

Polymatroids, closure operators and lattices

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Abstract

We study the closure operators of polymatroids from a lattice theoretic point of view. We show that polymatroid closure operators relate to lattices enriched with a generating set in the same way that matroids relate to geometric lattices. Through this relation we define a notion of minors for lattices enriched with a generating set. For the lattice of flats of a graphic matroid, the minors of the lattice are shown to correspond to simple minors of the graph when the vertices are labeled and the edges unlabeled. This correspondence is generalized to all polymatroids.

1 Introduction

Polymatroids were introduced by Edmonds in [4] as a generalization of matroids in connection with optimization theory. Edmonds introduced polymatroids as certain polytopes lying in the nonnegative orthant of the real vector space spanned by a ground set. Vectors in the nonnegative orthant can be viewed as nonnegative real weightings of the ground set, subsets of the ground set corresponding to 0,1 vectors. Conceptually the points in a polymatroid encapsulate independent weightings. Given a matroid M there is an associated polymatroid which is the convex hull of the 0,1 vectors which correspond to independent sets of M . In this sense polymatroids are a generalization of matroids. Edmonds also gave a description of polymatroids in terms of a rank function, see [4, Theorem 14]. We will prefer this rank function definition given below. For a set E we let B_E denote the poset of all subsets of E ordered by inclusion.

Definition 1.1. A polymatroid on the ground set E is a function $r : B_E \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following conditions for all $X, Y \subseteq E$.

$$r(\emptyset) = 0, \tag{1.1}$$

$$\text{If } X \subseteq Y \text{ then } r(X) \leq r(Y), \tag{1.2}$$

$$r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y). \tag{1.3}$$

Condition 1.3 is referred to as *submodularity* of the function r . When the polymatroid r is integer valued, and for all $X \subseteq E$ satisfies $r(X) \leq |X|$ then it is (the rank function of) a matroid.

Many notions from matroid theory carry over directly, or nearly so, when stated in terms of the rank function. Given a polymatroid r on E we say that an element $e \in E$ is a *loop* with respect to r when $r(\{e\}) = 0$. Two elements $e, f \in E$ are said to be *parallel* with respect to r when $r(\{e, f\}) = r(\{e\}) = r(\{f\})$. The *parallel class* of e with respect to r is the collection of elements in E which are parallel to e . A polymatroid is *simple* if it has no loops and all parallel classes are trivial.

Our main interest in the present work is the closure operator of polymatroids. The *closure operator* of a polymatroid r on E is the map $\overline{\cdot} : B_E \rightarrow B_E$ defined for $X \subseteq E$ by

$$\overline{X} = \{e \in E : r(X) = r(X \cup \{e\})\}.$$

The submodularity of r implies that $r(\overline{X}) = r(X)$ for any $X \subseteq E$. Sets of the form \overline{X} are referred to as *r -closed sets* or *flats* of r . Edmonds showed that the set of flats of a polymatroid is closed under intersection ([4, Theorem 25]). Since the set of flats of a polymatroid is finite and has a maximal element, namely E , this implies that the set of flats ordered under inclusion forms a lattice. The meet in this lattice is intersection and the join is given by $X \vee Y = \overline{X \cup Y}$.

Given a matroid r the closure operator uniquely determines r . For polymatroids this is not the case as the closure operator has no information about how much the rank of sets may differ. For example, define a polymatroid on the ground set $E = \{1, 2\}$ by assigning rank value 1 to $\{1\}$ and to $\{2\}$ and assigning any rank value in the interval $(1, 2]$ to the set $\{1, 2\}$. The resulting closure operator is the identity map on B_E regardless of the choice of the rank of $\{1, 2\}$.

2 Polymatroids and generator enriched lattices

We now introduce an object which will be seen to correspond to polymatroid closure operators in the same way geometric lattices correspond to matroids.

Definition 2.1. A generator enriched lattice is a pair (L, G) in which L is a finite lattice and $G \subseteq L \setminus \{\widehat{0}\}$ generates the lattice L via the join operation.

Note that if (L, G) is a generator enriched lattice, the set G necessarily contains the set of join irreducibles of L which we will denote as $\text{irr}(L)$. A generator enriched lattice of the form $(L, \text{irr}(L))$ will be said to be *minimally generated*.

A lattice is typically depicted via its Hasse diagram. The Hasse diagram is not enough information to specify a generator enriched lattice since it does not describe the generating set. Instead a generator enriched lattice may be depicted via a diagram analogous to Cayley graphs for groups with a generating set. Given a generator enriched lattice (L, G) the associated diagram has vertex set L , and directed edges $(\ell, \ell \vee g)$ for $\ell \in L$ and $g \in G$ such that $\ell \neq \ell \vee g$. Just as with Hasse diagrams all diagrams of generator enriched lattices will be depicted so that the edges are directed upwards. The diagram of a generator enriched lattice determines the underlying lattice: the order relation $\ell_1 \leq \ell_2$ holds when there is a directed path from ℓ_1 to ℓ_2 in the diagram. The minimal element $\widehat{0}$ is the unique source vertex. The generating set consists of the elements adjacent to $\widehat{0}$. See Figure 1 for examples of diagrams of generator enriched lattices.

