# HIGHER-DIMENSIONAL DELTA-SYSTEMS 

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#### Abstract

We investigate higher-dimensional $\Delta$-systems indexed by finite sets of ordinals, isolating a particular definition thereof and proving a higherdimensional version of the classical $\Delta$-system lemma. We focus in particular on systems that consist of sets of ordinals, in which case useful order-theoretic uniformities can be ensured. We then present three applications of these higherdimensional $\Delta$-systems to problems involving the interplay between forcing and partition relations on the reals.


## 1. Introduction

The starting point for this paper is one of the basic concepts of combinatorial set theory: the $\Delta$-system.

Definition 1.1. A family $\mathcal{U}$ of sets is a $\Delta$-system if there is a set $r$, known as the root of the $\Delta$-system, such that $u \cap v=r$ for all distinct $u, v \in \mathcal{U}$.

The uniformity provided by $\Delta$-systems can be quite useful, so it is no surprise that the $\Delta$-system lemma, which isolates conditions that guarantee that a given family of sets can be thinned out to form a large $\Delta$-system, is one of the foundational results of combinatorial set theory. The most commonly stated form of the lemma, introduced by Shanin [15], is the following.

Lemma 1.2. Suppose that $\mathcal{U}$ is an uncountable family of finite sets. Then there is an uncountable subfamily $\mathcal{U}^{*} \subseteq \mathcal{U}$ such that $\mathcal{U}^{*}$ is a $\Delta$-system.

The following is a less pithy but more general formulation. For a proof, we direct the reader to [13, Ch. II, §1].

Lemma 1.3. Suppose that $\kappa<\lambda$ are infinite cardinals such that $\lambda$ is regular and, for all $\nu<\lambda$, we have $\nu^{<\kappa}<\lambda$. Suppose also that $\mathcal{U}$ is a family of sets such that $|\mathcal{U}| \geq \lambda$ and $|u|<\kappa$ for all $u \in \mathcal{U}$. Then there is $\mathcal{U}^{*} \subseteq \mathcal{U}$ such that $\left|\mathcal{U}^{*}\right|=\lambda$ and $\mathcal{U}^{*}$ is a $\Delta$-system.
$\Delta$-systems are inherently one-dimensional objects, in practice often enumerated as sequences indexed by ordinals. When investigating higher-dimensional combinatorial objects, however, one frequently encounters families of sets indexed by $n$-element sets of ordinals for some $n>1$ and desires to find large subfamilies exhibiting certain uniformity properties analogous to the uniformities exhibited by $\Delta$-systems. In this context, higher-dimensional analogues of the $\Delta$-system lemma

[^0]come into play. Such analogues were first developed in work of Todorčević 19 and Shelah [16, 17, and have appeared with increasing frequency of late in works such as 2, 3], 5], 6, 12, 21], and 22].

The higher-dimensional $\Delta$-systems in the aforementioned works have taken a number of slightly different forms. In this paper, we isolate one particular definition, based most directly on the 2-dimensional $\Delta$-systems of 19 and $[2$ and on the $n$-dimensional $\Delta$-systems of 3. This definition generalizes the familiar 1dimensional definition and, in the case in which the higher-dimensional $\Delta$-system consists of sets of ordinals, it can be strengthened to incorporate some additional order-theoretic uniformities. The definition is presented in Section 2, where we also prove some basic properties of our higher-dimensional $\Delta$-systems. In Section 3. we prove our main result, Theorem 3.8 which is an $n$-dimensional analogue of the classical $\Delta$-system lemma, isolating conditions under which an $n$-dimensional $\Delta$-system of a particular size can be guaranteed to exist inside of an arbitrary collection of sets indexed by $n$-element sets of ordinals. Theorem 3.8 is naturally seen as an elaboration of the Erdős-Rado theorem and is closely connected to the work on canonical partition relations of Erdős and Rado 9 and of Baumgartner 11. After proving Theorem 3.8, we turn to a discussion of its optimality, proving that one of its parameters, the size of the arbitrary collection of sets inside of which we are guaranteed to find a large $n$-dimensional $\Delta$-system, cannot be improved and indicating precisely the extent to which another of its parameters, the upper bound on the size of the members of our arbitrary collection of sets, can consistently be improved. The results of this section are summarized in Corollaries 3.19 and 3.21 which incorporate Theorem 3.8, our discussion of its optimality, and its connections with the Erdős-Rado theorem.

The remaining sections of the paper present applications of our main result. Section 4 is a short section presenting a higher-dimensional analogue of the familiar use of $\Delta$-systems to prove that Cohen forcing satisfies the Knaster property. In Section 5, we present an application to a problem involving the interplay of forcing and polarized partition relations. In Section 6, we show that, in certain arguments, the $\Delta$-system lemma presented here can successfully replace a different lemma (from [16]) that, at least under the currently best known results, requires stronger assumptions. We apply this to a recent result of Zhang [22] regarding additive partition relations on the reals, obtaining a slight local improvement of his result.

Notation and conventions. For a class $X$ and a cardinal $\kappa,[X]^{\kappa}:=\{Y \subseteq X \mid$ $|Y|=\kappa\}$, and $[X]^{<\kappa}:=\{Y \subseteq X| | Y \mid<\kappa\}$. For a set $u$ of ordinals, otp $(u)$ denotes the order type of $u$. The class of ordinals is denoted by On. If $\rho$ is an ordinal and $X$ is a class of ordinals, then $[X]^{\rho}:=\{Y \subseteq X \mid \operatorname{otp}(Y)=\rho\}$. This is a slight abuse of notation given the previous definition of $[X]^{\kappa}$ and the customary identification of a cardinal with its initial ordinal, but in practice we will use the Greek letter $\rho$ precisely when the order-type definition of $[X]^{\rho}$ is intended, so no confusion will arise from this.

We will often think of sets of ordinals as increasing sequences of ordinals in the natural way. So, for instance, if $u$ is a set of ordinals, $\rho=\operatorname{otp}(u)$, and $i<\rho$, then $u(i)$ denotes the unique element $\alpha \in u$ such that $\operatorname{otp}(u \cap \alpha)=i$. If $\mathbf{i} \subseteq \rho$, then $u[\mathbf{i}]$ denotes $\{u(i) \mid i \in \mathbf{i}\}$. If $X$ is a set of ordinals and $n<\omega$, then we will use the notation $\left(\alpha_{0}, \ldots, \alpha_{n-1}\right) \in[X]^{n}$ to denote the conjunction of the statements
$\left\{\alpha_{0}, \ldots, \alpha_{n-1}\right\} \in[X]^{n}$ and $\alpha_{0}<\ldots<\alpha_{n-1}$. If $A$ and $B$ are nonempty sets of ordinals, then we write $A<B$ to assert that $\alpha<\beta$ for all $(\alpha, \beta) \in A \times B$. For improved readability, we will also sometimes omit commas and brackets when using small sets as subscripts or superscripts. For example, we may write $u_{\alpha \beta}^{0}$ instead of $u_{\{\alpha, \beta\}}^{\{0\}}$. For notational convenience, we will adopt the convention that $\max (\emptyset)=-1$.

If $\kappa$ is an infinite cardinal, then $\beth_{n}(\kappa)$ is defined by recursion on $n<\omega$ by setting $\beth_{0}(\kappa):=\kappa$ and $\beth_{n+1}(\kappa):=2^{\beth_{n}(\kappa)}$ for all $n<\omega$. As is customary, we will denote $\beth_{n}\left(\aleph_{0}\right)$ simply by $\beth_{n}$. Suppose that $\kappa<\lambda$ are cardinals. We say that $\lambda$ is $\kappa$-inaccessible if $\nu^{\kappa}<\lambda$ for all $\nu<\lambda$. Similarly, $\lambda$ is $<\kappa$-inaccessible if $\nu^{<\kappa}<\lambda$ for all $\nu<\lambda$.

If $\mu, \lambda$, and $\nu$ are cardinals and $n$ is a natural number, then the partition relation $\mu \rightarrow(\lambda)_{\nu}^{n}$ is the assertion that, for all $c:[\mu]^{n} \rightarrow \nu$, there is $H \in[\mu]^{\lambda}$ such that $c \upharpoonright[H]^{n}$ is constant. The negation of this partition relation is denoted by $\mu \nrightarrow(\lambda)_{\nu}^{n}$.

If $\mathbb{P}$ is a forcing notion and $p, q \in \mathbb{P}$, then $p \| q$ asserts that $p$ and $q$ are compatible, i.e., there is $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$, and $p \perp q$ asserts that $p$ and $q$ are incompatible.

## 2. Uniform $n$-DIMENSIONAL $\Delta$-SYSTEMS

In this section, we present the basic definitions of the paper and prove some of their basic properties. We begin by working towards our definition of an $n$ dimensional $\Delta$-system indexed by finite sets of ordinals. Most of the paper will focus on the case in which the elements of the $\Delta$-system are themselves sets of ordinals, in which case we can arrange for significant order-theoretic uniformities, but we first present a more general definition.

Our $n$-dimensional $\Delta$-systems will be indexed by sets of the form $[H]^{n}$, where $H$ is a set of ordinals, and for $n>1$ they will have not a single root witnessing the fact that they are $n$-dimensional $\Delta$-systems, but rather a family of roots. When first attempting to generalize $\Delta$-systems to higher dimensions, one might optimistically hope to require that, in an $n$-dimensional $\Delta$-system $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$, the intersection $u_{a} \cap u_{b}$ depends only on $a \cap b$ for all $a, b \in[H]^{n}$. In other words, one might hope to require the existence of a family of roots $\left\langle R_{a} \mid a \in[H]^{\leq n}\right\rangle$ such that, for all $b, b^{\prime} \in[H]^{n}$, we have $u_{b} \cap u_{b^{\prime}}=R_{b \cap b^{\prime}}$. However, this would be an overly restrictive requirement, even in the case of $n=2$. To see this, let $\mu$ be any infinite cardinal, and define a family of sets $\left\langle u_{b} \mid b \in[\mu]^{2}\right\rangle$ by letting $u_{\alpha \beta}:=\{\alpha, \beta+1\}$ for all $(\alpha, \beta) \in[\mu]^{2}$. Now observe that, if $\alpha<\beta<\gamma<\delta<\mu$, then

- $\beta \in u_{\beta \gamma} \cap u_{\beta \delta} ;$
- $\beta \notin u_{\alpha \beta} \cap u_{\beta \gamma}$.

Hence, $u_{\beta \gamma} \cap u_{\beta \delta} \neq u_{\alpha \beta} \cap u_{\beta \gamma}$, yet $\{\beta, \gamma\} \cap\{\beta, \delta\}=\{\beta\}=\{\alpha, \beta\} \cap\{\beta, \gamma\}$. Therefore, if one adopts the requirement that $u_{a} \cap u_{b}$ must depend only on $a \cap b$, then one could not even find a subset $H \subseteq \mu$ of size 4 for which $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is a 2-dimensional $\Delta$-system.

The starting point for what will become our actual definition is Todorčević's 2-dimensional double $\Delta$-system from [19]. According to Todorčević's definition, if $H$ is a set of ordinals, then a family of sets $\left\langle u_{b} \mid b \in[H]^{2}\right\rangle$ is a double $\Delta$-system if

- for all $\alpha \in H$, the family $\left\langle u_{\alpha \beta} \mid \beta \in H \backslash(\alpha+1)\right\rangle$ is a $\Delta$-system with root $r_{\alpha}^{0}$ (for simplicity, assume that $H$ has no maximal element);
- for all $\beta \in H \backslash\{\min (H)\}$, the family $\left\langle u_{\alpha \beta} \mid \alpha \in H \cap \beta\right\rangle$ is a $\Delta$-system with root $r_{\beta}^{1}$;
- both $\left\langle r_{\alpha}^{0} \mid \alpha \in H\right\rangle$ and $\left\langle r_{\beta}^{1} \mid \beta \in H \backslash\{\min (H)\}\right\rangle$ are $\Delta$-systems, with roots $r^{0}$ and $r^{1}$, respectively.
Note that, if $\left\langle u_{b} \mid b \in[H]^{2}\right\rangle$ is a double $\Delta$-system, as witnessed by sets $\left\langle r_{\alpha}^{0} \mid \alpha \in H\right\rangle$, $\left\langle r_{\beta}^{1} \mid \beta \in H \backslash\{\min (H)\}\right\rangle, r^{0}$, and $r^{1}$, then it is in fact the case that $r^{0}=r^{1}=$ $\bigcap_{b \in[H]^{2}} u_{b}$.

In order to succinctly generalize this definition to higher dimensions, and to help facilitate the later incorporation of further order-theoretic uniformities, the following notion will be useful.

Definition 2.1. Suppose that $a$ and $b$ are sets of ordinals.
(1) We say that $a$ and $b$ are aligned if $\operatorname{otp}(a)=\operatorname{otp}(b)$ and, for all $\gamma \in a \cap b$, we have $\operatorname{otp}(a \cap \gamma)=\operatorname{otp}(b \cap \gamma)$. In other words, if $\gamma$ is a common element of $a$ and $b$, then it occupies the same relative position in both $a$ and $b$.
(2) We let $\mathbf{r}(a, b):=\{i<\operatorname{otp}(a) \mid a(i) \in b\}$. Notice that $a \cap b=a[\mathbf{r}(a, b)]$ and, if $a$ and $b$ are aligned, then $a \cap b=a[\mathbf{r}(a, b)]=b[\mathbf{r}(a, b)]$.
Note that, in our counterexample to our initial overly restrictive attempt at a definition of a higher-dimensional $\Delta$-system at the beginning of this section, the problem came about when we considered the non-aligned sets $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$. As we will see shortly, it turns out that this is the only insurmountable problem with our definition, and if one only requires the family of roots in an $n$-dimensional $\Delta$ system to control the intersections of elements of the $\Delta$-system indexed by aligned sets, then one obtains a much more workable definition, which we adopt as our general definition of an $n$-dimensional $\Delta$-system indexed by $n$-element sets of ordinals.

Definition 2.2. Suppose that $H$ is a set of ordinals, $1 \leq n<\omega$, and, for each $b \in[H]^{n}, u_{b}$ is a set. We call $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ an $n$-dimensional $\Delta$-system if there is a family of roots $\left\langle R_{a}^{\mathbf{m}} \mid \mathbf{m} \subseteq n, a \in[H]^{|\mathbf{m}|}\right\rangle$ such that, for all $b, b^{\prime} \in[H]^{n}$, if $b$ and $b^{\prime}$ are aligned and $\mathbf{r}\left(b, b^{\prime}\right)=\mathbf{m}$, then $u_{b} \cap u_{b^{\prime}}=R_{b \cap b^{\prime}}^{\mathbf{m}}$.

We observe that, if $n=1$, then this is precisely the classical definition of a $\Delta$-system as given in Definition 1.1 modulo an enumeration of the $\Delta$-system via a set of ordinals; the root $r$ in Definition 1.1 corresponds to the root $R_{\emptyset}^{\emptyset}$ in Definition 2.2. When $n=2$, we obtain Todorčević's double $\Delta$-systems; the roots $r_{\alpha}^{0}, r_{\alpha}^{1}$, and $r^{0}\left(=r^{1}\right)$ of Todorčević's definition correspond to the roots $R_{\alpha}^{0}, R_{\alpha}^{1}$, and $R_{\emptyset}^{\emptyset}$, respectively.

We now turn to the special setting in which the elements of our $\Delta$-systems are sets of ordinals. In this setting, we can ask for our $\Delta$-systems to satisfy certain additional order-theoretic uniformities, and we will call $n$-dimensional $\Delta$-systems that satisfy these uniformities uniform $n$-dimensional $\Delta$-systems. Since any family of sets $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ can be transformed into a family of sets of ordinals via a bijection between $\bigcup_{b \in[H]^{n}} u_{b}$ and an ordinal, and since the proof of our higher dimensional analogue of the $\Delta$-system lemma (Theorem 3.8) in fact yields uniform $n$-dimensional $\Delta$-systems with no additional hypotheses, there will be no loss of generality for us in focusing on this setting. (However, for $n>1$ it will not in general be the case that every sufficiently large $n$-dimensional $\Delta$-system consisting of sufficiently small sets of ordinals can be refined to a uniform $n$-dimensional $\Delta$ system of the same size; see Remark 2.7.)

Let us first look at the 1-dimensional case to help us motivate our definition. In the context of families of sets of ordinals, the classical $\Delta$-system lemma can easily be strengthened to require that the root of the $\Delta$-system "sits inside" each of its elements in the same way, in the following sense.

