A POINT-FREE APPROACH TO CANONICAL EXTENSIONS OF BOOLEAN ALGEBRAS AND BOUNDED ARCHIMEDEAN ℓ-ALGEBRAS

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Abstract. In [13] an elegant choice-free construction of a canonical extension of a boolean algebra B was given as the boolean algebra of regular open subsets of the Alexandroff topology on the poset of proper filters of B. We make this construction point-free by replacing the Alexandroff space of proper filters of B with the free frame $\mathscr{L}$ generated by the bounded meet-semilattice of all filters of B (ordered by reverse inclusion) and prove that the booleanization of $\mathscr{L}$ is a canonical extension of B. Our main result generalizes this approach to the category $\boldsymbol{ba\ell}$ of bounded archimedean ℓ-algebras, thus yielding a point-free construction of canonical extensions in $\boldsymbol{ba\ell}$. We conclude by showing that the algebra of normal functions on the Alexandroff space of proper archimedean ℓ-ideals of A is a canonical extension of $A \in \boldsymbol{ba\ell}$, thus providing a generalization of the result of [13] to $\boldsymbol{ba\ell}$.

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1. Introduction

The theory of canonical extensions originates from the pioneering work of Jónsson and Tarski [32]. Originally it was defined for boolean algebras with operators, but was later generalized to distributive lattices with operators [20, 21], lattices with operators [18], and even to posets with operators [22, 19].

One of the most convenient (albeit neither choice-free nor point-free) ways to describe a canonical extension of a boolean algebra B is using Stone duality. If X is the Stone space of B, then B is isomorphic to the boolean algebra $\mathsf{Clop}(X)$ of clopen subsets of X,

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and the pair $(\wp(X), e)$ is a canonical extension of B where $\wp(X)$ is the powerset of X and $e : \mathsf{Clop}(X) \to \wp(X)$ is the identity embedding.

This approach generalizes naturally from Stone spaces to compact Hausdorff spaces. Let X be compact Hausdorff and $C(X)$ the ring of continuous (necessarily bounded) real-valued functions on X. In [11] a canonical extension of $C(X)$ was described as the pair $(B(X), e)$ where $B(X)$ is the ring of all bounded real-valued functions on X and $e : C(X) \to B(X)$ is the identity embedding. More generally, if A is a bounded archimedean ℓ-algebra, by Gelfand duality A embeds into $C(X)$, where X is the compact Hausdorff space of maximal ℓ-ideals of X, and the pair $(B(X), \zeta)$ is a canonical extension of A, where $\zeta : A \to C(X) \subseteq B(X)$ is the embedding of A into $C(X)$ (see Sections 3 and 5 for details).

This approach to canonical extensions is neither choice-free nor point-free. An elegant choice-free approach to canonical extensions of boolean algebras was developed in [13] where a canonical extension of a boolean algebra B was constructed as the boolean algebra of regular open sets of the Alexandroff space of proper filters of B (ordered by inclusion). Our first aim is to make the construction of [13] point-free. For this we utilize the well-known fact that the free frame on a meet-semilattice M with top is isomorphic to the downsets of M (see, e.g., [35, Prop. IV.2.3]). For a boolean algebra B, let $\mathsf{Filt}(B)$ be the co-frame of filters ordered by reverse inclusion. We view $\mathsf{Filt}(B)$ as a bounded meet-semilattice, and show that the free frame $\mathscr{L}$ on $\mathsf{Filt}(B)$ is isomorphic to the Alexandroff space of proper filters of B ordered by inclusion (see Corollary 2.7). From this we derive that the booleanization $\mathfrak{B}(\mathscr{L})$ of $\mathscr{L}$ is a canonical extension of B (see Theorem 2.9).

Our second aim is to generalize this point-free approach to the category $\boldsymbol{ba\ell}$ of bounded archimedean ℓ-algebras. The interest in this category stems from the fact that $\boldsymbol{ba\ell}$ provides an algebraic counterpart of the category KHaus of compact Hausdorff spaces. Indeed, by Gelfand duality, there is a dual adjunction between KHaus and $\boldsymbol{ba\ell}$, which restricts to a dual equivalence between KHaus and the full subcategory $\boldsymbol{uba\ell}$ of $\boldsymbol{ba\ell}$ consisting of uniformly complete algebras in $\boldsymbol{ba\ell}$ (see, e.g., [8]). Generalizing our point-free approach to $\boldsymbol{ba\ell}$ requires additional machinery. Let $A \in \boldsymbol{ba\ell}$. Following the work of Banaschewski [2, 3], we work with archimedean ℓ-ideals of A (see Section 4). Let $\mathsf{Arch}(A)$ be the frame of archimedean ℓ-ideals ordered by inclusion. Assuming the Axiom of Choice (AC), $\mathsf{Arch}(A)$ is isomorphic to the frame of opens of the space of maximal ℓ-ideals of A (see Remark 4.5).

Viewing $\mathsf{Arch}(A)$ as a bounded meet-semilattice, let $\mathscr{L}$ be the free frame generated by $\mathsf{Arch}(A)$, and let $\mathscr{B}(\mathscr{L})$ be the free boolean extension of $\mathscr{L}$ (see, e.g., [1, Sec. V.4]). We employ the concepts of Specker algebra and Dedekind completion (see Section 3 for details), which play an important role in the study of $\boldsymbol{ba\ell}$. We associate with $\mathscr{B}(\mathscr{L})$ the Specker algebra $\mathbb{R}[\mathscr{B}(\mathscr{L})]$ and prove that the Dedekind completion $D(\mathbb{R}[\mathscr{B}(\mathscr{L})])$ of $\mathbb{R}[\mathscr{B}(\mathscr{L})]$ is a canonical extension of A (see Theorem 5.15). This is our main result and yields a point-free construction of canonical extensions in $\boldsymbol{ba\ell}$. Its proof requires a number of technical calculations about archimedean ℓ-ideals. In order to not break the flow, we move these calculations to an appendix.

Finally, we show that the algebra of normal real-valued functions on the Alexandroff space of proper archimedean ℓ-ideals ordered by inclusion is a canonical extension of $A \in \boldsymbol{ba\ell}$ (see

Theorem 6.7). On the one hand, this provides a generalization of the construction of [13]. On the other hand, assuming (AC), this algebra of normal functions is isomorphic to the algebra of bounded real-valued functions on the set of maximal ℓ-ideals of A, thus yielding the result of [11].

2. Canonical extensions of boolean algebras point-free

In this section we show how to give a point-free description of canonical extensions of boolean algebras. Let BA be the category of boolean algebras and boolean homomorphisms. The next definition is well known (see, e.g., [18, Sec. 2]).

Definition 2.1. Let B be a boolean algebra, C a complete boolean algebra, and $e : B \to C$ a BA-monomorphism.

(1) We call e *compact* if whenever $S, T \subseteq B$ with $\bigwedge e[S] \leq \bigvee e[T]$, there are finite $S_0 \subseteq S$ and $T_0 \subseteq T$ with $\bigwedge S_0 \leq \bigvee T_0$.
(2) We call e *dense* if each element of C is a join of meets from $e[B]$.
(3) We say that the pair (C, e) is a *canonical extension* of B if e is dense and compact.

Remark 2.2. It is straightforward to see that the compactness condition is equivalent to each of the following two conditions.

(1) If $T \subseteq B$ with $\bigvee e[T] = 1$, then there is a finite $T_0 \subseteq T$ with $\bigvee T_0 = 1$.
(2) If $S \subseteq B$ with $\bigwedge e[S] = 0$, then there is a finite $S_0 \subseteq S$ with $\bigwedge S_0 = 0$.

In fact, (1) is the original definition of compactness in [32]. We will use (2) in the proof of Theorem 2.9.

Jónsson and Tarski [32] utilized Stone duality to show that each boolean algebra has a canonical extension, which is unique up to isomorphism. This requires the use of (AC). An elegant choice-free description of canonical extensions of boolean algebras was given in [13]. Let $B \in \mathsf{BA}$ and let X be the set of proper filters of B, ordered by inclusion. View X as an Alexandroff space where opens are the upsets of X (so U is open provided $x \in U$ and $x \leq y$ imply $y \in U$). Let $\mathsf{RO}(X)$ be the boolean algebra of regular open subsets of X. Define $e : B \to \mathsf{RO}(X)$ by $e(b) = \{x \in X \mid b \in x\}$. Then $(\mathsf{RO}(X), e)$ is a canonical extension of B [13, Thm. 8.27].

We give a point-free description of the construction in [13]. To do so we recall some basic notions from point-free topology. We refer the reader to [35] for the details.

A *frame* or *locale* is a complete distributive lattice L satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$ for each $a \in L$ and $S \subseteq L$. A *frame homomorphism* is a map $f : L \to M$ between two frames that preserves finite meets and arbitrary joins. For a frame L and $a \in L$, let $a^* := \bigvee\{s \in L \mid a \wedge s = 0\}$ be the *pseudocomplement* of a. The set $\mathfrak{B}(L) := \{a^{**} \mid a \in L\}$ is called the *booleanization* of L. It is a complete boolean algebra, where infinite joins and meets are calculated by

$$\bigsqcup S = \left(\bigvee S\right)^{**} \text{ and } \bigsqcap S = \bigwedge S.$$

The *well inside relation* $\prec$ is defined on a frame L by $a \prec b$ if $a^* \vee b = 1$. Then L is *regular* if $a = \bigvee\{s \in L \mid s \prec a\}$ for each $a \in L$. Also, L is *compact* if whenever $S \subseteq L$ with $\bigvee S = 1$, there is a finite $S_0 \subseteq S$ with $\bigvee S_0 = 1$.

Definition 2.3. Let $B \in \mathsf{BA}$. We denote by $\mathsf{Filt}(B)$ the set of filters of B, ordered by reverse inclusion.

Remark 2.4. As we will see below, we are using reverse inclusion on $\mathsf{Filt}(B)$ in order for the map $B \to \mathsf{Filt}(B)$ which sends b to ${\uparrow}b$ to be order preserving.

Let SLat^1 be the category of meet-semilattices with top with meet-semilattice morphisms preserving the top. The free frame on $M \in \mathsf{SLat}^1$ is isomorphic to the frame of downsets $\mathsf{Dn}(M)$ of M (see, e.g., [35, Prop. IV.2.3]), where we recall that D is a downset if whenever $x \le y$ and $y \in D$, we have $x \in D$. Let SLat^1_0 be the category of meet-semilattices with top and bottom, with meet-semilattice morphisms preserving both the top and bottom.

Proposition 2.5. *Let $M \in \mathsf{SLat}^1_0$. Then $\mathsf{Dn}(M \setminus \{0\})$ is isomorphic to the free frame on M.*

Proof. Define $i : M \to \mathsf{Dn}(M \setminus \{0\})$ by $i(m) = {\downarrow}m \setminus \{0\}$. It is straightforward to see that i is a SLat^1_0-morphism. Let L be a frame and $f : M \to L$ a SLat^1_0-morphism. Define $\varphi : \mathsf{Dn}(M \setminus \{0\}) \to L$ by $\varphi(D) = \bigvee\{f(m) \mid m \in D\}$. We show that φ is a frame homomorphism satisfying $\varphi \circ i = f$, and that φ is uniquely determined by these properties.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "f"'] & D(M \setminus \{0\}) \arrow[d, "\varphi"] \\ & L \end{tikzcd}$$

First, $\varphi(\varnothing) = \bigvee \varnothing = 0$ and $\varphi(M \setminus \{0\}) = f(1) = 1$. Next, let D_1, D_2 be downsets of $M \setminus \{0\}$. We have

$$\begin{aligned}
\varphi(D_1) \wedge \varphi(D_2) &= \bigvee\{f(m) \mid m \in D_1\} \wedge \bigvee\{f(n) \mid n \in D_2\} \\
&= \bigvee\{f(m) \wedge f(n) \mid m \in D_1, n \in D_2\} \\
&= \bigvee\{f(m \wedge n) \mid m \in D_1, n \in D_2\} \\
&= \bigvee\{f(p) \mid p \in D_1 \cap D_2\} = \varphi(D_1 \cap D_2).
\end{aligned}$$

Also, let $\{D_\gamma \mid \gamma \in \Gamma\}$ be a family of downsets. Then

$$\begin{aligned}
\varphi\left(\bigcup\{D_\gamma \mid \gamma \in \Gamma\}\right) &= \bigvee\left\{f(m) \mid m \in \bigcup D_\gamma\right\} \\
&= \bigvee\left\{\bigvee\{f(m) \mid m \in D_\gamma\} \mid \gamma \in \Gamma\right\} \\
&= \bigvee\{\varphi(D_\gamma) \mid \gamma \in \Gamma\}.
\end{aligned}$$

Therefore, φ is a frame homomorphism. It is clear from the definition that $\varphi(i(m)) = f(m)$ for each $m \in M$, so $\varphi \circ i = f$. Finally, since $D = \bigcup\{i(m) \mid m \in D\}$, it follows that φ is uniquely determined by the equation $\varphi \circ i = f$. Thus, $\mathsf{Dn}(M \setminus \{0\})$ is, up to isomorphism, the free frame on $M \in \mathsf{SLat}^1_0$. ∎

Remark 2.6. Let $M \in \mathsf{SLat}_0^1$ and let $(\mathscr{L}, i)$ be the free frame on M. Since i preserves finite meets, every element of $\mathscr{L}$ is a join of elements from $i[M]$.

Corollary 2.7. *Let $B \in \mathsf{BA}$. The frame of upsets of proper filters of B, ordered by reverse inclusion, is isomorphic the free frame on $\mathsf{Filt}(B) \in \mathsf{SLat}_0^1$.*

Proof. By Proposition 2.5, the free frame on $\mathsf{Filt}(B)$ is isomorphic to the frame of all downsets of $\mathsf{Filt}(B) \setminus \{B\}$, which is isomorphic to the frame of upsets of X. $\square$

Let $B \in \mathsf{BA}$ and let $\mathscr{L}$ be the free frame on the bounded meet-semilattice $\mathsf{Filt}(B)$ with the associated map $i : \mathsf{Filt}(B) \to \mathscr{L}$. Define $e : B \to \mathscr{L}$ by $e(b) = i({\uparrow}b)$.