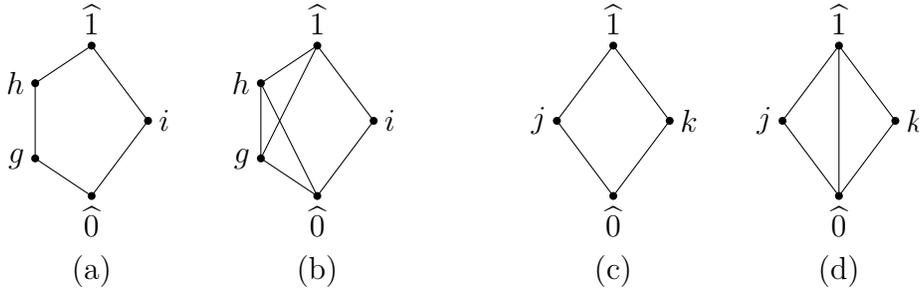


Figure 1: In (a) is the Hasse diagram of a lattice L with $\text{irr}(L) = \{g, h, i\}$, and in (b) is the diagram of the associated minimally generated lattice $(L, \text{irr}(L))$. In (c) is the Hasse diagram of the Boolean algebra B_2 , which is also the diagram of the minimally generated lattice $(B_2, \{j, k\})$, and in (d) is the diagram of the generator enriched lattice $(B_2, \{j, k, \widehat{1}\})$.

For every polymatroid we have an associated generator enriched lattice.

Definition 2.2. Given a polymatroid $r : E \rightarrow \mathbb{R}_{\geq 0}$ the generator enriched lattice of flats is the generator enriched lattice (L, G) where

$$L = \{\overline{X} : X \subseteq E\},$$

$$G = \{\overline{\{e\}} : e \in E, r(\{e\}) \neq 0\}.$$

See Figure 2 for examples of polymatroids and the associated generator enriched lattice of flats.

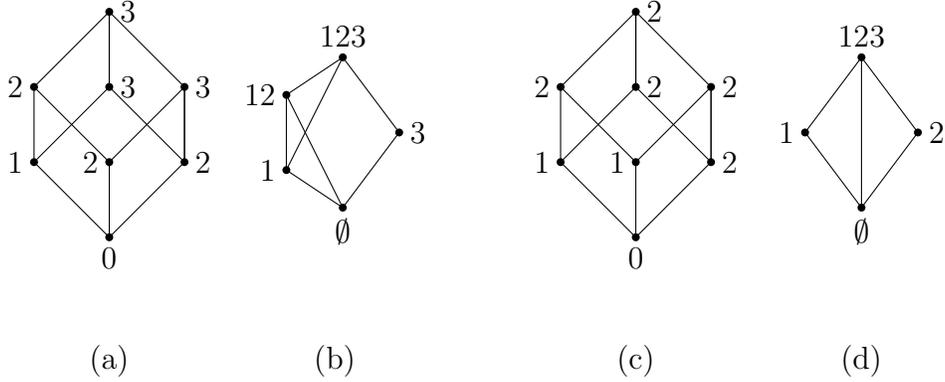


Figure 2: In (a) and (c) are polymatroids, and in (b) and (d) respectively are the diagrams of the generator enriched lattice of flats.

Let r and s be two simple matroids with lattice of flats L and K respectively. A *strong map* between r and s is a map $f : L \rightarrow K$ which is join-preserving and satisfies $f(\text{irr}(L)) \subseteq \text{irr}(K) \cup \{\widehat{0}_K\}$. Strong maps between simple matroids were introduced by Higgs in [6]. The notion of structure preserving maps between generator enriched lattices defined below generalizes strong maps between simple matroids.

Definition 2.3. *Let (L, G) and (K, H) be generator enriched lattices. A strong map from (L, G) to (K, H) is a map $f : L \rightarrow K$ which is join-preserving and satisfies $f(G) \subseteq H \cup \{\widehat{0}_K\}$. This will be abbreviated by saying that $f : (L, G) \rightarrow (K, H)$ is a strong map.*

A strong map $f : (L, G) \rightarrow (K, H)$ is said to be *injective* when it is injective as a map on the underlying lattices, and *surjective* when $f(G \cup \{\widehat{0}_L\}) = H \cup \{\widehat{0}_K\}$. Two generator enriched lattices are said to be isomorphic when there is a strong bijection between them.

Strong maps between matroids may be equivalently defined in several ways, for instance as join preserving maps which also preserve the relation “covers or equals”; see [1, Proposition 2]. This definition does not extend to the setting of generator enriched lattices, for example mapping atoms of a Boolean algebra to any elements of a chain will induce a strong map which need not preserve covers.

Let (L, G) be a generator enriched lattice and let E be a ground set. Let \mathcal{B}_E denote the generator enriched lattice $(B_E, \text{irr}(B_E))$. Given any map $f : E \rightarrow G \cup \{\widehat{0}_L\}$ we have an associated strong map $F : \mathcal{B}_E \rightarrow (L, G)$ defined by

$$F(X) = \bigvee_{x \in X} f(x),$$

for $X \subseteq E$. We refer to the map F as the *strong map induced by f* .

A certain nonstandard definition of matroids is useful for our lattice theoretic view of polymatroids. A matroid on a ground set E may be defined as a strong surjection f from the

Boolean algebra \mathcal{B}_E onto a generator enriched lattice of the form $(L, \text{irr}(L))$ for some geometric lattice L . In fact if one requires the map f to be strong in the sense of [1], the image is necessarily geometric; see [2, Proposition 9.12]. This view of matroids is briefly mentioned in [2, pp. 9.8-9.9]. Accordingly, we now turn our focus to strong surjections from Boolean algebras onto generator enriched lattices, and showing such maps are in bijection with polymatroid closure operators (when the codomain generator enriched lattice is considered up to isomorphism).