Definition 2.3. A family $\mathcal{U}$ of sets of ordinals is a uniform $\Delta$-system if there is a set $r$ such that, for all distinct $u, v \in \mathcal{U}, u$ and $v$ are aligned and $u \cap v=r$.

The following proposition indicates that Lemma 1.3 can be strengthened to yield a uniform $\Delta$-system in the case in which $\mathcal{U}$ is a family of sets of ordinals.

Proposition 2.4. Suppose that $\kappa<\lambda$ are infinite cardinals such that $\lambda$ is regular and $<\kappa$-inaccessible. Suppose also that $\mathcal{U}$ is a $\Delta$-system consisting of sets of ordinals, and that $|\mathcal{U}| \geq \lambda$ and $|u|<\kappa$ for all $u \in \mathcal{U}$. Then there is $\mathcal{U}^{*} \subseteq \mathcal{U}$ such that $\left|\mathcal{U}^{*}\right|=\lambda$ and $\mathcal{U}^{*}$ is a uniform $\Delta$-system.

Proof. Let $r$ be the root of $\mathcal{U}$. Since $\lambda$ is regular and $|u|<\kappa<\lambda$ for all $u \in \mathcal{U}$, by thinning out $\mathcal{U}$ if necessary, we may assume that there is an ordinal $\rho<\kappa$ such that $\operatorname{otp}(u)=\rho$ for all $u \in \mathcal{U}$. Define a function $g: \mathcal{U} \rightarrow \mathcal{P}(\rho)$ by letting $g(u):=\mathbf{r}(u, r)$. Since $\lambda$ is $<\kappa$-inaccessible, we can find a fixed set $\mathbf{r}^{*} \subseteq \rho$ and a set $\mathcal{U}^{*} \subseteq \mathcal{U}$ such that $\left|\mathcal{U}^{*}\right|=\lambda$ and $g(u)=\mathbf{r}^{*}$ for all $u \in \mathcal{U}^{*}$. Then, for all distinct $u, v \in \mathcal{U}^{*}$, it follows that $u$ and $v$ are aligned, with $\mathbf{r}(u, v)=\mathbf{r}^{*}$ and $u \cap v=r$.

We are now ready for our definition of a uniform $n$-dimensional $\Delta$-system. In the case $n=1$, this will coincide with Definition 2.3, and in the general case it will strengthen Definition 2.2 in the same way that Definition 2.3 strengthens Definition 1.1.

Definition 2.5. Suppose that $H$ is a set of ordinals, $1 \leq n<\omega$, and, for all $b \in[H]^{n}, u_{b}$ is a set of ordinals. We call $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ a uniform n-dimensional $\Delta$-system if there is an ordinal $\rho$ and, for each $\mathbf{m} \subseteq n$, a set $\mathbf{r}_{\mathbf{m}} \subseteq \rho$ satisfying the following statements.
(1) $\operatorname{otp}\left(u_{b}\right)=\rho$ for all $b \in[H]^{n}$.
(2) For all $a, b \in[H]^{n}$ and $\mathbf{m} \subseteq n$, if $a$ and $b$ are aligned with $\mathbf{r}(a, b)=\mathbf{m}$, then $u_{a}$ and $u_{b}$ are aligned with $\mathbf{r}\left(u_{a}, u_{b}\right)=\mathbf{r}_{\mathbf{m}}$.
(3) For all $\mathbf{m}_{0}, \mathbf{m}_{1} \subseteq n$, we have $\mathbf{r}_{\mathbf{m}_{0} \cap \mathbf{m}_{1}}=\mathbf{r}_{\mathbf{m}_{0}} \cap \mathbf{r}_{\mathbf{m}_{1}}$.

We now show that Definition 2.5 does indeed strengthen Definition 2.2 clause (1) of the following proposition will also be useful in a number of other situations.

Proposition 2.6. Suppose that $1 \leq n<\omega, H$ is a set of ordinals, and $\left\langle u_{b}\right| b \in$ $\left.[H]^{n}\right\rangle$ is a uniform n-dimensional $\Delta$-system as witnessed by an ordinal $\rho$ and sets $\left\langle\mathbf{r}_{\mathbf{m}} \mid \mathbf{m} \subseteq n\right\rangle$. Then the following statements hold.
(1) For all $\mathbf{m} \subseteq n$ and all $a, b \in[H]^{n}$, if $a[\mathbf{m}]=b[\mathbf{m}]$, then $u_{a}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{b}\left[\mathbf{r}_{\mathbf{m}}\right]$.
(2) The family $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is an $n$-dimensional $\Delta$-system in the sense of Definition 2.2.

Proof. (1) For all $a, b \in[H]^{n}$, let $\partial(a, b):=|\{\beta \in a \cap b| | a \cap \beta|\neq|b \cap \beta|\} \mid$. Our proof will be by induction on $\partial(a, b)$.

Fix $\mathbf{m}, a$, and $b$ as in the statement of clause (1) of the proposition. If $\partial(a, b)=0$, then $a$ and $b$ are aligned and $\mathbf{m} \subseteq \mathbf{r}(a, b)$. It follows from clauses (2) and (3) of Definition 2.5 that $u_{a}$ and $u_{b}$ are aligned and $\mathbf{r}\left(u_{a}, u_{b}\right) \supseteq \mathbf{r}_{\mathbf{m}}$. In particular, $u_{a}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{b}\left[\mathbf{r}_{\mathbf{m}}\right]$, as desired.

Now suppose that $\partial(a, b)>0$ and we have established all instances of clause (1) of the proposition for $a^{\prime}, b^{\prime} \in[H]^{n}$ for which $a^{\prime}[\mathbf{m}]=b^{\prime}[\mathbf{m}]$ and $\partial\left(a^{\prime}, b^{\prime}\right)<\partial(a, b)$. Let $\alpha \in a \cap b$ be least such that $|a \cap \alpha| \neq|b \cap \alpha|$. Let $k_{a}, k_{b}<n$ be such that $a\left(k_{a}\right)=\alpha=b\left(k_{b}\right)$. Without loss of generality, we may assume that $k_{a}<k_{b}$.

We now alter $a$ to form a new set $a^{\prime} \in[H]^{n}$. If $a \cap b \cap \alpha \neq \emptyset$, then let $\alpha^{*}:=$ $\max (a \cap b \cap \alpha)$. In this case, by our choice of $\alpha$, there must be $k^{*}<k_{a}$ such that $a\left(k^{*}\right)=b\left(k^{*}\right)=\alpha^{*}$. If $a \cap b \cap \alpha=\emptyset$, then let $k^{*}:=-1$. In either case, note that, for all $\ell \in\left(k^{*}, k_{a}\right]$, we have $b(\ell) \notin a$ and, if $k^{*} \geq 0$, then $a\left(k^{*}\right)<b(\ell)$. Moreover, $\mathbf{m} \cap\left(k^{*}, k_{a}\right]=\emptyset$. We define $a^{\prime}$ by specifying $a^{\prime}(\ell)$ for all $\ell<n$. If $\ell \leq k^{*}$ or $\ell>k_{a}$, then let $a^{\prime}(\ell):=a(\ell)$. If $\ell \in\left(k^{*}, k_{a}\right.$ ], then let $a^{\prime}(\ell):=b(\ell)$. The following observations are immediate.
(i) $a$ and $a^{\prime}$ are aligned, with $\mathbf{r}\left(a, a^{\prime}\right)=n \backslash\left(k^{*}, k_{a}\right]$. In particular, $\mathbf{m} \subseteq \mathbf{r}\left(a, a^{\prime}\right)$.
(ii) $\partial\left(a^{\prime}, b\right)=\partial(a, b)-1$, since

$$
\left\{\beta \in a^{\prime} \cap b| | a^{\prime} \cap \beta|\neq|b \cap \beta|\}=\{\beta \in a \cap b| | a \cap \beta|\neq|b \cap \beta|\} \backslash\{\alpha\} .\right.
$$

We can therefore invoke the inductive hypothesis together with (i) to conclude that $u_{a}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{a^{\prime}}\left[\mathbf{r}_{\mathbf{m}}\right]$ and together with (ii) to conclude that $u_{a^{\prime}}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{b}\left[\mathbf{r}_{\mathbf{m}}\right]$, so it follows that $u_{a}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{b}\left[\mathbf{r}_{\mathbf{m}}\right]$, as desired.
(2) To prove that $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ satisfies Definition 2.2, we must specify roots

$$
\left\langle R_{a}^{\mathbf{m}} \mid \mathbf{m} \subseteq n, a \in[H]^{|\mathbf{m}|}\right\rangle
$$

To this end, fix $\mathbf{m} \subseteq n$ and $a \in[H]^{|\mathbf{m}|}$. If there are no $b \in[H]^{n}$ for which $b[\mathbf{m}]=a$, then simply let $R_{a}^{\mathbf{m}}:=\emptyset$. Otherwise, choose $b \in[H]^{n}$ for which $b[\mathbf{m}]=a$ and set $R_{a}^{\mathbf{m}}:=u_{b}\left[\mathbf{r}_{\mathbf{m}}\right]$. By clause (1) of this proposition, the value of $R_{a}^{\mathbf{m}}$ is independent of our choice of $b$.

Now suppose that $b, b^{\prime} \in[H]^{n}$ are aligned and $\mathbf{r}\left(b, b^{\prime}\right)=\mathbf{m}$, so, in particular, $b[\mathbf{m}]=b \cap b^{\prime}$. Then $u_{b}$ and $u_{b^{\prime}}$ are aligned and $\mathbf{r}\left(u_{b}, u_{b^{\prime}}\right)=\mathbf{r}_{\mathbf{m}}$. Moreover, we defined $R_{b \cap b^{\prime}}^{\mathbf{m}}$ so that $R_{b \cap b^{\prime}}^{\mathrm{m}}=u_{b}\left[\mathbf{r}_{\mathbf{m}}\right]$. It follows that $u_{b} \cap u_{b^{\prime}}=R_{b \cap b^{\prime}}^{\mathbf{m}}$, so $\left\langle R_{a}^{\mathbf{m}}\right|$ $\left.\mathbf{m} \subseteq n, a \in[H]^{|\mathbf{m}|}\right\rangle$ witnesses the fact that $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ satisfies Definition 2.2

Remark 2.7. There is no direct analogue of Proposition 2.4 for $n$-dimensional $\Delta$-systems when $n>1$ unless $\lambda$ is weakly compact (see Corollary 3.17 for a positive result in case $\lambda$ is weakly compact). For a simple counterexample, suppose that $\lambda$ is a regular uncountable cardinal that is not weakly compact, let $\pi: \lambda \times \lambda \rightarrow \lambda$ be a bijection, and let $c:[\lambda]^{2} \rightarrow 2$ be a function such that $c^{"}[H]^{2}=2$ for all $H \in[\lambda]^{\lambda}$. Now, for all $\alpha<\beta<\lambda$, let

$$
u_{\alpha \beta}:= \begin{cases}\emptyset & \text { if } c(\alpha, \beta)=0 \\ \pi(\alpha, \beta) & \text { if } c(\alpha, \beta)=1\end{cases}
$$

Then $\left\langle u_{\alpha \beta} \mid \alpha<\beta<\lambda\right\rangle$ is a 2-dimensional $\Delta$-system (with $R_{\alpha}^{0}=R_{\alpha}^{1}=\emptyset$ for all $\alpha<\lambda$ ) consisting of finite sets of ordinals, yet whenever $H \in[\lambda]^{\lambda}$, the family $\left\langle u_{\alpha \beta} \mid(\alpha, \beta) \in[H]^{2}\right\rangle$ contains sets of cardinality 0 and of cardinality 1 and therefore cannot be a uniform 2-dimensional $\Delta$-system.

Nonetheless, as we shall see in Section 3, the cardinality hypotheses on the cardinal $\mu$ and the sizes of the sets $u_{b}$ that guarantee that a family $\left\langle u_{b}\right| b \in$ $\left.[\mu]^{n}\right\rangle$ of sets of ordinals can be refined to an $n$-dimensional $\Delta$-system of a specified cardinality are already sufficient to guarantee that the family can be refined to a uniform $n$-dimensional $\Delta$-system of the same cardinality.

## 3. A Higher-dimensional $\Delta$-SYSTEM LEMMA

In this section, we prove the main result of the paper (Theorem 3.8), a higherdimensional analogue of the $\Delta$-system lemma which asserts, roughly speaking, that inside every family of sets of ordinals indexed by $n$-element subsets of some sufficiently large cardinal $\mu$, we can find a subset $H$ of $\mu$ of some specified size such that $[H]^{n}$ indexes a uniform $n$-dimensional $\Delta$-system. In the absence of weakly compact cardinals, this $H$ will necessarily be smaller than $\mu$. In the same way that the $\Delta$-system lemma can fruitfully be seen as as an extension of the pigeonhole principle, this $n$-dimensional $\Delta$-system lemma can fruitfully be seen as an elaboration of the Erdős-Rado theorem, and in fact a version of the Erdős-Rado theorem will be folded into our statement to carry along as an inductive hypothesis.

The result is also closely related to results on canonical partition relations, introduced by Erdős and Rado in [9], and in particular to work done by Baumgartner on canonical partition relations [1], which can also be seen as an elaboration of the Erdős-Rado theorem. Indeed, in the cases in which $\kappa$ is a successor cardinal, much of our main result can be derived from the main result of [1]. When $\kappa$ is a limit cardinal (and in particular in the important case $\kappa=\aleph_{0}, \lambda=\aleph_{1}$ ), this approach does not seem to work, so we provide a single proof that covers all cases. We first introduce the following notation, from [1], that allows us to indicate precisely the size of the family needed to ensure the existence of a large uniform $n$-dimensional $\Delta$-system.

Definition 3.1. Given an infinite regular cardinal $\lambda$, recursively define $\sigma(\lambda, n)$ for $1 \leq n<\omega$ by letting $\sigma(\lambda, 1):=\lambda$ and, given $1 \leq n<\omega$, letting $\sigma(\lambda, n+1):=$ $\left(2^{<\sigma(\lambda, n)}\right)^{+}$.

Remark 3.2. To connect Definition 3.1 with the already familiar $\beth$-notation and to help clarify the choice of cardinals in the statements of Corollary 3.16. Theorem 5.4 and Corollary 6.2, we make the following observations, which we leave the reader to verify.
(1) If $\lambda=\kappa^{+}$and $1 \leq n<\omega$, then $\sigma(\lambda, n)=\left(\beth_{n-1}(\kappa)\right)^{+}$. In particular, $\sigma\left(\aleph_{1}, n\right)=\beth_{n-1}^{+}$and $\sigma\left(\beth_{1}^{+}, n\right)=\beth_{n}^{+}$.
(2) For every infinite regular $\lambda$, if $2 \leq n<\omega$, then $\sigma(\lambda, n)=\left(\beth_{n-2}\left(2^{<\lambda}\right)\right)^{+}$.

Note in particular that $\sigma(\lambda, n)$ is regular for each regular infinite $\lambda$ and each $1 \leq$ $n<\omega$.

We also remark that $\sigma(\lambda, n)$ is precisely the cardinal resource needed to ensure a monochromatic set of size $\lambda$ in the $n$-dimensional Erdős-Rado theorem, which can be formulated as follows: for every $1 \leq n<\omega$ and all infinite cardinals $\nu<\lambda$, with $\lambda$ regular, the partition relation $\sigma(\lambda, n) \rightarrow(\lambda+(n-1))_{\nu}^{n}$ holds ([10, Theorem 39]; cf. also [1, Proposition 1]). See Corollary 3.19 for a more precise formulation of the connection between our main result and the Erdős-Rado theorem.

In the proof of Theorem 3.8 we will make use of the following notion of the type of a sequence of sets of ordinals, which describes the order-relations existing among the sets.

Definition 3.3. Suppose that $I$ is a set and, for all $i \in I, u_{i}$ is a set of ordinals. Then $\operatorname{tp}\left(\left\langle u_{i} \mid i \in I\right\rangle\right)$ (the type of $\left.\left\langle u_{i} \mid i \in I\right\rangle\right)$ is a function from $\operatorname{otp}\left(\bigcup_{i \in I} u_{i}\right)$ to $\mathcal{P}(I)$ defined as follows. First, let $\bigcup_{i \in I} u_{i}$ be enumerated in increasing order as
$\left\langle\alpha_{\eta} \mid \eta<\operatorname{otp}\left(\bigcup_{i \in I} u_{i}\right)\right\rangle$. Then, for all $\eta<\operatorname{otp}\left(\bigcup_{i \in I} u_{i}\right)$, let $\operatorname{tp}\left(\left\langle u_{i} \mid i \in I\right\rangle\right)(\eta):=$ $\left\{i \in I \mid \alpha_{\eta} \in u_{i}\right\}$.