Lemma 2.8. *If $b \in B$, then $i({\uparrow}b)^* = i({\uparrow}\neg b)$. Consequently, $i({\uparrow}b) \in \mathfrak{B}(\mathscr{L})$.*

Proof. Since i is a SLat_0^1-morphism and the order on $\mathsf{Filt}(B)$ is reverse inclusion,

$$i({\uparrow}b) \wedge i({\uparrow}\neg b) = i({\uparrow}b \vee {\uparrow}\neg b) = i({\uparrow}(b \wedge \neg b)) = i(B).$$

Because B is the bottom of $\mathsf{Filt}(B)$, we obtain that $i({\uparrow}b) \wedge i({\uparrow}\neg b) = 0$. Let $x \in \mathscr{L}$ with $i({\uparrow}b) \wedge x = 0$. To show $x \le i({\uparrow}\neg b)$, by Remark 2.6, x is a join from $i[\mathsf{Filt}(B)]$. Since the order on $\mathsf{Filt}(B)$ is reverse inclusion, it suffices to show that if F is a filter of B, then $i({\uparrow}b) \wedge i(F) = 0$ implies ${\uparrow}\neg b \subseteq F$. If $i({\uparrow}b) \wedge i(F) = 0$, then $i({\uparrow}b \vee F) = 0$, so ${\uparrow}b \vee F = B$ as i is one-to-one. Therefore, there is $a \in F$ with $a \wedge b = 0$. Thus, $a \le \neg b$, and hence ${\uparrow}\neg b \subseteq F$, as desired. This shows that $i({\uparrow}b)^* = i({\uparrow}\neg b)$. From this we see that $i({\uparrow}b)^{**} = i({\uparrow}\neg b)^* = i({\uparrow}\neg\neg b) = i({\uparrow}b)$, so $i({\uparrow}b) \in \mathfrak{B}(\mathscr{L})$. $\square$

Theorem 2.9. *For $B \in \mathsf{BA}$, the pair $(\mathfrak{B}(\mathscr{L}), e)$ is a canonical extension of B.*

Proof. By Lemma 2.8, $e(b) \in \mathfrak{B}(\mathscr{L})$, so $e : B \to \mathfrak{B}(\mathscr{L})$ is well defined. If $b \ne c$, then ${\uparrow}b \ne {\uparrow}c$, so $i({\uparrow}b) \ne i({\uparrow}c)$. Therefore, e is one-to-one. To see that e is a BA-morphism, let $b, c \in B$. Then

$$e(b \wedge c) = i({\uparrow}(b \wedge c)) = i({\uparrow}b \vee {\uparrow}c) = i({\uparrow}b) \wedge i({\uparrow}c) = e(b) \wedge e(c),$$

so e preserves meet. Also, by Lemma 2.8, $e(\neg b) = i({\uparrow}\neg b) = i({\uparrow}b)^* = e(b)^*$. Thus, e preserves negation, and hence is a BA-morphism.

We next show that e is dense. Since every element of $\mathfrak{B}(\mathscr{L})$ is a join from $i[\mathsf{Filt}(B)]$, it is enough to show that if F is a filter of B, then $i(F)$ is a meet from $e[B]$. We show that $i(F) = \bigwedge\{e(b) \mid b \in F\}$. First, if $b \in F$, then ${\uparrow}b \subseteq F$, so $i(F) \le i({\uparrow}b) = e(b)$. Therefore, $i(F) \le \bigwedge\{e(b) \mid b \in F\}$. For the reverse inequality, suppose that G is a filter with $i(G) \le \bigwedge\{e(b) \mid b \in F\}$. Then $i(G) \le i({\uparrow}b)$, so ${\uparrow}b \subseteq G$, and hence $b \in G$ for each $b \in F$. This implies that $F \subseteq G$, and so $i(G) \le i(F)$. Therefore, if $x \in \mathscr{L}$ with $x \le \bigwedge\{e(b) \mid b \in F\}$, then $x \le i(F)$, showing that $i(F) = \bigwedge\{e(b) \mid b \in F\}$. Thus, $i(F)$ is a meet from $e[B]$, and so each element of $\mathscr{L}$ is a join of meets from $e[B]$. Consequently, e is dense.

Finally, we show that e is compact. Let $S \subseteq B$ with $\bigwedge e[S] = 0$. Then

$$0 = \bigwedge e[S] = \bigwedge\{i({\uparrow}s) \mid s \in S\}.$$

Let F be the filter generated by S. Then ${\uparrow}s \subseteq F$, and hence $i(F) \le i({\uparrow}s)$ for each $s \in S$. This forces $i(F) = 0$, so $F = B$. Therefore, there are $s_1, \dots, s_n \in S$ with $s_1 \wedge \cdots \wedge s_n = 0$. Thus, e is compact by Remark 2.2, and hence $(\mathfrak{B}(\mathscr{L}), e)$ is a canonical extension of B. $\square$

We can now derive the result of [13]. By Corollary 2.7, the Alexandroff topology $\mathsf{Up}(X)$ on X is isomorphic to $\mathscr{L}$. Consequently, $\mathsf{RO}(X) \cong \mathfrak{B}(\mathscr{L})$. Moreover, define $f : \mathsf{Filt}(B) \to \mathsf{Up}(X)$ by $f(F) = \{G \in X \mid F \subseteq G\}$. It is easy to see that f is a SLat_0^1-morphism, so induces a frame homomorphism $\varphi : \mathscr{L} \to \mathsf{Up}(X)$ satisfying $\varphi \circ i = f$.

$$\begin{tikzcd} \mathsf{Filt}(B) \arrow[r, "i"] \arrow[dr, "f"'] & \mathscr{L} \arrow[d, "\varphi"] \\ & \mathsf{Up}(X) \end{tikzcd}$$

If $b \in B$, then

$$\varphi(e(b)) = \varphi(i(\uparrow b)) = f(\uparrow b) = \{G \in X \mid \uparrow b \subseteq G\} = \{G \in X \mid b \in G\},$$

which is the map defined in [13]. This yields an explicit isomorphism between our construction and that in [13].

3. Bounded archimedean ℓ-algebras

In this section we recall several basic facts about bounded archimedean ℓ-algebras. We assume the reader's familiarity with ℓ-rings (lattice-ordered rings) and ℓ-algebras (lattice-ordered algebras). We use [14, Ch. XIII and onwards] as our main reference for ℓ-rings and [28, 8] as our main references for ℓ-algebras. All rings are assumed to be commutative and unital (have multiplicative identity 1).

Definition 3.1.

(1) An ℓ-algebra A is *bounded* if for each $a \in A$ there is an integer $n \geq 1$ such that $a \leq n \cdot 1$ (that is, 1 is a *strong order unit*).
(2) An ℓ-algebra A is *archimedean* if for each $a, b \in A$, whenever $n \cdot a \leq b$ for each $n \geq 1$, then $a \leq 0$.
(3) A $\boldsymbol{ba\ell}$*-algebra* is a bounded archimedean ℓ-algebra and a $\boldsymbol{ba\ell}$-morphism is a unital ℓ-algebra homomorphism. Let $\boldsymbol{ba\ell}$ be the category of $\boldsymbol{ba\ell}$-algebras and $\boldsymbol{ba\ell}$-morphisms.

Let $A \in \boldsymbol{ba\ell}$. If $A \neq 0$, then we view $\mathbb{R}$ as an ℓ-subalgebra of A by identifying $r \in \mathbb{R}$ with $r \cdot 1 \in A$.

Definition 3.2. Let $A \in \boldsymbol{ba\ell}$ and $a \in A$.

(1) Define the *positive* and *negative* parts of a by

$$a^+ = a \vee 0 \text{ and } a^- = (-a) \vee 0 = -(a \wedge 0).$$

(2) Define the *absolute value* of a by

$$|a| = a \vee (-a).$$

(3) Define the *norm* of $a \in A$ by

$$||a|| = \inf\{r \in \mathbb{R} : |a| \leq r\}.$$

We call A *uniformly complete* if the norm is complete. Let $\boldsymbol{uba\ell}$ be the full subcategory of $\boldsymbol{ba\ell}$ consisting of uniformly complete objects.

Theorem 3.3 (Gelfand duality [23, 37])**.** *There is a dual adjunction between* $\boldsymbol{ba\ell}$ *and* KHaus *which restricts to a dual equivalence between* KHaus *and* $\boldsymbol{uba\ell}$*.*

$$\begin{tikzcd}
\boldsymbol{uba\ell} \arrow[rr, hook] & & \boldsymbol{ba\ell} \arrow[ll, bend right] \arrow[dl, "(-)_*"] \\
& \mathsf{KHaus} \arrow[ul, "(-)^*"] &
\end{tikzcd}$$

The contravariant functors $(-)^* : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ and $(-)_* : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ establishing the dual adjunction of Theorem 3.3 are defined as follows. For a compact Hausdorff space X let X^* be the ring $C(X)$ of (necessarily bounded) continuous real-valued functions on X. For a continuous map $\varphi : X \to Y$ let $\varphi^* : C(Y) \to C(X)$ be defined by $\varphi^*(f) = f \circ \varphi$ for each $f \in C(Y)$. Then $(-)^* : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ is a well-defined contravariant functor.

For $A \in \boldsymbol{ba\ell}$, we recall that an ideal I of A is an *ℓ-ideal* if $|a| \le |b|$ and $b \in I$ imply $a \in I$, and that ℓ-ideals are exactly the kernels of ℓ-algebra homomorphisms. Let Y_A be the space of maximal ℓ-ideals of A, whose closed sets are exactly sets of the form

$$Z_\ell(I) = \{M \in Y_A \mid I \subseteq M\},$$

where I is an ℓ-ideal of A. The space Y_A is often referred to as the *Yosida space* of A, and it is well known that $Y_A \in \mathsf{KHaus}$. Let $A_* = Y_A$ and for a $\boldsymbol{ba\ell}$-morphism α let $\alpha_* = \alpha^{-1}$. Then $(-)_* : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ is a well-defined contravariant functor, and the functors $(-)_*$ and $(-)^*$ yield a dual adjunction between $\boldsymbol{ba\ell}$ and KHaus.

Moreover, for $X \in \mathsf{KHaus}$ we have that $\varepsilon_X : X \to Y_{C(X)}$ is a homeomorphism where

$$\varepsilon_X(x) = \{f \in C(X) \mid f(x) = 0\}.$$

Furthermore, for $A \in \boldsymbol{ba\ell}$ define $\zeta_A : A \to C(Y_A)$ by $\zeta_A(a)(M) = r$ where r is the unique real number satisfying $a + M = r + M$. Then ζ_A is a monomorphism in $\boldsymbol{ba\ell}$ separating points of Y_A. Thus, by the Stone-Weierstrass theorem, we have:

Theorem 3.4.

(1) *The uniform completion of* $A \in \boldsymbol{ba\ell}$ *is* $\zeta_A : A \to C(Y_A)$*. Therefore, if* A *is uniformly complete, then* ζ_A *is an isomorphism.*
(2) $\boldsymbol{uba\ell}$ *is a reflective subcategory of* $\boldsymbol{ba\ell}$*, and the reflector* $\zeta : \boldsymbol{ba\ell} \to \boldsymbol{uba\ell}$ *assigns to each* $A \in \boldsymbol{ba\ell}$ *its uniform completion* $C(Y_A) \in \boldsymbol{uba\ell}$*.*

Consequently, the dual adjunction restricts to a dual equivalence between $\boldsymbol{uba\ell}$ and KHaus, yielding Gelfand duality.

For the results in Section 5 we recall the definition of Dedekind algebras, Dedekind completions, and Specker algebras (see, e.g., [8, 9]).

Definition 3.5. Let $A \in \boldsymbol{ba\ell}$.

(1) A is *Dedekind complete* if every subset of A bounded above has a least upper bound (and hence every subset of A bounded below has a greatest lower bound).

(2) A is a *Dedekind* $\boldsymbol{ba\ell}$*-algebra* if A is Dedekind complete.
(3) Let $\boldsymbol{dba\ell}$ be the full subcategory of $\boldsymbol{ba\ell}$ consisting of Dedekind $\boldsymbol{ba\ell}$-algebras.

If $A \in \boldsymbol{ba\ell}$, then there is a unique up to isomorphism Dedekind $\boldsymbol{ba\ell}$-algebra $D(A) \in \boldsymbol{ba\ell}$ such that A embeds into $D(A)$ and the image is join dense in $D(A)$. This result was proved by Nakano [34, Sec. 31] in the setting of vector lattices and by Johnson [30, p. 493] in the setting of f-rings. It was adapted to $\boldsymbol{ba\ell}$ in [9].

Definition 3.6. For $A \in \boldsymbol{ba\ell}$, we call $D(A)$ the *Dedekind completion* of A. Throughout this paper we will identify A with its image in $D(A)$.

The following theorem provides a characterization of Dedekind completions in $\boldsymbol{ba\ell}$ (see, e.g., [9, Thm. 3.1]).

Theorem 3.7. *Let $A \in \boldsymbol{ba\ell}$ and $D \in \boldsymbol{dba\ell}$. If $\alpha : A \to D$ is a $\boldsymbol{ba\ell}$-monomorphism such that every element of D is a join from $\alpha[A]$, then there is a $\boldsymbol{ba\ell}$-isomorphism $\varphi : D(A) \to D$ with $\varphi|_A = \alpha$.*

Let $A \in \boldsymbol{ba\ell}$. Since A is a commutative ring, it is well known that the set $\mathrm{Id}(A)$ of idempotents of A is a boolean algebra under the operations

$$e \vee f = e + f - ef, \quad e \wedge f = ef, \quad \neg e = 1 - e.$$

The ordering on the boolean algebra $\mathrm{Id}(A)$ is the restriction of the ordering on A, and hence $e \in \mathrm{Id}(A)$ implies that $0 \le e \le 1$.

Definition 3.8. [8, Sec. 5] We call $A \in \boldsymbol{ba\ell}$ a *Specker* $\boldsymbol{ba\ell}$*-algebra* if A is generated by $\mathrm{Id}(A)$.

Remark 3.9. For the history of the notion of a Specker algebra see, e.g., [12].

Specker algebras can be characterized by the following construction, the origins of which go back to Bergman [5] and Rota [36].

Definition 3.10. [7, Def. 2.4] Let B be a Boolean algebra. We denote by $\mathbb{R}[B]$ the quotient ring $\mathbb{R}[\{x_e \mid e \in B\}]/I_B$ of the polynomial ring over $\mathbb{R}$ in variables indexed by the elements of B modulo the ideal I_B generated by the following elements, as e, f range over B:

$$x_{e\wedge f} - x_e x_f, \quad x_{e\vee f} - (x_e + x_f - x_e x_f), \quad x_{\neg e} - (1 - x_e), \quad x_0.$$

For $e \in B$ we abuse notation and identify x_e with its image in $\mathbb{R}[B]$. Considering the generators of I_B, for all $e, f \in B$, we have that

$$x_{e\wedge f} = x_e x_f, \quad x_{e\vee f} = x_e + x_f - x_e x_f, \quad x_{\neg e} = 1 - x_e, \quad x_0 = 0.$$

Theorem 3.11. *Let $A \in \boldsymbol{ba\ell}$.*
(1) [7, Lem. 3.2(4)] *The map sending $e \in B$ to $x_e \in \mathbb{R}[B]$ is a BA-isomorphism from B to $\mathrm{Id}(\mathbb{R}[B])$.*
(2) [7, Lem. 2.5] *For $B \in \mathsf{BA}$ and a BA-morphism $\tau : B \to \mathrm{Id}(A)$, there is a unique $\boldsymbol{ba\ell}$-morphism $\sigma : \mathbb{R}[B] \to A$ such that $\sigma(x_b) = \tau(b)$ for each $b \in B$.*

(3) [7, Thm. 2.7] *A is a Specker $\boldsymbol{ba\ell}$-algebra iff $A \cong \mathbb{R}[B]$ for some $B \in \mathsf{BA}$.*

Remark 3.12. Associating $\mathrm{Id}(A)$ with each $A \in \boldsymbol{ba\ell}$ extends to a covariant functor $\mathrm{Id} : \boldsymbol{ba\ell} \to \mathsf{BA}$. Definition 3.10 gives rise to a covariant functor $\mathrm{Sp} : \mathsf{BA} \to \boldsymbol{ba\ell}$ which is left adjoint to Id. The functors Id and Sp yield an equivalence between BA and the full subcategory of $\boldsymbol{ba\ell}$ consisting of Specker $\boldsymbol{ba\ell}$-algebras (see [7, Sec. 3] for details).

We will use the following two facts about $\mathrm{Id}(A)$. The proof of the first one can be found in [6, Lem. 4.9(6)], and we give a short proof of the second.

Remark 3.13. Let $A \in \boldsymbol{ba\ell}$.

(1) Let $0 \neq e, f \in \mathrm{Id}(A)$ and $0 < r, s \in \mathbb{R}$. If $re \le sf$, then $r \le s$ and $e \le f$.

