

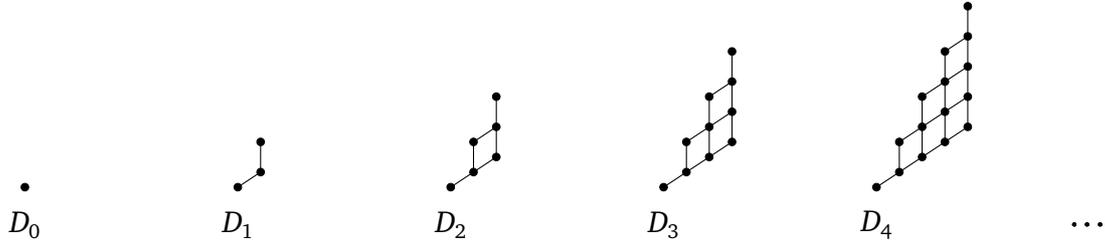
COMBINATORICS ARISING FROM LAX COLIMITS OF POSETS

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ABSTRACT. In this paper we study maximal chains in certain lattices constructed from powers of chains by iterated lax colimits in the 2-category of posets. Such a study is motivated by the fact that in lower dimensions, we get some familiar combinatorial objects such as Dyck paths and Kreweras walks.

INTRODUCTION

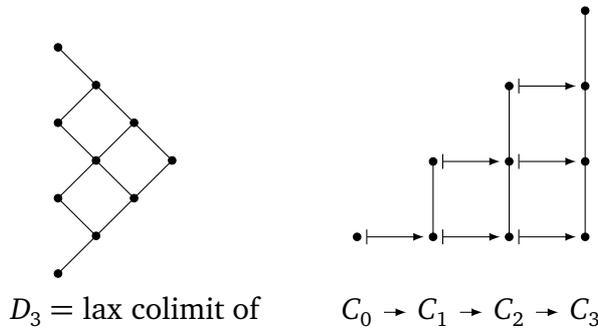
The well-known combinatorial objects, Dyck paths, can be interpreted as maximal chains in lattices (in the sense of [2]) given by the following Hasse diagrams:



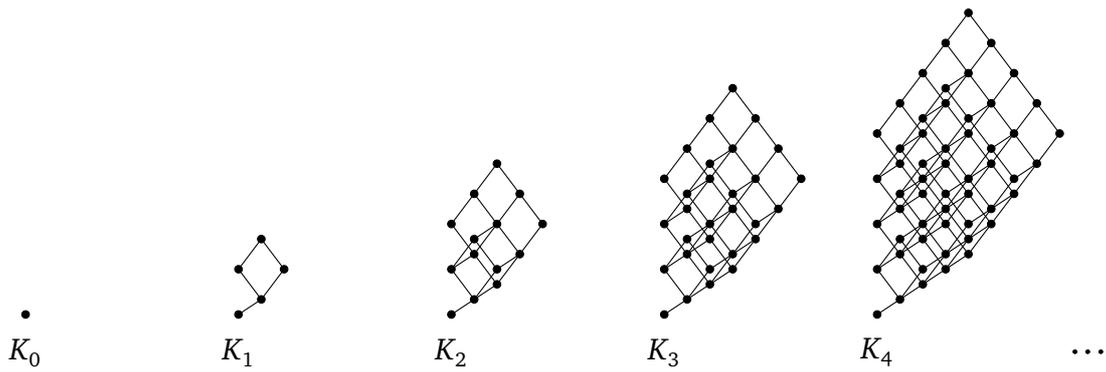
Each of the lattices D_m can be constructed as a ‘lax colimit’, in the 2-category of posets, of the diagram

$$C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow C_4 \longrightarrow \dots \longrightarrow C_m$$

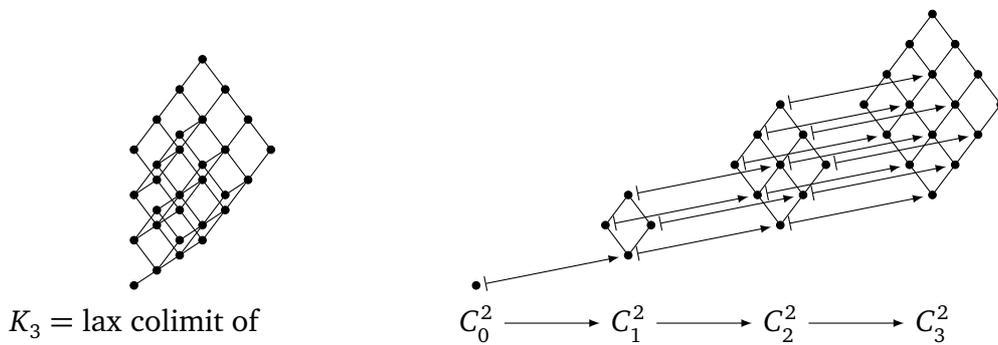
where C_i stands for the chain with $i + 1$ elements and each homomorphism $C_i \rightarrow C_{i+1}$ is an inclusion of down-closed sub-join-semilattices. A lax colimit ‘stacks’ the chains above each other, turning each assignment $x \mapsto y$ given by a homomorphism into the relation $x < y$ in the lax colimit. This process can be visualized in the case of the diagram above with $n = 3$ as follows:



One may now consider variations of this construction, where chains are replaced with other lattices. For instance, when we replace each C_i with its cartesian square C_i^2 , we get the following lattices:



Each of these lattices decomposes as a lax colimit as follows (the example shown is for $n = 3$):



Maximal chains in these lattices are in bijection with Kreweras walks [3, 9] — an observation originally due to Sarah Selkirk (private communication). This gives rise to the following question: *which combinatorial objects arise as maximal chains in lattices stacked by means of lax colimits?* For instance, noting that the two examples above correspond to the second and the third rows in a (commutative) diagram of powers of chains (see Figure 1), it becomes interesting to explore other sequences of homomorphisms arising from the same diagram.

Stacking lattices in the first row of the diagram in Figure 1 simply gives the sequence of chains (the second row) — not so interesting in its own right. However, stacking lattices in the sequence of chains, so lattices in the second row, gives the Dyck situation described above. Maximal chains in these lattices (i.e., the lattices D_m) can be counted, as it is well known, by the Catalan numbers:

$$\frac{(2m)!}{m!(m+1)!}$$

The third row gives the Kreweras case. This already is a highly nontrivial combinatorial situation; for example, a bijective proof for the formula

$$\frac{(3m)!4^m}{(m+1)!(2m+1)!}$$

that counts Kreweras walks (i.e., maximal chains in the lattices K_m) was found not long ago [1]. This might mean that getting similar numbers for stacking lattices along the subsequent rows can be vastly difficult. In this paper we solve the orthogonal problem: we find the maximal chain numbers for stacking lattices along the columns of the diagram in Figure 1. The first column is again trivial, as it is identical (or rather, isomorphic) to the first row. The sequence of number here is simply the sequence of all positive natural numbers $1, 2, 3, 4, \dots$. The second column stacks

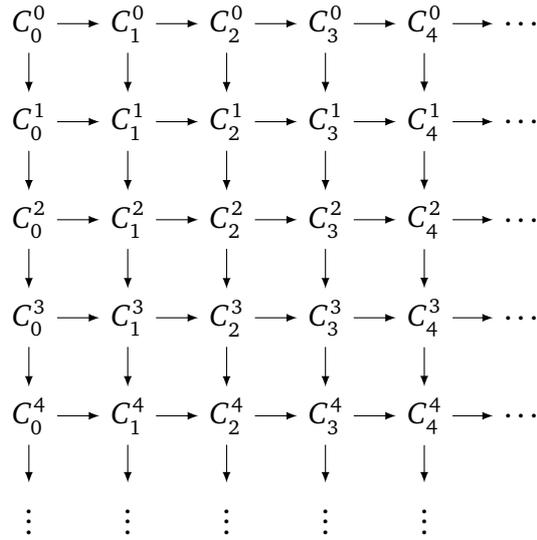


FIGURE 1. Homomorphism lattice of powers of chains

‘hypercubes’. We show that the corresponding numbers are given by the odd double factorials:

$$\frac{(2n)!}{2^n n!} = (2n - 1)!!$$

We first prove this by making use of the technique of representing weighted Dyck paths as involutions with no fixed points, from the lecture series of X. Viennot [12], which then suggests a direct bijection with maximal chains in stacks of hypercubes. After this, it becomes evident how to deal with the remaining columns: involutions with no fixed points, which are the same as 2-partitions, get replaced with m -partitions (of an mn element set), giving us the numbers

$$\frac{(mn)!}{(m!)^n n!}$$

for maximal chains in the stacks along the m -th column of Figure 1.