We will often slightly abuse notation and write, for instance, $\operatorname{tp}\left(u_{0}, u_{1}, u_{2}\right)$ instead of $\operatorname{tp}\left(\left\langle u_{0}, u_{1}, u_{2}\right\rangle\right)$.

Remark 3.4. To connect Definition 3.3 with the earlier definition of aligned sets, we note that, if $a$ and $b$ are sets of ordinals, then $a$ and $b$ are aligned if and only if $\operatorname{tp}(a \cap b, a)=\operatorname{tp}(a \cap b, b)$. We also observe the following useful facts about the tp operator, which can easily be verified:
(1) Suppose that $I$ is a set and, for all $i \in I, u_{i}$ and $u_{i}^{\prime}$ are sets of ordinals. Suppose also that $\operatorname{tp}\left(\left\langle u_{i} \mid i \in I\right\rangle\right)=\operatorname{tp}\left(\left\langle u_{i}^{\prime} \mid i \in I\right\rangle\right)$. Then the following statements hold.
(a) For all $i \in I$, we have $\operatorname{otp}\left(u_{i}\right)=\operatorname{otp}\left(u_{i}^{\prime}\right)$.
(b) For all $J \subseteq I$, we have $\operatorname{tp}\left(\left\langle u_{i} \mid i \in J\right\rangle\right)=\operatorname{tp}\left(\left\langle u_{i}^{\prime} \mid i \in J\right\rangle\right)$.
(2) Suppose that $u_{0}, u_{1}, u_{0}^{\prime}$, and $u_{1}^{\prime}$ are sets of ordinals. If $u_{0}$ and $u_{1}$ are aligned and $\operatorname{tp}\left(u_{0}, u_{1}\right)=\operatorname{tp}\left(u_{0}^{\prime}, u_{1}^{\prime}\right)$, then $u_{0}^{\prime}$ and $u_{1}^{\prime}$ are also aligned and $\mathbf{r}\left(u_{0}^{\prime}, u_{1}^{\prime}\right)=\mathbf{r}\left(u_{0}, u_{1}\right)$.

The higher-dimensional $\Delta$-systems that we isolate in our main result will have an additional technical uniformity (the "moreover" clause of Theorem 3.8) that allows us to control the relationship between $u_{a}$ and $u_{b}$ for certain non-aligned pairs $a, b \in[H]^{n}$ and is useful in some applications. In order to properly state it, we need some further definitions. Readers can safely skip these technical considerations and the "moreover" clause of the theorem on first read, if desired, as they are not needed in our applications in Sections 4 and 5. They are used in the proof of Corollary 6.2, which is presented not in this paper but in (14.

Definition 3.5. Suppose that $i<\rho$ are ordinals and $a, b \in[\mathrm{On}]^{\rho}$. We say that $a$ and $b$ are aligned above $i$ if $a[\rho \backslash i]$ and $b[\rho \backslash i]$ are aligned.

The following notion provides strictly less information than $\operatorname{tp}(a, b)$ but is sometimes easier to control.

Definition 3.6. Suppose that $a$ and $b$ are sets of ordinals. Then the intersection type of $a$ and $b$, denoted $\operatorname{tp}_{\text {int }}(a, b)$, is the set $\{(i, j) \in \operatorname{otp}(a) \times \operatorname{otp}(b) \mid a(i)=b(j)\}$.

Definition 3.7. Suppose that $a$ is a nonempty set of ordinals and $i<\operatorname{otp}(a)$.
(1) We say that an ordinal $\alpha$ is $i$-possible for $a$ if the following two statements hold:
(a) if $i>0$, then $\alpha>a(i-1)$;
(b) if $i+1<\operatorname{otp}(a)$, then $\alpha<a(i+1)$.

Intuitively, $\alpha$ is $i$-possible for $a$ if $a(i)$ can be replaced by $\alpha$ without changing the relative positions of the other elements of $a$.
(2) If $\alpha$ is $i$-possible for $a$, then $a_{i \mapsto \alpha}$ is the set $(a \backslash\{a(i)\}) \cup\{\alpha\}$, i.e., the set obtained by replacing the $i^{\text {th }}$ element of $a$ with $\alpha$.

We are now ready for our main result. As we will see at the end of this section, unless $\lambda$ is a weakly compact cardinal, the theorem is optimal in the sense that $\mu$ cannot be lowered. We also note that clause (1) of the following theorem is essentially the Erdős-Rado theorem. In response to a query from the referee, we note that our proof does not yield an essentially new proof of the Erdős-Rado
theorem; if one extracts the proof of just clause (1) from our proof, one obtains more or less a proof of the Erdős-Rado theorem originally given by Simpson in [18] (see also the proof of [4, Theorem 7.2.1]).

Theorem 3.8. Suppose that

- $1 \leq n<\omega$;
- $\kappa, \nu<\lambda$ are infinite cardinals, $\lambda$ is regular and $<\kappa$-inaccessible, and $\mu=$ $\sigma(\lambda, n)$;
- $g:[\mu]^{n} \rightarrow \nu$;
- for all $b \in[\mu]^{n}$, we are given a set $u_{b} \in[\mathrm{On}]^{<\kappa}$.

Then there are $H \in[\mu]^{\lambda}$ and $k<\nu$ such that
(1) $g(b)=k$ for all $b \in[H]^{n}$;
(2) $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is a uniform $n$-dimensional $\Delta$-system.

Moreover, we can arrange our choice of $H$ so that, for all $a, b \in[H]^{n}$ and all $m<n$, if it is the case that $a$ and $b$ are aligned above $m$ and $a(m)=b(m)$, then, for any ordinal $\alpha \in H$ that is m-possible for both $a$ and $b$, we have $\operatorname{tp}_{\mathrm{int}}\left(u_{a}, u_{b}\right)=$ $\operatorname{tp}_{\text {int }}\left(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}}\right)$.

Proof. The proof is by induction on $n$. When $n=1$, the result follows from Proposition 2.4 and the pigeonhole principle (note that the "moreover" clause of the theorem is trivial if $n=1$ ). So suppose that $1<n<\omega$ and we have established all instances of the theorem for $n-1$.

Set $\mu^{*}:=\sigma(\lambda, n-1)$. We will construct the desired set $H$ via a sequence of refinements, which we outline here at the start. We will first isolate an ordinal $\mu_{M}<\mu$, of size $2^{<\mu^{*}}$ and cofinality at least $\mu^{*}$. Next, we will build a set $A \subseteq \mu_{M}$ of order type $\mu^{*}$ exhibiting certain uniformities with respect to the family $\left\langle u_{b}\right| b \in$ $\left.[\mu]^{n}\right\rangle$ and the function $g$. An application of the inductive hypothesis for $n-1$ will then yield a set $H_{0} \subseteq A$ of cardinality $\lambda$. Finally, we will thin out $H_{0}$ one last time by recursively constructing an increasing sequence $\left\langle\beta_{\xi} \mid \xi<\lambda\right\rangle$ from $H_{0}$ and letting $H:=\left\{\beta_{\xi} \mid \xi<\lambda\right\}$. Together, this sequence of refinements is as follows:

$$
\mu \supseteq \mu_{M} \supseteq A \supseteq H_{0} \supseteq\left\{\beta_{\xi} \mid \xi<\lambda\right\}=H
$$

To begin, let $\theta$ be a sufficiently large regular cardinal, and let $M$ be an elementary substructure of $\left(H(\theta), \in, g,\left\langle u_{b} \mid b \in[\mu]^{n}\right\rangle\right)$ such that $M$ is closed under sequences of length less than $\mu^{*}$ and $\mu_{M}:=M \cap \mu \in \mu$. This is possible, since $\mu^{*}$ is regular and $\mu=\sigma(\lambda, n)=\left(2^{<\mu^{*}}\right)^{+}$. Note that $\operatorname{cf}\left(\mu_{M}\right) \geq \mu^{*}$.

Temporarily fix an arbitrary $a \in\left[\mu_{M}\right]^{n-1}$, and consider $u_{a \frown\left\langle\mu_{M}\right\rangle}$. Let $w_{a}:=$ $u_{a \frown\left\langle\mu_{M}\right\rangle} \cap M$ and $\rho_{a}=\operatorname{otp}\left(u_{a \frown\left\langle\mu_{M}\right\rangle}\right)$. Let $\mathbf{i}_{a}:=\mathbf{r}\left(u_{a \frown\left\langle\mu_{M}\right\rangle}, w_{a}\right)$, and let $\mathbf{j}_{a}:=$ $\rho_{a} \backslash \mathbf{i}_{a}$. Note that $u_{a \sim\left\langle\mu_{M}\right\rangle}\left[\mathbf{i}_{a}\right]=w_{a}$. For each $j \in \mathbf{j}_{a}$, let $\gamma_{a, j}$ be the least ordinal $\gamma$ in $M$ such that $u_{a-\left\langle\mu_{M}\right\rangle}(j)<\gamma$; to see that such an ordinal $\gamma$ exists, note that $\sup \left(\bigcup_{b \in[\mu]^{n}} u_{b}\right)$ is definable in $M$ and is therefore an element of $M$.
Claim 3.9. There is a set $A \subseteq \mu_{M}$ of order type $\mu^{*}$ such that:
(1) For every $a \in[A]^{n-1}$ and every $\beta \in A$ with $\max (a)<\beta$ :
(a) $g(a \frown\langle\beta\rangle)=g\left(a \frown\left\langle\mu_{M}\right\rangle\right)$;
(b) $\operatorname{otp}\left(u_{a}-\langle\beta\rangle\right)=\rho_{a}$;
(c) $u_{a}-\langle\beta\rangle\left[\mathbf{i}_{a}\right]=w_{a}$.
(2) For every $a \in[A]^{n-1}$, all $\alpha, \beta \in A$ with $\max (a)<\alpha<\beta$, and all $j \in \mathbf{j}_{a}$, we have $u_{a \frown\langle\beta\rangle}(j) \notin u_{a \frown\langle\alpha\rangle}$.
(3) For every $\beta \in A$, we have

$$
\operatorname{tp}\left(\left\langle u_{a \sim\langle\beta\rangle} \mid a \in[A \cap \beta]^{n-1}\right\rangle\right)=\operatorname{tp}\left(\left\langle u_{a \sim\left\langle\mu_{M}\right.}\right\rangle\left|a \in[A \cap \beta]^{n-1}\right\rangle\right)
$$

In particular, if $a_{0}, a_{1} \in[A \cap \beta]^{n-1}$, then

$$
\operatorname{tp}\left(u_{a_{0}-\langle\beta\rangle}, u_{a_{1}-\langle\beta\rangle}\right)=\operatorname{tp}\left(u_{a_{0}-\left\langle\mu_{M}\right\rangle}, u_{a_{1}-\left\langle\mu_{M}\right\rangle}\right) .
$$

Proof. We will recursively construct an increasing sequence $\left\langle\alpha_{\eta} \mid \eta<\mu^{*}\right\rangle$ of ordinals below $\mu_{M}$ and then let $A:=\left\{\alpha_{\eta} \mid \eta<\mu^{*}\right\}$. Our construction will maintain the hypothesis that, for all $\eta<\mu^{*}, A_{\eta}:=\left\{\alpha_{\xi} \mid \xi<\eta\right\}$ satisfies all of the items in the statement of the claim.

Begin by letting $\alpha_{\eta}=\eta$ for all $\eta<n-1$. Now suppose that $n-1 \leq \eta<\mu^{*}$ and we have defined $\left\langle\alpha_{\xi} \mid \xi<\eta\right\rangle$. By the closure of $M$ and the fact that $\left[A_{\eta}\right]^{n-1}$ has size less than $\mu^{*}$, we know that all of the following are elements of $M$ :

- $A_{\eta}$;
- $\left\langle g\left(a \frown\left\langle\mu_{M}\right\rangle\right) \mid a \in\left[A_{\eta}\right]^{n-1}\right\rangle ;$
- $\left\langle\left(w_{a}, \rho_{a}, \mathbf{i}_{a}, \mathbf{j}_{a}\right) \mid a \in\left[A_{\eta}\right]^{n-1}\right\rangle$;
- $\left\langle\gamma_{a, j} \mid a \in\left[A_{\eta}\right]^{n-1}, j \in \mathbf{j}_{\alpha}\right\rangle$.

Moreover, $\operatorname{tp}\left(\left\langle u_{a}-\left\langle\mu_{M}\right\rangle \mid a \in\left[A_{\eta}\right]^{n-1}\right\rangle\right)$ is a function from an ordinal less than $\mu^{*}$ to $\mathcal{P}\left(\left[A_{\eta}\right]^{n-1}\right)$, so again the closure of $M$ implies that $\operatorname{tp}\left(\left\langle u_{a \sim\left\langle\mu_{M}\right\rangle} \mid a \in\left[A_{\eta}\right]^{n-1}\right\rangle\right)$ is in $M$.

For each $a \in\left[A_{\eta}\right]^{n-1}$ and $j \in \mathbf{j}_{a}$, let

$$
\epsilon_{a, j}:=\sup \left\{\sup \left(u_{b} \cap \gamma_{a, j}\right) \mid b \in\left[A_{\eta}\right]^{n}\right\} .
$$

Note that $\operatorname{cf}\left(\gamma_{a, j}\right) \geq \mu^{*}$, since otherwise there would be a cofinal $x \subseteq \gamma_{a, j}$ such that $x \subseteq M$. Therefore, we have $\epsilon_{a, j} \in M \cap \gamma_{a, j}$ and, again by closure, $\left\langle\epsilon_{a, j}\right| a \in$ $\left.\left[A_{\eta}\right]^{n-1}, j \in \mathbf{j}_{a}\right\rangle \in M$.

In $H(\theta)$, the ordinal $\mu_{M}$ witnesses the truth of the statement asserting the existence of an ordinal $\beta$ such that:

- $\sup \left(A_{\eta}\right)<\beta<\mu$;
- $g\left(a^{\frown}\langle\beta\rangle\right)=g\left(a^{\frown}\left\langle\mu_{M}\right\rangle\right)$ for all $a \in\left[A_{\eta}\right]^{n-1}$;
- $\operatorname{tp}\left(\left\langle u_{a} \sim\langle\beta\rangle \mid a \in\left[A_{\eta}\right]^{n-1}\right\rangle\right)=\operatorname{tp}\left(\left\langle u_{a} \sim\left\langle\mu_{M}\right\rangle \mid a \in\left[A_{\eta}\right]^{n-1}\right\rangle\right)$;
- $u_{a}\left\langle\langle\beta\rangle\left[\mathbf{i}_{a}\right]=w_{a}\right.$ for all $a \in\left[A_{\eta}\right]^{n-1}$;
- $u_{a \sim\langle\beta\rangle}(j)$ is in the interval $\left(\epsilon_{a, j}, \gamma_{a, j}\right)$ for all $a \in\left[A_{\eta}\right]^{n-1}$ and all $j \in \mathbf{j}_{a}$.

All of the parameters in the above statement are in $M$ (note, for instance, that, in the second item, $a^{\frown}\left\langle\mu_{M}\right\rangle$ is not in $M$, but $\left\langle g\left(a^{\frown}\left\langle\mu_{M}\right\rangle\right) \mid a \in\left[A_{\eta}\right]^{n-1}\right\rangle$ is). Therefore, by elementarity, we can choose $\alpha_{\eta} \in M$ satisfying the statement. It is evident that this choice of $\alpha_{\eta}$ satisfies the requirements of the construction. In particular, notice that, as a consequence of clause (1a) of Remark 3.4 and the fact that $\alpha_{\eta}$ satisfies the third bullet point above, we have $\operatorname{otp}\left(u_{a} \frown\left\langle\alpha_{\eta}\right\rangle\right)=\operatorname{otp}\left(u_{a} \frown\left\langle\mu_{M}\right\rangle\right)=\rho_{a}$ for all $a \in\left[A_{\eta}\right]^{n-1}$. Also, the last bullet point above ensures that, for all $a \in\left[A_{\eta}\right]^{n-1}$, all $\alpha \in A_{\eta} \backslash(\max (a)+1)$, and all $j \in \mathbf{j}_{a}$, we have $u_{a \curvearrowleft\left\langle\alpha_{\eta}\right\rangle}(j) \notin u_{a \curvearrowleft\langle\alpha\rangle}$. Therefore, this completes the construction and the proof of the claim.