(2) If $e \in \mathrm{Id}(A)$ and $0 \le a \in A$ with $1 = e \vee a$, then $\neg e \le a$. Indeed, since $\neg e$ is an idempotent, $0 \le \neg e \le 1$, so

$$\neg e = \neg e \wedge (e \vee a) = (\neg e \wedge e) \vee (\neg e \wedge a) = 0 \vee (\neg e \wedge a) = \neg e \wedge a$$

because $0 \le a$. Thus, $\neg e \le a$.

In the next remark we collect together some elementary facts about $\mathbb{R}[B]$. We will use them frequently in our proofs.

Remark 3.14. Let $e, f \in B$.

(1) $x_e \vee x_f = x_{e \vee f}$ and $x_e \wedge x_f = x_{e \wedge f}$.

(2) If $e \wedge f = 0$, then $x_e + x_f = x_e \vee x_f$.

(3) $x_e \vee x_{\neg e} = 1$ and $x_e \wedge x_{\neg e} = 0$.

(4) $x_e = x_{e \wedge \neg f} + x_{e \wedge f}$.

In addition, if $a \in \mathbb{R}[B]$, we may write $a = r_1 x_{b_1} + \cdots + r_n x_{b_n}$ for some $r_i \in \mathbb{R}$ and $b_i \in B$ with $b_i \wedge b_j = 0$ whenever $i \neq j$ (see, e.g., [8, Lem. 5.4]).

We conclude this section by the following remark in which we collect together some well-known identities that hold in $\boldsymbol{ba\ell}$-algebras. They will be used throughout the paper. Most can be found in [14, Ch. XIII, XV, XVII], for (2), (6), and (8) see [33, Secs. 12, 13], and for (10) see [30, Lem. 1].

Remark 3.15. Let $A \in \boldsymbol{ba\ell}$, $a, b \in A$, and $C, D \subseteq A$.

(1) If $\bigvee C$ exists in A, then $\bigvee\{a + c \mid c \in C\}$ exists and is equal to $a + \bigvee C$. The dual property for meets also holds.

(2) A is a distributive lattice. Furthermore, if $\bigvee C$ exists in A, then $\bigvee\{a \wedge c \mid c \in C\}$ exists and is equal to $a \wedge \bigvee C$. The dual property for meets also holds.

(3) $-(a \vee b) = (-a) \wedge (-b)$ and $-(a \wedge b) = (-a) \vee (-b)$.

(4) $a^- = (-a)^+$.

(5) $a = a^+ - a^-$ and $|a| = a^+ + a^-$.

(6) $(a + b)^+ \le a^+ + b^+$ and $(a + b)^- \le a^- + b^-$.

(7) $a^+ \wedge a^- = 0 = a^+ a^-$.

(8) If $\bigvee C$ exists in A and $0 \le r \in \mathbb{R}$, then $\bigvee\{rc \mid c \in C\}$ exists and is equal to $r \bigvee C$. The dual property for meets also holds.

(9) If $a \wedge b = 0$, then $ra \wedge sb = 0$ for any $0 \leq r, s \in \mathbb{R}$.
(10) If C, D consist of nonnegative elements and the joins $\bigvee C, \bigvee D$ exist in A, then $(\bigvee C)(\bigvee D) = \bigvee \{cd \mid c \in C, d \in D\}$. The dual property for meets also holds.
(11) If α is a $\boldsymbol{bal}$-morphism, then $\alpha(|a|) = |\alpha(a)|$.

4. Archimedean ℓ-ideals

In this section we discuss archimedean ℓ-ideals in $\boldsymbol{ba\ell}$-algebras. This notion will be our main tool in the rest of the paper. Let $A \in \boldsymbol{ba\ell}$ and I be an ℓ-ideal of A. It is well known and easy to check that A/I is a bounded ℓ-algebra, but A/I may not be archimedean in general.

Definition 4.1. Let $A \in \boldsymbol{ba\ell}$. We call an ℓ-ideal I of A *archimedean* if A/I is archimedean. Let $\mathsf{Arch}(A)$ be the set of archimedean ℓ-ideals of A, ordered by inclusion.

Remark 4.2. Let $A \in \boldsymbol{ba\ell}$.

(1) If I is an ℓ-ideal of A, then A/I is archimedean iff $A/I \in \boldsymbol{ba\ell}$. Thus, I is archimedean iff $A/I \in \boldsymbol{ba\ell}$.
(2) If M is a maximal ℓ-ideal of A, then it is well known (see, e.g., [28, Cor. 27]) that $A/M \cong \mathbb{R}$. Thus, every maximal ℓ-ideal is archimedean.
(3) Assuming (AC), an ℓ-ideal I of $A \in \boldsymbol{ba\ell}$ is archimedean iff $I = \bigcap\{M \in Y_A \mid I \subseteq M\}$ (see, e.g., [8, p. 440]).

Remark 4.3. In [2] Banaschewski studied the ℓ-ideals in f-rings that are closed in the norm topology. If A is a $\boldsymbol{ba\ell}$-algebra, then it turns out that an ℓ-ideal I of A is archimedean iff it is closed in the norm topology. We skip the proof since this fact does not play an important role in the rest of the paper.

The next result is a consequence of Banaschewski's more general result.

Theorem 4.4. [2, App. 2] *If $A \in \boldsymbol{ba\ell}$, then $\mathsf{Arch}(A)$ is a compact regular frame.*

Remark 4.5. Since $\mathsf{Arch}(A)$ is compact regular, using (AC) it follows from [29] that $\mathsf{Arch}(A)$ is isomorphic to the frame of open subsets of a compact Hausdorff space (see also [4] or [31, Sec. III.1]). In fact, $\mathsf{Arch}(A)$ is isomorphic to the frame of open subsets of the Yosida space Y_A via the map that sends $I \in \mathsf{Arch}(A)$ to the open set $Z_\ell(I)^c$. This is an analogue of the well-known fact that for a commutative ring R, there is a bijection between the frame of radical ideals of R and the frame of open subsets of the prime spectrum of R with the Zariski topology.

It is straightforward to see that the intersection of a family of archimedean ℓ-ideals is archimedean. Thus, we can define the concept of the archimedean hull, which Banaschewski [3] referred to as the archimedean kernel.

Definition 4.6. Let $A \in \boldsymbol{ba\ell}$. The *archimedean hull* of $S \subseteq A$ is the intersection of all archimedean ℓ-ideals of A containing S. We denote the archimedean hull of S by $\langle S \rangle$.

Banaschewski [3, p. 321] showed that if A is an f-ring and I is an ℓ-ideal of A, then the archimedean hull of I is constructed as follows. Let

$$k(I) = \{a \in A \mid (n|a| - b)^+ \in I \text{ for some } b \geq 0 \text{ and for all } n \geq 1\}.$$

It is straightforward to see that $k(I)$ is an ℓ-ideal containing I. Set $k_1(I) = k(I)$. For each ordinal γ set $k_{\gamma+1}(I) = k(k_\gamma(I))$. If γ is a limit ordinal, define $k_\gamma(I) = \bigcup\{k_\delta(I) \mid \delta < \gamma\}$. Then there is a least γ with $k_\gamma(I) = k_{\gamma+1}(I)$, and the archimedean hull $\langle I\rangle$ of I is $k_\gamma(I)$.

We conclude this section by showing that when $A \in \boldsymbol{ba\ell}$ (and hence 1 is a strong order-unit), the construction of the archimedean hull simplifies. For this we need the following remark.

Remark 4.7.

(1) If A is a bounded ℓ-algebra, then it is archimedean iff for each $a \in A$, whenever $na \leq 1$ for each $n \geq 1$, then $a \leq 0$. To see that this condition implies that A is archimedean, suppose $na \leq b$ for each $a, b \in A$ and $n \geq 1$. Since A is bounded, there is $m \geq 1$ with $b \leq m$. Therefore, $(nm)a \leq b \leq m$, so $na \leq 1$ for all $n \geq 1$. This forces $a \leq 0$, and hence A is archimedean.

(2) Let $A \in \boldsymbol{ba\ell}$ and I be an ℓ-ideal of A. For $a \in A$, we have:

$$a + I \geq 0 + I \text{ iff } a^- \in I \text{ and } a + I \leq 0 + I \text{ iff } a^+ \in I \text{ (see, e.g., [10, Rem. 2.11]).}$$

Proposition 4.8. *Let $A \in \boldsymbol{ba\ell}$ and I be an ℓ-ideal of A. Then*

$$\langle I\rangle = \{x \in A \mid (n|x| - 1)^+ \in I \text{ for all } n \geq 1\}.$$

Proof. Set $K = \{x \in A \mid (n|x| - 1)^+ \in I \text{ for all } n \geq 1\}$. It is straightforward to see that K is an ℓ-ideal of A containing I. To see that A/K is archimedean, by Remark 4.7(1), it is enough to show that if $a \in A$ with $n(a + K) \leq 1 + K$ for each $n \geq 1$, then $a + K \leq 0 + K$. Since $na + K \leq 1 + K$ we get $na^+ + K \leq 1 + K$. By Remark 4.7(2) this implies that $(na^+ - 1)^+ \in K$ for each $n \geq 1$. To show $a + K \leq 0 + K$ it is sufficient to show that $a^+ \in K$. Thus, we may replace a by a^+ to assume $0 \leq a$. Since $(na - 1)^+ \in K$ for each $n \geq 1$, we have $(m(na - 1)^+ - 1)^+ \in I$ for each $n, m \geq 1$. Using several properties from Remark 3.15, we have

$$\begin{aligned} (m(na-1)^+ - 1)^+ &= (((mna - m) \vee 0) - 1) \vee 0 \\ &= ((mna - (m+1)) \vee -1) \vee 0 \\ &= (mna - (m+1)) \vee 0 \\ &= (mna - (m+1))^+, \end{aligned}$$

and so $(mna-(m+1))^+ \in I$ for each $m, n \geq 1$. Set $n = m+1$. Then $(m+1)(ma-1)^+ \in I$, so $(ma - 1)^+ \in I$ because $m + 1$ is a unit in A. Since this is true for each $m \geq 1$, we get $a \in K$, so $a + K = 0 + K$. Therefore, A/K is archimedean, and hence K is an archimedean ℓ-ideal of A.

Suppose $J \supseteq I$ is an archimedean ℓ-ideal. If $a \in K$, then $(n|a|-1)^+ \in J$ for each $n \geq 1$, so $(n|a|-1)+J \leq 0+J$, and hence $n|a|+J \leq 1+J$ for each $n \geq 1$. Since A/J is archimedean,

$|a| + J = 0 + J$, so $|a| \in J$, and hence $a \in J$. Therefore, $K \subseteq J$, and thus K is the least archimedean ℓ-ideal containing I, completing the proof. $\square$

5. Canonical extensions in $\boldsymbol{ba\ell}$ point-free

Canonical extensions of $\boldsymbol{ba\ell}$-algebras were introduced in [11], where it was shown that a canonical extension of $A \in \boldsymbol{ba\ell}$ is isomorphic to $(B(Y_A), \zeta_A)$. The proof that $(B(Y_A), \zeta_A)$ is a canonical extension of $A \in \boldsymbol{ba\ell}$ is neither choice-free nor point-free. However, the uniqueness part of the proof is point-free. In this section we give a point-free proof of the existence as well, generalizing our results from Section 2. The arguments are considerably more complicated than those of Section 2 and require a careful study of various properties of archimedean ℓ-ideals. To make the section easier to read we collect all these properties together in an appendix.

Definition 5.1. [11, Def. 1.6] Let A be a $\boldsymbol{ba\ell}$-algebra, D a Dedekind $\boldsymbol{ba\ell}$-algebra and $\zeta : A \to D$ a $\boldsymbol{ba\ell}$-monomorphism.

(1) We call ζ *compact* if whenever $S, T \subseteq A$ and $\varepsilon > 0$ with $\bigwedge \zeta[S] + \varepsilon \le \bigvee \zeta[T]$, there are finite $S_0 \subseteq S$ and $T_0 \subseteq T$ with $\bigwedge S_0 \le \bigvee T_0$.
(2) We call ζ *dense* if each element of D is a join of meets from $\zeta[A]$.
(3) We say that the pair (D, ζ) is a *canonical extension* of A if ζ is dense and compact.

Remark 5.2.

(1) In [11] canonical extensions were defined for the category of bounded archimedean vector lattices, but the same definition works for $\boldsymbol{ba\ell}$.
(2) By [11, Lem. 2.4], $\zeta : A \to D$ is compact iff for each $T \subseteq A$ and $\varepsilon > 0$, if $\varepsilon \le \bigvee \zeta[T]$, then there is a finite $T_0 \subseteq T$ with $0 \le \bigvee T_0$.

Let $A \in \boldsymbol{ba\ell}$. By [11, Thm. 1.8(2)], $(B(Y_A), \zeta_A)$ is a canonical extension of A. To motivate our new approach, we give a point-free description of $B(Y_A)$. Let Dist be the category of bounded distributive lattices. It is well known that the forgetful functor $\mathscr{U} : \mathsf{BA} \to \mathsf{Dist}$ has a left adjoint $\mathscr{B} : \mathsf{Dist} \to \mathsf{BA}$. For each $L \in \mathsf{Dist}$, the Boolean algebra $\mathscr{B}(L)$ together with the canonical embedding $i : L \to \mathscr{B}(L)$ is usually referred to as the *free Boolean extension* of L. It is characterized by the following universal mapping property: If $\lambda : L \to C$ is a bounded lattice homomorphism into a boolean algebra, then there is a unique BA-morphism $\tau : \mathscr{B}(L) \to C$ with $\tau \circ i = \lambda$ (see, e.g., [1, Sec. V.4]).

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[dr, "\lambda"'] & \mathscr{B}(L) \arrow[d, "\tau"] \\ & C \end{tikzcd}$$

Let $L = \mathsf{Arch}(A)$. Viewing L as a bounded distributive lattice, let (B, i) be the free boolean extension of L. Assuming (AC), L is isomorphic to the frame $\mathscr{O}(Y_A)$ of open sets of Y_A (see Remark 4.5) and B is isomorphic to the algebra $\mathsf{Con}(Y_A)$ of constructible sets of Y_A (see, e.g., [1, Sec. V.4]), where $\mathsf{Con}(Y_A)$ is the boolean subalgebra of $\wp(Y_A)$ generated by $\mathscr{O}(Y_A)$. The isomorphism $\lambda : L \to \mathscr{O}(Y_A)$ sends I to $Z_\ell(I)^c := \{M \in Y_A \mid I \not\subseteq M\}$

(see Remark 4.5). Since $\mathscr{O}(Y_A)$ is a sublattice of $\wp(Y_A)$, we view λ as a bounded lattice homomorphism into the boolean algebra $\wp(Y_A)$. As $\wp(Y_A)$ is isomorphic to $\mathrm{Id}(B(Y_A))$ by sending $U \subseteq Y_A$ to the characteristic function χ_U, the universal mapping property mentioned above yields a BA-morphism $\tau : B \to \mathrm{Id}(B(Y_A))$ with $\tau(i(I)) = \chi_{Z_\ell(I)^c}$. By Theorem 3.11(1), there is a $\boldsymbol{ba\ell}$-morphism $\sigma : \mathbb{R}[B] \to B(Y_A)$ extending τ. Since each $0 \le f \in B(Y_A)$ is equal to $\bigvee\{f(M)\chi_{\{M\}} \mid M \in Y_A\}$, each element of $B(Y_A)$ is a join from $\mathbb{R}[B]$ by Remark 3.15(1). Thus, by Theorem 3.7, there is an isomorphism $\theta : D(\mathbb{R}[B]) \to B(Y_A)$ satisfying $\theta(x_I) = \sigma(x_I) = \chi_{Z_\ell(I)^c}$ for each $I \in \mathsf{Arch}(A)$. Let $\alpha = \theta^{-1} \circ \zeta_A$.

$$\begin{tikzcd}
L \arrow[r, "i"] \arrow[dd, "\lambda"'] & B \arrow[r] \arrow[dd, "\tau"'] & \mathbb{R}[B] \arrow[r, hook] \arrow[ddr, "\sigma"'] & D(\mathbb{R}[B]) \arrow[dd, "\theta"] & \\
& & & & A \arrow[ul, "\alpha"'] \arrow[dl, "\zeta_A"] \\
\mathscr{O}(Y_A) \arrow[r] & \mathrm{Id}(B(Y_A)) \arrow[rr, hook] & & B(Y_A) &
\end{tikzcd}$$

FIGURE 1. The isomorphism $\theta : D(\mathbb{R}[B]) \to B(Y_A)$

If $0 \le a \in A$, we claim that

$$\zeta_A(a) = \bigvee\{r\chi_C \mid C \text{ closed in } Y_A \text{ and } r\chi_C \le \zeta_A(a)\}.$$

It is obvious that $\zeta_A(a)$ is above the join. Let $M \in Y_A$ and set $\zeta_A(a)(M) = r$. Then $r\chi_{\{M\}}(M) = r$ and $r\chi_{\{M\}} \le \zeta_A(a)$ since $0 \le a$. From this it follows that the equation above is true.