2.1 Strong surjections and closure operators

The following construction associates a closure operator to any strong surjection from \mathcal{B}_E onto a generator enriched lattice (L, G) . This is a standard construction in the theory of Galois connections. Let $\theta : \mathcal{B}_E \rightarrow (L, G)$ be a strong surjection. Define a right-sided inverse ϕ to θ by

$$\phi(\ell) = \bigcup_{X \in \theta^{-1}(\ell)} X.$$

The fact that $\theta \circ \phi$ is the identity follows directly from the fact that θ is join-preserving. We define the closure operator associated to θ to be the map $\text{cl}_\theta = \phi \circ \theta : B_E \rightarrow B_E$. One may associate a generator enriched lattice (K, H) to such a closure operator by setting

$$K = \{\text{cl}_\theta(X) : X \subseteq E\},$$

and

$$H = \{\text{cl}_\theta(\{e\}) : e \in E\} \setminus \{\text{cl}_\theta(\emptyset)\}.$$

This generator enriched lattice (K, H) is isomorphic to (L, G) via the isomorphism $\phi : (L, G) \rightarrow (K, H)$, which has inverse $\phi^{-1} = \theta|_K$.

The following result says that a polymatroid can be equivalently defined as a strong surjection from a Boolean algebra together with a strictly order-preserving and submodular function with nonnegative real values.

Proposition 2.4. *Let (L, G) be a generator enriched lattice and let $\theta : \mathcal{B}_E \rightarrow (L, G)$ be a strong surjection. For any strictly order-preserving and submodular function $r : L \rightarrow \mathbb{R}_{\geq 0}$ which maps $\widehat{0}_L$ to 0, the composition $r \circ \theta : B_E \rightarrow \mathbb{R}_{\geq 0}$ is a polymatroid whose generator enriched lattice of flats is isomorphic to (L, G) . Furthermore the polymatroid is simple if and only if $\theta|_{\text{irr}(B_E) \cup \{\emptyset\}}$ is injective.*

Conversely given a polymatroid $s : B_E \rightarrow \mathbb{R}_{\geq 0}$ with generator enriched lattice of flats (L, G) , let $\theta : \mathcal{B}_E \rightarrow (L, G)$ be the strong map induced by the map $e \mapsto \overline{\{e\}}$. There is a strictly order-preserving and submodular function $r : L \rightarrow \mathbb{R}_{\geq 0}$ such that $s = r \circ \theta$.

Proof. Let s be the composition $r \circ \theta$. By assumption $s(\emptyset) = r(\widehat{0}) = 0$. The maps θ and r are order-preserving, hence s must be as well. To show that s is submodular, let X and Y be subsets of E . Since θ is join-preserving we have $s(X \cup Y) = r(\theta(X) \vee \theta(Y))$. On the other hand since θ is order-preserving, the image $\theta(X \cap Y)$ is a lower bound for both $\theta(X)$ and $\theta(Y)$, hence $\theta(X \cap Y) \leq \theta(X) \wedge \theta(Y)$. Thus $s(X \cap Y) \leq r(\theta(X) \wedge \theta(Y))$. Summing these two values results in the inequality

$$s(X \cap Y) + s(X \cup Y) \leq r(\theta(X) \wedge \theta(Y)) + r(\theta(X) \vee \theta(Y)).$$

Applying the submodularity of the function r leads to the inequality

$$s(X \cap Y) + s(X \cup Y) \leq r(\theta(X)) + r(\theta(Y)) = s(X) + s(Y).$$

Therefore the function s is a polymatroid.

To show that the generator enriched lattice of flats of s is isomorphic to (L, G) , it will suffice to show that the closure operator cl_θ is the closure operator of s . The closure of two sets X and Y with respect to s is the same if and only if $s(X) = s(X \cup Y) = s(Y)$. Since r is strictly order-preserving, this holds if and only if $\theta(X) = \theta(Y)$, which holds if and only if $\text{cl}_\theta(X) = \text{cl}_\theta(Y)$. By the same argument we see that s has a loop or a nontrivial parallel class precisely when $\theta|_{\text{irr}(B_E) \cup \{\emptyset\}}$ is not injective.

To prove the converse, consider a polymatroid $s : B_E \rightarrow \mathbb{R}_{\geq 0}$ with generator enriched lattice of flats (L, G) . Let $\theta : \mathcal{B}_E \rightarrow (L, G)$ be the strong map induced by the map $e \mapsto \overline{\{e\}}$, and let $r = s|_L$. If $A \subsetneq B \in L$ are flats then $s(A) < s(B)$ so r is strictly order-preserving on L . Since $A \vee B = \overline{A \cup B}$ we have $s(A \vee B) = s(A \cup B)$. Therefore we have that $r(A \wedge B) + r(A \vee B) = s(A \cap B) + s(A \cup B)$, which by submodularity of s is less than or equal to $s(A) + s(B)$. This of course equals $r(A) + r(B)$ so the function r is submodular. \square

Lemma 2.5. *For any lattice L there exists a strictly order preserving submodular function $r : L \rightarrow \mathbb{Z}_{\geq 0}$ with $r(\widehat{0}) = 0$.*

Proof. It will suffice to construct such a function with values in $\mathbb{Q}_{\geq 0}$. Afterwards one can scale by a sufficiently large positive integer to clear denominators. Define a function $r : L \rightarrow \mathbb{Q}_{\geq 0}$ by, for $\ell \in L$ such that the largest chain in L from $\widehat{0}$ to ℓ is length k , setting $r(\ell) = 1 - 2^{-k}$. The map r is strictly order-preserving and maps $\widehat{0}$ to 0. To show r satisfies the submodularity condition, let $x, y \in L$. It may be assumed that $x \wedge y$ is neither x nor y , otherwise the submodularity inequality holds trivially for x and y . Let $r(x) = 1 - 2^{-n}$ and $r(y) = 1 - 2^{-m}$. It may also be assumed that $n \leq m$. Observe that $r(x \wedge y) \leq 1 - 2^{-n+1}$ and $r(x \vee y) \leq 1$. Adding these terms gives,

$$r(x \wedge y) + r(x \vee y) \leq 2 - 2^{-n+1} \leq 2 - 2^{-n} - 2^{-m} = r(x) + r(y).$$

Thus r is submodular and can be used to construct the desired function. \square

It is known that every lattice is isomorphic to the lattice of flats of some polymatroid that is integer valued. This result is attributed to Dilworth in [7, pp. 26] and follows from Dilworth's embedding theorem [3, Theorem 14.1], which states that any finite lattice can be embedded into a geometric lattice. Below is a somewhat stronger result.