Let $A$ be as given by Claim 3.9. Define a function $g^{*}$ on $[A]^{n-1}$ by letting $g^{*}(a):=\left\langle g\left(a^{\frown}\left\langle\mu_{M}\right\rangle\right), \rho_{a}, \mathbf{i}_{a}, \mathbf{j}_{a}\right\rangle$ for all $a \in[A]^{n-1}$. Since we know that

- $g:[\mu]^{n} \rightarrow \nu$;
- $\rho_{a}<\kappa$; and
- $\mathbf{i}_{a}, \mathbf{j}_{a} \subseteq \rho_{a}$;
it follows that $g^{*}$ can be coded as a function from $[A]^{n-1}$ to $\max \left\{\nu, 2^{<\kappa}\right\}$ which, by the hypothesis of the theorem, is less than $\lambda$. Recalling that $\mu^{*}=\sigma(\lambda, n-1)=|A|$, apply the induction hypothesis to $g^{*}$ and $\left\langle u_{a \sim\left\langle\mu_{M}\right\rangle} \mid a \in[A]^{n-1}\right\rangle$ to find $H_{0} \subseteq A$, $k<\nu, \rho<\kappa$, and sets $\mathbf{i}, \mathbf{j} \subseteq \rho$ such that the following statements all hold:
- otp $\left(H_{0}\right)=\lambda$;
- $g\left(a^{\frown}\left\langle\mu_{M}\right\rangle\right)=k$ for all $a \in\left[H_{0}\right]^{n-1}$;
- $\left\langle\rho_{a}, \mathbf{i}_{a}, \mathbf{j}_{a}\right\rangle=\langle\rho, \mathbf{i}, \mathbf{j}\rangle$ for all $a \in\left[H_{0}\right]^{n-1}$;
- $\left\langle u_{a} \curvearrowleft\left\langle\mu_{M}\right\rangle \mid a \in\left[H_{0}\right]^{n-1}\right\rangle$ is a uniform $(n-1)$-dimensional $\Delta$-system, as witnessed by $\rho$ and by sets $\mathbf{s}_{\mathbf{m}} \subseteq \rho$ for each $\mathbf{m} \subseteq n-1$;
- $\left\langle u_{a \sim\left\langle\mu_{M}\right\rangle} \mid a \in\left[H_{0}\right]^{n-1}\right\rangle$ satisfies the "moreover" clause in the statement of the theorem.
We will thin out $H_{0}$ to a further unbounded subset $H \subseteq H_{0}$ before the end of the proof. For now, let us begin verifying clauses (1) and (2) in the statement of the theorem, noting that what we verify for $H_{0}$ will remain true after further thinning out.

We first take care of clause (1) of the theorem, simultaneously showing that $\operatorname{otp}(b)=\rho$ for all $b \in\left[H_{0}\right]^{n}$. To this end, fix $b \in\left[H_{0}\right]^{n}$. Then $b$ is of the form $a^{\frown}\langle\beta\rangle$ for some $\beta \in H_{0}$ and $a \in\left[H_{0} \cap \beta\right]^{n-1}$. Since $A$ satisfies Clause (1) of Claim 3.9 and $H_{0} \subseteq A$, we have $g(a \frown\langle\beta\rangle)=g\left(a \frown\left\langle\mu_{M}\right\rangle\right)$, and $\operatorname{otp}\left(u_{a \frown\langle\beta\rangle}\right)=\operatorname{otp}\left(u_{a \frown\left\langle\mu_{M}\right\rangle}\right)=\rho_{a}$. Then, by our choice of $H_{0}, k$, and $\rho$, we have $g\left(a^{\frown}\left\langle\mu_{M}\right\rangle\right)=k$ and $\rho_{a}=\rho$. Therefore, $g(b)=k$ and $\operatorname{otp}\left(u_{b}\right)=\rho$, as desired.

We now turn our attention to clause (2). The value of $\rho$ that we isolated above is the order type that will eventually witness that $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is a uniform $n$-dimensional $\Delta$-system; indeed, by the previous paragraph we have $\operatorname{otp}\left(u_{b}\right)=\rho$ for all $b \in\left[H_{0}\right]^{n}$. We next specify the values for $\left\langle\mathbf{r}_{\mathbf{m}} \mid \mathbf{m} \subseteq n\right\rangle$ that will witness that $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is a uniform $n$-dimensional $\Delta$-system. For each $\mathbf{m} \subseteq n$, let $\mathbf{m}^{-}:=\mathbf{m} \cap(n-1)$. If $n-1 \in \mathbf{m}$, then set $\mathbf{r}_{\mathbf{m}}:=\mathbf{s}_{\mathbf{m}^{-}}$. If $n-1 \notin \mathbf{m}$, then set $\mathbf{r}_{\mathbf{m}}:=\mathbf{s}_{\mathbf{m}^{-}} \cap \mathbf{i}$. Note that, in either case, we do indeed have $\mathbf{r}_{\mathbf{m}} \subseteq \rho$.

Claim 3.10. For all $\mathbf{m}_{0}, \mathbf{m}_{1} \subseteq n$, we have $\mathbf{r}_{\mathbf{m}_{0} \cap \mathbf{m}_{1}}=\mathbf{r}_{\mathbf{m}_{0}} \cap \mathbf{r}_{\mathbf{m}_{1}}$.
Proof. This follows immediately from the fact that $\mathbf{s}_{\mathbf{m}_{0}^{-} \cap \mathbf{m}_{1}^{-}}=\mathbf{s}_{\mathbf{m}_{0}^{-}} \cap \mathbf{s}_{\mathbf{m}_{1}^{-}}$for all $\mathbf{m}_{0}, \mathbf{m}_{1} \subseteq n$.

It remains to verify clause (2) of Definition 2.5 i.e., if $a, b \in[H]^{n}$ are aligned and $\mathbf{r}(a, b)=\mathbf{m}$, then $u_{a}$ and $u_{b}$ are aligned, and $\mathbf{r}\left(u_{a}, u_{b}\right)=\mathbf{r}_{\mathbf{m}}$. We split this verification into two cases, depending on whether or not $n-1$ is in $\mathbf{m}$.
Claim 3.11. Suppose that $b_{0}, b_{1} \in\left[H_{0}\right]^{n}$ are aligned and $n-1 \in \mathbf{m}=\mathbf{r}\left(b_{0}, b_{1}\right)$. Then $u_{b_{0}}$ and $u_{b_{1}}$ are aligned and $\mathbf{r}\left(u_{b_{0}}, u_{b_{1}}\right)=\mathbf{r}_{\mathbf{m}}$.
Proof. Since $n-1 \in \mathbf{m}$, we have $\mathbf{r}_{\mathbf{m}}=\mathbf{s}_{\mathbf{m}^{-}}$. It also follows from the fact that $n-1 \in \mathbf{m}$ that there is $\beta \in H_{0}$ such that $b_{0}$ and $b_{1}$ are of the form $a_{0} \frown\langle\beta\rangle$ and $a_{1} \frown\langle\beta\rangle$ respectively, where $a_{0}, a_{1} \in\left[H_{0} \cap \beta\right]^{n-1}$ are aligned and $\mathbf{r}\left(a_{0}, a_{1}\right)=$ $\mathbf{m}^{-}$. By our choice of $H_{0}$ and $\mathbf{s}_{\mathbf{m}^{-}}$, it follows that $u_{a_{0}-\left\langle\mu_{M}\right\rangle}$ and $u_{a_{1} \smile\left\langle\mu_{M}\right\rangle}$ are aligned and $\mathbf{r}\left(u_{a_{0}-\left\langle\mu_{M}\right\rangle}, u_{a_{1}-\left\langle\mu_{M}\right\rangle}\right)=\mathbf{s}_{\mathbf{m}^{-}}$. The fact that $A$ satisfies Clause (3) of Claim 3.9 then implies that $\operatorname{tp}\left(u_{b_{0}}, u_{b_{1}}\right)=\operatorname{tp}\left(u_{a_{0}-\left\langle\mu_{M}\right\rangle}, u_{a_{1} \neg\left\langle\mu_{M}\right\rangle}\right)$, and therefore, recalling Remark 3.4 that $u_{b_{0}}$ and $u_{b_{1}}$ are aligned, with $\mathbf{r}\left(u_{b_{0}}, u_{b_{1}}\right)=\mathbf{s}_{\mathbf{m}^{-}}=\mathbf{r}_{\mathbf{m}}$, as desired.

We next deal with the case in which $\mathbf{m} \subseteq n-1$. This will take a bit more work. We first establish the following claim.

Claim 3.12. Suppose that $b_{0}, b_{1} \in\left[H_{0}\right]^{n}, \mathbf{m} \subseteq n-1$, and $b_{0}[\mathbf{m}]=b_{1}[\mathbf{m}]$. Then $u_{b_{0}}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{b_{1}}\left[\mathbf{r}_{\mathbf{m}}\right]$.
Proof. Since $\mathbf{m} \subseteq n-1$, we have $\mathbf{m}^{-}=\mathbf{m}$ and $\mathbf{r}_{\mathbf{m}}=\mathbf{s}_{\mathbf{m}} \cap \mathbf{i}$. We also know that $b_{0}$ and $b_{1}$ are of the form $a_{0} \wedge\langle\alpha\rangle$ and $a_{1} \smile\langle\beta\rangle$, respectively, where $\alpha, \beta \in H_{0}$, $a_{0}, a_{1} \in\left[H_{0}\right]^{n-1}$, and $a_{0}[\mathbf{m}]=a_{1}[\mathbf{m}]$. By Proposition 2.6(1) applied to $\left\langle u_{a-\left\langle\mu_{M}\right\rangle}\right|$ $\left.a \in\left[H_{0}\right]^{n-1}\right\rangle, \mathbf{m}, a_{0}$, and $a_{1}$, we know that $u_{a_{0}-\left\langle\mu_{M}\right\rangle}\left[\mathbf{s}_{\mathbf{m}}\right]=u_{a_{1} \sim\left\langle\mu_{M}\right\rangle}\left[\mathbf{s}_{\mathbf{m}}\right]$. Now fix $i \in \mathbf{r}_{\mathbf{m}}$. Since $i \in \mathbf{s}_{\mathbf{m}}$, it follows that $\left.u_{a_{0}-\left\langle\mu_{M}\right\rangle}(i)=u_{a_{1} \leftharpoonup\left\langle\mu_{M}\right\rangle}\right\rangle$. Since $i \in \mathbf{i}$, the fact that $A$ satisfies Clause (1c) of Claim 3.9 implies that $u_{b_{0}}(i)=u_{a_{0}} \sim\left\langle\mu_{M}\right\rangle(i)$ and $\left.u_{b_{1}}(i)=u_{a_{1}-\left\langle\mu_{M}\right\rangle}\right\rangle$. Together, this implies that $u_{b_{0}}(i)=u_{b_{1}}(i)$, and hence $u_{b_{0}}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{b_{1}}\left[\mathbf{r}_{\mathbf{m}}\right]$.

As an immediate consequence of Claim 3.12, if $b_{0}, b_{1} \in\left[H_{0}\right]^{n}$ are aligned and $\mathbf{r}\left(b_{0}, b_{1}\right)=\mathbf{m} \subseteq n-1$, then $u_{b_{0}}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{b_{1}}\left[\mathbf{r}_{\mathbf{m}}\right]$. Showing that $u_{b_{0}}$ and $u_{b_{1}}$ are disjoint outside of $u_{b_{0}}\left[\mathbf{r}_{\mathbf{m}}\right]$ will take some more work and possibly a thinning out of $H_{0}$. For $m<n$ and $a \in\left[H_{0}\right]^{m}$, choose any $b \in\left[H_{0}\right]^{n}$ with $a=b[m]$ (i.e., $b$ is an end-extension of $a$ ), and define $u_{a}:=u_{b}\left[\mathbf{r}_{m}\right]$. By Claim 3.12 this definition is independent of our choice of $b$.

For the following claim, recall our convention that $\max (\emptyset)=-1$.
Claim 3.13. Suppose that $m<n$ and $a \in\left[H_{0}\right]^{m}$. Then

$$
\left\langle u_{a \frown\langle\beta\rangle} \mid \beta \in H_{0} \backslash(\max (a)+1)\right\rangle
$$

is a $\Delta$-system with root $u_{a}$.
Proof. Suppose first that $m=n-1$, in which case $\mathbf{r}_{m}=\mathbf{i}$. Fix $(\alpha, \beta) \in\left[H_{0}\right]^{2}$ with $\alpha>\max (a)$, and consider $u_{a \prec\langle\alpha\rangle} \cap u_{a \sim\langle\beta\rangle}$. By Claim [3.12] we have $u_{a \prec\langle\alpha\rangle}[\mathbf{i}]=$ $u_{a}\ulcorner\langle\beta\rangle[\mathbf{i}]$. Furthermore, for all $j \in \mathbf{j}$, the fact that $A$ satisfies Clause (2) of Claim 3.9 implies that $u_{a \curvearrowleft\langle\beta\rangle}(j) \notin u_{a \curvearrowleft\langle\alpha\rangle}$. It follows that

$$
u_{a \frown\langle\alpha\rangle} \cap u_{a \frown\langle\beta\rangle}=u_{a \leftharpoonup\langle\alpha\rangle}[\mathbf{i}]=u_{a},
$$

as desired.
Next, suppose that $m<n-1$. Fix $\left(\beta_{0}, \beta_{1}\right) \in\left[H_{0}\right]^{2}$ with $\beta_{0}>\max (a)$, and consider $u_{a} \prec\left\langle\beta_{0}\right\rangle \cap u_{a} \frown\left\langle\beta_{1}\right\rangle$. Fix $c \in\left[H_{0}\right]^{n-m-1}$ with $\min (c)>\beta_{1}$ and set $b_{\ell}:=$ $a \curvearrowleft\left\langle\beta_{\ell}\right\rangle \subset c$ for $\ell<2$. Note that $b_{\ell} \in\left[H_{0}\right]^{n}$, that $u_{a \frown\left\langle\beta_{\ell}\right\rangle}=u_{b_{\ell}}\left[\mathbf{r}_{m+1}\right]$, and that $u_{a}=u_{b_{\ell}}\left[\mathbf{r}_{m}\right]$. Observe also that $b_{0}$ and $b_{1}$ are aligned and that $\mathbf{r}\left(b_{0}, b_{1}\right)=n \backslash\{m\}$, so, by Claim [3.11] we have $u_{b_{0}} \cap u_{b_{1}}=u_{b_{0}}\left[\mathbf{r}_{n \backslash\{m\}}\right]=u_{b_{1}}\left[\mathbf{r}_{n \backslash\{m\}}\right]$. Putting this together, we obtain

$$
\begin{aligned}
u_{a-\left\langle\beta_{0}\right\rangle} \cap u_{a-\left\langle\beta_{1}\right\rangle} & =u_{b_{0}}\left[\mathbf{r}_{m+1}\right] \cap u_{b_{1}}\left[\mathbf{r}_{m+1}\right] \\
& =u_{b_{0}}\left[\mathbf{r}_{m+1}\right] \cap u_{b_{1}}\left[\mathbf{r}_{m+1}\right] \cap u_{b_{0}}\left[\mathbf{r}_{n \backslash\{m\}}\right] \cap u_{b_{1}}\left[\mathbf{r}_{n \backslash\{m\}}\right] \\
& =u_{b_{0}}\left[\mathbf{r}_{m}\right] \cap u_{b_{1}}\left[\mathbf{r}_{m}\right] \\
& =u_{a},
\end{aligned}
$$

where the passage from the second to the third line in the above sequence of equations follows from Claim 3.10 and the observation that $(m+1) \cap(n \backslash\{m\})=m$.

We are now ready to thin out $H_{0}$ to our final set $H$ witnessing the conclusion of the theorem. We will recursively construct an increasing sequence $\left\langle\beta_{\xi} \mid \xi<\lambda\right\rangle$ of ordinals from $H_{0}$ and then define $H:=\left\{\beta_{\xi} \mid \xi<\lambda\right\}$.