To make this description of $\zeta_A(a)$ point-free, let C be closed in Y_A and $I \in \mathsf{Arch}(A)$ with $Z_\ell(I) = C$. We show that $r\chi_C \le \zeta_A(a)$ for some $r \ge 0$ iff $(a-r)^- \in I$.

First suppose $r\chi_C \le \zeta_A(a)$ for some $r \ge 0$. If $M \in C$, then $r \le \zeta_A(a)(M)$, so $a + M \ge r + M$, which means $(a-r)^- \in M$ (see Remark 4.7(2)). Since this is true for all $M \in C$, we have $(a-r)^- \in \bigcap Z_\ell(I) = I$ (see Remark 4.2(3)).

Conversely, let $(a-r)^- \in I$ and $M \in C$. Then $I \subseteq M$, so $(a-r)^- \in M$ which gives $a + M \ge r + M$, so $r \le \zeta_A(a)(M)$. Since $0 \le a$, this yields $r\chi_C \le \zeta_A(a)$. Consequently,

$$\begin{aligned}
\zeta_A(a) &= \bigvee\{r\chi_C \mid C \text{ closed in } Y_A \text{ and } r\chi_C \le \zeta_A(a)\} \\
&= \bigvee\{r\chi_{Z_\ell(I)} \mid I \in \mathsf{Arch}(A) \text{ and } (a-r)^- \in I\}.
\end{aligned}$$

For $a \in A$ arbitrary, let $s \in \mathbb{R}$ be such that $a + s \ge 0$. Then

$$\begin{aligned}
\zeta_A(a) &= -s + \zeta_A(a+s) \\
&= -s + \bigvee\{r\chi_{Z_\ell(I)} \mid I \in \mathsf{Arch}(A), (a+s-r)^- \in I\}.
\end{aligned}$$

Definition 5.3. Let (B, i) be the free boolean extension of $\mathsf{Arch}(A)$. For ease of notation we assume that $\mathsf{Arch}(A) \subseteq B$, and so i is the identity. Thus, for $I \in \mathsf{Arch}(A)$ we have that x_I and $x_{\neg I}$ are idempotents of $\mathbb{R}[B]$.

The discussion above motivates the following point-free definition.

Definition 5.4. Define $\alpha : A \to D(\mathbb{R}[B])$ by

$$\alpha(a) = -s + \bigvee\{rx_{\neg I} \mid I \in \mathsf{Arch}(A), (a+s-r)^- \in I\}$$

where $s \in \mathbb{R}$ with $a + s \geq 0$.

Remark 5.5. Let $a \in A$ and $s \in \mathbb{R}$ with $a + s \geq 0$. Then $(a+s)^- = 0$. Thus, for each $I \in \mathsf{Arch}(A)$, we have that $0x_{\neg I}$ is part of the join defining $\alpha(a)$. Consequently, we may assume $r \geq 0$ in the definition of $\alpha(a)$.

To show that α is well defined we need to show that the join in Definition 5.4 exists and that the expression does not depend on the choice of s. Showing that the join exists is straightforward, but independence of s requires some work. In particular, we will utilize Lemma A.3 given in the appendix.

Proposition 5.6. *$\alpha : A \to D(\mathbb{R}[B])$ is well defined.*

Proof. Let $a \in A$ and $s \in \mathbb{R}$ with $a + s \geq 0$. We first show that

$$\{rx_{\neg I} \mid I \in \mathsf{Arch}(A), (a+s-r)^- \in I\}$$

is bounded above, so the join exists in $D(\mathbb{R}[B])$. Let $I \in \mathsf{Arch}(A)$. If $I = A$, then $x_{\neg I} = 0$, so $rx_{\neg I} = 0$. Suppose that $I \neq A$. If $r > \|a\| + s + 1$, then $a + s - r \leq \|a\| + s - r < -1$, so $(a+s-r)^- > 1$. Therefore, $(a+s-r)^- \notin I$. Thus, if $(a+s-r)^- \in I$, then $r \leq \|a\| + s + 1$, and so $rx_{\neg I} \leq \|a\| + s + 1$ as $x_{\neg I}$ is an idempotent. From this it follows that the set above is bounded by $\|a\| + s + 1$, and hence the join defining $\alpha(a)$ exists.

We next show that $\alpha(a)$ does not depend on s. Let $0 \leq s, t \in \mathbb{R}$ with $a+s, a+t \geq 0$. Set $f = \bigvee\{rx_{\neg I} \mid I \in \mathsf{Arch}(A), (a+s-r)^- \in I\}$ and $g = \bigvee\{rx_{\neg I} \mid I \in \mathsf{Arch}(A), (a+t-r)^- \in I\}$. Then $f, g \in D(\mathbb{R}[B])$ by the previous paragraph. By Lemma A.3(3) and Remark 3.15(1),

$$\begin{aligned} f + t &= \bigvee\{(r+t)x_{\neg I} \mid (a+s-r)^- \in I\} \\ &= \bigvee\{ux_{\neg I} \mid (a+s+t-u)^- \in I\} \\ &= \bigvee\{(v+s)x_{\neg I} \mid (a+t-v)^- \in I\} \\ &= s + g. \end{aligned}$$

Therefore, $-s + f = -t + g$, which proves that the formula defining $\alpha(a)$ does not depend on the choice of s. Thus, α is well defined. $\square$

We next show that α preserves order and addition by a scalar.

Lemma 5.7.

(1) *If $0 \leq a \in A$, then $\alpha(a) = \bigvee\{rx_{\neg I} \mid 0 \leq r, I \in \mathsf{Arch}(A), (a-r)^- \in I\}$.*
(2) *α is order preserving.*
(3) *If $a \in A$ and $t \in \mathbb{R}$, then $\alpha(a+t) = \alpha(a) + t$.*

Proof. (1) Set $s = 0$ and apply Remark 5.5.

(2) Suppose that $a \leq b$. Choose $0 \leq s$ with $a + s \geq 0$. Then $b + s \geq 0$. Therefore, $(a+s-r)^- \in I$ implies $(b+s-r)^- \in I$ because $(b+s-r)^- \leq (a+s-r)^-$. From this it follows that $\alpha(a) \leq \alpha(b)$.

(3) Let $s_1, s_2 \in \mathbb{R}$ be such that $a + s_1 \geq 0$ and $t + s_2 \geq 0$. Set $s = s_1 + s_2$. By Remark 5.5,

$$\alpha(a) = -s_1 + \bigvee\{rx_{\neg I} \mid 0 \leq r, I \in \mathsf{Arch}(A), (a + s_1 - r)^- \in I\},$$

so by Lemma A.3(3), we have

$$\begin{aligned}\alpha(a) + s_2 + t &= -s_1 + \bigvee\{(s_2 + t + r)x_{\neg I} \mid (a + s_1 - r)^- \in I\}\\ &= -s_1 + \bigvee\{(s_2 + t + r)x_{\neg I} \mid (a + s + t - (s_2 + t + r))^- \in I\}\\ &= s_2 - s + \bigvee\{ux_{\neg I} \mid (a + s + t - u)^- \in I\}\\ &= s_2 + \alpha(a + t)\end{aligned}$$

since $a + t + s \geq 0$. Thus, $\alpha(a + t) = \alpha(a) + t$. □

We are ready to prove that $(D(\mathbb{R}[B]), \alpha)$ is a canonical extension of A. This we do in the next three propositions.

Proposition 5.8. *$\alpha : A \to D(\mathbb{R}[B])$ is a $\boldsymbol{ba\ell}$-monomorphism.*

Proof. We first show that α is a $\boldsymbol{ba\ell}$-morphism. This we do by showing that α preserves addition, meet, scalar multiplication, join, and multiplication.

Claim 5.9. *α preserves addition.*

Proof of the Claim. Let $a, b \in A$. We first assume that $0 \leq a, b$. Because α is order preserving (see Lemma 5.7(2)) and $a, b \leq a + b$, we have that $\alpha(a), \alpha(b) \leq \alpha(a + b)$. In addition, by Lemma 5.7(1) (and Remarks 3.15(1,9) and 3.14(2)),

$$\begin{aligned}\alpha(a) + \alpha(b) &= \bigvee\{rx_{\neg I} \mid (a - r)^- \in I\} + \bigvee\{sx_{\neg J} \mid (b - s)^- \in J\}\\ &= \bigvee\{rx_{\neg I} + sx_{\neg J} \mid (a - r)^- \in I, (b - s)^- \in J\}\\ &= \bigvee\{rx_{\neg I \wedge J} + sx_{\neg J \wedge I} + (r + s)x_{\neg I \wedge \neg J} \mid (a - r)^- \in I, (b - s)^- \in J\}\\ &= \bigvee\{rx_{\neg I \wedge J} \vee sx_{\neg J \wedge I} \vee (r + s)x_{\neg(I \vee J)} \mid (a - r)^- \in I, (b - s)^- \in J\}.\end{aligned}$$

We use this to show that $\alpha(a) + \alpha(b) \leq \alpha(a + b)$. We have $rx_{\neg I \wedge J} \leq rx_{\neg I} \leq \alpha(a) \leq \alpha(a + b)$ and $sx_{\neg J \wedge I} \leq sx_{\neg J} \leq \alpha(b) \leq \alpha(a + b)$. It remains to show that $(r + s)x_{\neg(I \vee J)} \leq \alpha(a + b)$. This is trivial if $I \vee J = A$ since then $x_{\neg(I \vee J)} = 0$. Otherwise, set $K = I \vee J$. We have $(a - r)^-, (b - s)^- \in K$, and since $0 \leq (a + b - (r + s))^- \leq (a - r)^- + (b - s)^-$ by Remark 3.15(6), we see that $(a + b - (r + s))^- \in K$. Therefore, $(r + s)x_{\neg K} \leq \alpha(a + b)$ by Lemma 5.7(1). This completes the proof that $\alpha(a) + \alpha(b) \leq \alpha(a + b)$.

We use Lemma A.6 to show the reverse inequality. Suppose that $tx_{\neg I} \leq \alpha(a + b)$. We may assume that $t = \sup\{r \mid rx_{\neg I} \leq \alpha(a + b)\}$. Then $J := I \vee \langle(a + b - t)^+\rangle \neq A$ and $(a + b - t)^- \in I$ by Lemmas A.2 and A.7, so $(a + b - t)^+ - (a + b - t)^- \in J$, and hence $a + b - t \in J$ by Remark 3.15(5). Let $s = \sup\{r \mid rx_{\neg J} \leq \alpha(a)\}$. Then $sx_{\neg J} \leq \alpha(a)$, $(a - s)^- \in J$, and $K := J \vee \langle(a - s)^+\rangle \neq A$, again by Lemmas A.2 and A.7. We have $a - s \in K$, so $b - (t - s) = (a + b - t) - (a - s) \in K$. Therefore, $(b - (t - s))^- \in K$, so $(t - s)x_{\neg K} \leq \alpha(b)$ by Lemma 5.7(1). Since $J \subseteq K$, we have $sx_{\neg K} \leq sx_{\neg J} \leq \alpha(a)$. From this we see that $tx_{\neg K} \le \alpha(a)+\alpha(b)$. Consequently, by Lemma A.6, $\alpha(a+b) \le \alpha(a)+\alpha(b)$. This shows that $\alpha(a+b) = \alpha(a)+\alpha(b)$ for $0 \le a, b$.

To complete the argument, let $a, b \in A$ be arbitrary and choose $t \in \mathbb{R}$ with $a+t, b+t \ge 0$. Then $\alpha(a+b+2t) = \alpha(a+t)+\alpha(b+t)$, so $\alpha(a+b)+2t = \alpha(a)+\alpha(b)+2t$ by Lemma 5.7(3). Therefore, $\alpha(a+b) = \alpha(a)+\alpha(b)$. $\square$

Claim 5.10. *α preserves meet.*

Proof of the Claim. Let $a, b \in A$. Since α is order preserving, $\alpha(a \wedge b) \le \alpha(a) \wedge \alpha(b)$. Because α preserves addition (by a scalar), we may assume $0 \le a, b$. By Lemmas 5.7(1) and A.1(1) (and Remarks 3.15(2) and 3.14(1)), we have

$$\begin{aligned}
\alpha(a) \wedge \alpha(b) &= \bigvee\{rx_{\neg I} \mid (a-r)^- \in I\} \wedge \bigvee\{sx_{\neg J} \mid (b-s)^- \in J\} \\
&= \bigvee\{rx_{\neg I} \wedge sx_{\neg J} \mid (a-r)^- \in I, (b-s)^- \in J\} \\
&= \bigvee\{\min(r,s)x_{\neg I \wedge \neg J} \mid (a-r)^- \in I, (b-s)^- \in J\} \\
&= \bigvee\{\min(r,s)x_{\neg(I \vee J)} \mid (a-r)^- \in I, (b-s)^- \in J\}.
\end{aligned} \tag{1}$$

Suppose $(a-r)^- \in I$, $(b-s)^- \in J$, and assume without loss of generality that $r \le s$. Since $(b-r)^- \le (b-s)^-$, we have $(a-r)^-, (b-r)^- \in I \vee J$. Therefore, $(a-r)^- \vee (b-r)^- \in I \vee J$ (because ℓ-ideals are closed under $\vee$). Since

$$\begin{aligned}
(a-r)^- \vee (b-r)^- &= [(r-a) \vee 0] \vee [(r-b) \vee 0] = [(r-a) \vee (r-b)] \vee 0 \\
&= [r + (-a \vee -b)] \vee 0 = [r - (a \wedge b)] \vee 0 = ((a \wedge b) - r)^-,
\end{aligned}$$

we see that $((a \wedge b) - r)^- \in I \vee J$. Therefore, $rx_{\neg(I \vee J)} \le \alpha(a \wedge b)$, and hence (1) implies that $\alpha(a) \wedge \alpha(b) \le \alpha(a \wedge b)$. Thus, $\alpha(a) \wedge \alpha(b) = \alpha(a \wedge b)$. $\square$