These results are obtained in Section 1 of this paper (see Theorems 1 and 2). In Section 2, we turn our attention to iterating the process of stacking and find the following:

- Maximal chain numbers for k -th iteration of stacking lattices in the first column of Figure 1 are the k -dimensional Catalan numbers from [10]:

$$\frac{(k - 1)!(kn)!}{n!(n + 1)! \dots (n + k - 1)!}$$

- Maximal chains in k -th iteration of stacking hypercubes (i.e., lattices in the second column of Figure 1) are in bijection with the combinatorial objects discussed in [4] in the case of $k = 2$ (see the entry A213275 by Alois P. Heinz in the OEIS for the general k): words w of length $(k + 1)n$ in an alphabet $\{a_1, \dots, a_n\}$ with n distinct letters, such that each letter occurs $k + 1$ times in the word, and for each prefix z of w , either a_i does not occur in z or if it does, then for each $j > i$, it occurs more or the same number of times as a_j . We establish this in Theorem 4. No explicit formulas for counting these combinatorial objects seem to be known.
- Maximal chains in iteration of stacking lattices in the third column onwards in Figure 1 appear to be new combinatorial objects. At least, we could not find the corresponding numbers, generated on a computer, in the OEIS (see Figure 5).

All three of these results arise as applications of a ‘representation theorem’ (Theorem 3), which gives an embedding of k -iterated stacking of m -fold hypercubes into $(k + m)$ -fold hypercubes,

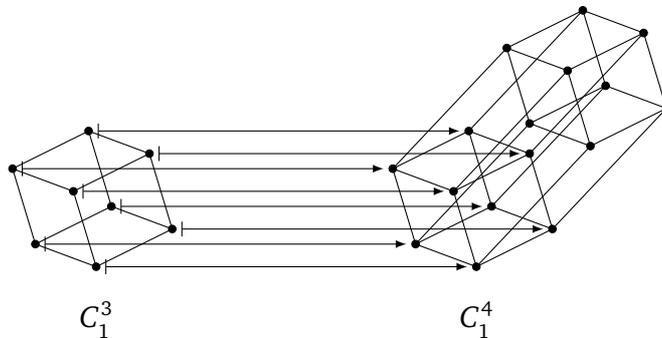
where by an r -fold hypercube we mean a lattice of the form C_r^n (i.e., a lattice in the r -th column of Figure 1). We formulate this theorem in Section 2, but defer its proof to Section 4. The representation theorem also helped with the visualization of the corresponding stacked lattices — see Figures 3 and 4, where $\Sigma_n^k C_m^n$ denotes the k -iterated stacking of m -fold hypercubes up to dimension n .

In Section 3, we expand on the link with lax colimits in the 2-category of posets, and establish some algebraic properties of this construction to get an insight as to *what kinds of lattices may arise when considering stacking of lattices in Figure 1?* In particular, we show that first of all, they are indeed lattices, and secondly, they are in fact distributive lattices. We do not know, however, whether distributivity of these lattices can play a role in the combinatorial investigation of their maximal chains. Section 3 also prepares a way to Section 4, which is devoted to the proof of the representation theorem mentioned earlier. The proof is based on decomposing lax colimits of chains of homomorphisms into another kind of (weighted) colimits in category theory, which we call ‘lax pushouts’ (although they do not form a particular type of lax colimits). This method enables one to almost trivialize the geometric complexity of the stacked lattices of m -fold hypercubes. A similar method can be used to prove a representation theorem (Theorem 11) for iterated stacking of lattices along the rows of Figure 1, which we formulate in the last Section 5. As did the previous representation theorem for stacked lattices along the columns of Figure 1, this representation theorem allows one to easily generate on a computer the first few terms in the corresponding sequences of maximal chain numbers, and just as before, we quickly encounter new sequences — see Figure 7.

This paper is aimed at readers of diverse background. While we bring together combinatorics, lattice theory and category theory, we took particular care in the presentation to make the paper accessible to non-experts of each of these fields (moreover, the paper uses only basic concepts from these fields). We hope that our work will entice further research on combinatorics of maximal chains of stacked lattices (or posets, more generally). Among questions left unresolved in this paper is the question of finding explicit formulas for those integer sequences from Figures 5 and 7 that do not appear in OEIS, or proving that they do not exist. We anticipate these questions to be highly non-trivial.

1. STACKING HYPERCUBES

Consider the sequence $C_1^\infty = (C_1^0, C_1^1, C_1^2, C_1^3, \dots)$ of hypercubes: each C_1^n denotes the n -th cartesian power of the 1-chain C_1 . Thus, each C_1^n is an n -dimensional cube. We will stack these along the obvious embeddings $C_1^i \rightarrow C_1^{i+1}$, visualized below in the case when $i = 3$:



The numbers $\#C_1^n$ of maximal chains in the members of the sequence C_1^∞ are of course given by the factorials

$$\#C_1^n = n! \quad (n \geq 0).$$

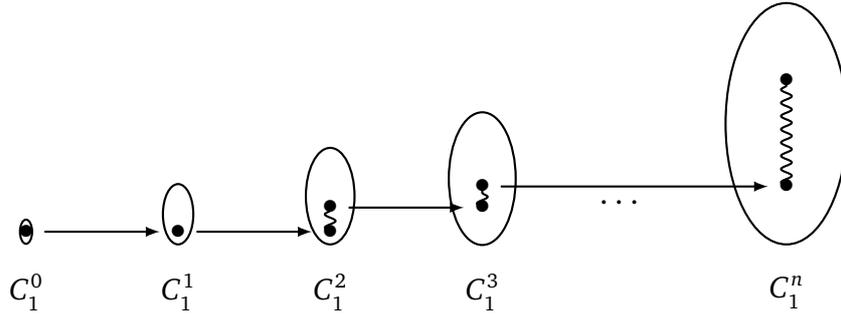
Theorem 1. *The number of maximal chains in the lattice $\Sigma_n C_1^n$ obtained by stacking the hypercubes $C_1^0 \rightarrow C_1^1 \rightarrow \cdots \rightarrow C_1^n$ is given by odd double factorials (where $(-1)!! = 1$):*

$$\#\Sigma_n C_1^n = (2n - 1)!!$$

Proof. It is well known (see e.g. [12]) that odd double factorials count the number of involutions on a set with $2n$ elements with no fixed points (which are the same as 2-partitions). To see how, notice that such an involution can be represented as a list of distinct elements of $2n$ (permutation), where each consecutive pair of odd and neighboring even entry in the list is an involuted pair. There are $(2n)!$ such lists, but we get them too often: for each pair, we must divide by 2, so altogether by 2^n , and then the order of the pairs does not matter, so we must divide by $n!$. This gives

$$\frac{(2n)!}{2^n n!} = (2n - 1)!!.$$

We will now establish a bijection between maximal chains in $\Sigma_n C_1^n$ and involutions of a $2n$ element set having no fixed points. A walk along such chain visits each cube, makes some steps in the cube, and moves on to the next cube:



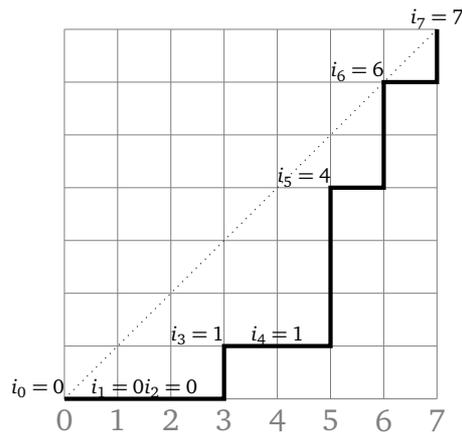
Let i_j represent height reached in the cube C_1^j before the exit ($i_0 = 0$). Then i_j is also the height at which the next cube C_1^{j+1} is entered. The maximum height that can be reached in C_1^j is $j + 1$. The number of possible paths that can be taken inside C_1^{j+1} after entering it at height i_j and before existing it at height i_{j+1} is the falling factorial

$$(j + 1 - i_j)(j + 1 - i_j - 1) \cdots (j + 1 - i_{j+1} - (i_{j+1} - i_j - 1)) = (j + 1 - i_j)^{\overline{i_{j+1} - i_j}}.$$

The number of maximal chains in $\Sigma_n C_1^n$ is thus given by

$$\#\Sigma_n C_1^n = \sum_{0=i_0 \leq i_1 \leq \cdots \leq i_n=n} \prod_{j=0}^{n-1} (j + 1 - i_j)^{\overline{i_{j+1} - i_j}}.$$