Begin by letting $\beta_{0}:=\min \left(H_{0}\right)$. Next, suppose that $0<\zeta<\lambda$ and $\left\langle\beta_{\xi} \mid \xi<\zeta\right\rangle$ has been defined. Let $B_{\zeta}:=\left\{\beta_{\xi} \mid \xi<\zeta\right\}$. Suppose that $a_{0} \in\left[B_{\zeta}\right]^{<n}$ and $a_{1} \in$
$\left[B_{\zeta}\right] \leq n$. By Claim 3.13, the sequence $\left\langle u_{a_{0}-\langle\beta\rangle} \backslash u_{a_{0}} \mid \beta \in H_{0} \backslash\left(\sup \left(B_{\zeta}\right)+1\right)\right\rangle$ consists of pairwise disjoint sets. Since $\left|u_{a_{1}}\right|<\kappa$, it follows that, letting $C_{a_{0}, a_{1}}$ be the set of $\beta \in H_{0} \backslash\left(\sup \left(B_{\zeta}\right)+1\right)$ such that $u_{a_{0}-\langle\beta\rangle} \backslash u_{a_{0}}$ has nonempty intersection with $u_{a_{1}}$, we have $\left|C_{a_{0}, a_{1}}\right|<\kappa$. Since the number of such pairs $\left(a_{0}, a_{1}\right)$ is less than $\lambda$, we can find $\beta \in H_{0} \backslash\left(\sup \left(B_{\zeta}\right)+1\right)$ such that, for all $a_{0} \in\left[B_{\zeta}\right]^{<n}$ and all $a_{1} \in\left[B_{\zeta}\right]^{\leq n}$, we have $\beta \notin C_{a_{0}, a_{1}}$. Let $\beta_{\zeta}$ be the least such $\beta$, and continue to the next step of the construction.

To verify that $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is a uniform $n$-dimensional $\Delta$-system as witnessed by $\rho$ and $\left\langle\mathbf{r}_{\mathbf{m}} \mid \mathbf{m} \subseteq n\right\rangle$, we must show that, for all $b_{0}, b_{1} \in[H]^{n}$, if $b_{0}$ and $b_{1}$ are aligned and $\mathbf{m}=\mathbf{r}\left(b_{0}, b_{1}\right)$, then $u_{b_{0}}$ and $u_{b_{1}}$ are aligned with $\mathbf{r}\left(u_{b_{0}}, u_{b_{1}}\right)=\mathbf{r}_{\mathbf{m}}$. To this end, fix $b_{0}, b_{1} \in[H]^{n}$ such that $b_{0}$ and $b_{1}$ are aligned, and let $\mathbf{m}=\mathbf{r}\left(b_{0}, b_{1}\right)$. If $n-1 \in \mathbf{m}$, then the desired conclusion already follows from Claim 3.11 so assume that $n-1 \notin \mathbf{m}$.

Without loss of generality, assume that $\max \left(b_{0}\right)<\max \left(b_{1}\right)$. By Claim 3.12, we know that $u_{b_{0}}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{b_{1}}\left[\mathbf{r}_{\mathbf{m}}\right]$. It will therefore suffice to show that, for all $i<\rho$, if $u_{b_{1}}(i) \in u_{b_{0}}$, then $i \in \mathbf{r}_{\mathbf{m}}$.

To this end, fix $i<\rho$ such that $\gamma:=u_{b_{1}}(i) \in u_{b_{0}}$. Let $m^{*}<n$ be least such that $b_{1}\left(m^{*}\right)>\max \left(b_{0}\right)$. Notice that this $m^{*}$ exists, since $\max \left(b_{1}\right)>\max \left(b_{0}\right)$.

Claim 3.14. $\gamma \in u_{b_{1}\left[m^{*}\right]}$.
Proof. We will prove by induction on $\ell \leq n-m^{*}$ that $\gamma \in u_{b_{1}[n-\ell]}$. First, if $\ell=0$, then $b_{1}[n-\ell]=b_{1}[n]=b_{1}$, and, by assumption, we have $\gamma \in u_{b_{1}}$. Next, suppose that $\ell<n-m^{*}$ and we have proven that $\gamma \in u_{b_{1}[n-\ell]}$. Then $b_{1}(n-\ell-1)>\max \left(b_{0}\right)$, so, by our thinning out of $H_{0}$ to $H$, we know that $u_{b_{1}[n-\ell]} \backslash u_{b_{1}[n-\ell-1]}$ is disjoint from $u_{b_{0}}$. Since $\gamma \in u_{b_{0}}$, it follows that $\gamma \in u_{b_{1}[n-\ell-1]}$.

Claim 3.15. $\gamma \in u_{b_{0}[n-1]}$.
Proof. Because $b_{0}$ and $b_{1}$ are aligned and $\max \left(b_{1}\right)>\max \left(b_{0}\right)$, we know that $\max \left(b_{0}\right) \notin b_{1}$. Since $m^{*}$ was least with $b_{1}\left(m^{*}\right)>\max \left(b_{0}\right)$, it follows that $\max \left(b_{0}\right)>$ $\max \left(b_{1}\left[m^{*}\right]\right)$. Therefore, by our thinning out of $H_{0}$ to $H$, we know that $u_{b_{0}} \backslash u_{b_{0}[n-1]}$ is disjoint from $u_{b_{1}\left[m^{*}\right]}$. Since $\gamma \in u_{b_{1}\left[m^{*}\right]}$ by the previous claim, it follows that $\gamma \in u_{b_{0}[n-1]}$.

For $\ell<2$, let $a_{\ell}=b_{\ell}[n-1]$ and $\beta_{\ell}=b_{\ell}(n-1)$. By the two previous claims and our choice of $\beta_{0}$ and $\beta_{1}$, we know that

$$
\begin{aligned}
\gamma \in u_{b_{0}[n-1]} \cap u_{b_{1}\left[m^{*}\right]} & =u_{b_{0}}\left[\mathbf{r}_{n-1}\right] \cap u_{b_{1}}\left[\mathbf{r}_{m^{*}}\right] \\
& \subseteq u_{b_{0}}[\mathbf{i}] \cap u_{b_{1}}[\mathbf{i}] \\
& =u_{a_{0}-\left\langle\mu_{M}\right\rangle}[\mathbf{i}] \cap u_{a_{1}-\left\langle\mu_{M}\right\rangle}[\mathbf{i}] .
\end{aligned}
$$

In particular, we have $i \in \mathbf{i}$ and, since $u_{b_{1}}[\mathbf{i}]=u_{a_{1}-\left\langle\mu_{M}\right\rangle}[\mathbf{i}]$, we also know that $u_{a_{1}-\left\langle\mu_{M}\right\rangle}(i)=\gamma$.

Since $b_{0}$ and $b_{1}$ are aligned, we know that $a_{0}$ and $a_{1}$ are aligned, and, since $n-1 \notin \mathbf{m}$, we also have $\mathbf{r}\left(a_{0}, a_{1}\right)=\mathbf{m}$. Therefore, by our choice of $H_{0}$, it follows that $u_{a_{0} \neg\left\langle\mu_{M}\right\rangle}$ and $u_{a_{1} \frown\left\langle\mu_{M}\right\rangle}$ are aligned and $\mathbf{r}\left(u_{a_{0} \neg\left\langle\mu_{M}\right\rangle}, u_{a_{1} \neg\left\langle\mu_{M}\right\rangle}\right)=\mathbf{s}_{\mathbf{m}}$. Since $\gamma \in u_{a_{0} \sim\left\langle\mu_{M}\right\rangle} \cap u_{a_{1} \smile\left\langle\mu_{M}\right\rangle}$, it follows that $i \in \mathbf{s}_{\mathbf{m}}$. But since $i \in \mathbf{i}$ and $\mathbf{r}_{\mathbf{m}}=\mathbf{s}_{\mathbf{m}} \cap \mathbf{i}$, it follows that $i \in \mathbf{r}_{\mathbf{m}}$, which finishes the proof of clause (2).

We finally turn our attention to the "moreover" clause. To this end, fix $m<n$ and $a, b \in[H]^{n}$ such that $a$ and $b$ are aligned above $m$ and $a(m)=b(m)$. Fix $\alpha \in H$ such that $\alpha$ is $m$-possible for both $a$ and $b$. We must show that $\operatorname{tp}_{\text {int }}\left(u_{a}, u_{b}\right)=$
$\operatorname{tp}_{\text {int }}\left(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}}\right)$. Let $a^{-}=a[n-1]$ and $b^{-}=b[n-1]$, and let $a^{+}:=a^{-}\left\langle\left\langle\mu_{M}\right\rangle\right.$ and $b^{+}:=b^{-\frown}\left\langle\mu_{M}\right\rangle$.

Suppose first that $m=n-1$, so $a(n-1)=b(n-1)$. By the fact that A satisfies Clause (3) of Claim 3.9 we know that $\operatorname{tp}\left(u_{a}, u_{b}\right)=\operatorname{tp}\left(u_{a+}, u_{b^{+}}\right)=$ $\operatorname{tp}\left(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}}\right)$, and hence $\operatorname{tp}_{\text {int }}\left(u_{a}, u_{b}\right)=\operatorname{tp}_{\text {int }}\left(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}}\right)$.

Suppose next that $m<n-1$. By the fact that $\left\langle u_{a \frown\left\langle\mu_{M}\right\rangle} \mid a \in[H]^{n-1}\right\rangle$ satisfies the "moreover" clause in the statement of the theorem, we know that

$$
\begin{equation*}
\operatorname{tp}_{\mathrm{int}}\left(u_{a^{+}}, u_{b^{+}}\right)=\operatorname{tp}_{\mathrm{int}}\left(u_{a_{m \mapsto \alpha}^{+}}, u_{b_{m \mapsto \alpha}^{+}}\right) . \tag{*}
\end{equation*}
$$

Suppose in addition that $a(n-1)=b(n-1)$. Then, again by the fact that $A$ satisfies Clause (3) of Claim 3.9, we know that

$$
\operatorname{tp}\left(u_{a}, u_{b}\right)=\operatorname{tp}\left(u_{a^{+}}, u_{b^{+}}\right) \text {and } \operatorname{tp}\left(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}}\right)=\operatorname{tp}\left(u_{a_{m \mapsto \alpha}^{+}}, u_{b_{m \mapsto \alpha}^{+}}\right) .
$$

Putting this together yields $\operatorname{tp}_{\mathrm{int}}\left(u_{a}, u_{b}\right)=\operatorname{tp}_{\text {int }}\left(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}}\right)$, as desired.
The remaining case is that in which $a(n-1) \neq b(n-1)$. We show that $\operatorname{tp}_{\text {int }}\left(u_{a}, u_{b}\right) \subseteq \operatorname{tp}_{\text {int }}\left(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}}\right)$. A symmetric argument will yield the reverse inclusion. To this end, fix $(i, j) \in \operatorname{tp}_{\text {int }}\left(u_{a}, u_{b}\right)$. Thus, we have $u_{a}(i)=u_{b}(j)=\gamma$ for some ordinal $\gamma$. Since $a$ and $b$ are aligned above $m, a(m)=b(m)$, and $a(n-1) \neq b(n-1)$, it follows that $b(n-1) \notin a$ and $a(n-1) \notin b$. An argument exactly as in the proofs of Claims 3.14 and 3.15 then shows that $\gamma \in u_{a-} \cap u_{b^{-}}=u_{a}[\mathbf{i}] \cap u_{b}[\mathbf{i}]$, and hence we have $i, j \in \mathbf{i}$.

Since $i, j \in \mathbf{i}$, the fact that $A$ satisfies Clause (1c) of Claim 3.9 implies that $u_{a}(i)=u_{a^{+}}(i)$ and $u_{b}(j)=u_{b^{+}}(j)$, and hence $(i, j) \in \operatorname{tp}_{\text {int }}\left(u_{a^{+}}, u_{b^{+}}\right)$. By equation $(*)$ above, we have $(i, j) \in \operatorname{tp}_{\text {int }}\left(u_{a_{m \mapsto \alpha}^{+}}, u_{b_{m \mapsto \alpha}^{+}}\right)$. Again by the facts that $A$ satisfies Clause (1c) of Claim 3.9 and that $i, j \in \mathbf{i}$, we have $u_{a_{m \mapsto \alpha}}(i)=u_{a_{m \mapsto \alpha}^{+}}(i)$ and $u_{b_{m \mapsto \alpha}}(j)=u_{b_{m \mapsto \alpha}^{+}}(j)$, so $(i, j) \in \operatorname{tp}_{\text {int }}\left(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}}\right)$, thus finishing the proof.

The following corollary gives an important special case, obtained from setting $\kappa=\aleph_{0}$ and $\lambda=\aleph_{1}$ in Theorem 3.8,

Corollary 3.16. Suppose that $1 \leq n<\omega$, and let $\mu:=\beth_{n-1}^{+}$. If $\left\langle u_{b} \mid b \in[\mu]^{n}\right\rangle$ is a family of finite sets of ordinals and $g:[\mu]^{n} \rightarrow \omega$ is a function, then there is $H \in[\mu]^{\aleph_{1}}$ such that $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is a uniform $n$-dimensional $\Delta$-system and $g \upharpoonright[H]^{n}$ is constant.

We end this section with a discussion of the optimality of Theorem 3.8. It can be argued that, if $\kappa<\lambda \leq \mu$ are infinite cardinals, $1 \leq n<\omega$, and $\mu \rightarrow(\lambda)_{2<\kappa}^{2 n}$, then any sequence $\left\langle u_{a} \mid a \in[\mu]^{n}\right\rangle$ consisting of elements of [On] ${ }^{<\kappa}$ can be thinned out to a uniform $n$-dimensional $\Delta$-system of size $\lambda$ (see [3] for such an argument).

In general, $\mu \rightarrow(\lambda)_{2<\kappa}^{2 n}$ is a stronger assertion than $\mu \geq \sigma(\lambda, n)$, which is our assumption in Theorem 3.8, so this argument yields weaker results than those of Theorem 3.8. However, if $\lambda$ is weakly compact, then we have $\lambda \rightarrow(\lambda)_{2<\kappa}^{2 n}$ for all $1 \leq n<\omega$ and all $\kappa<\lambda$, so we obtain the following corollary.

Corollary 3.17. Suppose that $1 \leq n<\omega$ and that $\kappa<\lambda$ are infinite cardinals, with $\lambda$ being weakly compact. Suppose also that $\left\langle u_{a} \mid a \in[\lambda]^{n}\right\rangle$ is a sequence consisting of elements of $[\mathrm{On}]^{<\kappa}$. Then there is $H \in[\lambda]^{\lambda}$ such that $\left\langle u_{a} \mid a \in[H]^{n}\right\rangle$ is a uniform $n$-dimensional $\Delta$-system.

If $\lambda$ is not weakly compact, though, then our result is optimal in the sense that the value of $\mu$ cannot be decreased. This is true even disregarding clause (1) or the "moreover clause" of Theorem 3.8 and focusing only on the higher-dimensional $\Delta$-systems (and not even requiring that the $\Delta$-systems be uniform), for essentially the same reason that the Erdős-Rado theorem is optimal.

Proposition 3.18. Suppose that $1 \leq n<\omega$ and $\lambda$ is a regular uncountable cardinal that is not weakly compact, and suppose that $\mu<\sigma(\lambda, n)$. Then there is a sequence $\left\langle u_{a} \mid a \in[\mu]^{n}\right\rangle$ consisting of finite sets of ordinals such that there is no $H \in[\mu]^{\lambda}$ for which $\left\langle u_{a} \mid a \in[H]^{n}\right\rangle$ is an $n$-dimensional $\Delta$-system.

Proof. If $n=1$, then we have $\mu<\lambda$, so the result is trivial. So assume that $n>1$. Since $\lambda$ is uncountable, regular, and not weakly compact, [8, Corollary 21.5] implies that $2^{<\lambda} \nrightarrow(\lambda)_{2}^{2}$. Therefore, by successive applications of [7, Lemma 5A], which is the lemma establishing the optimality of the Erdős-Rado theorem, we have, for all $m<\omega, \beth_{m}\left(2^{<\lambda}\right) \nrightarrow(\lambda)_{2}^{2+m}$. By Remark $3.2(2), \sigma(\lambda, n)=\left(\beth_{n-2}\left(2^{<\lambda}\right)\right)^{+}$. Therefore, we have $\mu \leq \beth_{n-2}\left(2^{<\lambda}\right)$, so there is a function $c:[\mu]^{n} \rightarrow 2$ that is not constant on $[H]^{n}$ for any $H \in[\mu]^{\lambda}$. For each $a \in[\mu]^{n}$, simply let $u_{a}:=c(a)$. Now suppose that $H \in[\mu]^{\lambda}$, and suppose for sake of contradiction that $\left\langle u_{a} \mid a \in[H]^{n}\right\rangle$ is an $n$-dimensional $\Delta$-system, as witnessed by roots $\left\langle R_{a}^{\mathbf{m}} \mid \mathbf{m} \subseteq n, a \in[H]^{|\mathbf{m}|}\right\rangle$. Using the fact that $c$ is not constant on $\left[H^{\prime}\right]^{n}$ for any unbounded $H^{\prime} \subseteq H$, we can fix three sets $a_{0}<a_{1}<a_{2}$ in $[H]^{n}$ such that $c\left(a_{0}\right)=0$ and $c\left(a_{1}\right)=c\left(a_{2}\right)=1$. By the definition of an $n$-dimensional $\Delta$-system, we should have $u_{a_{0}} \cap u_{a_{1}}=R_{\emptyset}^{\emptyset}=u_{a_{1}} \cap u_{a_{2}}$. However, we actually have $u_{a_{0}} \cap u_{a_{1}}=\emptyset$ and $u_{a_{1}} \cap u_{a_{2}}=1$, which is our desired contradiction.