Claim 5.11. *α preserves scalar multiplication and join.*

Proof of the Claim. Since α is a group homomorphism, to show that it preserves scalar multiplication it suffices to show $\alpha(sa) = s\alpha(a)$ for each $a \in A$ and $0 < s \in \mathbb{R}$. Moreover, by Lemma 5.7(3), it suffices to assume $0 \le a$. Since $(sa-r)^- \in I$ iff $(a - r/s)^- \in I$, by Lemma 5.7(1) and Remark 3.15(8), we have

$$\begin{aligned}
\alpha(sa) &= \bigvee\{rx_{\neg I} \mid (sa-r)^- \in I\} = \bigvee\{rx_{\neg I} \mid (a-r/s)^- \in I\} \\
&= s\bigvee\{(r/s)x_{\neg I} \mid (a-r/s)^- \in I\} = s\alpha(a).
\end{aligned}$$

Since α preserves meet and scalar multiplication, it preserves join by Remark 3.15(3). $\square$

Claim 5.12. *α preserves multiplication.*

Proof of the Claim. First suppose that $0 \le a, b$. By Lemma 5.7(1) and Remarks 3.15(10) and 3.14(1), we have

$$\begin{aligned}\alpha(a)\alpha(b) &= \bigvee\{rx_{\neg I} \mid 0 \le r, (a-r)^- \in I\} \cdot \bigvee\{sx_{\neg J} \mid 0 \le s, (b-s)^- \in J\} \\ &= \bigvee\{rsx_{\neg I}x_{\neg J} \mid (a-r)^- \in I, (b-s)^- \in J\} \\ &= \bigvee\{rsx_{\neg(I\vee J)} \mid (a-r)^- \in I, (b-s)^- \in J\}.\end{aligned}$$

Also $\alpha(ab) = \bigvee\{tx_{\neg K} \mid (ab-t)^- \in K\}$. To see that $\alpha(a)\alpha(b) \le \alpha(ab)$, it suffices to show that $rsx_{\neg(I\vee J)} \le \alpha(ab)$, where $0 \le r, s$, $(a-r)^- \in I$, and $(b-s)^- \in J$. Set $K = I \vee J$. Then $(a-r)^-, (b-s)^- \in K$. Therefore, $r + K \le a + K$ and $s + K \le b + K$. Because $0 \le r, s$, we have

$$rs + K = (r + K)(s + K) \le (a + K)(b + K) = ab + K.$$

This gives $(ab - rs)^- \in K$. Thus, $rsx_{\neg(I\vee J)} = rsx_{\neg K} \le \alpha(ab)$ by Lemma 5.7(1). Consequently, $\alpha(a)\alpha(b) \le \alpha(ab)$.

For the reverse inequality, we use Lemma A.6 and argue as in the proof of Claim 5.9. Let $I \in \mathsf{Arch}(A)$ and $0 \le t$ with $tx_{\neg I} \le \alpha(ab)$. Then $(ab-t)^- \in I$ by Lemma A.7(2). We may assume that $t = \sup\{r \mid (ab-r)^- \in I\}$ by Lemma A.7(1). Therefore, $J := I \vee \langle (ab-t)^+\rangle \ne A$ by Lemma A.7(3). Set $r = \sup\{p \mid (a-p)^- \in J\}$. Then $(a-r)^- \in J$ and $K := J \vee \langle (a-r)^+\rangle \ne A$. We have $ab-t \in J$ and $a-r \in K$. Therefore, $ab-t, (a-r)b \in K$. Thus, $rb - t \in K$. If $r = 0$, then $t \in K$, which implies $t = 0$ since $K \ne A$ and nonzero real numbers are units in A. It is then clear that $tx_{\neg K} \le \alpha(a)\alpha(b)$ since $0 \le \alpha(a)\alpha(b)$. If $r \ne 0$, then $b - t/r \in K$. Therefore, $(t/r)x_{\neg K} \le \alpha(b)$ and $rx_{\neg K} \le \alpha(a)$, so $tx_{\neg K} \le \alpha(a)\alpha(b)$ since $x_{\neg K}$ is an idempotent. Thus, by Lemma A.6, $\alpha(ab) \le \alpha(a)\alpha(b)$. This shows that $\alpha(ab) = \alpha(a)\alpha(b)$ for $0 \le a, b$.

For a, b arbitrary, since $a = a^+ - a^-$ and $b = b^+ - b^-$ (see Remark 3.15(5)), we have

$$ab = (a^+ - a^-)(b^+ - b^-) = (a^+b^+ + a^-b^-) - (a^+b^- + a^-b^+)$$

By the previous case and Claim 5.9,

$$\begin{aligned}\alpha(ab) &= \alpha(a^+b^+ + a^-b^-) - \alpha(a^+b^- + a^-b^+) \\ &= (\alpha(a^+b^+) + \alpha(a^-b^-)) - (\alpha(a^+b^-) + \alpha(a^-b^+)) \\ &= (\alpha(a^+)\alpha(b^+) + \alpha(a^-)\alpha(b^-)) - (\alpha(a^+)\alpha(b^-) + \alpha(a^-)\alpha(b^+)) \\ &= (\alpha(a^+) - \alpha(a^-))(\alpha(b^+) - \alpha(b^-)) = \alpha(a^+ - a^-)\alpha(b^+ - b^-) \\ &= \alpha(a)\alpha(b).\end{aligned}$$

□

This completes the proof that α is a $\boldsymbol{ba\ell}$-morphism. It is left to show that α is a monomorphism. This we do by showing that $\alpha(a) \ne 0$ for $a \ne 0$. Because $\alpha(|a|) = |\alpha(a)|$ (see Remark 3.15(11)) and $a \ne 0$ iff $|a| \ne 0$, it suffices to assume $a \ge 0$. Since $a \ne 0$, we have $\|a\| > 0$. Choose $r \in \mathbb{R}$ with $0 < r \le \|a\|$ and set $I = \langle (a-r)^-\rangle$. We claim that $I \ne A$. To see this, if $I = A$, by Lemma A.2(2) there is n with $1 \le n(a-r)^-$, so $1/n \le (a-r)^- = (r-a) \vee 0$.

Therefore, $1/n \le r - a$ by Lemma A.1(2). This implies $a \le r - 1/n < r$, which contradicts the inequality $r \le \|a\|$ since $\|a\|$ is the smallest real number above a. Thus, $\neg I \neq 0$, so $x_{\neg I} \neq 0$, and hence $0 < rx_{\neg I} \le \alpha(a)$. This shows that $\alpha(a) \neq 0$. ☐

Proposition 5.13. *$\alpha : A \to D(\mathbb{R}[B])$ is dense.*

Proof. We first show that $x_{\neg I}$ is a meet from $\alpha[A]$ for each $I \in \mathsf{Arch}(A)$. By Lemma 5.7(1), if $(a-1)^- \in I$, then $x_{\neg I} \le \alpha(a)$, so

$$x_{\neg I} \le \bigwedge\{\alpha(a) \mid (a-1)^- \in I\} =: f.$$

In the inequality above we may assume that $0 \le a \le 1$ since $x_{\neg I}$ is an idempotent, so $0 \le x_{\neg I}$ and $x_{\neg I} \le \alpha(a)$ implies that $x_{\neg I} \le \alpha(a) \wedge 1 = \alpha(a \wedge 1)$. Therefore, we have $0 \le f \le 1$. If $I \subseteq J$, then $x_{\neg J} \le x_{\neg I}$. Because $0 \le f$, it is a join of elements of the form $rx_{\neg J}$ by Lemma A.3(2). To show $f = x_{\neg I}$ we show that if $rx_{\neg J} \le f$ with $r > 0$, then $J \supseteq I$ and $r \le 1$, so $rx_{\neg J} \le x_{\neg I}$. First, if $rx_{\neg J} \le f$, then $rx_{\neg J} \le 1$, so $r \le 1$ by Remark 3.13(1). Next, suppose that $I \not\subseteq J$. Then there is $K \neq A$ with $J \subseteq K$ and $K + I = A$ by Lemma A.2(2). Therefore, there are $0 \le a, b$ with $a \in K$, $b \in I$, and $1 = a + b$ by Lemma A.5(1). So $(a-1)^- = b \in I$, and hence $x_{\neg I} \le \alpha(a)$. Suppose that there is $r > 0$ with $(a-r)^- \in J$. Then $A = \langle a, (a-r)^- \rangle$ by Lemma A.5(1). This is impossible since $K \neq A$ but $a, (a-r)^- \in K$. This implies that $r \le 0$. This contradiction shows that $x_{\neg I} = f$ and hence is a meet from $\alpha[A]$.

If $0 \le f \in D(\mathbb{R}[B])$, then f is a join of nonnegative elements from $\mathbb{R}[B]$. If $0 \le c \in \mathbb{R}[B]$, we can write c as a sum of terms of the form rx_b with $0 \le r \in \mathbb{R}$ and $b \in B$, and so f is a join of such terms by Remark 3.14. Since each b can be written as a join of terms of the form $\neg I \wedge J$ with $I, J \in \mathsf{Arch}(A)$, we see that f is a join of elements of the form $rx_{\neg I \wedge J}$. Therefore, by Lemma A.3(1), f is a join of elements of the form $rx_{\neg I}$. Thus, f is a join of meets from $\alpha[A]$. For f arbitrary, if $f + n \ge 0$, then $f + n$ is a join of meets from $\alpha[A]$, and so f is also a join of meets from $\alpha[A]$ by Remark 3.15(1). Consequently, α is dense. ☐

Proposition 5.14. *$\alpha : A \to D(\mathbb{R}[B])$ is compact.*

Proof. Let $0 < \varepsilon \in \mathbb{R}$ and $T \subseteq A$ with $\varepsilon \le \bigvee \alpha[T]$. By Remark 5.2(2), it suffices to show that there is a finite $T_0 \subseteq T$ with $\bigvee T_0 \ge 0$. Set $T' = \{(a+\varepsilon) \vee 0 \mid a \in T\}$. Since $\alpha((a+\varepsilon) \vee 0) \ge \alpha(a+\varepsilon) = \alpha(a) + \varepsilon$, we have $2\varepsilon \le \bigvee \alpha[T']$ by Remark 3.15(1). As $0 \le b$ for each $b \in T'$, Lemma 5.7(1) implies

$$2\varepsilon \le \bigvee\{rx_{\neg I} \mid (b-r)^- \in I, b \in T'\}.$$

We next consider the archimedean ℓ-ideal $L = \bigvee\{\langle (b-\varepsilon)^+ \rangle \mid b \in T'\}$ and show that $L = A$. If not, then $x_{\neg L} \neq 0$, so using Remarks 3.14(1) and 3.15(10), we have

$$\begin{aligned} 2\varepsilon x_{\neg L} &\le x_{\neg L} \cdot \bigvee\{rx_{\neg I} \mid (b-r)^- \in I, b \in T'\} \\ &= \bigvee\{rx_{\neg L}x_{\neg I} \mid (b-r)^- \in I, b \in T'\} \qquad (2) \\ &= \bigvee\{rx_{\neg(I \vee L)} \mid (b-r)^- \in I, b \in T'\}. \end{aligned}$$

Observe that if $I \vee L = A$, then $x_{\neg(I\vee L)} = 0$. Suppose that $r \leq 3\varepsilon/2$ for all r, b, I in the join above with $I \vee L \neq A$. Because $x_{\neg(I\vee L)} \leq x_{\neg L}$ for each I, the join above is then bounded by $(3\varepsilon/2)x_{\neg L}$, a contradiction to the inequality (2). Therefore, there are r, b, I in the join above with $I \vee L \neq A$ and $r > 3\varepsilon/2$. Since $(b-3\varepsilon/2)^- \leq (b-r)^-$, we have $(b-3\varepsilon/2)^- \in I \subseteq I \vee L$, and $(b-\varepsilon)^+ \in L \subseteq I \vee L$ by definition of L. Therefore, $I \vee L = A$ by Lemma A.5(1). This contradiction yields $L = A$.

Since $\mathsf{Arch}(A)$ is a compact frame (see Theorem 4.4), there are $b_1, \dots, b_n \in T'$ with $\langle (b_1-\varepsilon)^+ \rangle \vee \cdots \vee \langle (b_n-\varepsilon)^+ \rangle = A$. For each i there is $a_i \in T$ with $b_i = (a_i+\varepsilon) \vee 0$. Then, using Remark 3.15(1), we have

$$(b_i-\varepsilon)^+ = [((a_i+\varepsilon)\vee 0) - \varepsilon] \vee 0 = (a_i \vee -\varepsilon) \vee 0 = a_i \vee 0 = a_i^+$$

for each i. Therefore, $\langle a_1^+ \rangle \vee \cdots \vee \langle a_n^+ \rangle = A$. Set $c = a_1 \vee \cdots \vee a_n$. Then $a_i^+ \leq c^+$ for each i, so $\langle c^+ \rangle = A$. Thus, there is $m \geq 1$ with $1 \leq mc^+$, and hence $1/m \leq c^+$. By Lemma A.1(2), $1/m \leq c$ which yields $0 \leq 1/m \leq a_1 \vee \cdots \vee a_n$. This shows that α is compact. $\square$

Propositions 5.8, 5.13, and 5.14 yield our main result.

Theorem 5.15. *For each $A \in \boldsymbol{ba\ell}$, the pair $(D(\mathbb{R}[B]), \alpha)$ is a canonical extension of A.*

6. Canonical extensions and normal functions

In the previous section we gave a point-free description of a canonical extension of $A \in \boldsymbol{ba\ell}$ as the pair $(D(\mathbb{R}[B]), \alpha)$. In this section we show that $D(\mathbb{R}[B])$ can be described as the algebra $N(X)$ of (bounded) normal real-valued functions on the space X of proper archimedean ℓ-ideals of A. The idempotents of $N(X)$ are exactly the characteristic functions of regular opens of X. Thus, we obtain a generalization of a result of [13] that a canonical extension of a boolean algebra B is isomorphic to the boolean algebra of regular open subsets of the space of proper filters of B. Assuming (AC), we show that $N(X)$ is isomorphic to the algebra of bounded real-valued functions on the Yosida space of A, thus obtaining a result of [11]. We conclude the paper by drawing a connection between our results and those in point-free topology describing normal functions on an arbitrary frame [24, 26, 27, 25].

For a topological space X, we recall that $B(X)$ is the set of all bounded real-valued functions on X. It is straightforward to see that under pointwise operations $B(X) \in \boldsymbol{dba\ell}$. Recall that $f \in B(X)$ is *lower semicontinuous* if $f^{-1}(r, \infty)$ is open, and f is *upper semicontinuous* if $f^{-1}(-\infty, r)$ is open for each $r \in \mathbb{R}$ (see, e.g., [15, p. 361]). For each $x \in X$, let $\mathscr{N}_x$ be the collection of open neighborhoods of x. For each $f \in B(X)$ define

$$f_*(x) = \sup\{\inf f[U] \mid U \in \mathscr{N}_x\}$$
$$f^*(x) = \inf\{\sup f[U] \mid U \in \mathscr{N}_x\}.$$

It is well known that f is lower semicontinuous iff $f = f_*$ and f is upper semicontinuous iff $f = f^*$ (see, e.g., [15, p. 360-362]).

Since we will be interested in the poset and the corresponding Alexandroff space of proper archimedean ℓ-ideals, we will utilize the following lemma.

Lemma 6.1. *Let X be an Alexandroff space and $f \in B(X)$.*

(1) *f is lower semicontinuous iff f is order preserving.*
(2) *f is upper semicontinuous iff f is order reversing.*

Proof. We only prove (1) as (2) is proved similarly. First suppose that f is order preserving. Let $s \in \mathbb{R}$. Since (s, ∞) is an upset in $\mathbb{R}$ and f is order preserving, $f^{-1}(s, \infty)$ is an upset in X. Therefore, $f^{-1}(s, \infty)$ is open in X. Thus, f is lower semicontinuous.