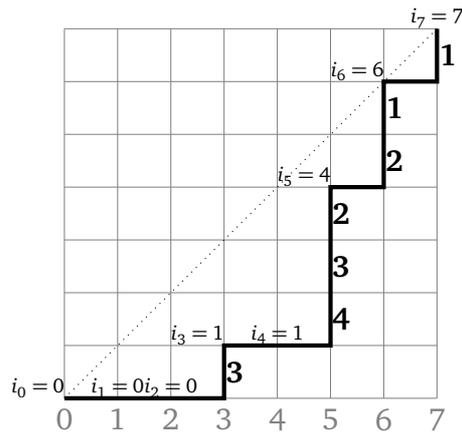
The indices in this summation form a Dyck path. Here is an example (with $n = 7$):



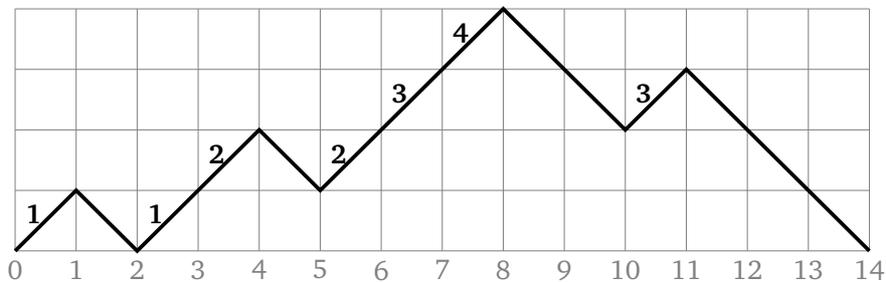
This is a ‘weighted’ Dyck path, with the weight given by the terms from the above summation:

$$1^0 2^0 3^1 3^0 4^3 2^2 1^1 = 3 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1.$$

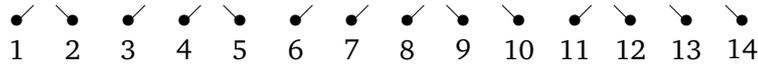
Here is a drawing with the weights distributed on the Dyck path:



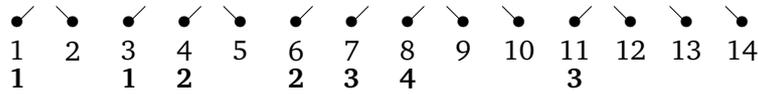
Each bold-face number in the display above represents number of choices when going upward in the given cube. A more traditional way of drawing this is:



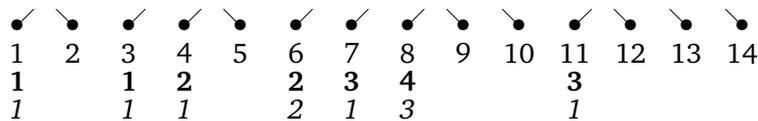
Notice that the weights at each level match with the height of the level. In his video book, X. Viennot (*n!–garden*, part (b)) constructs a fixed-point free involution on the set $\{1, 2, \dots, 14\}$ from such a weighted Dyck path (which he calls ‘Hermite history’). First, he draws the following, which indicates whether at each position the Dyck path makes an ‘up-step’ or a ‘down-step’.



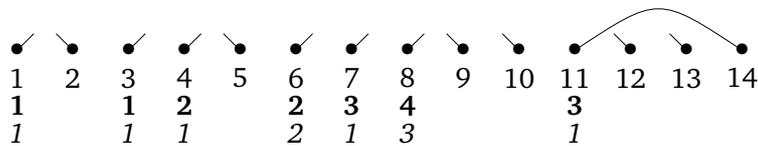
He will pair each up-step-node with a down-step-node, but which one? We have choices! Copy the weights from the Dyck path:



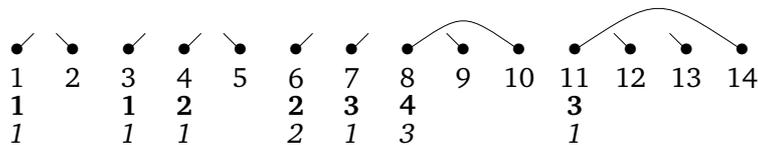
As we can see, one may connect 11 with one of 12, 13, 14. The boldface number **3** now tells us that we have 3 options for such a pairing. Write below the bold-face numbers which options will be selected:



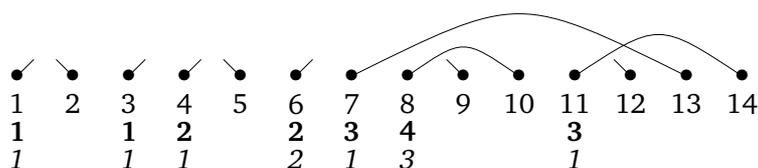
Now we work off the up-step-nodes from right to left, and connect with our choice. For 11 we choose 1, therefore our first choice (from the right), which is node 14:



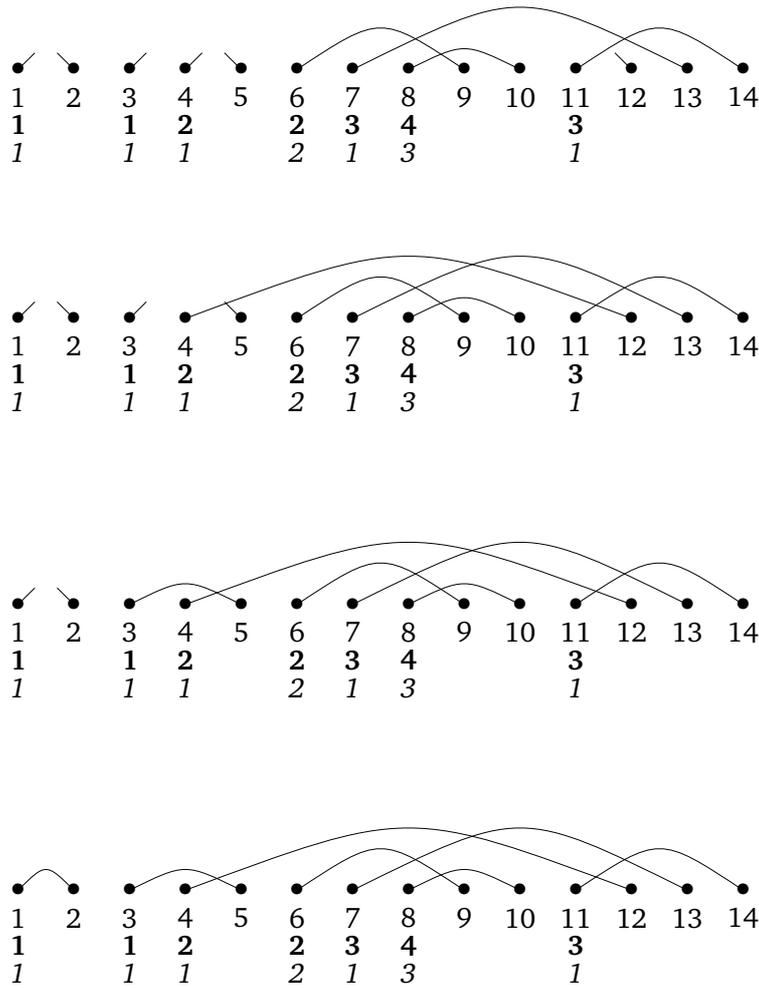
We then move to the next node, which is labeled 8, and connect it with the 3rd option (from the right), which is node 10:



Now we move to the next node 7 and connect it with the first option (from the right), which is node 13:

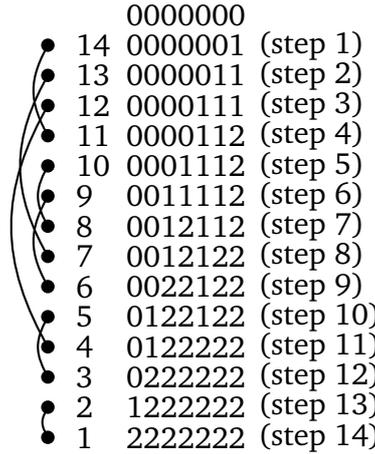


And so forth:



The goal is achieved; we constructed an involution. □

To see the bijection between 2-partitions of a $2n$ -element set and maximal chains in the lattice $\Sigma_n C_1^n$ more directly, we first embed $\Sigma_n C_1^n$ in C_2^n as follows. Represent elements of C_2 as 0, 1 and 2, in the increasing order. Then elements of C_2^n can be represented as strings of 0, 1, 2 of length n . On the other hand, if we write 1 and 2 for the elements of C_1 , then elements of each C_1^j can be represented as strings of 1, 2 of length j . Embed C_1^j into C_2^n by adding in front 0's to fill up each string of length j to a string of length n . A walk along a maximal chain in $\Sigma_n C_1^n$ now becomes a walk along a maximal chain in C_2^n which passes through only those strings where a zero never follows a nonzero entry. Each time we make a step along such walk, we have two choices: either to increment any of the nonzero entries in the string, or to increment the right-most 0 entry to 1. This second choice corresponds to moving to the next cube, whereas the first one, moving up in the same cube. There are altogether $2n$ steps to make. Pair each j -th step of the first type to the j' -th step of the second type, at both steps i -th entry was incremented. Note that each step can only be incremented twice and the step of the first type incrementing i -th entry will always succeed step of the second type incrementing the i -th entry (first I would have to move to the cube that has the dimension along which I want to make a step, before such step can be made). The 2-partition displayed above, in the case when $n = 14$, will then give rise to the following walk:



This argument easily generalizes, by replacing C_1 with C_{m-1} , to get a bijection between maximal chains in $\Sigma_n C_{m-1}^n$ and m -partitions of a set with mn elements. We therefore get:

Theorem 2. *For any natural number $m \geq 1$, there is a bijection between the set of maximal chains in the lattice $\Sigma_n C_{m-1}^n$ obtained by stacking $C_{m-1}^0 \rightarrow C_{m-1}^1 \rightarrow \dots \rightarrow C_{m-1}^n$, and the set of m -partitions of a set with mn elements. Therefore,*

$$\#\Sigma_n C_{m-1}^n = \frac{(mn)!}{(m!)^n n!}.$$

2. ITERATED STACKING

The process of stacking lattices in a sequence can be iterated: the stacked lattices produce a sequence of lattices, whose members can be stacked. We write $\Sigma_n^k L_n$ for the result of k -th iteration, with $\Sigma_n^1 L_n = \Sigma_n L_n$ and $\Sigma_n^0 L_n = L_n$. Each homomorphism $\Sigma_n^k L_n \rightarrow \Sigma_{n+1}^k L_{n+1}$ used for the next iteration is given by the universal property of lax colimit and the homomorphism $\Sigma_n^{k-1} L_n \rightarrow \Sigma_{n+1}^{k-1} L_{n+1}$ from the previous iteration (see Section 3). The sequence C_1, C_2, \dots of chains, with the inclusions $C_i \rightarrow C_{i+1}$ we have been considering, can be obtained by stacking copies of the trivial chain C_0 along the identity maps $C_0 \rightarrow C_0 \rightarrow \dots \rightarrow C_0$. Thus,

$$\Sigma_n C_0 = C_n.$$

We then get

$$\#\Sigma_n^2 C_0 = \#\Sigma_n C_n = \frac{(2n)!}{n!(n+1)!}.$$

A natural question arises: what happens if we go higher in iteration? Geometric inspection of $\Sigma_n^k C_0$ shows that it is isomorphic to the portion of C_n^k consisting of points with non-increasing coordinates; that is, all points whose coordinates satisfy

$$x_1 \leq x_2 \leq \dots \leq x_n.$$

Hence maximal chains in $\Sigma_n^k C_0$ are counted by

$$\#\Sigma_n^k C_0 = \frac{(k-1)!(kn)!}{n!(n+1)! \dots (n+k-1)!},$$

nothing other than k -dimensional Catalan numbers [10].

So then, what do we get if we iterate stacking of hypercubes? Adopting ideas from the discussion before Theorem 2, we can represent each $\Sigma_n^k C_m^n$ as a subposet of C_{k+m}^n , consisting of those elements whose coordinates (x_1, \dots, x_n) satisfy the following condition:

(*) for each $i \in \{1, \dots, n-1\}$, if $x_{i+1} \in \{0, \dots, k-1\}$ then $x_i \leq x_{i+1}$.

This representation turns out to be an embedding of lattices:

Theorem 3 (representation of k -iterated stacking of m -fold hypercubes). *The poset $\Sigma_n^k C_m^n$ is isomorphic to the sublattice of C_{k+m}^n consisting of those elements that satisfy (*).*

The formal proof of this theorem is given in Section 4.

Note that the Dyck situation and the hypercube one come together with $\Sigma_n^k C_m^n$: we get the first by letting $m = 0$ and $k = 2$, and the second by letting $k = 1$ and $m = 1$. The representation from the theorem above has been used to create drawings of stacked lattices in Figures 2, 3 and 4. In each row of each figure, the hollow vertices represent points mapped from the previous term.

The case $k = 0$ is trivial. This is when no stacking is taking place. So in this case,

$$\#\Sigma_n^0 C_m^n = \frac{(mn)!}{(m!)^n}$$

is the well-known number of maximal chains of C_m^n . The proof is a straightforward. Walking up in the lattice C_m^n requires mn many steps. Write the steps out in a sequence where the first m many steps are from walk in the first dimension, the second m many steps in the second dimension, etc. In each group, the order of steps is insignificant, so for each group, divide $(mn)!$ by all possible permutations of the group – that is, by $m!$. There are n many such divisions that need to take place.

It appears that except those cases discussed in this paper, no explicit formulas are known for $\Sigma_n^k C_m^n$, and moreover, when $k > 1$ and $m > 1$, these combinatorial objects have not been considered in the literature. Figure 5 gives some of the numbers $\#\Sigma_n^k C_m^n$ generated by a computer. As indicated there (as well as in the Introduction), the numbers corresponding to the iterated stacking of hypercubes, i.e., the case when $m = 1$, do show up on the On-Line Encyclopedia of Integer Sequences with an interpretation given in the following theorem. In the case when $k = 2$, these combinatorial objects are in bijection with certain tree-child networks, as explained in [4].

Theorem 4. *There is a bijection between maximal chains in the lattice $\Sigma_n^k C_1^n$ and words w of length $(k + 1)n$ in an alphabet $\{a_1, \dots, a_n\}$ with n distinct letters, such that each letter occurs $k + 1$ times in the word, and for each prefix z of w , either a_i does not occur in z or if it does, then for each $j > i$, it occurs more or the same number of times as a_j .*

Proof. Let w be such a word. As we read letters encountered in w from left to right, record the letters that remain unused as an n -tuple (x_1, \dots, x_n) , where x_i is the number of remaining occurrences of a_i in the rest of the word. The following example (where $n = 2$ and $k = 2$) illustrates this, where letters left to the dot are those that have been read.

$$\begin{array}{l|l} .a_2a_1a_1a_2a_1a_2 & (3, 3) \\ a_2.a_1a_1a_2a_1a_2 & (3, 2) \\ a_2a_1.a_1a_2a_1a_2 & (2, 2) \\ a_2a_1a_1.a_2a_1a_2 & (1, 2) \\ a_2a_1a_1a_2.a_1a_2 & (1, 1) \\ a_2a_1a_1a_2a_1.a_2 & (0, 1) \\ a_2a_1a_1a_2a_1a_2. & (0, 0) \end{array}$$

With every next reading, exactly one of the terms in (x_1, \dots, x_n) decrements. We get a bijection between all words w of length $(k + 1)n$ in the alphabet $\{a_1, \dots, a_n\}$ and maximal chains in the lattice C_{k+1}^n — the points of the chain, as we descend down the chain, are given by the sequence of n -tuples (x_1, \dots, x_n) . If we could show that the condition (*) on (x_1, \dots, x_n) is equivalent to the requirement on the prefix z read (which resulted in the tuple (x_1, \dots, x_n)) given in the theorem, then we would be done. However, as we will now see, this equivalence actually fails. First, note the following:

- $x_i = k + 1$ if and only if a_i does not occur in z .