Before turning to the optimality of the value of $\kappa$ in Theorem 3.8, we pause to summarize the results of this section thus far in a corollary connecting Theorem 3.8 and Proposition 3.18 with the Erdős-Rado theorem.

Corollary 3.19. Suppose that $1 \leq n<\omega$ and that $\lambda$ and $\mu$ are infinite regular cardinals such that $\lambda$ is uncountable but not weakly compact. Then the following are equivalent:
(1) $\mu \geq \sigma(\lambda, n)$;
(2) $\mu \rightarrow(\lambda)_{2}^{n}$;
(3) $\mu \rightarrow(\lambda+(n-1))_{\nu}^{n}$ for every $\nu<\lambda$;
(4) for every sequence $\left\langle u_{b} \mid b \in[\mu]^{n}\right\rangle$ such that each $u_{b}$ is a finite set, there is $H \in[\mu]^{\lambda}$ such that $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is an n-dimensional $\Delta$-system;
(5) the conclusion of Theorem 3.8 holds for $n, \lambda$, and $\mu$, and for any choice of $\kappa, \nu, g:[\mu]^{n} \rightarrow \nu$, and $\left\langle u_{b} \mid b \in[\mu]^{n}\right\rangle$ such that
(a) $\nu<\lambda$;
(b) $\lambda$ is $<\kappa$-inaccessible; and
(c) $u_{b} \in[\mathrm{On}]^{<\kappa}$ for every $b \in[\mu]^{n}$.

Proof. (1) $\Rightarrow(3)$ is the Erdős-Rado theorem, or the pigeonhole principle if $n=1$ (it can also be extracted from our proof of Theorem 3.8), and (3) $\Rightarrow(2)$ is immediate. $(1) \Rightarrow(5)$ is Theorem 3.8 and $(5) \Rightarrow(4)$ follows by setting $\kappa=\aleph_{0}$ in Theorem 3.8 and invoking Proposition [2.6(2). (4) $\Rightarrow(2)$ is precisely the second half of the proof of Proposition 3.18. Finally, $(2) \Rightarrow(1)$ follows from the optimality of the Erdős-Rado theorem (the argument in the first half of the proof of Proposition (3.18).

We now turn to the optimality of $\kappa$ in Theorem 3.8 in other words, we investigate the necessity of the requirement that $\lambda$ be $<\kappa$-inaccessible in the statement of the theorem. It turns out that the optimality of $\kappa$ is slightly more complicated than the optimality of $\mu$, since even if $\lambda$ is not $<\kappa$-inaccessible, it could be the case that $\sigma(\lambda, n)=\sigma\left(\lambda^{*}, n\right)$ for some $\lambda^{*}>\lambda$ such that $\lambda^{*}$ is $<\kappa$-inaccessible. For example, suppose that $2^{\aleph_{0}}=\aleph_{2}$ and $2^{\aleph_{1}}=2^{\aleph_{2}}=\aleph_{3}$. Then $\sigma\left(\aleph_{2}, n\right)=\sigma\left(\aleph_{3}, n\right)$ for all $n \geq 2$. Also, $\aleph_{3}$ is $<\aleph_{1}$-inaccessible, so Theorem 3.8 holds for $\lambda=\aleph_{3}$ and $\kappa=\nu=\aleph_{1}$ (and any value of $n$ ). This immediately implies that the conclusion of Theorem 3.8 holds for $\lambda=\aleph_{2}, \kappa=\nu=\aleph_{1}$, and $2 \leq n<\omega$, despite the fact that $\aleph_{2}$ is not $<\aleph_{1}$-inaccessible. We can show however, that this is essentially the only way in which the value of $\kappa$ in Theorem 3.8 can fail to be optimal.

Proposition 3.20. Suppose that $1 \leq n<\omega$ and $\kappa<\lambda$ are infinite cardinals such that $\lambda$ is regular and not $<\kappa$-inaccessible. Let $\lambda^{*}=\left(\lambda^{<\kappa}\right)^{+}$, and suppose that $\mu<\sigma\left(\lambda^{*}, n\right)$. Then there is a sequence $\left\langle u_{a} \mid a \in[\mu]^{n}\right\rangle$ consisting of elements of $[\lambda]^{<\kappa}$ such that there is no $H \in[\mu]^{\lambda}$ for which $\left\langle u_{a} \mid a \in[H]^{n}\right\rangle$ is an n-dimensional $\Delta$-system.

Proof. Fix a cardinal $\nu<\lambda$ such that $\nu \geq \kappa$ and $\nu^{<\kappa} \geq \lambda$. Next, fix an injective sequence $\left\langle x_{\eta} \mid \eta<\lambda\right\rangle$ of elements of $[\nu]^{<\kappa}$ such that, for all distinct $\eta, \xi<\lambda$, neither of $x_{\eta}$ nor $x_{\xi}$ is a subset of the other. One way to see that this can be done is the following. Let $\left\langle\kappa_{i} \mid i<\theta\right\rangle$ be such that

- if $\kappa$ is a successor cardinal, then $\theta=1$ and $\kappa_{0}$ is its immediate predecessor (so $\nu^{<\kappa}=\nu^{\kappa_{0}}$ ); or
- if $\kappa$ is a limit cardinal, then $\theta=\operatorname{cf}(\kappa)$ and $\left\langle\kappa_{i} \mid i<\theta\right\rangle$ is a strictly increasing sequence of cardinals converging to $\kappa$.
Now let $\left\langle f_{\eta} \mid \eta<\lambda\right\rangle$ be an injective sequence of elements of $\bigcup_{i<\theta}{ }^{\kappa_{i}} \nu$. Partition $\nu$ into pairwise disjoint pieces $\left\langle A_{i} \mid i<\theta\right\rangle$, each of size $\nu$ and, for each $i<\theta$, let $\pi_{i}: \kappa_{i} \times \nu \rightarrow A_{i}$ be a bijection. Now, viewing elements of $\kappa_{i} \nu$ as subsets of $\kappa_{i} \times \nu$, for each $\eta<\lambda$, let $i_{\eta}$ be the unique $i<\theta$ such that $f_{\eta} \in{ }^{\kappa_{i}} \nu$, and let $x_{\eta}:=\pi_{i_{\eta}}$ " $f_{\eta}$. Then $\left\langle x_{\eta} \mid \eta<\lambda\right\rangle$ is as desired. Similarly, fix an injective sequence $\left\langle y_{\alpha} \mid \alpha<\lambda^{<\kappa}\right\rangle$ of elements of $[\lambda \backslash \nu]^{<\kappa}$ such that, for all $\alpha<\beta<\lambda^{<\kappa}$, neither of $y_{\alpha}$ nor $y_{\beta}$ is a subset of the other. For all $\alpha<\lambda^{<\kappa}$, let $\eta_{\alpha}=\sup \left(y_{\alpha}\right)$. Since $\lambda$ is regular, we have $\eta_{\alpha}<\lambda$.

Suppose first that $n=1$. Then $\sigma\left(\lambda^{*}, 1\right)=\lambda^{*}=\left(\lambda^{<\kappa}\right)^{+}$, so we can assume that $\mu=\lambda^{<\kappa}$. For all $\alpha<\mu$, let $u_{\alpha}:=y_{\alpha} \cup x_{\eta_{\alpha}}$. Fix $H \in[\mu]^{\lambda}$, and suppose for sake of contradiction that $\left\langle u_{\alpha} \mid \alpha \in H\right\rangle$ is a $\Delta$-system, with root $r$. Let $r^{-}:=r \cap \nu$ and $r^{+}:=r \backslash \nu$. Note that, for all distinct $\alpha, \beta \in H$, we have $y_{\alpha} \cap y_{\beta}=r^{+}$and $x_{\eta_{\alpha}} \cap x_{\eta_{\beta}}=r^{-}$.

There are now two cases to consider, depending on whether or not $\left\{\eta_{\alpha} \mid \alpha \in H\right\}$ is unbounded in $\lambda$. Suppose first that $\eta^{*}:=\sup \left\{\eta_{\alpha} \mid \alpha \in H\right\}$ is less than $\lambda$. Then $\left\langle y_{\alpha} \backslash r^{+} \mid \alpha \in H\right\rangle$ is an injective sequence of pairwise disjoint nonempty subsets of $\eta^{*}+1$, contradicting the fact that $|H|=\lambda>\eta^{*}+1$. Suppose next that $\eta^{*}=\lambda$. Then $\left\{x_{\eta_{\alpha}} \backslash r^{-} \mid \alpha \in H\right\}$ is a set of size $\lambda$ consisting of pairwise disjoint nonempty subsets of $\nu$, contradicting the fact that $\lambda>\nu$.

Now suppose that $n>1$. Then, by Remark 3.2(1), we know that $\sigma\left(\lambda^{*}, n\right)=$ $\left(\beth_{n-1}\left(\lambda^{<\kappa}\right)\right)^{+}$, so we can assume that $\mu=\beth_{n-1}\left(\lambda^{<\kappa}\right)$. For any infinite cardinal $\chi$, the coloring $d:\left[{ }^{\chi} 2\right]^{2} \rightarrow \chi$ defined by letting $d(f, g)$ be the least $\xi<\chi$ for which $f(\xi) \neq g(\xi)$ for all distinct $f, g \in{ }^{\chi} 2$ witnesses the negative partition relation
$2^{\chi} \nrightarrow(3)_{\chi}^{2}$. Therefore, setting $\chi=\lambda^{<\kappa}$ and repeatedly applying [7, Lemma 5A], we have $\mu \nrightarrow\left(\aleph_{0}\right)_{\lambda<\kappa}^{n}$. Let $c:[\mu]^{n} \rightarrow \lambda^{<\kappa}$ witness this negative partition relation.

Now define $\left\langle u_{a} \mid a \in[\mu]^{n}\right\rangle$ by letting $u_{a}:=y_{c(a)} \cup x_{\eta_{c(a)}}$ for all $a \in[\mu]^{n}$. Fix $H \in[\mu]^{\lambda}$, and suppose for sake of contradiction that $\left\langle u_{a} \mid a \in[H]^{n}\right\rangle$ is an $n$ dimensional $\Delta$-system, as witnessed by roots $\left\langle R_{a}^{\mathbf{m}} \mid \mathbf{m} \subseteq n, a \in[H]^{|\mathbf{m}|}\right\rangle$. Let $r:=R_{\emptyset}^{\emptyset}, r^{-}:=r \cap \nu$, and $r^{+}:=r \backslash \nu$. Since $c$ witnesses $\mu \nrightarrow\left(\aleph_{0}\right)_{\lambda<\kappa}^{n}$, we can find disjoint sets $a_{0}, a_{1} \in[H]^{n}$ such that $c\left(a_{0}\right) \neq c\left(a_{1}\right)$. Now arbitrarily fix a set $a_{\gamma} \in[H]^{n}$ for each $2 \leq \gamma<\lambda$ in such a way that $\left\langle a_{\gamma} \mid \gamma<\lambda\right\rangle$ is an injective sequence of pairwise disjoint sets. By the definition of $\left\langle u_{a} \mid a \in[H]^{n}\right\rangle$ and our choice of $r, r^{+}$, and $r^{-}$, we know that, for all $\gamma<\delta<\lambda$, we have $u_{a_{\gamma}} \cap u_{a_{\delta}}=r$, and hence $y_{c\left(a_{\gamma}\right)} \cap y_{c\left(a_{\delta}\right)}=r^{+}$and $x_{\eta_{c\left(a_{\gamma}\right)}} \cap x_{\eta_{c\left(a_{\delta}\right)}}=r^{-}$.

There are now two possibilities. First, suppose that there are $\gamma<\delta<\lambda$ for which $u_{a_{\gamma}}=u_{a_{\delta}}$. Then we can find $\ell<2$ for which $u_{a_{\ell}} \neq u_{a_{\gamma}}$ (and hence $u_{a_{\gamma}} \nsubseteq u_{a_{\ell}}$ ). But now we are in the same situation as in the proof of Proposition 3.18 we must have $u_{a_{\gamma}} \cap u_{a_{\delta}}=r=u_{a_{\ell}} \cap u_{a_{\gamma}}$, but $u_{a_{\gamma}} \cap u_{a_{\delta}}=u_{a_{\gamma}}$, and since $u_{a_{\gamma}} \nsubseteq u_{a_{\ell}}$, we have $u_{a_{\ell}} \cap u_{a_{\gamma}} \neq u_{a_{\gamma}}$, which is a contradiction.

The other possibility is that the sets $\left\langle u_{a_{\gamma}} \mid \gamma<\lambda\right\rangle$ are all pairwise disjoint. There are now two subcases, depending on whether or not $\eta^{*}:=\sup \left\{\eta_{c\left(a_{\gamma}\right)} \mid \gamma<\lambda\right\}$ is equal to $\lambda$. If $\eta^{*}<\lambda$, then $\left\langle u_{a_{\gamma}} \backslash r \mid \gamma<\lambda\right\rangle$ is an injective sequence of pairwise disjoint nonempty subsets of $\max \left\{\eta^{*}+1, \nu\right\}$. If $\eta^{*}=\lambda$, then $\left\{x_{\eta_{c\left(a_{\gamma}\right)}} \backslash r^{-} \mid \gamma<\lambda\right\}$ is a set of size $\lambda$ consisting of pairwise disjoint nonempty subsets of $\nu$. In either case, we contradict the fact that $\lambda>\max \left\{\eta^{*}+1, \nu\right\}$.

If $\kappa<\lambda$ are both regular infinite cardinals and $\lambda$ is not $<\kappa$-inaccessible, then $\left(\lambda^{<\kappa}\right)^{+}$is the least $<\kappa$-inaccessible cardinal greater than or equal to $\lambda$ (it can fail to be $<\kappa$-inaccessible if $\kappa$ is singular). We can therefore combine the results of this section in the following equivalence.

Corollary 3.21. Suppose that $1 \leq n<\omega$ and $\kappa<\lambda$ are infinite regular cardinals. Let $\lambda^{*}$ be the least $<\kappa$-inaccessible cardinal greater than or equal to $\lambda$, and suppose that $\mu$ is an infinite cardinal. Then the following are equivalent.
(1) $\mu \geq \sigma\left(\lambda^{*}, n\right)$;
(2) the conclusion of Theorem 3.8 holds for $n, \kappa$, $\lambda$, and $\mu$ with any choice of $\nu<\lambda, g:[\mu]^{n} \rightarrow \nu$, and $\left\langle u_{b} \mid b \in[\mu]^{n}\right\rangle$ with each $u_{b}$ in $[\mathrm{On}]^{<\kappa}$;
(3) for every sequence $\left\langle u_{a} \mid a \in[\mu]^{n}\right\rangle$ such that each $u_{a}$ is a set of cardinality less than $\kappa$, there is $H \in[\mu]^{\lambda}$ such that $\left\langle u_{a} \mid a \in[H]^{n}\right\rangle$ is an n-dimensional $\Delta$-system.

Proof. (1) $\Rightarrow$ (2) follows from Theorem 3.8, and $(2) \Rightarrow(3)$ is immediate. If $\lambda$ is $<\kappa$-inaccessible, then $\lambda^{*}=\lambda$, in which case $(3) \Rightarrow(1)$ follows from Proposition 3.18. If $\lambda$ is not $<\kappa$-inaccessible, then $(3) \Rightarrow(1)$ follows from Proposition 3.20 and the observation that $\lambda^{*}=\left(\lambda^{<\kappa}\right)^{+}$in this case.

## 4. Chain conditions

One of the primary uses of the classical $\Delta$-system lemma is in proving that certain forcing notions satisfy chain conditions. For example, one of the first applications that many people learn is in the proof that the forcing notion to add any number of Cohen reals is $\kappa$-Knaster for every regular uncountable $\kappa$ :

Lemma 4.1. Let $\chi$ be any infinite cardinal, and let $\mathbb{P}=\operatorname{Add}(\omega, \chi)$ be the forcing to add $\chi$-many Cohen reals. Suppose that $\kappa$ is a regular uncountable cardinal and $\left\langle p_{\alpha} \mid \alpha<\kappa\right\rangle$ is a sequence of conditions from $\mathbb{P}$. Then there is an unbounded $A \subseteq \kappa$ such that $\left\langle p_{\alpha} \mid \alpha \in A\right\rangle$ consists of pairwise compatible conditions.