Conversely, suppose that f is lower semicontinuous. Let $x, y \in X$ with $x \leq y$. If $f(x) = s$, then for each $r < s$, we have $x \in f^{-1}(r, \infty)$, which is an open subset of X since f is lower semicontinuous. Therefore, $y \in f^{-1}(r, \infty)$. Thus, for each $r \in \mathbb{R}$ we have $r < f(x)$ implies $r < f(y)$. This forces $f(x) \leq f(y)$, and hence f is order preserving. $\square$

Remark 6.2. It is well known that a map between Alexandroff spaces is continuous iff it is order preserving. Therefore, $f \in B(X)$ is lower semicontinuous iff f is continuous with respect to the Alexandroff topology on $\mathbb{R}$, and f is upper semicontinous iff f is continuous with respect to the topology of downsets of $\mathbb{R}$.

The following definition is motivated by Dilworth [17].

Definition 6.3. Let X be a topological space and $f \in B(X)$. We call $f^{\#} := (f^*)_*$ the *normalization* of f and we call f *normal* if $f = f^{\#}$. Let

$$N(X) = \{f \in B(X) \mid f = f^{\#}\}.$$

Theorem 6.4. *$N(X) \in \boldsymbol{dba\ell}$ and the operations on $N(X)$ are normalizations of the corresponding operations on $B(X)$.*

Proof. It follows from [17] that $N(X)$ is a Dedekind complete lattice where bounded joins and meets are normalizations of pointwise bounded joins and meets. By [16], $N(X)$ is a lattice-ordered vector space, where addition and scalar multiplication are normalizations of pointwise addition and scalar multiplication. Finally, by [10, Sec. 8], $N(X) \in \boldsymbol{ba\ell}$, where multiplication is the normalization of pointwise multiplication. Thus, $N(X) \in \boldsymbol{dba\ell}$. We point out that in [17, 16] X is assumed to be completely regular and in [10] compact Hausdorff, but the same proofs work for an arbitrary topological space. $\square$

We next show that idempotents of $N(X)$ correspond to regular opens of X.

Lemma 6.5. *For a topological space X, the idempotents of $N(X)$ are precisely the characteristic functions χ_U for U a regular open subset of X. Consequently,* $\mathrm{Id}(N(X)) \cong \mathsf{RO}(X)$.

Proof. Let $e \in N(X)$ be an idempotent. Then $e = 2e \wedge 1$ since $(2e \wedge 1) - e = e \wedge (1 - e) = 0$. Since positive scalar multiplication and meet in $N(X)$ are pointwise (see, e.g., [16, Thm. 5.1]), the equation $e = 2e \wedge 1$ yields that $e(x) \in \{0, 1\}$ for each $x \in X$. Therefore, e is a characteristic function. If $e = \chi_U$ for $U \subseteq X$, then it is straightforward to see that $e^* = \chi_{\mathsf{cl}(U)}$ and $e_* = \chi_{\mathsf{int}(U)}$. Thus, $e^{\#} = \chi_{\mathsf{int}\,\mathsf{cl}(U)}$, and so $e = e^{\#}$ iff U is regular open in X. It is then straightforward to check that the map $U \mapsto \chi_U$ is an order preserving and order reflecting bijection, and hence a boolean isomorphism between $\mathsf{RO}(X)$ and $\mathrm{Id}(N(X))$. $\square$

For a poset X and $S \subseteq X$, we use the standard notation

$$\begin{aligned}{\uparrow}S &= \{x \in S \mid s \leq x \text{ for some } s \in S\}\\ {\downarrow}S &= \{x \in S \mid x \leq s \text{ for some } s \in S\}.\end{aligned}$$

If $S = \{x\}$ is a singleton, we write ${\uparrow}x$ for ${\uparrow}S$ and ${\downarrow}x$ for ${\downarrow}S$. The closure and interior operators of the Alexandroff topology on X are given by

$$\mathsf{cl}(S) = {\downarrow}S \quad\text{and}\quad \mathsf{int}(S) = \{x \in X \mid {\uparrow}x \subseteq S\}.$$

Lemma 6.6. *Let $A \in \boldsymbol{ba\ell}$, $X = \mathsf{Arch}(A) \setminus \{A\}$, and $I \in X$. Then ${\uparrow}I$ and $U_I := \{J \in X \mid J \vee I = A\}$ are regular open subsets of X, and U_I is the complement of ${\uparrow}I$ in $\mathsf{RO}(X)$.*

Proof. Since ${\uparrow}I$ is an upset, hence open in X, the inclusion ${\uparrow}I \subseteq \mathsf{int}\,\mathsf{cl}({\uparrow}I)$ is clear. For the reverse inclusion, suppose that $J \notin {\uparrow}I$. By Lemma A.5(2), there is $K \supseteq J$ with $K + I = A$. Therefore, $K \notin {\downarrow}{\uparrow}I$, so ${\uparrow}J \not\subseteq {\downarrow}{\uparrow}I = \mathsf{cl}({\uparrow}I)$, showing that $J \notin \mathsf{int}\,\mathsf{cl}({\uparrow}I)$. Thus, ${\uparrow}I \in \mathsf{RO}(X)$.

Since U_I is an upset, the inclusion $U_I \subseteq \mathsf{int}\,\mathsf{cl}(U_I)$ is clear. Suppose that $J \notin U_I$. Then $K := J \vee I \neq A$. If $K \subseteq L$ with $L \vee I = A$, then $L = A$ since $I \subseteq K \subseteq L$. Therefore, ${\uparrow}K \not\subseteq {\downarrow}U_I$, so ${\uparrow}J \not\subseteq \mathsf{cl}(U_I)$, and hence $J \notin \mathsf{int}\,\mathsf{cl}(U_I)$. This shows $U_I = \mathsf{int}\,\mathsf{cl}(U_I)$, so $U_I \in \mathsf{RO}(X)$.

Finally, to see that U_I is the complement of ${\uparrow}I$ in $\mathsf{RO}(X)$, it is clear that ${\uparrow}I \cap U_I = \varnothing$. Let $V \in \mathsf{RO}(X)$ with ${\uparrow}I \cap V = \varnothing$. If $J \in V$, then $I \not\subseteq J$. Therefore, by Lemma A.5(2), there is $K \in X$ with $J \subseteq K$ and $K + I = A$, so $K \vee I = A$ as $K + I \subseteq K \vee I$. Thus, $K \in U_I$, and hence $J \in {\downarrow}U_I$. This shows that $V \subseteq {\downarrow}U_I$, so $V \subseteq \mathsf{int}\,\mathsf{cl}(U_I) = U_I$. Consequently, U_I is the complement of ${\uparrow}I$ in $\mathsf{RO}(X)$. ☐

From now on we will assume that X is the set of proper archimedean ℓ-ideals of $A \in \boldsymbol{ba\ell}$ ordered by inclusion. The proof of the next theorem is choice-free.

Theorem 6.7. *There is a $\boldsymbol{ba\ell}$-isomorphism $\varphi : D(\mathbb{R}[B]) \to N(X)$ such that $\varphi(x_I) = \chi_{U_I}$ for each $I \in X$.*

Proof. We first define $\lambda : \mathsf{Arch}(A) \to \mathrm{Id}(N(X))$ by setting $\lambda(I) = \chi_{U_I}$. By Lemmas 6.5 and 6.6, $\chi_{U_I} \in \mathrm{Id}(N(X))$, so λ is well defined. We show that λ is a bounded lattice homomorphism. It is clear that $U_0 = \varnothing$ and $U_A = X$. We show that $U_{I\cap J} = U_I \cap U_J$. It is obvious that $I \subseteq J$ implies $U_I \subseteq U_J$. Therefore, $U_{I\cap J} \subseteq U_I \cap U_J$. For the reverse inclusion, suppose that $K \in U_I \cap U_J$. Then $K \vee I = K \vee J = A$, so $(K \vee I) \cap (K \vee J) = A$. Since $\mathsf{Arch}(A)$ is a frame, $K \vee (I \cap J) = A$, and so $K \in U_{I\cap J}$. We next show that $U_{I\vee J} = U_I \vee U_J$. The inclusion $U_I \vee U_J \subseteq U_{I\vee J}$ is obvious. For the reverse inclusion, suppose that $K \in U_{I\vee J}$. Then $K \vee (I \vee J) = A$. Let $L \in X$ with $K \subseteq L$. If $L \vee I = A$, then $L \in U_I$. If not, then as $(L \vee I) \vee J = A$, we have $L \vee I \in U_J$ so $L \in {\downarrow}U_J$. Therefore, in any case, $L \in {\downarrow}U_I \cup {\downarrow}U_J$, and so

$$K \in \mathsf{int}({\downarrow}U_I \cup {\downarrow}U_J) = \mathsf{int}({\downarrow}(U_I \cup U_J)) = \mathsf{int}\,\mathsf{cl}(U_I \cup U_J) = U_I \vee U_J.$$

Thus, λ is a bounded lattice homomorphism, and hence it extends to a BA-morphism $\tau : B \to \mathrm{Id}(N(X))$ (see, e.g., [1, Sec. V.4]). By Theorem 3.11(1), there is a $\boldsymbol{ba\ell}$-morphism $\sigma : \mathbb{R}[B] \to N(X)$ with $\sigma(x_I) = \tau(I) = \chi_{U_I}$.

To simplify notation, set $e_I = \chi_{\uparrow I}$ and $f_I = \chi_{U_I}$. Then e_I, f_I are complementary idempotents of $N(X)$ by Lemma 6.6. We show that σ is one-to-one. Let $a \in \mathbb{R}[B]$ with $\sigma(a) = 0$. By Remark 3.14, we may write $a = r_1 x_{b_1} + \cdots + r_n x_{b_n}$ for some $r_i \in \mathbb{R}$ and $b_i \in B$ with $b_i \wedge b_j = 0$ whenever $i \neq j$. From this we see that $ax_{b_i} = r_i x_{b_i} \in \ker(\sigma)$. Therefore, it suffices to show that $\sigma(rx_b) = 0$ implies $rx_b = 0$. If $r = 0$, this is clear, so suppose $r \neq 0$. Then $x_b \in \ker(\sigma)$. Since B is generated by $\mathsf{Arch}(A)$, we may write $b = (I_1 \wedge \neg J_1) \vee \cdots \vee (I_n \wedge \neg J_n)$ for some $I_k, J_k \in \mathsf{Arch}(A)$. Then $0 \leq x_{I_k \wedge \neg J_k} \leq x_b$, so each $x_{I_k \wedge \neg J_k} \in \ker(\sigma)$. Suppose that $\sigma(x_{I \wedge \neg J}) = 0$. We have

$$0 = \sigma(x_{I \wedge \neg J}) = \sigma(x_I) \wedge \sigma(x_{\neg J}) = \tau(I) \wedge \neg\tau(J) = f_I \wedge e_J,$$

where the last equality follows from Lemma 6.6. Therefore, $e_J \leq \neg f_I = e_I$, so $\uparrow J \subseteq \uparrow I$. This yields $I \subseteq J$, so $I \wedge \neg J = 0$ in B, and hence $x_{I \wedge \neg J} = 0$. This shows that σ is one-to-one.

We next show that each element of $N(X)$ is a join from $\sigma[\mathbb{R}[B]]$. To do this we first show that each nonnegative element of $N(X)$ is a join of scalar multiples of the $e_I = \chi_{\uparrow I}$. Let $0 \leq f \in N(X)$. We show that f is the join of those re_I for $r \in \mathbb{R}$ with $re_I \leq f$. Clearly f is above this join. Let $I \in X$ and set $r = f(I)$. Then $f(J) \geq r$ for each $J \in \uparrow I$ since f is order preserving. Because $0 \leq f$, this shows $re_I \leq f$. But $(re_I)(I) = r$. Therefore, $\bigvee\{re_I \mid r \in \mathbb{R}, re_I \leq f\} = f$, and so $\bigsqcup\{re_I \mid r \in \mathbb{R}, re_I \leq f\} = f$, where $\bigsqcup$ is the normalization of $\bigvee$ (see Theorem 6.4). For an arbitrary $f \in N(X)$, there is $s \in \mathbb{R}$ with $f + s \geq 0$. Thus, $f + s$ is a join from the image of $\mathbb{R}[B]$, and hence so is f by Remark 3.15(1). Consequently, by Theorem 3.7, there is a $\boldsymbol{ba\ell}$-isomorphism $\varphi : D(\mathbb{R}[B]) \to N(X)$ with $\varphi(x_I) = \sigma(x_I) = \chi_{U_I}$. $\square$

Theorem 6.8. *Assuming* (AC), *there is a* $\boldsymbol{ba\ell}$*-isomorphism* $\psi : N(X) \to B(Y_A)$ *such that* $\psi(f) = f|_{Y_A}$ *is the restriction of* f *to* Y_A.

Proof. To see that ψ is a $\boldsymbol{ba\ell}$-morphism, we first observe that if $f \in B(X)$ and $I \in X$, then since $\uparrow I$ is the least open neighborhood of I in X, we have

$$f^*(I) = \inf\{\sup f[U] \mid U \in \mathscr{N}_I\} = \sup\{f(J) \mid I \subseteq J\}.$$

This yields that $f^*(M) = f(M)$ for each $M \in Y_A$. A similar calculation gives $f_*(I) = \inf\{f(J) \mid I \subseteq J\}$. Therefore, since f^* is order reversing by Lemma 6.1(2),

$$\begin{aligned} f^\#(I) = (f^*)_*(I) &= \inf\{f^*(J) \mid I \subseteq J\} = \inf\{f^*(M) \mid M \in Y_A, I \subseteq M\} \\ &= \inf\{f(M) \mid M \in Y_A, I \subseteq M\}. \end{aligned}$$

Consequently, $f^\#|_{Y_A} = f|_{Y_A}$.

Denoting the sum in $N(X)$ by $\oplus$, we have for $f, g \in N(X)$

$$\psi(f \oplus g) = \psi((f+g)^\#) = (f+g)^\#|_{Y_A} = (f+g)|_{Y_A} = f|_{Y_A} + g|_{Y_A} = \psi(f) + \psi(g).$$

A similar calculation shows that ψ preserves the other operations. Thus, ψ is a $\boldsymbol{ba\ell}$-morphism.

We next show that ψ is onto. Let $h \in B(Y_A)$ and define h^u on X by

$$h^u(I) = \inf\{h(M) \mid M \in Y_A, I \subseteq M\}.$$

Then $h^u \in B(X)$ and

$$(h^u)^\#(I) = \inf\{h^u(M) \mid M \in Y_A, I \subseteq M\} = \inf\{h(M) \mid M \in Y_A, I \subseteq M\} = h^u(I).$$

This implies that $h^u \in N(X)$. By definition of h^u we have $\psi(h^u) = h^u|_{Y_A} = h$. Thus, ψ is onto.

