1, 2\}$ and $\{3, 4\}$ as follows: $\{1, 2, 3\}$ and $\{1, 3, 4\}$ are yellow, $\{2, 3\}$ is green, $\{1, 4\}$ is green, $\{1, 2, 4\}$ and $\{2, 3, 4\}$ are yellow, and $\{2, 4\}$ is green. Regardless of the colors of $\{1, 2\}$ and $\{3, 4\}$, the automorphism that swaps 1 and 2 and also swaps 3 and 4 preserves colors, contradicting our assumption that ϕ is distinguishing.

Now consider the instance that one color appears on three of the rank 1 points and the other appears on one. Without loss of generality we may assume $\{1\}$ and $\{2\}$ and $\{3\}$ are red, $\{4\}$ is blue, and $\{3, 4\}$ is green. Since ϕ is proper, the colors are forced on all points except $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$ as follows: $\{1, 3, 4\}$ and $\{2, 3, 4\}$ are yellow, $\{2, 4\}$ and $\{1, 4\}$ are green, and $\{1, 2, 4\}$ is yellow. The remaining points $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ must be green or blue, and two of them must be the same color. Without loss of generality, $\{1, 2\}$ and $\{1, 3\}$ are the same color, but then the automorphism that swaps 2 and 3 preserves colors, contradicting our assumption that ϕ is distinguishing.

Case 4: The rank 1 points use one color.

If the rank 3 points use two or more colors, then we reach a contradiction using previous cases since ϕ is also a proper and distinguishing coloring of the dual of B_4 . Thus without loss of generality we may assume $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$ are red, and $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$ are blue, and the rank 2 points are green and yellow. Any such coloring is proper but not distinguishing. Coloring the rank 2 points using green and yellow is equivalent to giving a distinguishing coloring to the edges of the K_4 graph using green and yellow. We show this is impossible.

If such a coloring were possible, without loss of generality, at least three edges are green. First suppose there is a triangle of green edges, say $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ are green. If the remaining edges are yellow, the automorphism $(123)(4)$ preserves colors. If one additional edge is green, say $\{1, 4\}$, then the automorphism $(23)(1)(4)$ preserves colors. If two additional edges are green, say $\{1, 4\}$ and $\{2, 4\}$ then the automorphism $(12)(3)(4)$ preserves colors. If there is no triangle of green edges, then without loss of generality, $\{1, 2\}$, $\{2, 3\}$ and $\{3, 4\}$ are green and the remaining three edges are yellow. In this instance, the automorphism $(14)(23)$ preserves colors. $\square$

5 Rank-Connected Planar Posets

In this final section of results, we consider rank-connected planar lattices.

In Section 2.1 we defined planar posets and ranked posets. Note that a planar poset is a lattice if it has both a minimal and maximal element.

Definition 31. A ranked poset is *rank-connected* if every pair of consecutive ranks, considered as a vertex-induced subgraph is connected.

Definition 32. Incomparable points x and y are *twins* if they have the same relationship to all other points of the poset. A poset is *twin-free* if it has no twins.

Cohen-Macaulay posets, are a well-known class of posets in the study of flag f -vectors of simplicial complexes, for example, see [19]. Although it is difficult to obtain a complete characterization of the set of flag f -vectors of Cohen-Macaulay posets, many subclasses of this set are lexicographically shellable and thus have an explicit shelling. Collins [7] has shown that for ranked, planar lattices, being Cohen-Macaulay is the same as being rank-connected. In this section, we show that the distinguishing number of a rank-connected, twin-free, planar lattice is less than or equal to 2. The proof is completely different from the proof of Theorem 14, which uses the regular structure of a distributive lattice. However, the 2-coloring of the points is again a chain in the poset.

We say that a Hasse diagram for ranked planar poset is a *standard diagram* if it is planar, all points at a given rank have the same y -coordinate, and all edges are straight line segments. The following result appears to be part of the folklore of the field [20]. We include a proof for completeness.

Proposition 33. *If P is a ranked planar poset with $\hat{0}$ and $\hat{1}$ then P has a standard diagram.*

Proof. Partition the points of P by rank so that R_i is the set of points of rank i for $0 \leq i \leq n$. Since P is ranked, each covering edge in P is between points at consecutive ranks. Suppose we have a planar Hasse diagram for P in which the points of each rank have the same y -coordinate for ranks k and lower and every edge between points of rank at most k is a straight line segment. If $k = n$ we are done, so assume $k < n$.