Proposition 2.6. *Every generator enriched lattice is isomorphic to the generator enriched lattice of flats of some polymatroid, which may be chosen to have integer values.*

Proof. Let (L, G) be a generator enriched lattice. By Lemma 2.5 there is an integer valued strictly order-preserving submodular function r on L . Let $\theta : \mathcal{B}_G \rightarrow (L, G)$ be the strong surjection induced by the identity map on G . By Proposition 2.4 the map $r \circ \theta$ is a polymatroid whose lattice of flats is isomorphic to (L, G) . \square

Theorem 2.7. *Let E be a set. A function from B_E to B_E is the closure operator of a polymatroid if and only if it is the closure operator of a strong surjection $\theta : \mathcal{B}_E \rightarrow (L, G)$ onto some generator enriched lattice (L, G) .*

Proof. Let $r : B_E \rightarrow \mathbb{R}_{\geq 0}$ be a polymatroid, and let (L, G) be the generator enriched lattice of flats of r . Let $\theta : \mathcal{B}_E \rightarrow (L, G)$ be the strong map induced by the map $e \mapsto \overline{\{e\}}$ from E to L . The image $\theta(X)$ is by definition

$$\theta(X) = \bigvee_{x \in X} \overline{\{x\}},$$

in other words the smallest flat including $\overline{\{x\}}$ for all $x \in X$. If Y is a flat including $\overline{\{x\}}$ for all $x \in X$, then $Y \supseteq X$. Taking the closure $Y \supseteq \overline{X}$. Thus the image $\theta(X)$ equals the closure \overline{X} . Since $L \subseteq B_E$ the closure operator cl_θ takes the same values as θ so we have shown that $\overline{\cdot} = \text{cl}_\theta$.

Conversely consider a generator enriched lattice (L, G) and a strong surjection $\theta : \mathcal{B}_E \rightarrow (L, G)$. We wish to construct a polymatroid whose closure operator coincides with the closure operator cl_θ of θ . By Lemma 2.5 there is a strictly order-preserving submodular function $r : L \rightarrow \mathbb{R}_{\geq 0}$. By Proposition 2.4 the function $s = r \circ \theta : B_E \rightarrow \mathbb{R}_{\geq 0}$ is a polymatroid on E . Furthermore the generator enriched lattice of flats of s is isomorphic to (L, G) via the isomorphism $\overline{X} \mapsto \theta(X)$. From this it is evident that $\overline{\cdot} = \text{cl}_\theta$, hence the closure operator cl_θ of θ is the closure operator of the polymatroid s . \square

3 Minors

In this section we discuss minors of polymatroids in regards to the associated closure operators and generator enriched lattices. The underlying generator enriched lattice of a minor does not fully depend on the original polymatroid if it is simple, only the underlying generator

enriched lattice. In Section 3.2 we discuss minors of generator enriched lattices themselves (with no structure of a strong surjection). In Theorem 3.9 we show that for a graphic matroid the minors of the generator enriched lattice of flats are in bijection with the minors of the graph when the vertices are labeled and the edges are unlabeled. In Theorem 3.11 we prove a generalization of this result to polymatroids.

Let r be a polymatroid with ground set E . The *deletion by* $X \subseteq E$ is the polymatroid $r \setminus X : B_{E \setminus X} \rightarrow \mathbb{R}_{\geq 0}$ defined as the usual function restriction $r \setminus X = r|_{B_{E \setminus X}}$. The *contraction by* X is the polymatroid $r/X : B_{E \setminus X} \rightarrow \mathbb{R}_{\geq 0}$ defined for $Y \subseteq E \setminus X$ by setting $(r/X)(Y) = r(Y \cup X) - r(X)$. These operations correspond to restricting to a lower and upper interval of the Boolean algebra B_E respectively. Any polymatroid obtained from r via deletion and contraction operations is said to be a *minor* of r .

3.1 Minors of strong surjections

We begin by observing that minors of a polymatroid closure operator are well defined as the closure operator of the corresponding minor of any associated polymatroid.

Lemma 3.1. *Let (r, E) and (s, E) be two polymatroids with the same closure operator. For any two disjoint sets $X, Y \subseteq E$ the closure operators of the minors $(r/X) \setminus Y$ and $(s/X) \setminus Y$ are the same.*

Proof. Let $r' = (r/X) \setminus Y$ and $s' = (s/X) \setminus Y$. Let $Z \subseteq E$ and $e \in E \setminus Z$. By assumption $r(Z) = r(Z \cup \{e\})$ if and only if $s(Z) = s(Z \cup \{e\})$. The minor r' is the function defined on $E \setminus (X \cup Y)$ by $r'(Z) = r(Z) - r(X)$, and similarly for s and s' . Thus we have that $r'(Z) = r'(Z \cup \{e\})$ if and only if $s'(Z) = s'(Z \cup \{e\})$ which shows that the closure operators of r' and s' are the same. \square

We now turn to defining deletion and contraction operations on strong surjections from a lattice theoretic viewpoint. In Proposition 3.2 we prove that these operations agree with the same operations on polymatroids. First we set up some notation.