Finally, we show that ψ is one-to-one. Let $f, g \in N(X)$ with $\psi(f) = \psi(g)$. Then $f|_{Y_A} = g|_{Y_A}$ and for each $I \in X$ we have

$$\begin{aligned} f(I) = f^\#(I) &= \inf\{f(M) \mid M \in Y_A, I \subseteq M\} = \inf\{f|_{Y_A}(M) \mid M \in Y_A, I \subseteq M\} \\ &= \inf\{g|_{Y_A}(M) \mid M \in Y_A, I \subseteq M\} = \inf\{g(M) \mid M \in Y_A, I \subseteq M\} = g^\#(I) = g(I), \end{aligned}$$

which yields $f = g$. Thus, ψ is one-to-one. $\square$

Recalling the isomorphism $\theta : D(\mathbb{R}[B]) \to B(Y_A)$ in the beginning of Section 5 (see Figure 1) and putting Theorems 6.7 and 6.8 together, we obtain:

Theorem 6.9. *Assuming* (AC), *for* $A \in \boldsymbol{ba\ell}$, *the algebras* $D(\mathbb{R}[B])$, $N(X)$, *and* $B(Y_A)$ *are all isomorphic. Moreover, if* $\gamma = \varphi \circ \alpha$, *then*

$$\gamma(a)(I) = \sup\{r \in \mathbb{R} \mid (a - r)^- \in I\}$$

and the following diagram commutes.

$$\begin{tikzcd}
 & D(\mathbb{R}[B]) \arrow[d, "\varphi"'] \arrow[dd, bend left, "\theta"] \\
A \arrow[ur, "\alpha"] \arrow[r, "\gamma"] \arrow[dr, "\zeta_A"'] & N(X) \arrow[d, "\psi"'] \\
 & B(Y_A)
\end{tikzcd}$$

Proof. The isomorphism $\theta : D(\mathbb{R}[B]) \to B(Y_A)$ satisfies $\theta(x_I) = \chi_{Z_\ell(I)^c}$ for each $I \in \mathsf{Arch}(A)$ and $\alpha = \theta^{-1} \circ \zeta_A$, where $Z_\ell(I)^c$ denotes the complement of $Z_\ell(I)$ in Y_A. We show that $\theta = \psi \circ \varphi$. Since all three maps are $\boldsymbol{dba\ell}$-isomorphisms and so preserve arbitrary joins, it is enough to show that they agree on $\mathbb{R}[B]$. For the latter it is enough to show that they agree on each x_I for $I \in X$. We have $\theta(x_I) = \chi_{Z_\ell(I)^c}$ and $\varphi(x_I) = \chi_{U_I}$. Since $\psi(\chi_{U_I}) = \chi_{U_I}|_{Y_A}$, we then need to show that $Z_\ell(I)^c = U_I \cap Y_A$. To see this, if $M \in Y_A$, then $M \in Z_\ell(I)^c$ iff $I \not\subseteq M$. Since M is maximal, $I \not\subseteq M$ iff $M + I = A$. But $M + I = A$ iff $M \vee I = A$ by Lemma A.2(4). Since $M \vee I = A$ iff $M \in U_I$, it follows that $Z_\ell(I)^c = U_I \cap Y_A$, which completes the proof that $\theta = \psi \circ \varphi$. Thus,

$$\psi \circ \gamma = \psi \circ \varphi \circ \alpha = \theta \circ \alpha = \zeta_A,$$

which shows that the diagram is commutative.

It is left to show that the formula for γ is valid. Suppose that $0 \le a \in A$. Since φ preserves arbitrary joins, by Lemmas 5.7(1), 6.6, and Theorem 6.7, we have

$$\begin{aligned}\gamma(a) = \varphi(\alpha(a)) &= \bigsqcup\{\varphi(rx_{\neg I}) \mid (a-r)^- \in I\} \\ &= \bigsqcup\{r\varphi(x_{\neg I}) \mid (a-r)^- \in I\} \\ &= \bigsqcup\{r\chi_{\uparrow I} \mid (a-r)^- \in I\}.\end{aligned}$$

Let $f \in B(X)$ be the pointwise join of $\{r\chi_{\uparrow I} \mid (a-r)^- \in I\}$. Then $\gamma(a) = f^\#$ by Theorem 6.4. We claim that $f(J) = \sup\{r \mid (a-r)^- \in J\}$ for each $J \in X$, and that $f \in N(X)$. We have

$$\begin{aligned}f(J) &= \sup\{r\chi_{\uparrow I}(J) \mid (a-r)^- \in I\} \\ &= \sup\{r \mid (a-r)^- \in I, J \in {\uparrow}I\} \\ &= \sup\{r \mid (a-r)^- \in J\}.\end{aligned}$$

To see the last equality, if $(a-r)^- \in I$ and $J \in {\uparrow}I$, then $(a-r)^- \in J$. Conversely, if $(a-r)^- \in J$, then setting $I = J$, we have $(a-r)^- \in I$ and $J \in {\uparrow}I$.

To show that $f \in N(X)$, by [17, Thm. 3.2] it is enough to show that f is lower semicontinuous and $f^{-1}(-\infty, s)$ is a union of regular closed sets for each $s \in \mathbb{R}$. First, f is clearly order preserving, so f is lower semicontinuous by Lemma 6.1(1). Next, Let $I \in f^{-1}(-\infty, s)$. Set $t = f(I) < s$, so $(a-t)^- \in I$ by Lemma A.7(1). In addition, $J := I \vee \langle (a-t)^+ \rangle \ne A$ by Lemma A.7(3). Let $U = {\uparrow}J$, an open subset of X. We claim that $I \in \mathsf{cl}(U) \subseteq f^{-1}(-\infty, s)$. Since $I \subseteq J$, we have $I \in {\downarrow}({\uparrow}J) = \mathsf{cl}(U)$. Because f is order preserving, $f^{-1}(-\infty, s)$ is a downset. Therefore, to show that $\mathsf{cl}(U) \subseteq f^{-1}(-\infty, s)$, it suffices to show that ${\uparrow}J \subseteq f^{-1}(-\infty, s)$. Let $K \in X$ with $J \subseteq K$. We have $(a-t)^- \in I \subseteq J$, so $(a-t)^- \in K$, and $(a-t)^+ \in J$, so $(a-t)^+ \in K$. Thus, $a-t \in K$ by Remark 3.15(5). Because $(a-t)^- \in K$, we have $t \le f(K)$. Let $f(K) = r$. Then $(a-r)^- \in K$ by Lemma A.7(1). If $r > t$, we have $\langle (a-t)^+, (a-r)^- \rangle = A$ by Lemma A.5(1), so $K = A$. This contradiction shows that $f(K) \le t$, so $f(K) < s$. Therefore, ${\uparrow}J \subseteq f^{-1}(-\infty, s)$, and hence $f^{-1}(-\infty, s)$ is a union of regular closed sets. Thus, we conclude that $f \in N(X)$.

Since $f \in N(X)$, we have $f^\# = f$, so $\gamma(a) = f^\# = f$. This shows that if $0 \le a$, then $\gamma(a)(I) = \sup\{r \mid (a-r)^- \in I\}$ for all $I \in X$. If a is arbitrary, then there is $n \ge 1$ with $a + n \ge 0$. Since γ preserves addition, by the above argument we have:

$$\begin{aligned}\gamma(a)(J) = (\gamma(a+n) - n)(J) &= \sup\{r \mid (a+n-r)^- \in J\} - n \\ &= \sup\{r - n \mid (a+n-r)^- \in J\} = \sup\{s \mid (a-s)^- \in J\},\end{aligned}$$

completing the proof. □

Consequently, we have three equivalent ways to think about canonical extensions of $\boldsymbol{ba\ell}$-algebras:

(1) The simplest is as $(B(Y_A), \zeta_A)$ which is a direct generalization of viewing the powerset of its Stone space as a canonical extension of a boolean algebra [32]. However, this requires (AC).

(2) A choice-free description is as $(N(X), \gamma)$, which generalizes the choice-free description of a canonical extension of a boolean algebra as regular opens of the Alexandroff space of its proper filters given in [13].

(3) Finally, a point-free description is as $(D(\mathbb{R}[B]), \alpha)$, which is technically the most challenging. It is this description that generalizes the point-free description of a canonical extension of a boolean algebra given in Section 2.

In point-free topology there is a description of normal functions on an arbitrary frame [24, 26, 27, 25]. We finish the article by the following remark, which connects our results to that line of research.

Remark 6.10. We recall that Frm is the category of frames and frame homomorphisms. If L, K are frames, then we write $\hom_{\mathsf{Frm}}(L, K)$ for the set of frame homomorphisms from L to K. Let L be a frame. It is well known that homomorphic images of L are characterized by nuclei on L (see, e.g., [35, Sec. III.5.3]). For a frame L we write $\mathrm{Nuc}(L)$ for the frame of nuclei on L. We also write $\mathscr{L}(\mathbb{R})$ for the frame of opens of $\mathbb{R}$. A point-free description of $\mathscr{L}(\mathbb{R})$ is due to Banaschewski [2] (see also [35, Sec. XIV.1]). The role of $C(X)$ is then played by the ℓ-algebra $\mathscr{C}(L) = \hom_{\mathsf{Frm}}(\mathscr{L}(\mathbb{R}), L)$.

As was shown in [24, Sec. 5] the role of the algebra of all real-valued functions on X is played by the ℓ-algebra $F(L) = \hom_{\mathsf{Frm}}(\mathscr{L}(\mathbb{R}), \mathrm{Nuc}(L))$, and that of $B(X)$ by the bounded subalgebra $F^*(L)$ of $F(L)$. Then the operators $(-)^*, (-)_* : B(X) \to B(X)$ generalize to $(-)^*, (-)_* : F^*(L) \to F^*(L)$ [25, Sec. 3], yielding the notion of normal function on L. We write $N(L) = \{f \in F^*(L) \mid (f^*)_* = f\}$ for the set of (bounded) normal functions on L. It follows from [16] and [10, Sec. 8] that $N(L) \in \boldsymbol{dba\ell}$.

Let $A \in \boldsymbol{ba\ell}$, $L = \mathsf{Arch}(A)$, and (B, i) be the free boolean extension of L. Then B is isomorphic to $\mathrm{Id}(N(L))$, yielding that $D(\mathbb{R}[B])$ is isomorphic to $N(L)$. Thus, our point-free description of a canonical extension of A can alternatively be described using the algebra of normal functions in point-free topology.

Appendix: Technical lemmas

In this appendix we present the technical lemmas used in Section 5 to prove that the pair $(D(\mathbb{R}[B]), \alpha)$ is a canonical extension of $A \in \boldsymbol{ba\ell}$. We start by recalling that $0 \le u \in A$ is a *weak order-unit* if $a \wedge u = 0$ implies $a = 0$ for each $a \in A$. It is well known that a strong order-unit is a weak order-unit (see, e.g., [14, Lem. XIII.11.4]). It is easy to see that any positive multiple of a strong order-unit is again a strong order-unit. Thus, every positive multiple of a strong order-unit is a weak order-unit. We will use this in the proof of next lemma.

Lemma A.1. *Let $A \in \boldsymbol{ba\ell}$.*

(1) *If $e, f \in \mathrm{Id}(A)$, and $0 \le r, s \in \mathbb{R}$, then $re \wedge sf = \min(r, s)(e \wedge f)$.*

(2) *Let $a \in A$ and $r, s \in \mathbb{R}$ with $r < s$. If $a \vee r \ge s$, then $a \ge s$.*

Proof. (1) Without loss of generality suppose that $r \le s$. Since $(e-(e\wedge f))\wedge f = (e-f)\wedge f = 0$, we have $r(e-(e\wedge f))\wedge sf = 0$ by Remark 3.15(9). Therefore,

$$0 \le (re \wedge sf) - r(e\wedge f) \le r(e-(e\wedge f))\wedge sf = 0$$

by Remark 3.15(1). Thus, $re\wedge sf = r(e\wedge f) = \min(r,s)(e\wedge f)$.

(2) If $a\vee r \ge s$, then $-s \ge -(a\vee r) = (-a)\wedge(-r)$, so $0 \ge s+[(-a)\wedge(-r)] = (s-a)\wedge(s-r)$. This yields

$$0 = [(s-a)\wedge(s-r)]\vee 0 = [(s-a)\vee 0]\wedge(s-r).$$

Because $s-r>0$, it is a weak order-unit. Therefore, $(s-a)\vee 0 = 0$, and so $s-a\le 0$. Thus, $s\le a$. $\square$

Lemma A.2. *Let $A \in \boldsymbol{ba\ell}$.*

1. *If $I+J=A$, then there are $a\in I$, $b\in J$ with $0\le a,b$ and $a+b=1$.*
2. *$\langle I\rangle = A$ implies $I=A$.*
3. *If $I,J\in \mathsf{Arch}(A)$, then $I\vee J = \langle I+J\rangle$.*
4. *If $I,J\in\mathsf{Arch}(A)$ and $I\vee J=A$, then $I+J=A$.*

Proof. (1) Since $I+J=A$, there are $x\in I$ and $y\in J$ with $1=x+y$. Since $y\in J$, we have $1+J = x+J \le |x|+J$. Set $a = 1\wedge|x|$. Then $0\le a\le 1$ and $a\in I$ because $x\in I$, so $|x|\in I$. Therefore, $a+J = (1+J)\wedge(|x|+J) = 1+J$. Thus, $b := 1-a\in J$. Clearly $a+b=1$ and $0\le b$ since $a\le 1$.

(2) Suppose $\langle I\rangle = A$. Then $1\in\langle I\rangle$, so $(n\cdot 1-1)^+\in I$ for each $n\ge 1$ by Proposition 4.8. In particular, $(2\cdot 1-1)^+\in I$. Thus, $1\in I$, and so $I=A$.

(3) Since $I+J\subseteq I\vee J$ and $I\vee J$ is archimedean, we have $\langle I+J\rangle\subseteq I\vee J$. On the other hand, $\langle I+J\rangle$ is an archimedean ideal which contains both I and J, so it contains $I\vee J$. Thus, $I\vee J = \langle I+J\rangle$.

(4) This follows from (2) and (3). $\square$

Let L be a frame and B its free boolean extension. We recall that for $a\in L$, we write a^* for the pseudocomplement of a in L. On the other hand, we write $\neg a$ for the complement of a in B.

Lemma A.3. *Let $A\in\boldsymbol{ba\ell}$ and B be the free boolean extension of $\mathsf{Arch}(A)$.*

1. *If $I\in\mathsf{Arch}(A)$, then $x_I = \bigvee\{x_{\neg J}\mid J\vee I = A\}$.*
2. *If $0\le f\in D(\mathbb{R}[B])$, then there are $0\le r_I\in\mathbb{R}$ with $f=\bigvee\{r_I x_{\neg I}\mid I\in\mathsf{Arch}(A)\}$.*
3. *Let $0\le f\in D(\mathbb{R}[B])$ and $0\le t\in\mathbb{R}$. If $f=\bigvee\{r_I x_{\neg I}\mid I\in\mathsf{Arch}(A)\}$ with $r_I\ge 0$, then $f+t = \bigvee\{(t+r_I)x_{\neg I}\mid I\in\mathsf{Arch}(A)\}$.*

Proof. (1) Since $\mathsf{Arch}(A)$ is a regular frame (see Theorem 4.4),

$$I = \bigvee\{K \mid K\prec I\} = \bigvee\{K\mid K^*\vee I = A\}.$$

We show that $I=\bigvee\{\neg J\mid J\vee I = A\}$. The right-to-left inclusion is clear. For the left-to-right inclusion it is sufficient to show that if $K^*\vee I = A$, then $K\le\neg J$ for some J with $J\vee I = A$. But if we set $J=K^*$, then $K\subseteq K^{**} = J^*\le\neg J$. Thus, $I=\bigvee\{\neg J\mid J\vee I=A\}$, and hence $x_I = \bigvee\{x_{\neg J}\mid J\vee I = A\}$.