Let $w_1, w_2, w_3, \dots, w_r$ be the elements of R_k listed from left to right in the Hasse diagram. If $n = k + 1$ then $R_{k+1} = \{\hat{1}\}$ and we can draw a straight line segment from each element of R_k to $\hat{1}$, and this completes the proof. Otherwise, $n \geq k + 2$.

Let L_k be the horizontal line containing the points of R_k and L_{k+1} the horizontal line containing the point(s) of R_{k+1} with lowest y -coordinate. In Figure 4, the only point of R_{k+1} with lowest y -coordinate is z_2 . Order the edges between R_k and R_{k+1} from left to right as $e_1, e_2, \dots, e_t$ by their order in the strip between lines L_k and L_{k+1} . This order is uniquely determined because the diagram is planar. This ordering of edges induces an ordering of the points in R_{k+1} as follows: for any $u, v \in R_{k+1}$, we order u before v if the leftmost edge e_i incident to u is to the left of the leftmost edge e_j incident to v . The resulting order is $z_1, z_2, z_3, \dots, z_s$ and this is illustrated by the example in Figure 4.

For each z_i that is above line L_{k+1} , choose an edge e_j incident to z_i and relocate z_i to the point z'_i where edge e_j meets line L_{k+1} . In Figure 4, in each case the first such edge was selected. For each $z_i \in R_{k+1}$, let $N(z_i)$ be the set of points in R_k covered by z_i . The points in $N(z_i)$ have consecutive indices, for otherwise, there would be a point in R_k with no upward route to $\hat{1}$. Thus we can think of z_i and its edges to $N(z_i)$ as forming a cone. If $N(z_i) = \{w_r, w_{r+1}, \dots, w_t\}$, then z_i is the only member of R_{k+1} that covers any of the internal points $w_{r+1}, w_{j+2}, \dots, w_{t-1}$, for any other such element in R_{k+1} would have no upward path to $\hat{1}$. Thus cones intersect only at their outermost points. Indeed, if $w_r \neq w_t$, then no other z_j of R_{k+1} can cover *both* w_r and w_t , because this would imply that one of z_i, z_j would have no upward path to $\hat{1}$. Thus for all $z_i, z_j \in R_{k+1}$ with $i \neq j$ we have, $|N(z_i) \cap N(z_j)| \leq 1$. Furthermore, if $i < j$ and $N(z_i) \cap N(z_j) \neq \emptyset$, then by the planarity assumption and the way we indexed the z_i 's, the unique point in $N(z_i) \cap N(z_j)$ is the rightmost element of R_k incident to z_i and also is the leftmost element of R_k incident to z_j . Starting with edges incident to z_1 and continuing rightward, we can draw the edges between R_{k+1} and R_k as straight line segments from the points $z'_1, z'_2, \dots, z'_s$ on L_{k+1} to the points $w_1, w_2, \dots, w_r$ on L_k and by the properties above we know that none of these segments cross. This is illustrated in the right portion of Figure 4.

It remains to reroute the edges between R_{k+1} and R_{k+2} . For each $z'_i \in R_{k+1}$, form a narrow band from z'_i to z_i along the original edge e_j that is incident to z'_i (see Figure 4). Each edge between z_i and R_{k+2} will now start at z'_i , travel through this band to z_i and continue on its original path to its destination in R_{k+2} . The result is a planar Hasse diagram for P in which points at each rank are each located on a

horizontal line for ranks $k + 1$ and lower, and edges between points of ranks at most $k + 1$ are straight line segments. The result follows by induction. $\square$

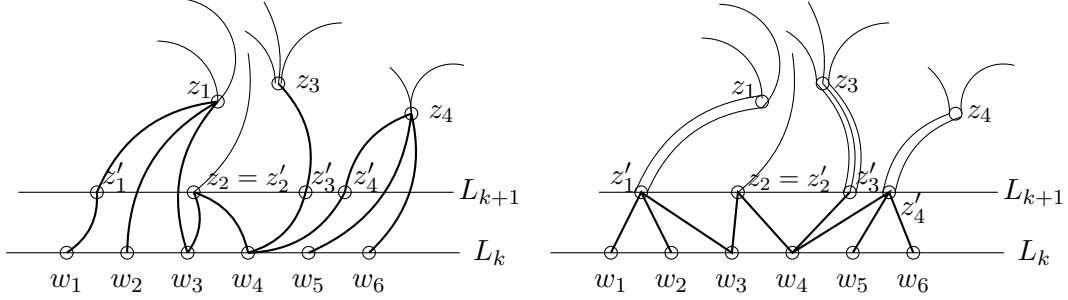


Figure 4: An illustration of the induction step in the proof of Proposition 33.