Given a lattice L , let $H \subseteq L$ and let $z \in L$ be an element such that $z < h$ for all $h \in H$. Define the *generator enriched lattice with generating set H and minimal element z* to be

$$\begin{aligned} \langle H|z \rangle &= \left(\{z \vee \bigvee_{x \in X} x : X \subseteq H\}, H \right) \\ &= \left(\{z\} \cup \{ \bigvee_{x \in X} x : \emptyset \neq X \subseteq H \}, H \right). \end{aligned}$$

Usually when listing H explicitly the set brackets will be repressed.

Let E be a ground set, let (L, G) be a generator enriched lattice and let $\theta : \mathcal{B}_E \rightarrow (L, G)$ be a strong surjection. For $X \subseteq E$ define the *deletion of θ by X* to be the strong surjection

$$\theta \setminus X : \mathcal{B}_{E \setminus X} \rightarrow \langle \{\theta(\{e\}) : e \in E \setminus X\} \setminus \{\widehat{0}_L\} | \widehat{0}_L \rangle,$$

defined for $Z \subseteq E \setminus X$ by $(\theta \setminus X)(Z) = \theta(Z)$. Define the *contraction of θ by X* to be the strong surjection

$$\theta/X : \mathcal{B}_{E \setminus X} \rightarrow \langle \{\theta(X \cup \{e\}) : e \in E \setminus X\} \setminus \{\theta(X)\} | \theta(X) \rangle,$$

defined for $Z \subseteq E \setminus X$ by $(\theta/X)(Z) = \theta(X \cup Z)$.

Conceptually the deletion by X of a strong surjection θ is obtained by restricting θ to the lower interval $[\emptyset, E \setminus X] \subseteq B_E$, and then restricting the codomain to ensure the resulting function is a surjection. Similarly contracting by X corresponds to restricting to the upper interval $[X, E] \subseteq B_E$.

Proposition 3.2. *Let $r : E \rightarrow \mathbb{R}_{\geq 0}$ be a polymatroid with generator enriched lattice of flats (L, G) , and let $X, Y \subseteq E$ be disjoint sets. If $\theta : \mathcal{B}_E \rightarrow (L, G)$ is the strong surjection associated to r then the closure operator of the polymatroid $(r/X) \setminus Y$ is equal to $\text{cl}_{(\theta/X) \setminus Y}$.*

Proof. Set $r' = (r/X) \setminus Y$ and $\theta' = (\theta/X) \setminus Y$. Let $\overline{}$ denote the closure operator of r' . By definition $\overline{Z_1} = \overline{Z_2}$ if and only if $r'(Z_1) = r'(Z_1 \cup Z_2) = r'(Z_2)$ which occurs if and only if $r(Z_1 \cup X) = r(Z_1 \cup Z_2 \cup X) = r(Z_2 \cup X)$. Since cl_θ is the closure operator of r this occurs if and only if $\theta(Z_1 \cup X) = \theta(Z_1 \cup Z_2 \cup X) = \theta(Z_2 \cup X)$. This is in turn equivalent to the condition $\theta'(Z_1) = \theta'(Z_1 \cup Z_2) = \theta'(Z_2)$. Therefore $\overline{Z_1} = \overline{Z_2}$ if and only if $\theta'(Z_1) = \theta'(Z_2)$, and thus $\text{cl}_{\theta'}$ is the closure operator of r' . \square

3.2 Minors of generator enriched lattices

Let (L, G) be a generator enriched lattice and $\theta : \mathcal{B}_E \rightarrow (L, G)$ be a strong surjection. When θ is simple, that is, when $\theta|_{\text{irr}(B_E) \cup \{\emptyset\}}$ is injective, the codomain of the deletion $\theta \setminus X$ depends only on the set $\{\theta(\{x\}) : x \in X\}$. Similarly the codomain of the contraction θ/X depends only on the image $\theta(X)$. Thus viewing generator enriched lattices as encoding closure operators of simple polymatroids we have a notion of deletion and contraction operations, the result of which is another generator enriched lattice.

Let (L, G) be a generator enriched lattice and let $I \subseteq G$. The *deletion of (L, G) by I* is the generator enriched lattice

$$(L, G) \setminus I = \langle G \setminus I | \widehat{0}_L \rangle.$$

Let $i_0 = \bigvee_{i \in I} i$ and set $J = \{g \vee i_0 : g \in G\} \setminus \{i_0\}$. The *contraction of (L, G) by I* is the generator enriched lattice

$$(L, G)/I = \langle J | i_0 \rangle.$$

For convenience we also define the *restriction of (L, G) to I* as

$$(L, G)|_I = (L, G) \setminus (G \setminus I).$$

The operations of deletion and contraction on generator enriched lattices correspond to first choosing a simple strong surjection, performing the operations as previously defined for strong surjections, and taking a simplification of the result.

It will be convenient at times to index deletions and contractions by subsets of some ground set E , or by elements of L instead. To define the former choose a labeling of G by E so that $G = \{g_e : e \in E\}$. Given $X \subseteq E$ the deletion and contraction by X are defined as

$$\begin{aligned} (L, G) \setminus X &= (L, G) \setminus \{g_x : x \in X\}, \\ (L, G)/X &= (L, G)/\{g_x : x \in X\}. \end{aligned}$$

Given $\ell \in L$ the deletion and contraction by ℓ are defined as

$$\begin{aligned} (L, G) \setminus \ell &= (L, G) \setminus \{g \in G : g \leq \ell\}, \\ (L, G)/\ell &= (L, G)/\{g \in G : g \leq \ell\}. \end{aligned}$$

The result of any sequence of deletions and contractions applied to (L, G) is called a *minor* of (L, G) .

A few basic remarks for minors of a generator enriched lattice (L, G) are in order.