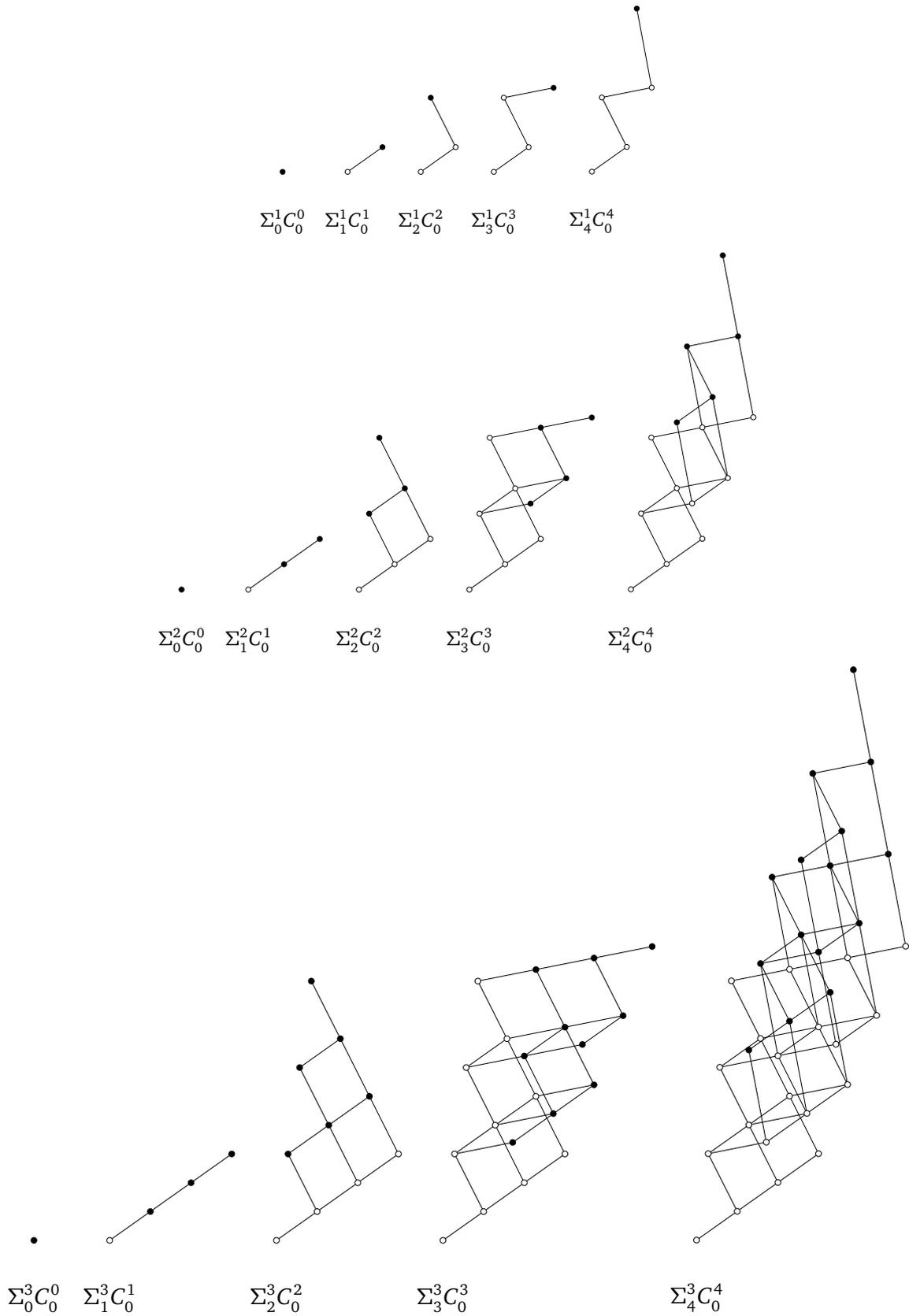


FIGURE 2. First, second and third iteration of stacking the trivial lattice — in the second and third iteration, maximal chains in these lattices are given by Catalan numbers and 3-dimensional Catalan numbers, respectively.

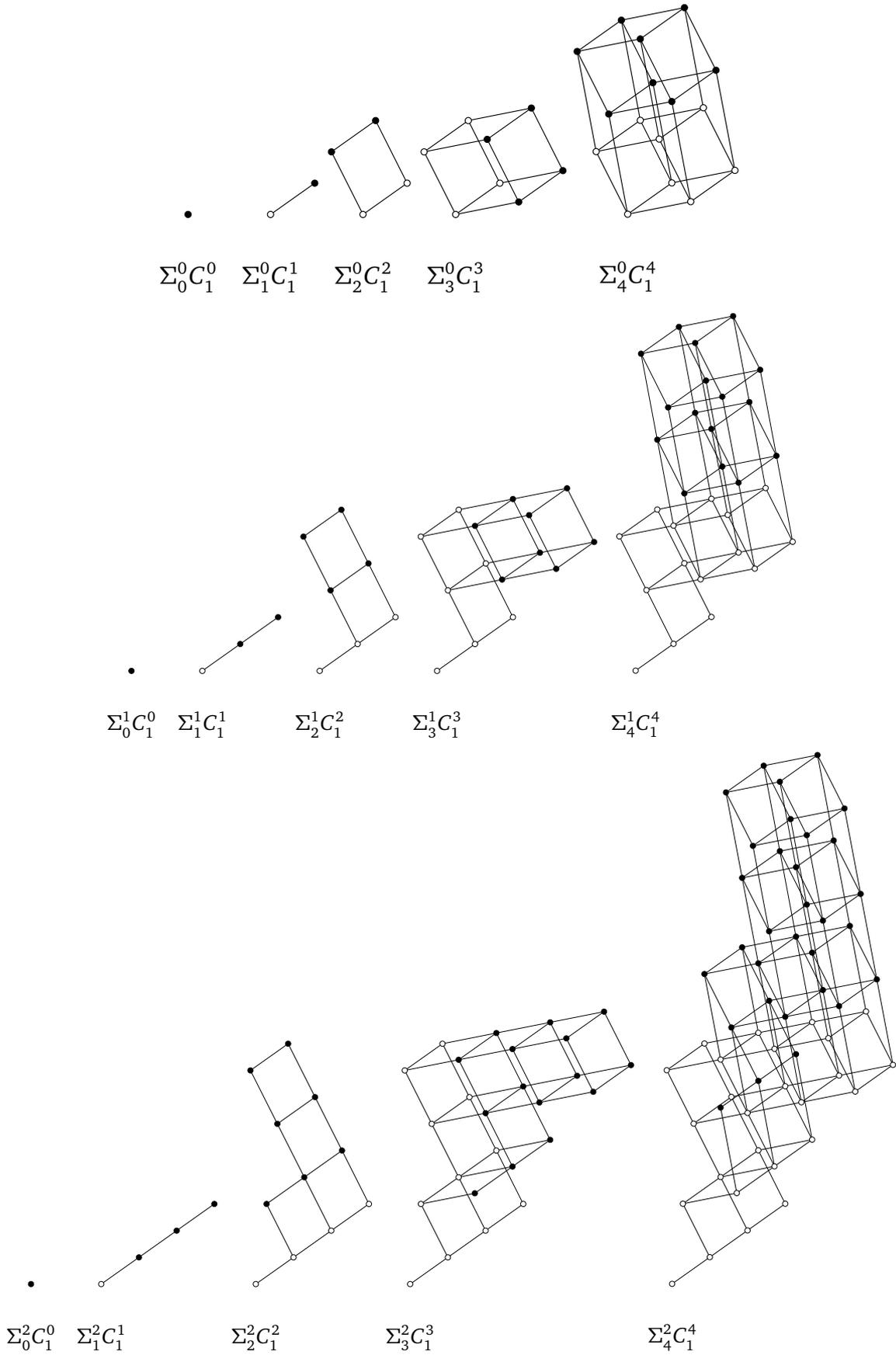


FIGURE 3. Stacking hypercubes — zeroth, first and second iteration.