Remark 34. Using Proposition 33, it is not hard to show that the poset L_{pqr} in Figure 1 is not planar.

The next lemma appears in [7] and is helpful to us in the induction proof of Theorem 36.

Lemma 35. *Given a standard diagram of a planar rank-connected poset that has a $\hat{0}$ and a $\hat{1}$, there exists a rank in which the leftmost element x covers exactly one element, a , and is covered by exactly one element, b . Furthermore, if there is an element immediately to the right of x , it also covers a and is covered by b .*

Theorem 36. *If P is a twin-free, rank-connected planar lattice, then $D(P) \leq 2$.*

Proof. Let r be the maximum rank of points in P . In any automorphism of P , rank is preserved. If there exists a point x in P (other than $\hat{0}$ and $\hat{1}$) for which x is the only element of its rank, then x is pinned. Thus we need only pin the elements below x and separately the elements above x . So without loss of generality, we may assume P has at least two points at each rank other the lowest and highest ranks.

By Proposition 33, we may fix a standard diagram of P . At each rank, there is a leftmost point in the diagram, and because the diagram is planar and P is rank-connected, the union of these points forms maximal chain C_0 from $\hat{0}$ to $\hat{1}$. Color the points on chain C_0 red, except for its minimal and maximal elements. Since there is at most one red point at each rank, these points are pinned. Color the remaining points blue. We show this coloring is distinguishing.

We apply Lemma 35, repeatedly to obtain a sequence of chains $C_0, C_1, \dots, C_n$, each from $\hat{0}$ to $\hat{1}$, so that C_i and C_{i+1} are identical except for two points $x_i \in C_i$ and $x_{i+1} \in C_{i+1}$ where x_i and x_{i+1} have the same rank in P and x_{i+1} is immediately

to the right of x_i in the diagram. We know that x_0 is pinned since it is red and we proceed by induction.

Assume the points $x_0, x_1, \dots, x_{j-1}$ are pinned, thus the points in C_{j-1} are pinned. If there are no points at x_j 's rank that lie to the right of x_j then x_j is pinned since all remaining points at that rank are already pinned. Otherwise, there exists one or more points at x_j 's rank that lie to the right of x_j .

Let a be the point immediately below x_j on chain C_j and b the point immediately above x_j on C_j . Suppose x_j is not pinned, so thus there exists a nontrivial automorphism ϕ of P with $\phi(x_j) = w_j$ for some $w_j \neq x_j$. We know w_j has the same rank as x_j and is located to the right of x_j since the points to the left of x_j are already pinned. Since ϕ is an automorphism, $\phi(a) \prec \phi(x_j) \prec \phi(b)$ and since a and b are pinned we have $a \prec w_j \prec b$. Partition the set of points with b 's rank as $B_1 \cup B_2 \cup \{b\}$ where the points in B_1 lie to the left of b and the points in B_2 lie to the right of b . By planarity, w_j is not adjacent to any point in B_1 . However, the points in $B_1 \cup \{b\}$ are pinned by our induction hypothesis, and $\phi(x_j) = w_j$, so x_j cannot be adjacent to any point of B_1 either. Also by planarity, x_j is not adjacent to any points in B_2 , thus the only point at b 's rank that is adjacent to x_j is the point b . Similarly, the only point at a 's rank adjacent to x_j is a . So x_j is adjacent only to a and b . The same must be true of w_j since $\phi(x_j) = w_j$ and a and b are pinned. This means x_j and w_j are twins in P , a contradiction. Thus x_j is pinned and this completes the induction. $\square$

6 Open Questions

We conclude with some open questions.