Remark 3.3. *By definition the underlying lattice of a minor of (L, G) is a join subsemilattice of L . In general the underlying lattice of a minor of (L, G) may not be a sublattice of L . For example, consider the partition lattice Π_4 with minimal generating set*

$$\text{irr}(\Pi_4) = \{12/3/4, 13/2/4, 14/2/3, 1/23/4, 1/24/3, 1/2/34\}.$$

Deleting the atom $13/2/4$ results in a minor which is not a sublattice of Π_4 ; in said minor the meet of $123/4$ and $134/2$ is the minimal partition $1/2/3/4$ as opposed to $13/2/4$ when computed in L .

Remark 3.4. *Any interval of L is the underlying lattice of a minor of (L, G) . If $a \leq b$ in L then the minor $((L, G)/a)|_b$ has underlying lattice the interval $[a, b]$ of L . The example given in Remark 3.3 shows the converse is false, that in general not all minors of (L, G) have as underlying lattice an interval of L .*

Remark 3.5. *The deletion and contraction operations of generator enriched lattices do not in general commute. See Figure 3 for an example.*

The following observation will be useful.

Lemma 3.6. *Any minor of a generator enriched lattice (L, G) may be expressed as the result of a contraction followed by a deletion. Namely, a minor (K, H) of (L, G) may be expressed as $(K, H) = ((L, G)/\widehat{0}_K)|_H$.*

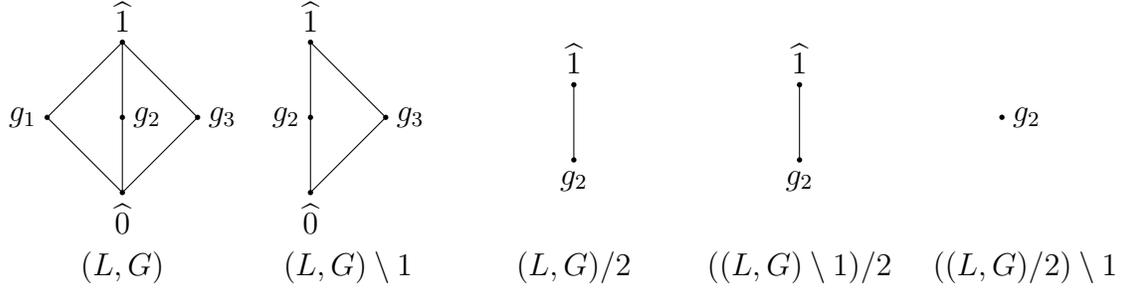


Figure 3: A generator enriched lattice (L, G) , where $G = \{g_1, g_2, g_3\}$ for which deletions and contractions do not commute, along with the relevant minors.

Proof. Let (K, H) be a minor of (L, G) . By definition (K, H) may be expressed as the result of a sequence of contractions and deletions. That is, for some possibly empty sets of generators $I_1, J_1, \dots, I_r, J_r$, that

$$(K, H) = (((\dots(((L, G)/I_1) \setminus J_1) \dots /I_r) \setminus J_r).$$

For $1 \leq j \leq r$ let i_j be the join of all elements in I_j . Set $i_0 = i_1 \vee \dots \vee i_r$. By definition of deletion and contraction, the minimal element $\widehat{0}_K$ of K is i_0 . Furthermore, the generators of (K, H) can each be expressed as $g \vee i_1 \vee \dots \vee i_r = g \vee i_0$ for some $g \in G$. Thus each generator of (K, H) is a generator of $(L, G)/i_0$, hence $(K, H) = ((L, G)/i_0)|_H$. \square

The lemma below gives an explicit description of the generating sets of minors.

Lemma 3.7. *For any generator enriched lattice (L, G) the minors are precisely generator enriched lattices of the form $\langle \ell \vee g_1, \dots, \ell \vee g_k | \ell \rangle$ for $\ell \in L$ and $\{g_1, \dots, g_k\} \subseteq G$ such that $g_j \not\leq \ell$ for $1 \leq j \leq k$.*

Proof. Consider a minor $(K, H) = ((L, G)/I)|_J$ of (L, G) , where I and J are sets of generators. Let ℓ be the join of all elements of I and let $J = \{j_1, \dots, j_k\}$. By definition

$$(K, H) = \langle \ell \vee j_1, \dots, \ell \vee j_k | \ell \rangle.$$

Conversely consider a generator enriched lattice $(K, H) = \langle \ell \vee g_1, \dots, \ell \vee g_k | \ell \rangle$ for some $\ell \in L$ and $g_j \in G$ with $g_j \not\leq \ell$ for $1 \leq j \leq k$. The generators of the contraction $(L, G)/\ell$ are all elements $\ell \vee g$ for $g \in G$ with $g \not\leq \ell$. Thus $\ell \vee g_1, \dots, \ell \vee g_k$ are generators of $(L, G)/\ell$, so setting $I = \{\ell \vee g_1, \dots, \ell \vee g_k\}$ we have that $(K, H) = ((L, G)/\ell)|_I$. \square

Lemma 3.8. *If L is a geometric lattice then the minors of $(L, \text{irr}(L))$ are the generator enriched lattices of the form $\langle \ell_1, \dots, \ell_k | \ell \rangle$ such that $\ell_i \succ \ell \in L$ for $1 \leq i \leq k$. In particular, every minor of $(L, \text{irr}(L))$ is minimally generated and geometric.*

Proof. Since L is geometric, for any $x, y \in L$ we have $x \prec y$ if and only if $y = x \vee i$ for some $i \in \text{irr}(L)$. Thus Lemma 3.7 specializes to the claimed form of the generating sets of