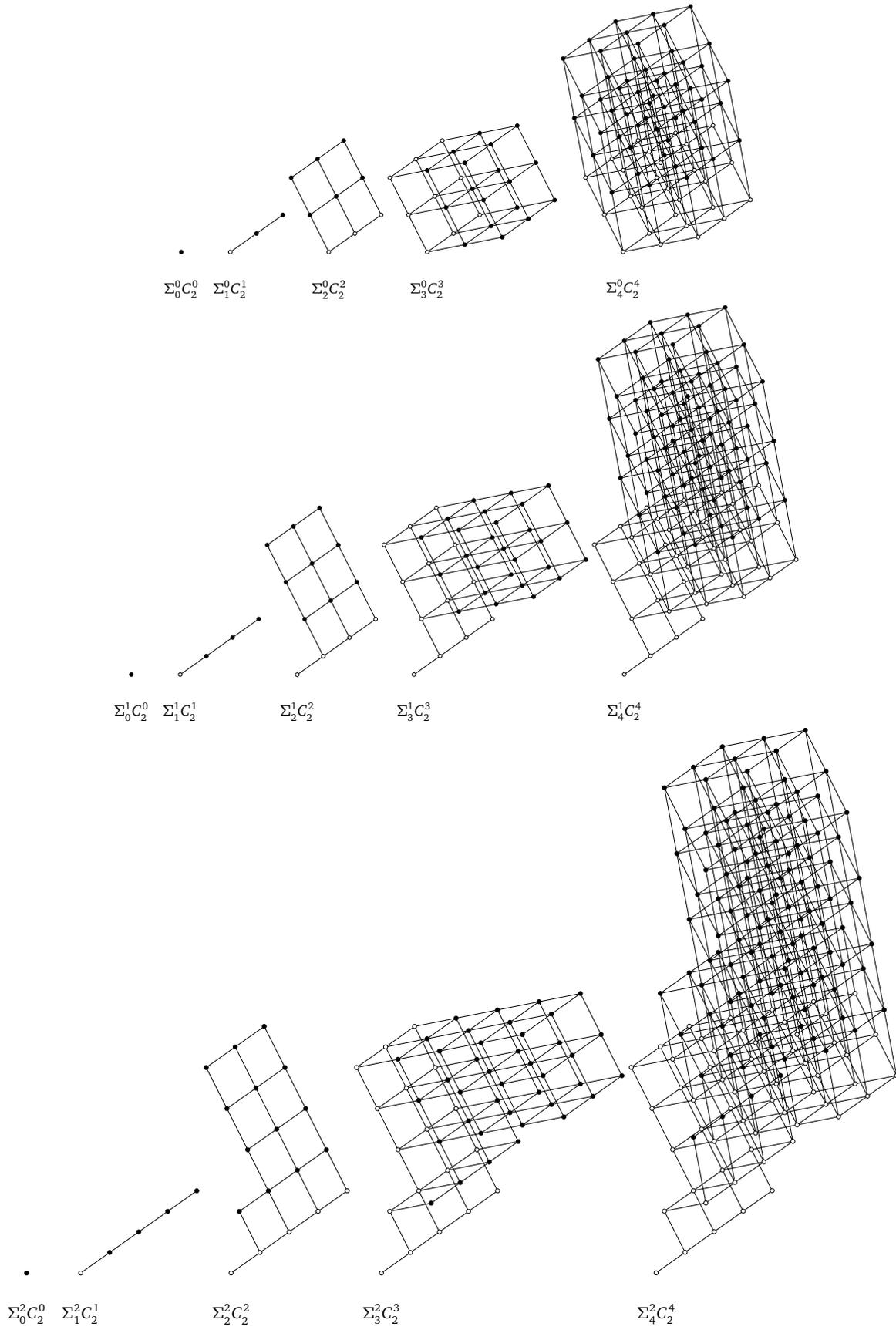


FIGURE 4. Lattices $\Sigma_n^k C_2$ for $k \in \{0, 1, 2\}$ and $n \in \{0, 1, 2, 3, 4\}$.

$n = 0, 1, 2, 3, 4, 5 \dots$	
$m = 0 :$	
$k = 0$	1, 1, 1, 1, 1, 1, ... (A000012)
$k = 1$	1, 1, 1, 1, 1, 1, ... (A000012)
$k = 2$	1, 1, 2, 5, 14, 42, ... (A000108 – Catalan numbers)
$k = 3$	1, 1, 5, 42, 462, 6006, ... (A005789 – 3-dimensional Catalan numbers)
$k = 4$	1, 1, 14, 462, 24024, 1662804, ... (A005790 – 4-dimensional Catalan numbers)
$m = 1 :$	
$k = 0$	1, 1, 2, 6, 24, 120, ... (A000142 – factorial numbers)
$k = 1$	1, 1, 3, 15, 105, 945, ... (A001147 – double factorial numbers)
$k = 2$	1, 1, 7, 106, 2575, 87595, ... (A213863 – no explicit formula)
$k = 3$	1, 1, 19, 1075, 115955, 19558470, ... (A213864 – no explicit formula)
$k = 4$	1, 1, 56, 13326, 7364321, 7236515981, ... (A213865 – no explicit formula)
$m = 2 :$	
$k = 0$	1, 1, 6, 90, 2520, 113400, ... (A000680 – $(2n)!/2^n$)
$k = 1$	1, 1, 10, 280, 15400, 1401400, ... (A025035 – $(3n)!/(3!)^n n!$)
$k = 2$	1, 1, 25, 2305, 482825, 183500625, ... (no entry)
$k = 3$	1, 1, 71, 25911, 25754021, 52213860026, ... (no entry)
$k = 4$	1, 1, 216, 345651, 1848745731, 23070700145026, ... (no entry)
$m = 3 :$	
$k = 0$	1, 1, 20, 1680, 369600, 168168000, ... (A014606 – $(3n)!/(3!)^n$)
$k = 1$	1, 1, 35, 5775, 2627625, 2546168625, ... (A025036 – $(4n)!/(4!)^n n!$)
$k = 2$	1, 1, 91, 51821, 94597041, 404793761526, ... (no entry)
$k = 3$	1, 1, 266, 621831, 5616763761, 134269580611026, ... (no entry)
$k = 4$	1, 1, 827, 8721245, 438307511209, 66953592509190248, ... (no entry)

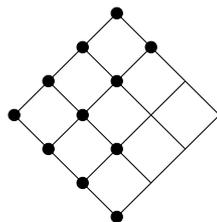
FIGURE 5. The maximal chain numbers $\#\Sigma_n^k C_m^n$ and the corresponding entry code (indicated in brackets) in the On-Line Encyclopedia of Integer Sequences (accessed 9/08/2020).

- $x_i \leq x_j$ if and only if the number of occurrences of a_i in z is more or the same as the number of occurrences of a_j .

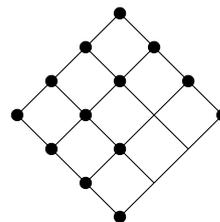
So the requirement in the theorem on a prefix z translates to the following requirement on the corresponding tuple (x_1, \dots, x_n) .

(*)' For each $i \in \{1, \dots, n\}$, either $x_i = k + 1$ or $x_i \leq x_j$ for each $j > i$.

To see that (*) is not equivalent to (*)', consider the tuple $(3, 0)$ (for $n = 2$ and $k = 2$). For this tuple, (*)' holds, but (*) does not. To see what to do next, we look at an illustration of each of the two types of tuples, seen as points in C_{k+1}^n , in the case when $n = 2$ and $k = 2$:



Tuples allowed by (*)



Tuples allowed by (*)'

To prove the theorem it is sufficient to show what these pictures suggest: that $(*)$ implies $(*)'$ for individual points (tuples), and points satisfying $(*)'$ but not $(*)$, can never be encountered on a maximal chain whose all points satisfy $(*)'$.

Suppose $(*)$ holds for a point (x_1, \dots, x_n) and $x_i \neq k + 1$ for some $i \in \{1, \dots, n\}$. Assume the contrary to what $(*)'$ requires: that $x_i > x_j$ for some $j > i$. Since $x_i \leq k$, we get $x_j < k$ and iteratively applying $(*)$ results in $x_i \leq x_{i+1} \leq \dots \leq x_j$. This contradicts the assumption $x_i > x_j$. So $(*)'$ follows from $(*)$. Conversely, suppose $(*)'$ holds. Let $x_{i+1} \in \{0, \dots, k-1\}$ and suppose unlike what $(*)$ requires, we have $x_i > x_{i+1}$. Then $(*)'$ forces $x_i = k + 1$. Suppose the point (x_1, \dots, x_n) is encountered in some maximal chain. Then at some descend along the chain there is a point (y_1, \dots, y_n) with either $y_{i+1} = x_{i+1} - 1$ and $y_i = x_i$, or $y_{i+1} = x_{i+1}$ and $y_i = x_i - 1 = k$. The second case would violate $(*)'$. For the first case, we repeat the same argument. Eventually, we end up with the second case, and so a maximal chain containing a point that satisfies $(*)'$ but not $(*)$, will also contain a point that does not satisfy $(*)'$. This completes the proof. \square

3. SOME CONCEPTUAL REMARKS ON STACKING LATTICES

As remarked in the Introduction, the process of stacking lattices in a sequence comes from a construction of ‘lax colimit’ in category theory. In this section we elaborate a bit on this remark. We will be concerned with the 2-category of posets, which we denote by **Pos**. Note that this is equivalent to a subcategory of the 2-category of categories, consisting of those categories where any two parallel morphisms are equal and any isomorphism is an identity morphism. More explicitly:

- Objects in **Pos** are posets — partially ordered sets, i.e., sets equipped with a reflexive, transitive and antisymmetric binary relation.
- Morphisms are monotone maps between posets, i.e., maps which preserve the relation.
- Composition of morphisms is defined by composition of maps.
- For any two posets L and M , the category structure on the set of morphisms from L to M is a poset structure given by setting $f \leq g$ when $f(x) \leq g(x)$ for each $x \in L$. In other words, a 2-cell between two morphisms f and g is a relation $f \leq g$ (it either does not exist, or is unique, for a given f and g).