(2) Each element of $D(\mathbb{R}[B])$ is a join from $\mathbb{R}[B]$. A nonnegative element of $\mathbb{R}[B]$ can be written in the form $r_1x_{b_1} + \cdots + r_nx_{b_n} = r_1x_{b_1} \vee \cdots \vee r_nx_{b_n}$ for some $0 \le r_i \in \mathbb{R}$ and $b_1, \ldots, b_n \in B$ with $b_i \wedge b_j = 0$ whenever $i \neq j$ (see Remark 3.14). Since each $b \in B$ is a finite join of elements of the form $J \wedge \neg I$ with $I, J \in \mathsf{Arch}(A)$, we may write a nonnegative element of $\mathbb{R}[B]$ as a join of elements of the form $r(x_J \wedge x_{\neg I})$. Thus, by (1), if $0 \le f \in D(\mathbb{R}[B])$, we may write f as a join of elements of the form $rx_{\neg I}$ with $I \in \mathsf{Arch}(A)$.

(3) By Remark 3.14(2,3), $x_I + x_{\neg I} = x_I \vee x_{\neg I} = 1$. Therefore, by Remark 3.15(1),

$$f + t = \bigvee\{t + r_Ix_{\neg I} \mid I \in \mathsf{Arch}(A)\} = \bigvee\{(t + r_I)x_{\neg I} + tx_I \mid I \in \mathsf{Arch}(A)\}.$$

Because $x_{\neg I} \wedge x_I = 0$, Remark 3.15(9) implies $(t + r_I)x_{\neg I} \wedge tx_I = 0$, so $(t + r_I)x_{\neg I} + tx_I = (t + r_I)x_{\neg I} \vee tx_I$. Thus, by (1),

$$\begin{aligned} f + t &= \bigvee\{(t + r_I)x_{\neg I} \vee tx_I \mid I \in \mathsf{Arch}(A)\} \\ &= \bigvee\{(t + r_I)x_{\neg I} \vee tx_{\neg J} \mid I, J \in \mathsf{Arch}(A), I \vee J = A\}. \end{aligned}$$

Now, $t \le t+r_J$, so $tx_{\neg J} \le (t+r_J)x_{\neg J}$. Consequently, $f+t = \bigvee\{(t+r_I)x_{\neg I} \mid I \in \mathsf{Arch}(A)\}$. □

Remark A.4. Let $A \in \boldsymbol{ba\ell}$. For $S \subseteq A$ we let $[S]$ be the ℓ-ideal of A generated by S. It is well known (see, e.g., [33, p. 96]) that

$$[S] = \{x \in A : |x| \le n|a| \text{ for some } n \ge 1, a \in S\}.$$

If $S = \{a\}$, we write $[a]$ for $[S]$.

Lemma A.5. *Let $A \in \boldsymbol{ba\ell}$ and let $X = \mathsf{Arch}(A) \setminus \{A\}$.*

1. *If $a, b \in A$ with $a < b$ and $b - a \in \mathbb{R}$, then $\langle b^+, a^-\rangle = A$.*
2. *If $I, J \in X$ with $I \not\subseteq J$, then there is $K \in X$ with $J \subseteq K$ and $K + I = A$.*

Proof. (1) Set $I = \langle b^+, a^-\rangle$. Since $0 \le a^+ \le b^+$, we have $a^+ \in I$, so $a = a^+ - a^- \in I$. Also, as $0 \le b^- \le a^-$, we have $b^- \in I$, and so $b \in I$. Thus, $b - a \in I$, and since $b - a$ is a nonzero real number, it is a unit in A, and hence $I = A$.

(2) Since $I \not\subseteq J$ there is $a \in I$ with $a \notin J$. Because J is archimedean, by Proposition 4.8, there is $n \ge 1$ with $(n|a| - 1)^+ \notin J$. Let $K = J \vee \langle (n|a| - 1)^-\rangle$. Then $J \subseteq K$, and $I \vee K = A$ by (1). We show that $K \in X$. Otherwise $1 = x + y$ with $0 \le x, y$, $x \in J$, and $y \in \langle (n|a| - 1)^-\rangle$. We claim that $y(n|a| - 1)^+ = 0$. To see this, we set $b = n|a| - 1$. Since $y \in \langle b^-\rangle$, we have $(y - 1/p)^+ \in [b^-]$ for each $p \ge 1$ by Proposition 4.8. Therefore, by Remark A.4, for each p there is m with $(y - 1/p)^+ \le mb^-$. Thus, $0 \le (y - 1/p)^+b^+ \le mb^-b^+ = 0$ by Remark 3.15(7), and so $(y - 1/p)^+b^+ = 0$. Because $y - 1/p \le (y - 1/p)^+$, we have $(y - 1/p)b^+ \le (y - 1/p)^+b^+ = 0$, so $yb^+ \le (1/p)b^+$, which yields $pyb^+ \le b^+$. Since this is true for all $p \ge 1$, it follows that $yb^+ = y(|n|a - 1)^+ = 0$ as A is archimedean. This verifies the claim. Therefore, $(n|a| - 1)^+ = (n|a| - 1)^+(x + y) = (n|a| - 1)^+x$, and so $(n|a| - 1)^+ \in J$, which is a contradiction. Thus, $K \in X$. □

Lemma A.6. *Let $A \in \boldsymbol{ba\ell}$, B be the free boolean extension of $\mathsf{Arch}(A)$, $X = \mathsf{Arch}(A) \setminus \{A\}$, $I \in X$, $0 \le f, g \in D(\mathbb{R}[B])$, and $0 \le t \in \mathbb{R}$. Suppose that whenever $tx_{\neg I} \le f$, there is $K \in X$ with $I \subseteq K$ and $tx_{\neg K} \le g$. Then $f \le g$.*

Proof. To show $f \le g$, by Lemma A.3(2) we need to show that $tx_{\neg I} \le f$ implies $tx_{\neg I} \le g$. Given $tx_{\neg I} \le f$, there is $K \supseteq I$ with $tx_{\neg K} \le g$. If $K = I$, then we are done. Suppose $I \subset K$. For each $J \supseteq I$ with $J \vee K = A$ we have $tx_{\neg J} \le tx_{\neg I} \le f$, so there is $J' \supseteq J$ with $tx_{\neg J'} \le g$. We have $J' \vee K = A$ since $J' \supseteq J$. We claim that

$$I = K \cap \bigcap \{J' \in X \mid J' \supseteq J, J' \vee K = A\}.$$

The inclusion $I \subseteq K \cap \bigcap\{J' \in X \mid J' \supseteq J, J' \vee K = A\}$ is clear since $I \subseteq J \subseteq J'$. For the reverse inclusion, let $a \in K \setminus I$. Then $|a| \in K \setminus I$, so we assume $0 \le a$. Since I is archimedean, there is $n \ge 1$ with $(a - 1/n)^+ \notin I$. We show that $I \vee \langle (a-1/n)^- \rangle \ne A$. For, if $I \vee \langle (a-1/n)^- \rangle = A$, then $I + [(a-1/n)^-] = A$ by Lemma A.2. Therefore, by Lemma A.2(1), there are $0 \le x, y$ with $x \in I$, $y \in [(a-1/n)^-]$, and $x + y = 1$. Thus, by Remark A.4, we have $y \le m(a-1/n)^-$ for some $m \ge 1$, and hence $y(a-1/n)^+ = 0$ by Remark 3.15(7). Consequently, $(a-1/n)^+ = (a-1/n)^+ x \in I$, a contradiction. Set $J = I \vee \langle (a-1/n)^- \rangle$. Because $\langle a, (a-1/n)^- \rangle = A$ by Lemma A.5(1), we have $J \vee K = A$ since $a \in K$, and $a \notin J'$ because J' is proper. Therefore, a is not in the intersection. Thus, $I = K \cap \bigcap\{J' \in X \mid J' \supseteq J, J' \vee K = A\}$ as desired. From this we obtain that in B we have $\neg I = \neg K \vee \bigvee\{\neg J' \mid J' \supseteq J, J' \vee K = A\}$, and so

$$tx_{\neg I} = tx_{\neg K} \vee \bigvee\{tx_{\neg J'} \mid J' \supseteq J, J' \vee K = A\} \le g.$$

□

We arrive at our final auxiliary lemma, item (2) of which has the most involved proof.

Lemma A.7. *Let $A \in \boldsymbol{ba\ell}$, $X = \mathsf{Arch}(A) \setminus \{A\}$, $0 \le a \in A$, and $I \in X$.*

1. *If $s_I = \sup\{r \mid (a-r)^- \in I\}$, then $(a - s_I)^- \in I$.*
2. *$rx_{\neg I} \le \alpha(a)$ iff $(a-r)^- \in I$.*
3. *$s_I = \sup\{r \mid rx_{\neg I} \le \alpha(a)\}$ and $I \vee \langle (a - s_I)^+ \rangle \ne A$.*

Proof. (1) If $(a-r)^- \in I$, then $a + I \ge r + I$ in A/I by Remark 4.7(2). We use this to show that $(a - s_I)^- \in I$. For each $n \ge 1$ there is r with $(a-r)^- \in I$ and $s_I - 1/n \le r$. Therefore, $(s_I - 1/n) + I \le r + I \le a + I$, and so $(s_I - a) + I \le 1/n + I$. Since this is true for all n, we have $(s_I - a) + I \le 0 + I$ as A/I is archimedean. Thus, $s_I + I \le a + I$. Applying Remark 4.7(2) again yields $(a - s_I)^- \in I$.

(2) If $(a-r)^- \in I$, then $rx_{\neg I} \le \alpha(a)$ by Lemma 5.7(1). Conversely, suppose that $rx_{\neg I} \le \alpha(a)$. The result is clear if $r \le 0$ since then $(a-r)^- = 0 \in I$, so assume $r > 0$. By (1) and Lemma 5.7(1), we may write $\alpha(a) = \bigvee\{s_J x_{\neg J} \mid J \in \mathsf{Arch}(A)\}$, where s_J is given in (1). To show $(a-r)^- \in I$ we then need to show $r \le s_I$. We have $rx_{\neg I} \le \bigvee\{s_J x_{\neg J} \mid J \in \mathsf{Arch}(A)\}$ and $rx_{\neg I} \le r$. Therefore,

$$\begin{aligned} rx_{\neg I} &\le \bigvee\{s_J x_{\neg J} \mid J \in \mathsf{Arch}(A)\} \wedge r \\ &= \bigvee\{s_J x_{\neg J} \wedge r \mid J \in \mathsf{Arch}(A)\} \\ &= \bigvee\{\min(s_J, r) x_{\neg J} \mid J \in \mathsf{Arch}(A)\} \end{aligned}$$

by Remark 3.15(2) and Lemma A.1(1). To simplify notation set $t_J = \min(s_J, r)$. Then $(a - t_J)^- \le (a - s_J)^-$, so $(a - t_J)^- \in J$. From this and Remark 3.14(3) we get

$$r = rx_I \vee rx_{\neg I} \le rx_I \vee \bigvee\{t_J x_{\neg J} \mid J \in \mathsf{Arch}(A)\} \le r$$

since the join is bounded by r, so equality holds throughout. Fix $\varepsilon > 0$. Then

$$\begin{aligned} r &= \Big(rx_I \vee \bigvee\{t_J x_{\neg J} \mid J \in \mathsf{Arch}(A)\}\Big) \vee (r - \varepsilon) \\ &= rx_I \vee \bigvee\{t_J x_{\neg J} \vee (r-\varepsilon) \mid J \in \mathsf{Arch}(A)\}. \end{aligned}$$

If $t_J \le r - \varepsilon$, then $t_J x_{\neg J} \le r - \varepsilon$. Therefore,

$$\begin{aligned} r &= rx_I \vee \bigvee\{t_J x_{\neg J} \vee (r-\varepsilon) \mid J \in \mathsf{Arch}(A)\} \\ &= \Big(rx_I \vee \bigvee\{t_J x_{\neg J} \mid r - \varepsilon < t_J\}\Big) \vee (r - \varepsilon). \end{aligned}$$

Thus, $rx_I \vee \bigvee\{t_J x_{\neg J} \mid r - \varepsilon < t_J\} = r$ by Lemma A.1(2). Multiplying both sides by r^{-1} yields $x_I \vee \bigvee\{r^{-1} t_J x_{\neg J} \mid r - \varepsilon < t_J\} = 1$. Consequently, $x_{\neg I} \le \bigvee\{r^{-1} t_J x_{\neg J} \mid r - \varepsilon < t_J\}$ by Remark 3.13(2), and hence $rx_{\neg I} \le \{t_J x_{\neg J} \mid r - \varepsilon < t_J\}$. Let $S = \{J \in \mathsf{Arch}(A) \mid r - \varepsilon < t_J\}$. We then have $rx_{\neg I} \le \bigvee\{t_J x_{\neg J} \mid J \in S\}$ and $(a - (r-\varepsilon))^- \in J$ for each $J \in S$ since $(a - (r - \varepsilon))^- \le (a - t_J)^-$ and $(a - t_J)^- \in J$. Because $t_J \le r$ for each J by definition and $r > 0$, we have $rx_{\neg I} \le \bigvee\{rx_{\neg J} \mid J \in S\}$, so $x_{\neg I} \le \bigvee\{x_{\neg J} \mid J \in S\}$. Therefore, $\neg I \le \bigvee\{\neg J \mid J \in S\}$ in B. Since B is a boolean algebra, $\bigwedge_B S \le I$. Because $\mathsf{Arch}(A)$ is a sublattice of B, we have $\bigwedge_{\mathsf{Arch}(A)} S \le \bigwedge_B S$. But $\bigwedge_{\mathsf{Arch}(A)} S = \bigcap S$, so $\bigcap S \subseteq I$. As $(a - (r - \varepsilon))^- \in J$ for each $J \in S$, we see that $(a - (r - \varepsilon))^- \in I$. Since this is true for all ε, we have $a + I \ge (r - \varepsilon) + I$ for each ε, so $a + I \ge r + I$ because I is archimedean. Thus, $(a - r)^- \in I$.

(3) We write $s = s_I$ for convenience. The first part of the statement follows from (1) and (2). Suppose that $I \vee \langle (a-s)^+ \rangle = A$. Then $I + [(a-s)^+] = A$ by Lemma A.2. By Lemma A.2(1) and Remark A.4, there are $0 \le x, y$ with $x \in I$, $y \le n(a-s)^+$ for some n, and $x + y = 1$. Then $1/n - y/n = x/n \in I$ and $1/n - y/n \ge 1/n - (a-s)^+$. Therefore, $1/n - y/n \ge (1/n - (a-s)^+) \vee 0$, so $(1/n - (a-s)^+)^+ \in I$. Using items (3), (1), (2), and (4) of Remark 3.15, we have

$$\begin{aligned} \big(1/n - (a-s)^+\big)^+ &= (1/n - ((a-s) \vee 0))^+ = (1/n + ((s-a) \wedge 0))^+ \\ &= ((s + 1/n - a) \wedge 1/n) \vee 0 \\ &= ((s + 1/n - a) \vee 0) \wedge 1/n \\ &= (a - (s + 1/n))^- \wedge 1/n. \end{aligned}$$

Thus, $(a - (s + 1/n))^- \wedge 1/n \in I$. Let $m \ge 1$ be such that $(a - (s+1/n))^- \le m$. Applying Remark 3.15(8) yields

$$\begin{aligned} (a - (s + 1/n))^- &\le mn(a - (s+1/n))^- \wedge m \\ &= mn[(a - (s+1/n))^- \wedge 1/n] \in I. \end{aligned}$$

Therefore, $(a - (s + 1/n))^- \in I$, so $(s + 1/n)x_{\neg I} \le \alpha(a)$ by Lemma 5.7(1). This is a contradiction to the definition of $s = s_I$. Thus, $I \vee \langle (a - s_I)^+ \rangle \ne A$. $\square$

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