Question 6.1. Is Theorem 28 tight for all $n \geq 5$?

Question 6.2. Many theorems about 2-distinguishability can be proven using the Motion Lemma, proved by Russell and Sundaram [17]. Is there a proof of Theorem 14 using the Motion Lemma?

Question 6.3. Hadjicostas [11] has found the generating function for the number of distinguishing 2-colorings of an n -cycle. Given a distributive lattice L , what is the generating function of the number of distinguishing 2-colorings of L ?

Question 6.4. The *cost* of 2-distinguishing a graph G is the minimum size of a color class in any 2-distinguishing labeling of G , see [6]. What is the cost of 2-distinguishing a distributive lattice? The cost may be smaller than the minimum sizes of color classes in the 2-labelings we have used in our proofs. For example, the cost of 2-distinguishing $L_{p^2q^2}$ is 1, as can be seen by coloring point p red and the remaining points blue, whereas the cost of 2-distinguishing L_{pqr} is 2, because there will still be an automorphism that preserves the colors even if one point is fixed.

References

- [1] M. O. Albertson and K. L. Collins, Symmetry breaking in graphs, *Electron. J. of Combin.*, **3** (1996) #R18.
- [2] S. Alikhani and S. Soltani, The distinguishing number and distinguishing index of the lexicographic product of two graphs, *Discuss. Math. Graph Theory* **38(3)** (2018) 853-865.
- [3] N. Balachandran and S. Padinhatteeri, Distinguishing chromatic number of random Cayley graphs, *Discrete Math.* **340(10)** (2017) 2447-2455.
- [4] N. Balachandran and S. Padinhatteeri, $\chi_D(G)$, $|Aut(G)|$ and a variant of the motion lemma, *Ars Math. Contemp.* **12(1)** (2017) 89-109.
- [5] G. Birkhoff, On the combination of subalgebras, *Proc. Cambridge Phil. Soc.* **29** (1933) 441-464.
- [6] D. L. Boutin, The cost of 2-distinguishing Cartesian powers, *Electron. J. Combin.* **20(1)** (2013) Paper 74, 13 pp.
- [7] K. L. Collins, Planar lattices are lexicographically shellable, *Order* **8** (1992) 375-381.
- [8] K. L. Collins, M. Hovey, A. N. Trenk, Bounds on the distinguishing chromatic number, *Electron. J. Combin.* **16(1)** (2009) Research Paper 88, 14 pp.
- [9] K. L. Collins and A. N. Trenk, The distinguishing chromatic number, *Electron. J. of Combin.*, **13** (2006) #R16.
- [10] D. W. Cranston, Proper distinguishing colorings with few colors for graphs with girth at least 5, *Electron. J. Combin.* **25(3)** (2018) Paper 3.5, 9 pp.
- [11] P. Hadjicostas, On the number of aperiodic chiral bracelets of two colors, *OEIS entry A032239 link* 2019.
- [12] W. Imrich, R. Kalinowski, M. Pilśniak, M. H. Shekarriz, Bounds for distinguishing invariants of infinite graphs, *Electron. J. Combin.* **24(3)** (2017) Paper 3.6, 14 pp.
- [13] W. Imrich, S. Smith, T. W. Tucker and M. E. Watkins, Infinite motion and 2-distinguishability of graphs and groups, *J. Algebraic Combin.* **41(1)** (2015) 109-122.
- [14] I. Kaplansky and J. Riordan, The problem of the rooks and its applications, *Duke Math. J.* **13** (1946) 259-268.

- [15] F. Lehner, Breaking graph symmetries by edge colourings, *J. Combin. Theory Ser. B* **127** (2017) 205-214.
- [16] M. Pilśniak, Edge motion and the distinguishing index, *Theoret. Comput. Sci.* **678** (2017) 56-62.
- [17] A. Russell and R. Sundaram, A note on the asymptotics and computational complexity of graph distinguishability, *Electron. J. Combin.* **5** (1998) #R23, 1-7.
- [18] R. P. Stanley, *Enumerative combinatorics, Vol. I*, second edition, Cambridge Univ. Press, New York, NY, 2012.
- [19] R. P. Stanley, *Combinatorics and Commutative Algebra*, second edition, Birkhäuser, Boston MA, 1996.
- [20] W. T. Trotter, personal communication.