During forcing constructions involving higher-dimensional combinatorial statements, one frequently encounters sequences of conditions indexed not by single ordinals but by $n$-element sets of ordinals for some $n>1$. One would then like to find a large set such that the restriction of the sequence to that set satisfies certain uniformities analogous to the uniformities exhibited by $\left\langle p_{\alpha} \mid \alpha \in A\right\rangle$ in Lemma 4.1. A first, naïve attempt at formulating a statement to this effect, similar to our overly optimistic first attempt to define higher-dimensional $\Delta$-systems at the start of Section 2 might look vaguely as follows:

Let $\chi$ be an infinite cardinal and $1 \leq n<\omega$, and let $\mathbb{P}$ be the forcing to add $\chi$-many Cohen reals. Then there is a sufficiently large regular cardinal $\mu \leq \chi$ such that, for every sequence $\left\langle p_{a}\right| a \in$ $\left.[\mu]^{n}\right\rangle$ of conditions in $\mathbb{P}$, there is a "large" set $H \subseteq \mu$ such that $\left\langle p_{a} \mid a \in[H]^{n}\right\rangle$ consists of pairwise compatible conditions.

It is easily seen that such a statement cannot possibly hold if $n>1$, however. Indeed suppose that $n=2$ and, for all $(\alpha, \beta) \in[\mu]^{2}$, define a condition $p_{\alpha \beta} \in \mathbb{P}$ by letting $\operatorname{dom}\left(p_{\alpha \beta}\right):=\{\alpha, \beta\}, p_{\alpha \beta}(\alpha):=0$, and $p_{\alpha \beta}(\beta):=1$ (we are thinking of conditions in $\mathbb{P}$ as being finite partial functions from $\chi$ to 2 ). Then $p_{\alpha \beta} \perp p_{\beta \gamma}$ for all $(\alpha, \beta, \gamma) \in[\mu]^{3}$, so we could not even find a set $H$ of size 3 as in the above statement. The obvious problem here is that the sets $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$ are not aligned, and it turns out that this is the only obstacle. By requiring the compatibility of $p_{a}$ and $p_{b}$ only when $a$ and $b$ are aligned, we obtain a consistent statement. For example:

Lemma 4.2. Suppose that $\lambda$ is a regular uncountable cardinal, $1 \leq n<\omega$, and $\mu=\sigma(\lambda, n)$, and suppose that $\mathbb{P}$ is the forcing notion to add $\chi$-many Cohen reals for some infinite cardinal $\chi$. Then, for every sequence $\left\langle p_{a} \mid a \in[\mu]^{n}\right\rangle$ of conditions in $\mathbb{P}$, there is a set $H \in[\mu]^{\lambda}$ such that, for all $a, b \in[H]^{n}$, if $a$ and $b$ are aligned, then $p_{a} \| p_{b}$.

Proof. Fix a sequence $\left\langle p_{a} \mid a \in[\mu]^{n}\right\rangle$ consisting of conditions in $\mathbb{P}$. For each $a \in[\mu]^{n}$, let $u_{a}:=\operatorname{dom}\left(p_{a}\right)$ and $k_{a}:=\operatorname{otp}\left(u_{a}\right)$, and let $\bar{p}_{a}: k_{a} \rightarrow 2$ denote the condition isomorphic to $p_{a}$, i.e., $\bar{p}_{a}(i)=p_{a}\left(u_{a}(i)\right)$ for all $i<k_{a}$. Now apply Theorem 3.8 to $\left\langle u_{a} \mid a \in[\mu]^{n}\right\rangle$ and the function $a \mapsto \bar{p}_{a}$ to find an $H \in[\mu]^{\lambda}$, a $k<\omega$, and a function $\bar{p}: k \rightarrow 2$ such that $\left\langle u_{a} \mid a \in[H]^{n}\right\rangle$ is a uniform $n$-dimensional $\Delta$-system and $\bar{p}_{a}=\bar{p}$ for all $a \in[H]^{n}$.

We claim that $p_{a} \| p_{b}$ for all aligned $a, b \in[H]^{n}$. To this end, fix $a, b \in[H]^{n}$ such that $a$ and $b$ are aligned. The only way we could have $p_{a} \perp p_{b}$ is if there is $\alpha \in u_{a} \cap u_{b}$ such that $p_{a}(\alpha) \neq p_{b}(\alpha)$. Since $\left\langle u_{a} \mid a \in[H]^{n}\right\rangle$ is a uniform $n$ dimensional $\Delta$-system, we know that $u_{a}$ and $u_{b}$ are aligned. Moreover, we know that $\bar{p}_{a}=\bar{p}_{b}=\bar{p}$. Therefore, if $\alpha \in u_{a} \cap u_{b}$, then there is $i<k$ such that $\alpha=u_{a}(i)=u_{b}(i)$. But then $p_{a}(\alpha)=\bar{p}(i)=p_{b}(\alpha)$. Therefore, we have $p_{a} \| p_{b}$.

Remark 4.3. If $\lambda$ is weakly compact, then, by Corollary 3.17 Lemma 4.2 still holds with $\mu=\lambda$ rather than $\mu=\sigma(\lambda, n)$.

## 5. An application to polarized partition relations

In this section, we give a relatively simple application illustrating a typical use of Theorem 3.8 in a forcing argument. The following definition was introduced by Todorčević.

Definition 5.1 ([20, Remark 9.3.3]). Let $1 \leq n<\omega$. Then $\Theta_{n}$ is the least cardinal $\theta$ such that, for every function $f: \theta^{n} \rightarrow \omega$, there is a sequence $\left\langle A_{i} \mid i<n\right\rangle$ of infinite subsets of $\theta$ such that $f \upharpoonright \prod_{i<n} A_{i}$ is constant.

We clearly have $\Theta_{1}=\aleph_{1}$. The next proposition establishes lower bounds for $\Theta_{n}$ for $n>1$.

Proposition 5.2. Suppose that $1 \leq n<\omega$, $\kappa$ is a cardinal, and $\Theta_{n}>\kappa$. Then $\Theta_{n+1}>\kappa^{+}$.

Proof. Since $\Theta_{n}>\kappa$, we can fix a function $g: \kappa^{n} \rightarrow \omega$ such that $g$ is not constant on any product of $n$ infinite subsets of $\kappa$. For each $\beta<\kappa^{+}$, fix an injective function $e_{\beta}: \beta \rightarrow \kappa$. Then the function $g_{\beta}: \beta^{n} \rightarrow \omega$ defined by letting

$$
g_{\beta}\left(\left\langle\alpha_{0}, \ldots, \alpha_{n-1}\right\rangle\right):=g\left(\left\langle e_{\beta}\left(\alpha_{0}\right), \ldots, e_{\beta}\left(\alpha_{n-1}\right)\right\rangle\right)
$$

for all $\left\langle\alpha_{0}, \ldots, \alpha_{n-1}\right\rangle \in \beta^{n}$ has the property that $g_{\beta}$ is not constant on any product of $n$ infinite subsets of $\beta$.

We now define a function $f:\left(\kappa^{+}\right)^{n+1} \rightarrow(n+2) \times \omega$ that will not be constant on any product of $(n+1)$ infinite subsets of $\kappa^{+}$. This can easily be coded as a function into $\omega$, so this suffices to prove the proposition.

Given $\vec{\alpha}=\left\langle\alpha_{0}, \ldots, \alpha_{n}\right\rangle \in\left(\kappa^{+}\right)^{n+1}$ and $i \leq n$, let $\vec{\alpha}^{i}$ denote the sequence formed by removing $\alpha_{i}$ from $\vec{\alpha}$, i.e., $\vec{\alpha}^{i}:=\left\langle\alpha_{0}, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n}\right\rangle$. Let us now define $f(\vec{\alpha})$. If there are $i<j \leq n$ such that $\alpha_{i}=\alpha_{j}$, then let $f(\vec{\alpha}):=(n+1,0)$. Otherwise, let $i \leq n$ be such that $\alpha_{j}<\alpha_{i}$ for all $j \in(n+1) \backslash\{i\}$, and let $f(\vec{\alpha}):=\left(i, g_{\alpha_{i}}\left(\vec{\alpha}^{i}\right)\right)$.

Suppose for sake of contradiction that $\left\langle A_{i} \mid i \leq n\right\rangle$ is a sequence of infinite subsets of $\kappa^{+}$such that $f \upharpoonright \prod_{i \leq n} A_{i}$ is constant, taking value $(m, k)$. First note that we can always find a sequence $\vec{\alpha} \in \prod_{i<n} A_{i}$ whose coordinates are all distinct, so it cannot be the case that $m=n+1$. Thus, $m \leq n$, so, by our definition of $f$, it follows that $A_{j}<A_{m}$ for all $j \in(n+1) \backslash\{m\}$. Fix $\beta \in A_{m}$, and define $\left\langle A_{j}^{*} \mid j<n\right\rangle$ by letting $A_{j}^{*}:=A_{j}$ for $j<m$ and $A_{j}^{*}:=A_{j+1}$ for $m \leq j<n$. Then each $A_{j}^{*}$ is an infinite subset of $\beta$ and, by our definition of $f$, it follows that $g_{\beta} \upharpoonright \prod_{j<n} A_{j}^{*}$ is constant, taking value $k$, contradicting our assumptions about $g_{\beta}$.

In particular, we immediately obtain the following corollary, answering a part of Question 9.3.4 from [20.

Corollary 5.3. $\Theta_{n} \geq \aleph_{n}$ for all $1 \leq n<\omega$.
It follows easily from the Erdős-Rado theorem that $\Theta_{n} \leq \beth_{n-1}^{+}$for all $1 \leq$ $n<\omega$. In particular, if GCH holds, then $\Theta_{n}=\aleph_{n}$ for all $1 \leq n<\omega$. We now apply Theorem 3.8 to prove that adding any number of Cohen reals preserves the inequality $\Theta_{n} \leq\left(\beth_{n-1}^{+}\right)^{V}$. In fact, we will prove that a slightly stronger partition relation, which easily implies $\Theta_{n} \leq\left(\beth_{n-1}^{+}\right)^{V}$, holds after forcing to add the Cohen reals.

Theorem 5.4. Suppose that $1 \leq n<\omega$ and $\chi$ is an infinite cardinal. Let $\mu=\beth_{n-1}^{+}$, and let $\mathbb{P}$ be the forcing to add $\chi$-many Cohen reals. Then the following statement holds in $V^{\mathbb{P}}$ :

For every function $c:[\mu]^{n} \rightarrow \omega$, there is a sequence $\left\langle A_{m} \mid m<n\right\rangle$ such that

- for all $m<n, A_{m}$ is a subset of $\mu$ of order type $\omega+1$;
- for all $m<m^{\prime}<n$, we have $A_{m}<A_{m^{\prime}}$;
- $c \prod_{m<n} A_{m}$ is constant.

Proof. We think of conditions in $\mathbb{P}$ as being finite partial functions from $\chi$ to 2 , ordered by reverse inclusion. Given a condition $p \in \mathbb{P}$, let $\bar{p}$ denote the function from $|\operatorname{dom}(p)|$ to 2 defined by letting $\bar{p}(i):=p(\operatorname{dom}(p)(i))$ for all $i<|\operatorname{dom}(p)|$.

Since the conclusion of the theorem is trivial if $n=1$, we may assume that $n>1$. Fix a condition $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{c}$ forced by $p$ to be a function from $[\mu]^{n}$ to $\omega$. For each $b \in[\mu]^{n}$, find a condition $q_{b} \leq p$ and a color $k_{b}<\omega$ such that $q_{b} \Vdash " \dot{c}(b)=k_{b} "$. Let $u_{b}:=\operatorname{dom}\left(q_{b}\right)$, and define a function $g:[\mu]^{n} \rightarrow{ }^{<\omega} 2 \times \omega$ by letting $g(b):=\left\langle\bar{q}_{b}, k_{b}\right\rangle$ for all $b \in[\mu]^{n}$. Apply Theorem 3.8 to find $H \in[\mu]^{\aleph_{1}}$ such that $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is a uniform $n$-dimensional $\Delta$-system and $g \upharpoonright[H]^{n}$ is constant, taking value $\langle\bar{q}, k\rangle$. By taking an initial segment if necessary, assume that we in fact have $\operatorname{otp}(H)=\omega_{1}$. Note that, if $b$ and $b^{\prime}$ are aligned elements of $[H]^{n}$, then $q_{b}$ and $q_{b^{\prime}}$ are compatible in $\mathbb{P}$.

Let $\rho:=|\bar{q}|$, and let $\left\langle\mathbf{r}_{\mathbf{m}} \subseteq \rho \mid \mathbf{m} \subseteq n\right\rangle$ witness the fact that $\left\langle u_{b} \mid b \in[H]^{n}\right\rangle$ is a uniform $n$-dimensional $\Delta$-system. For each $m<n$ and each $a \in[H]^{m}$, define $u_{a}$ by letting $b$ be any element of $[H]^{n}$ such that $b[m]=a$ and then letting $u_{a}:=u_{b}\left[\mathbf{r}_{m}\right]$ (we are thinking of $m$ as an initial subset of $n$ here). Then set $q_{a}:=q_{b} \upharpoonright u_{a}$. By Proposition [2.6 and the fact that $\bar{q}_{b}=\bar{q}$ for all $b \in[H]^{n}$, it follows that our definition of $u_{a}$ and $q_{a}$ is independent of our choice of $b$.

By the arguments of Claim 3.13, we know that, for every $m<n$ and every $a \in[H]^{m}$, the sequence $\left\langle u_{a \frown\langle\beta\rangle} \mid \beta \in H \backslash(\max (a)+1)\right\rangle$ is a 1 -dimensional $\Delta$ system, with root $u_{a}$. Since $q_{b} \leq p$ for all $b \in[H]^{n}$, it follows that $\operatorname{dom}(p) \subseteq u_{\emptyset}$ and $q_{\emptyset} \leq p$. We will show that $q_{\emptyset}$ forces the existence of a sequence $\left\langle A_{m} \mid m<n\right\rangle$ in $V^{\mathbb{P}}$ such that

- each $A_{m}$ is a subset of $\mu$ of order type $\omega+1$;
- $A_{m}<A_{m^{\prime}}$ for all $m<m^{\prime}<n$;
- the realization of $\dot{c}$ is constant when restricted to $\prod_{m<n} A_{m}$, with value $k$.

Since $p$ was arbitrary, this suffices to prove the theorem. We first need the following claim.

Claim 5.5. Suppose that $m<n, a \in[H]^{m}$, and $\gamma \in H \backslash(\max (a)+1)$. Then the set $D_{a, \gamma}:=\left\{q_{a} \sim\langle\beta\rangle \mid \beta \in H \backslash \gamma\right\}$ is predense below $q_{a}$ in $\mathbb{P}$.

Proof. Fix a condition $r \leq q_{a}$. We will find an element of $D_{a, \gamma}$ compatible with $r$. Since $\left\langle u_{a-\langle\beta\rangle} \mid \beta \in H \backslash \gamma\right\rangle$ is an infinite 1-dimensional $\Delta$-system with root $u_{a}$, and since $\operatorname{dom}(r)$ is finite, we can find $\beta \in H \backslash \gamma$ such that $u_{a} \wedge\langle\beta\rangle \backslash u_{a}$ is disjoint from $\operatorname{dom}(r)$. But then $q_{a-\langle\beta\rangle} \upharpoonright \operatorname{dom}(r)=q_{a}$, so, since $r \leq q_{a}$, it follows that $r \cup q_{a} \frown\langle\beta\rangle$ is a condition in $\mathbb{P}$, so $q_{a} \frown\langle\beta\rangle$ is an element of $D_{a, \gamma}$ compatible with $r$.

Now suppose that $G$ is $\mathbb{P}$-generic over $V$ with $q_{\emptyset} \in G$, and let $c$ be the realization of $\dot{c}$ in $V[G]$. By applying Claim 5.5 $n$ times, working in $V[G]$, we can recursively
choose an increasing sequence $\left\langle\delta_{m} \mid m<n\right\rangle$ of elements of $H$ such that, letting $d=\left\{\delta_{m} \mid m<n\right\}$, we have

- $q_{d} \in G$;
- $H \cap \delta_{0}$ is infinite;
- for all $m<n-1, H \cap\left(\delta_{m+1} \backslash\left(\delta_{m}+1\right)\right)$ is infinite.