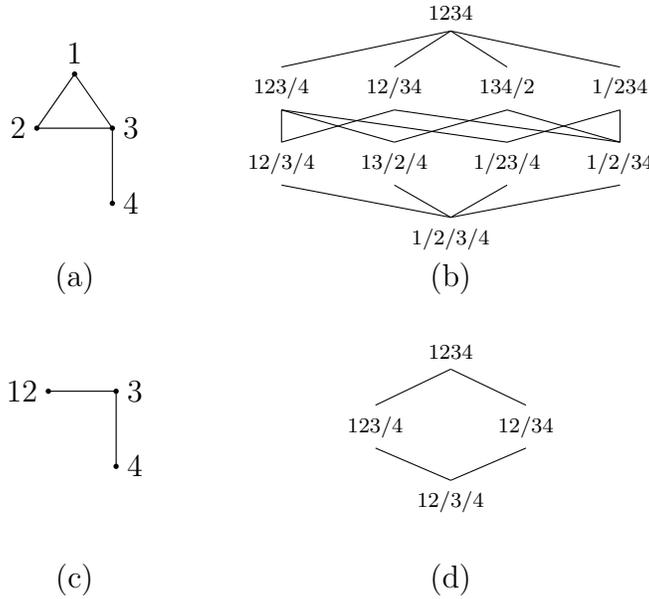


Figure 4: At (a) a graph G and at (b) the lattice of flats $L(G)$. At (c) the simplification of the contraction $G/\{1, 2\}$ and at (d) the contraction $L(G)/(12/3/4)$.

minors of $(L, \text{irr}(L))$. In particular, for any minor (K, H) the generating set H is the set of atoms of K .

Let $(K, \text{irr}(K)) = \langle \ell_1, \dots, \ell_k | \ell \rangle$ be a minor of $(L, \text{irr}(L))$. In order to show that K is semi-modular we claim that if $x \prec y$ in K then $x \prec y$ in L as well. Since $x \prec y$ there exists i such that $y = x \vee \ell_i$. We have that $\ell_i = \ell \vee a$ for some atom a of L , hence $y = x \vee a$. Since L is geometric this implies that $x \prec y$ in L .

Now let $x, y \in K$ such that $x \wedge_K y \prec x$ and $x \wedge_K y \prec y$ in K . Since K is a subposet of L we have that $x \wedge_K y \leq x \wedge_L y$. On the other hand, since $x \wedge_K y$ is covered by x and y in K , hence in L , we must have $x \wedge_K y = x \wedge_L y$. Thus $x \wedge_L y$ is covered by x and y in L , and since L is semimodular $x \vee_L y$ covers x and y in L . Then since $x \vee_L y = x \vee_K y$ this implies that $x \vee_K y$ covers x and y in K , and therefore K is geometric. \square

Given a graph G , the lattice of flats L may be viewed as a lattice of partitions of the vertices of G . Each flat is associated to the partition whose blocks consist of the connected components of said flat considered as a subgraph of G . Let $L(G)$ denote the generator enriched lattice of flats of G labeled as partitions. See Figure 4.

The minors of the graph G inherit a vertex labeling by blocks of a partition of the vertices of G . When an edge is contracted, the label of the new vertex is obtained by merging the two blocks labeling the vertices of the contracted edge. In this way the minors of a vertex labeled graph are considered to be themselves vertex labeled graphs.

Theorem 3.9. *Let G be a vertex labeled graph with unlabeled edges. The vertex labeled minors of G which are simple are in bijection with the minors of the minimally generated lattice of flats $L(G)$ via the map $H \mapsto L(H)$.*

Proof. Let L be the lattice of flats of the graph G . It may be assumed that the graph G is simple, that is, that G has no loops or multiple edges. This only changes the labeling of elements in the lattice of flats and does not change the collection of simple minors of G . The inverse of the map $H \mapsto L(H)$ will be constructed. Let $(K, \text{irr}(K)) = \langle \ell_1, \dots, \ell_k | \ell \rangle$ be a minor of $(L, \text{irr}(L))$. Construct a graph H as follows. Each atom of L corresponds to an edge of G . The element ℓ corresponds to a set of edges of G ; namely, those edges corresponding to an atom which is less than or equal to ℓ . Let H' be the minor of G obtained by contracting this set of edges corresponding to ℓ . The vertices of H' are labeled by the blocks of the partition ℓ . Each atom ℓ_i in K is obtained from the partition ℓ by merging two blocks, and corresponds to an edge in H' . Let H'' be the graph obtained from H' by restricting to these edges which correspond to an atom of K . The graph H is defined to be the simplification of H'' .

It remains to show that the map $K \mapsto H$ constructed above and the map $H \mapsto L(H)$ are inverses. A vertex labeled graph H is determined by the labeling of its vertices and its edges. The associated lattice minor $(K, \text{irr}(K))$ of $(L, \text{irr}(L))$ records this same information as the minimal partition and the atoms, which in turn determines K . \square

The above result generalizes to polymatroids with the appropriate notion replacing vertex labeled minors.

Definition 3.10. *Let r be a polymatroid with ground set E . A parallel closed pair is a pair (F, s) such that $F \subseteq E$ is a flat of r and s is a polymatroid that may be obtained as a deletion of the polymatroid r/F satisfying the following condition:*

If $e \in E \setminus F$ is parallel with respect to r/F to an element f of the ground set of s then e is an element of the ground set of s as well. In other words, the ground set of s must be a union of parallel classes with respect to r/F .

For a graphic matroid the parallel closed pairs are in bijection with the vertex labeled minors of the graph obtained by first contracting, and then deleting entire parallel classes of edges. The vertex labeling naturally encodes the flat in the parallel closed pair. Such graphs are in bijection with the simple minors when the edges are unlabeled. Without an edge labeling each such graph has one simplification, obtained by identifying parallel edges, and no two such graphs have the same simplification. Thus the following theorem is an analogue of Theorem 3.9.