Let $A_{0}^{*}$ denote the set of the first $\omega$-many elements of $H \cap \delta_{0}$ and, for all $m<n-1$, let $A_{m+1}^{*}$ denote the set of the first $\omega$-many elements of $H \cap\left(\delta_{m+1} \backslash\left(\delta_{m}+1\right)\right)$.

We now construct an $n \times \omega$ matrix $\left\langle\alpha_{m, \ell} \mid m<n, \ell<\omega\right\rangle$ such that

- for all $m<n,\left\langle\alpha_{m, \ell} \mid \ell<\omega\right\rangle$ is an increasing sequence of elements of $A_{m}^{*}$;
- letting $A_{m}=\left\{\alpha_{m, \ell} \mid \ell<\omega\right\} \cup\left\{\delta_{m}\right\}$ for each $m<n$, we have $q_{b} \in G$ for all $b \in \prod_{m<n} A_{m}$.
The construction is by recursion on the anti-lexicographical order on $n \times \omega$, i.e., we set $(m, \ell)<\left(m^{\prime}, \ell^{\prime}\right)$ if $\ell<\ell^{\prime}$ or $\left(\ell=\ell^{\prime}\right.$ and $\left.m<m^{\prime}\right)$. During the construction, at stage $(m, \ell)$, for all $m^{\prime}<n$, we will let $A_{m^{\prime}} \upharpoonright(m, \ell)$ denote the set $\left\{\alpha_{m^{\prime}, \ell^{\prime}} \mid \ell^{\prime} \leq \ell\right\} \cup$ $\left\{\delta_{m^{\prime}}\right\}$ if $m^{\prime}<m$ and $\left\{\alpha_{m^{\prime}, \ell^{\prime}} \mid \ell^{\prime}<\ell\right\} \cup\left\{\delta_{m^{\prime}}\right\}$ if $m \leq m^{\prime}$. In other words, $A_{m^{\prime}} \upharpoonright$ $(m, \ell)$ is simply the portion of $A_{m^{\prime}}$ that we have specified before stage ( $m, \ell$ ) of the construction. Our recursion hypothesis will be the assumption that, when we reach stage $(m, \ell)$, for all $b \in \prod_{m^{\prime}<n} A_{m^{\prime}} \upharpoonright(m, \ell)$, we have $q_{b} \in G$. It will then follow that $q_{m, \ell}^{*}:=\bigcup\left\{q_{b} \mid b \in \prod_{m^{\prime}<n} A_{m^{\prime}} \upharpoonright(m, \ell)\right\}$ is also an element of $G$.

To begin the construction, note that, for all $m^{\prime}<n$, we have $A_{m^{\prime}} \upharpoonright(0,0)=$ $\left\{\delta_{m^{\prime}}\right\}$, so $q_{0,0}^{*}=q_{d} \in G$. Thus, our recursion hypothesis is initially satisfied. Now suppose that $(m, \ell) \in n \times \omega$ and we have defined $\left\langle\alpha_{m^{\prime}, \ell^{\prime}} \mid\left(m^{\prime}, \ell^{\prime}\right)<(m, \ell)\right\rangle$ so that the resulting condition $q_{m, \ell}^{*}$ is in $G$. Temporarily move back to $V$, noting that each $A_{m^{\prime}} \upharpoonright(m, \ell)$ is finite and hence in $V$, and $A_{m}^{*}$ is also in $V$, as it is definable from $H$ (and $\delta_{m-1}$, if $m>0$ ).

Let $B_{0}:=\prod_{m^{\prime}<m}\left(A_{m^{\prime}} \upharpoonright(m, \ell)\right)$ and $B_{1}:=\prod_{m<m^{\prime}<n}\left(A_{m^{\prime}} \upharpoonright(m, \ell)\right)$. If $\ell>0$, then let $\gamma:=\alpha_{m, \ell-1}+1$; if $\ell=0$, then let $\gamma:=0$. For each $\alpha \in A_{m}^{*} \backslash \gamma$, let $q_{\alpha}^{*}:=\bigcup\left\{q_{b_{0}-\langle\alpha\rangle-b_{1}} \mid b_{0} \in B_{0}, b_{1} \in B_{1}\right\}$. Notice that, if $b_{0}, b_{0}^{\prime} \in B_{0}$ and $b_{1}, b_{1}^{\prime} \in B_{1}$, then $b_{0} \frown\langle\alpha\rangle \frown b_{1}$ and $b_{0}^{\prime} \frown\langle\alpha\rangle \frown b_{1}^{\prime}$ are aligned, and hence $q_{b_{0}} \frown\langle\alpha\rangle \frown b_{1}$ and $q_{b_{0}^{\prime}} \frown\langle\alpha\rangle \frown b_{1}^{\prime}$ are compatible. It follows that $q_{\alpha}^{*}$ is a condition in $\mathbb{P}$.

Claim 5.6. The set $E:=\left\{q_{\alpha}^{*} \mid \alpha \in A_{m}^{*} \backslash \gamma\right\}$ is predense below $q_{m, \ell}^{*}$ in $\mathbb{P}$.
Proof. Fix $r \leq q_{m, \ell}^{*}$. We will find an element of $E$ compatible with $r$. Let $\mathbf{m}=$ $n \backslash\{m\}$. For each $\left(b_{0}, b_{1}\right) \in B_{0} \times B_{1}$, the sequence $\left\langle u_{b_{0}-\langle\alpha\rangle-b_{1}} \mid \alpha \in A_{m}^{*} \backslash \gamma\right\rangle$ forms a 1-dimensional $\Delta$-system whose root is equal to $u_{b_{0}-\langle\alpha\rangle-b_{1}}\left[\mathbf{r}_{\mathbf{m}}\right]$ for all $\alpha \in A_{m}^{*} \backslash \gamma$. Since $A_{m}^{*} \backslash \gamma$ is infinite and since $\operatorname{dom}(r), B_{0}$, and $B_{1}$ are all finite, we can find $\alpha \in A_{m}^{*} \backslash \gamma$ such that, for all $\left(b_{0}, b_{1}\right) \in B_{0} \times B_{1}$, the set $u_{b_{0}-\langle\alpha\rangle-b_{1}} \backslash\left(u_{b_{0}-\langle\alpha\rangle-b_{1}}\left[\mathbf{r}_{\mathbf{m}}\right]\right)$ is disjoint from $\operatorname{dom}(r)$.

We claim that $q_{\alpha}^{*}$ and $r$ are compatible. To see this, it suffices to show that $q_{b_{0}}-\langle\alpha\rangle-b_{1}$ and $r$ are compatible for every $\left(b_{0}, b_{1}\right) \in B_{0} \times B_{1}$. Thus, fix $\left(b_{0}, b_{1}\right) \in$ $B_{0} \times B_{1}$. We know that $u_{b_{0}-\langle\alpha\rangle \smile b_{1}} \cap \operatorname{dom}(r) \subseteq u_{b_{0}-\langle\alpha\rangle \frown b_{1}}\left[\mathbf{r}_{\mathbf{m}}\right]$. Since $b_{0} \frown\langle\alpha\rangle \frown b_{1}$ and $b_{0} \frown\left\langle\delta_{m}\right\rangle \frown b_{1}$ are aligned with $\mathbf{r}\left(b_{0} \frown\langle\alpha\rangle \frown b_{1}, b_{0} \frown\left\langle\delta_{m}\right\rangle \frown b_{1}\right)=\mathbf{m}$, we also know that $q_{b_{0} \frown\langle\alpha\rangle \frown b_{1}} \| q_{b_{0} \frown\left\langle\delta_{m}\right\rangle-b_{1}}$ and $u_{b_{0} \frown\langle\alpha\rangle \smile b_{1}}\left[\mathbf{r}_{\mathbf{m}}\right]=u_{b_{0} \frown\left\langle\delta_{m}\right\rangle \smile b_{1}}\left[\mathbf{r}_{\mathbf{m}}\right]$. Then

But we have $q_{m, \ell}^{*} \leq q_{b_{0}} \frown\left\langle\delta_{m}\right\rangle \frown b_{1}$, since $b_{0} \frown\left\langle\delta_{m}\right\rangle \frown b_{1} \in \prod_{m^{\prime}<n} A_{m^{\prime}} \upharpoonright(m, \ell)$. It follows that $r \leq q_{m, \ell}^{*} \leq q_{b_{0}-\langle\alpha\rangle-b_{1}} \upharpoonright \operatorname{dom}(r)$. Therefore, $r$ and $q_{b_{0}-\langle\alpha\rangle-b_{1}}$ are compatible.

Returning to $V[G]$, we can find $\alpha \in A_{m}^{*} \backslash \gamma$ such that $q_{\alpha}^{*} \in G$. But notice that, if we were to set $\alpha_{m, \ell}:=\alpha$, then, letting $(m, \ell)^{+}$denote the anti-lexicographic successor of $(m, \ell)$, we would have $q_{(m, \ell)^{+}}^{*}=q_{m, \ell}^{*} \cup q_{\alpha}^{*} \in G$. We can therefore set $\alpha_{m, \ell}:=\alpha$ while maintaining the recursion hypothesis, and continue to the next step of the construction.

At the end of the construction, we have built sets $\left\langle A_{m} \mid m<n\right\rangle$ such that

- for each $m<n, A_{m}$ is a subset of $H$ and $\operatorname{otp}\left(A_{m}\right)=\omega+1$;
- for each $m<m^{\prime}<n, A_{m}<A_{m^{\prime}}$;
- for each $b \in \prod_{m<n} A_{m}$, we have $q_{b} \in G$, and hence $c(b)=k$.

Therefore, $\left\langle A_{m} \mid m<n\right\rangle$ witnesses this instance of the theorem.
Remark 5.7. With some appropriate bookkeeping, the order type $\omega+1$ in the statement of Theorem 5.4 can be replaced by any countable ordinal $\alpha$.

Corollary 5.8. The statement " $\forall n \in[1, \omega)\left(\Theta_{n}=\aleph_{n}\right)$ " is compatible with an arbitrarily large value of the continuum. In particular, if $V$ is a model of $\mathrm{GCH}, \chi$ is an infinite cardinal, and $\mathbb{P}$ is the forcing to add $\chi$-many Cohen reals, then, in $V^{\mathbb{P}}, \Theta_{n}=\aleph_{n}$ for all $1 \leq n<\omega$.
Proof. In $V$, since GCH holds, we have $\beth_{n-1}^{+}=\aleph_{n}$ for all $1 \leq n<\omega$. Therefore, Theorem 5.4 implies that $\Theta_{n} \leq \aleph_{n}$ in $V^{\mathbb{P}}$ for all $1 \leq n<\omega$. By Corollary 5.3, it follows that $\Theta_{n}=\aleph_{n}$ for all $1 \leq n<\omega$ in $V^{\mathbb{P}}$.

## 6. A Variation, and monochromatic sumsets of reals

In this section, we discuss an alternative form of higher-dimensional $\Delta$-system that has appeared in the literature. The following theorem is due to Shelah and follows from the proof of [16, Lemma 4.1] (cf. also [6, Claim 7.2.a] and [21, Lemma 3.6] for more complete proofs of similar statements).

Theorem 6.1. Suppose that $\nu \leq \lambda \leq \mu$ are infinite cardinals, $1 \leq n<\omega$, and $\mu \rightarrow(\lambda)_{2^{\nu}}^{2 n}$. Suppose moreover that $\left\langle u_{a} \mid a \in[\mu]^{n}\right\rangle$ is a sequence of elements from $[\mathrm{On}]^{\leq \nu}$. Then there is $H \in[\mu]^{\lambda}$ and a sequence $\left\langle u_{a}^{*} \mid a \in[H]^{\leq n}\right\rangle$ of elements from $[\mathrm{On}] \leq \nu$ such that
(1) $u_{a}^{*} \supseteq u_{a}$ for all $a \in[H]^{n}$;
(2) for all $a, b \in[H]^{n}$, we have $\operatorname{tp}\left(u_{a}^{*}, u_{a}\right)=\operatorname{tp}\left(u_{b}^{*}, u_{b}\right)$;
(3) for all $a, b \in[H]^{\leq n}$, we have $u_{a}^{*} \cap u_{b}^{*}=u_{a \cap b}^{*}$;
(4) for all $a_{0} \subseteq a_{1}$ and $b_{0} \subseteq b_{1}$, where $a_{1}, b_{1} \in[H] \leq n$, if $\operatorname{tp}\left(a_{1}, a_{0}\right)=\operatorname{tp}\left(b_{1}, b_{0}\right)$, then $\operatorname{tp}\left(u_{a_{1}}^{*}, u_{a_{0}}^{*}\right)=\operatorname{tp}\left(u_{b_{1}}^{*}, u_{b_{0}}^{*}\right)$.
It is currently unclear whether arguments similar to those in the proof of Theorem 3.8 can be used to obtain the conclusion of Theorem6.1 from a weaker assumption on $\mu$, such as $\mu \geq \sigma(\lambda, n)$. It is the case, however, that certain results that have been proven using Theorem 6.1 can be proven by instead using Theorem 3.8. This can yield some improvements, since Theorem 3.8 places weaker assumptions on the cardinal $\mu$. We give one example of such a result here.

In [22], Zhang uses Theorem 6.1 to prove that, in the forcing extension obtained by adding $\beth_{\omega}$-many Cohen reals, we have $\mathbb{R} \rightarrow^{+}\left(\aleph_{0}\right)_{r}$ for every $r<\omega$, i.e., for every $r<\omega$ and every function $f: \mathbb{R} \rightarrow r$, there is an infinite set $X \subseteq \mathbb{R}$ such that $f \upharpoonright(X+X)$ is constant. We remark that, by a result of Hindman, Leader, and Strauss [11], if $2^{\aleph_{0}}<\aleph_{\omega}$, then there is $r<\omega$ such that $\mathbb{R} \nrightarrow^{+}\left(\aleph_{0}\right)_{r}$, so, over a
model of GCH, it is necessary to add at least $\beth_{\omega}$-many reals to obtain $\mathbb{R} \rightarrow^{+}\left(\aleph_{0}\right)_{r}$ for every $r<\omega$.

Let us examine, though, the number of reals that must be added to obtain $\mathbb{R} \rightarrow^{+}\left(\aleph_{0}\right)_{r}$ for some fixed $r<\omega$. Zhang in fact proves that $\mathbb{R} \rightarrow^{+}\left(\aleph_{0}\right)_{2}$ holds in ZFC and, for a fixed $r>2$, in proving that $\mathbb{R} \rightarrow^{+}\left(\aleph_{0}\right)_{r}$ holds in the forcing extension, Theorem6.1 is employed with $\nu=\aleph_{0}, \lambda=\aleph_{1}$, and $n=2 r$. Hence, $\mu$ can be taken to be least such that $\mu \rightarrow\left(\aleph_{1}\right)_{2^{\aleph_{0}}}^{4 r}$. By the Erdős-Rado theorem, then, we can take $\mu=\beth_{4 r}^{+}$. Zhang's proof uses the fact that we have added at least $\mu$-many Cohen reals and therefore shows that, for this fixed value of $r>2$, the statement $\mathbb{R} \rightarrow^{+}\left(\aleph_{0}\right)_{r}$ holds in the forcing extension obtained by adding $\beth_{4 r}^{+}$-many Cohen reals.

Inspection of Zhang's proof reveals that Theorem 3.8, with $\kappa=\aleph_{1}, \lambda=\beth_{1}^{+}$, and $n=2 r$, can be used in place of Theorem 6.1. We can therefore take $\mu=$ $\sigma\left(\beth_{1}^{+}, 2 r\right)=\beth_{2 r}^{+}$, obtaining the following corollary:

Corollary 6.2. Suppose that $2<r<\omega$ and $\mathbb{P}$ is the forcing to add at least $\beth_{2 r}^{+}$many Cohen reals. Then, in $V^{\mathbb{P}}$, we have $\mathbb{R} \rightarrow^{+}\left(\aleph_{0}\right)_{r}$.

This is an improvement on the bound of $\beth_{4 r}^{+}$given by Zhang's proof, though of course it does not improve on Zhang's bound for obtaining $\mathbb{R} \rightarrow^{+}\left(\aleph_{0}\right)_{r}$ simultaneously for all $r<\omega$. We omit the adaptation of Zhang's proof using Theorem 3.8 instead of Theorem 6.1 here, as it would entail introducing a considerable number of definitions and only involves very minor changes to Zhang's proof. Instead, we direct the reader to [22] and [14], in which Zhang's original proof and the adaptation using Theorem 3.8 are spelled out in detail.

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