Theorem 3.11. *Let r be a polymatroid and let (L, G) be the associated generator enriched lattice of flats. The minors of (L, G) are in bijection with the parallel closed pairs of r .*

Proof. Let E be the ground set of r . Let $\overline{\cdot} : B_E \rightarrow B_E$ be the closure operator of r . Let $\theta : \mathcal{B}_E \rightarrow (L, G)$ be the strong surjection induced by the ground set map $e \mapsto \overline{\{e\}}$.

Let (F, s) be a parallel closed pair of r and let $Y \subseteq E$ be the ground set of s . Define a map f from the set of parallel closed pairs of the polymatroid r to the set of minors of the generator enriched lattice (L, G) by

$$f(F, s) = ((L, G)|_{Y \cup F})/F.$$

To show f is a bijection construct the inverse map g . Let (K, H) be a minor of (L, G) and let Y be the set

$$Y = \{y \in E : \overline{\{y\} \cup \widehat{0}_K} \in H\}.$$

Let $g(K, H)$ be the pair $(\widehat{0}_K, (r/\widehat{0}_K)|_Y)$. Observe that $g(K, H)$ is a parallel closed pair of r . Furthermore g is the inverse of f so the map f is a bijection. \square

3.3 Minors of distributive lattices

In the remainder of this section we examine minors of minimally generated distributive lattices. Recall that the fundamental theorem of finite distributive lattices states that every finite distributive lattice L is isomorphic to the lattice of lower order ideals of the subposet $\text{irr}(L)$ of L . The minors of a minimally generated distributive lattice have an alternative description in terms of certain pairs of subsets of the poset of irreducibles.

Definition 3.12. *Let P be a poset. An order minor of P is a pair (I, J) of disjoint subsets of P such that J is a lower order ideal of P .*

The poset P itself corresponds to the order minor (P, \emptyset) . The set of order minors of P is shown below to be in bijection with the minors of the minimally generated lattice of lower order ideals of P . To prove this bijection, the following lemma is needed.

Lemma 3.13. *If L is a distributive lattice, for any $\ell \in L$ and any distinct join irreducibles i and j such that $i \vee \ell \neq \ell$ and $j \vee \ell \neq \ell$ the elements $i \vee \ell$ and $j \vee \ell$ are distinct.*

Proof. Let L be the lattice of lower order ideals of a poset P , necessarily isomorphic to $\text{irr}(L)$. It may be assumed without loss of generality that $i \not\leq j$ in L . Let $p \in P$ be the element such that the principal lower order ideal of P generated by p is the join irreducible i of L . The fact that $i \vee \ell \neq \ell$ implies that p is not contained in the ideal ℓ . Since $i \not\leq j$ the element p is not contained in the ideal j . As a consequence p is not contained in the ideal $j \vee \ell$ since the join in L corresponds to the union of lower order ideals. This establishes that $i \vee \ell \neq j \vee \ell$. \square

When $(L, \text{irr}(L))$ is the minimally generated lattice of lower order ideals of a poset P there is an implicit bijection between P and the generating set $\text{irr}(L)$. Through this bijection deletions and contractions of $(L, \text{irr}(L))$ may be indexed by subsets of P .

Proposition 3.14. *Let L be the lattice of lower order ideals of a poset P . The order minors of P and the minors of $(L, \text{irr}(L))$ are in bijection via the map*

$$(I, J) \mapsto ((L, \text{irr}(L))|_{I \cup J})/J.$$

Proof. Define a map from minors of $(L, \text{irr}(L))$ to order minors of P as follows. Given a minor (K, H) of $(L, \text{irr}(L))$ define J to be the subset of P corresponding to the join irreducibles in L which are less than or equal to $\widehat{0}_K$. Define I to be the subset of P consisting of all elements whose corresponding join irreducible i of L satisfies $i \vee \widehat{0}_K \in H$. Lemma 3.13 implies that I is the unique set satisfying $(K, H) = ((L, \text{irr}(L))|_{I \cup J})/J$. The inverse of this map is given by $(I, J) \mapsto ((L, \text{irr}(L))|_{I \cup J})/J$ so we have a bijection. \square

Not only do the order minors of a poset index the minors of the minimally generated lattice of lower order ideals, the order minors also describe the isomorphism types of the lattice minors.

Proposition 3.15. *Let P be a poset and L the lattice of lower order ideals of P , and let (I, J) be an order minor of P . The minor $((L, \text{irr}(L))|_{I \cup J})/J$ of $(L, \text{irr}(L))$ is isomorphic to the minimally generated lattice of lower order ideals of I .*

Proof. Let (I, J) be an order minor of P and let $(K, H) = ((L, \text{irr}(L))|_{I \cup J})/J$. It is claimed that K consists of the lower order ideals of P whose maximal elements are all contained in $I \cup J$ and which include J . Observe that $(L, \text{irr}(L))|_{I \cup J}$ is generated by the principal lower order ideals which are themselves generated by an element of $I \cup J$. Hence this lattice consists of all lower order ideals of P whose maximal elements are contained in $I \cup J$. The generators of $((L, \text{irr}(L))|_{I \cup J})/J$ are thus each the union of the lower order ideal J of P with a principal lower order ideal of P which is generated by an element of I . Such lower order ideals as a join subsemilattice of L generate the set of lower order ideals of P which include J and whose maximal elements are contained in $I \cup J$.

Let $(M, \text{irr}(M))$ be the minimally generated lattice of lower order ideals of the subposet I of P . Define a map $f : (K, H) \rightarrow (M, \text{irr}(M))$ by $f(\Lambda) = \Lambda \cap I$. Define a map $g : (M, \text{irr}(M)) \rightarrow (K, H)$ by letting $g(\Lambda)$ be the lower order ideal of P generated by $\Lambda \cup J$. Observe this is the inverse of f since every ideal in L includes J and has maximal elements which are contained in $I \cup J$. Therefore K is isomorphic to M as claimed. \square

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