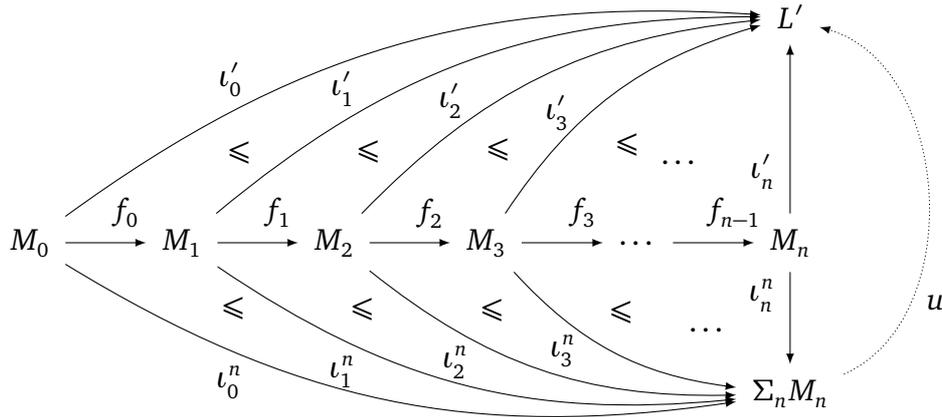
Given a sequence

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \xrightarrow{f_4} \dots \xrightarrow{f_{n-1}} M_n$$

of objects and morphisms in **Pos**, we define its *lax sum* as a specialization of the notion of a lax colimit of a diagram in a 2-category. It is given by an object $L = \Sigma_n M_n$ in **Pos**, equipped with morphisms $\iota_j^n: M_j \rightarrow L$, where $j \in \{0, \dots, n\}$, such that the following conditions hold:

- $\iota_j^n \leq \iota_{j+1}^n \circ f_j$ for each $j \in \{1, \dots, n-1\}$.
- For any object L' and morphisms ι'_j (where $j \in \{0, \dots, n\}$) such that $\iota'_j \leq \iota'_{j+1} \circ f_j$ for each $j \in \{1, \dots, n-1\}$, there exists a unique morphism $u: L \rightarrow L'$ such that $u \circ \iota_j^n = \iota'_j$ for each j .

This property can be pictured as follows:



In the picture, each occurrence of the symbol \leqslant stands for a 2-cell between the composites of the surrounding diagram, indicating that the surrounding diagram *lax commutes*. In other words, they represent the relations $\iota_{j+1}^n \circ f_j = \iota_j^n$ and $\iota'_{j+1} \circ f_j = \iota'_j$.

As any object defined by a universal property, lax sum is unique up to an isomorphism: two lax sums of the same diagram will be connected by morphisms in both directions from the above universal property, which will turn out to be inverses of each other.

Concretely, a lax sum of a sequence displayed above can be constructed as follows:

- Start by taking the disjoint union of all the posets M_j . Let (x, j) denote the representative of $x \in M_j$ in the disjoint union.
- To turn the disjoint union of posets into a poset, equip it with the relations $(x, j) \leqslant (y, j+i)$ for each

$$(f_{j+i-1} \circ \cdots \circ f_j)(x) \leqslant y.$$

This includes the possibility $i = 0$, in which case $(f_{j+i-1} \circ \cdots \circ f_i)(x)$ is defined as x .

- Each map ι_j^n is then defined by $\iota_j^n(x) = (x, j)$.

It is not difficult to show that this construction has the universal property required from a lax sum. We call this the *concrete lax sum*. Note that a concrete lax sum always exists and any lax sum is canonically isomorphic to a concrete one.

There is also another (equivalent) conceptual interpretation of a lax sum. Thinking of the given sequence of posets as a functor from the chain C_n seen as a category, into the category of categories, the lax sum is nothing other than the category (which in this case happens to be a poset) arising from the (dual of) standard Grothendieck construction in the theory of fibrations [7].

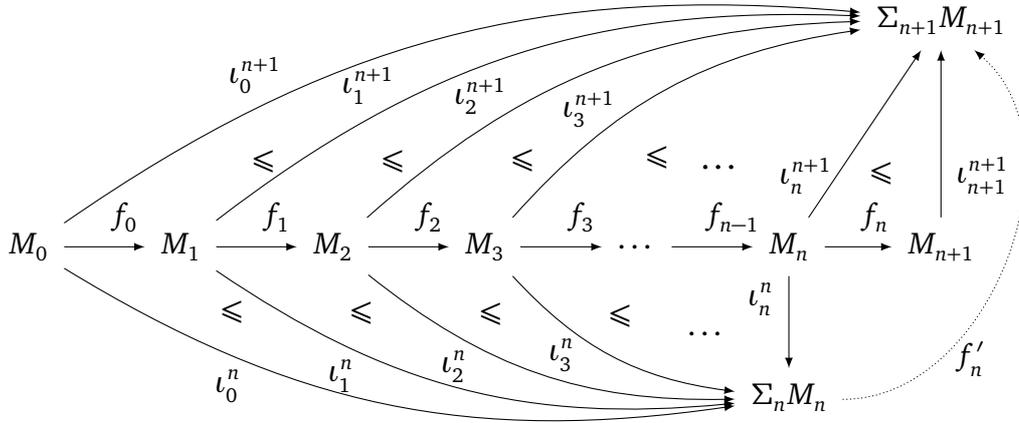
A sequence

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \xrightarrow{f_4} \cdots$$

of posets and monotone maps (can be an infinite sequence or a finite one) gives rise to another such sequence,

$$\Sigma_0 M_0 \xrightarrow{f'_0} \Sigma_1 M_1 \xrightarrow{f'_1} \Sigma_2 M_2 \xrightarrow{f'_2} \Sigma_3 M_3 \xrightarrow{f'_3} \Sigma_4 M_4 \xrightarrow{f'_4} \cdots$$

where each f'_n arises from the universal property of lax sum $\Sigma_n M_n$, as shown in the following picture:



As a morphism between concrete lax sums, the definition of f'_n is simple:

$$f'_n(x, n) = (x, n),$$

for all $x \in M_n$.

Iterating the process above gives the diagram in Figure 6, where we do not distinguish in notation the ι 's arising at different iterations.

We may think of $\Sigma_n M_n$ as the n -th 'partial sum' of the infinite 'series' of posets. Note that $\Sigma_0 M_0 = M_0$. The process of stacking lattices considered in this paper is given by this notion of 'partial sum'. Recall that a lattice is a poset having binary joins and meets. The fact that in all situations considered in this paper, stacking lattices results in a lattice, comes from the following result:

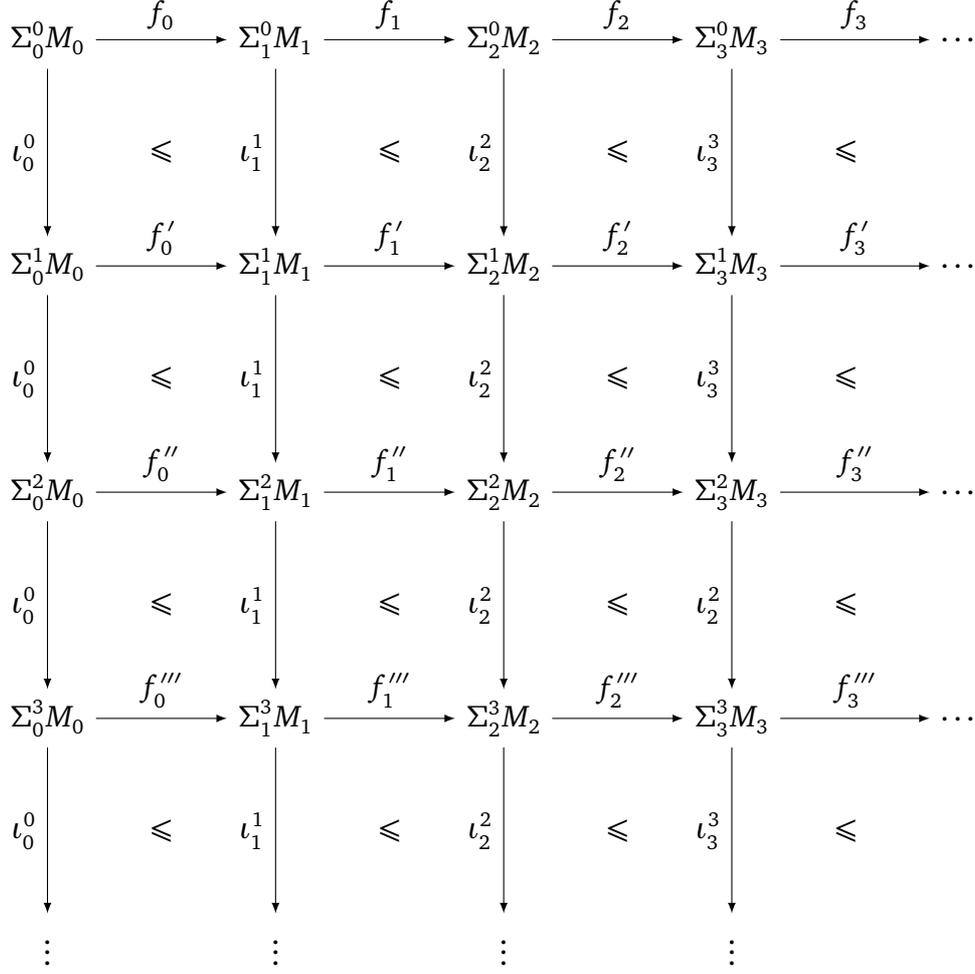
Theorem 5. For any sequence $f_i: M_i \rightarrow M_{i+1}$ of posets, the following hold:

- (1) Each $f'_i: \Sigma_i M_i \rightarrow \Sigma_{i+1} M_{i+1}$ is order-reflecting (hence injective) and has down-closed image. Moreover, each $\iota_j^n: M_j \rightarrow \Sigma_n M_n$ is order-reflecting (and hence injective).
- (2) If M_n has empty meet (top element), then so does $\Sigma_n M_n$, and $\iota_n^n: M_n \rightarrow \Sigma_n M_n$ preserves empty meet.
- (3) If each M_i has empty join (bottom element) and each $f_i: M_i \rightarrow M_{i+1}$ preserves it, then each $\Sigma_n M_n$ has empty join and $\iota_0^n: M_0 \rightarrow \Sigma_n M_n$, as well as each $f'_i: \Sigma_i M_i \rightarrow \Sigma_{i+1} M_{i+1}$, preserves empty join.
- (4) If each M_i has binary joins and each $f_i: M_i \rightarrow M_{i+1}$ preserves them, then each $\Sigma_n M_n$ has binary joins, and moreover, each $f'_i: \Sigma_i M_i \rightarrow \Sigma_{i+1} M_{i+1}$, as well as each $\iota_j^n: M_j \rightarrow \Sigma_n M_n$, preserves binary joins.
- (5) If each M_i has binary meets and each $f_i: M_i \rightarrow M_{i+1}$ is order-reflecting with down-closed image, then each $\Sigma_n M_n$ has binary meets and each $\iota_j^n: M_j \rightarrow \Sigma_n M_n$ preserves binary meets (as does each f_i and f'_i).

Proof. Recall that $\iota_j^n(x) = (x, j)$, and: $(x, j) \leq (y, k)$, when $j \leq k$ and $(f_{k-1} \circ \dots \circ f_j)(x) \leq y$. Furthermore, recall that each $f'_i: \Sigma_i M_i \rightarrow \Sigma_{i+1} M_{i+1}$ is defined by

$$f'_i(x, j) = (x, j), \quad j \leq i.$$

It is then clear that the image of each f'_i is down-closed and order-reflecting, as well as that each ι_j^n is order-reflecting, so we have (1) — note that an order-reflective monotone map is always injective.


 FIGURE 6. Iterated lax sums for a sequence $f_i: M_i \rightarrow M_{i+1}$ of poset morphisms

When M_n has top element t , the top element in $\Sigma_n M_n$ is given by (t, n) . This yields (2).

Suppose M_0 has bottom element. If each f_i preserves the bottom element, then $(b, 0)$ is the bottom element of $\Sigma_n M_n$, where b is the bottom element of M_0 . This yields (3).

Suppose now each M_i has binary joins and each f_i preserves them. Consider two elements (x, j) and (y, k) of $\Sigma_n M_n$, with $j \leq k$. Define

$$(x, j) \vee (y, k) = ((f_{k-1} \circ \dots \circ f_j)(x) \vee y, k),$$

where the join appearing on the right hand side of the equality is the join in M_k . We will now show that $(x, j) \vee (y, k)$ is the join of (x, j) and (y, k) in $\Sigma_n M_n$. It is easy to see that $(x, j) \leq (x, j) \vee (y, k)$ and $(y, k) \leq (x, j) \vee (y, k)$. Suppose $(x, j) \leq (z, l)$ and $(y, k) \leq (z, l)$. Then $k \leq l$ and

$$(f_{l-1} \circ \dots \circ f_j)(x) \vee (f_{l-1} \circ \dots \circ f_k)(y) \leq z.$$

On the other hand, since each f preserves joins,

$$(f_{l-1} \circ \dots \circ f_k)((f_{k-1} \circ \dots \circ f_j)(x) \vee y) = (f_{l-1} \circ \dots \circ f_j)(x) \vee (f_{l-1} \circ \dots \circ f_k)(y).$$

This shows $(x, j) \vee (y, k) \leq (z, l)$. We have thus proved that $\Sigma_n M_n$ has binary joins. The formula for join in $\Sigma_n M_n$ established above guarantees that each f'_i , as well as each ι_j^n , preserve joins. This proves (4).

Suppose each M_i has binary meets, and, each f_i is order-reflecting with down-closed image. Then, each f_i is injective and preserves binary meets. Thanks to (1), the same is true for each

f'_i . We now show that each $\Sigma_n M_n$ has binary meets. Consider two elements (x, j) and (y, k) in $\Sigma_n M_n$. Then $x \in M_j$ and $y \in M_k$, where $j \leq n$ and $k \leq n$. Without loss of generality, assume $j \leq k$. Consider the meet

$$z = (f_{k-1} \circ \cdots \circ f_j)(x) \wedge y$$

in M_k . Let z' be the unique element of M_j such that

$$(f_{k-1} \circ \cdots \circ f_j)(z') = z.$$

We will prove

$$(z', j) = (x, j) \wedge (y, k).$$

Note that $(z', j) \leq (y, k)$. Moreover, since $z \leq (f_{k-1} \circ \cdots \circ f_j)(x)$, we get $z' \leq x$ by the order-reflection property of the composite $f_{k-1} \circ \cdots \circ f_j$. So $(z', j) \leq (x, j)$. Now suppose $(w, l) \leq (x, j)$ and $(w, l) \leq (y, k)$. Then $l \leq j$. Moreover,

$$(f_{j-1}, \dots, f_l)(w) \leq x \text{ and } (f_{k-1}, \dots, f_l)(w) \leq y.$$

This implies $(f_{k-1}, \dots, f_l)(w) \leq z$, which by order-reflection gives

$$(f_{j-1}, \dots, f_l)(w) \leq z'.$$

Then $(w, l) \leq (z', j)$, as desired. So (z', j) is indeed the meet of (x, j) and (y, k) . In the case when $j = k$, we get $z' = x \wedge y$, from which it follows at once that u_j^n preserves binary meets. This proves (5). \square

The sequences of lattices that we dealt with in the paper satisfy all assumptions of the theorem above. Let us call such sequence of lattices a *lattice series*. Thus, a lattice series is an infinite sequence

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \xrightarrow{f_4} \cdots$$

of lattices and homomorphisms of join semi-lattices, such that f_i 's are order-reflecting and have down-closed images, and hence also preserve binary meets and are injective. A lattice series is a model of the self-dual axiomatic context for isomorphism theorems described in [5]: it is an example of a 'noetherian form' [11]. The following construction, used already in the proof of the theorem above, is related to 'diagram chasing' in a noetherian form. We introduce below a short-hand notation for it.

For $x \in M_j$ and $k \geq j$, define

$$x^k = \begin{cases} x, & k = j, \\ (f_{k-1} \circ \cdots \circ f_j)(x), & k > j. \end{cases}$$

For $y \in M_k$, if $y = (f_{k-1} \circ \cdots \circ f_j)(x)$ for $x \in M_j$ and $j \leq k$, then this x , necessarily unique, will be denoted by $x = y^{-j}$. According to Theorem 5 above, the sequence of partial sums of a lattice series is again a lattice series. The proof of Theorem 5 shows that the joins and the meets in the poset $\Sigma_n M_n$ can be expressed in terms of joins and meets in each M_i as follows:

- $(x, j) \vee (y, k) = (x^k \vee y, k)$ when $j \leq k$,
- $(x, j) \wedge (y, k) = ((x^k \wedge y)^{-j}, j)$ when $j \leq k$.

We furthermore have the following rules; the last three of these rules are consequences of the properties of the homomorphisms f_i in a lattice series:

- $(x^j)^k = x^k$ when $j \leq k$ and $x \in M_i$, where $i \leq j$,
- $(x^{-i})^j = x^{-j}$ when $i \leq j \leq k$ and $x \in M_k$ is such that x^{-i} is defined,
- $(x^k)^{-j} = x^j$ when $i \leq j \leq k$ and $x \in M_i$,
- $(x^k)^{-i} = x^{-i}$ when $i \leq j \leq k$ and $x \in M_j$ with x^{-i} defined,

- $(x \wedge y)^j = x^j \wedge y^j$ when $x, y \in M_i$ with $i \leq j$,
- $(x \vee y)^j = x^j \vee y^j$ when $x, y \in M_i$ with $i \leq j$,
- $(x \vee y)^{-j} = x^{-j} \vee y^{-j}$ when $x, y \in M_k$ with $j \leq k$, and $(x \vee y)^{-j}$ is defined.

With these rules we can establish the following:

Theorem 6. *If in a lattice series, each lattice is a distributive lattice, i.e., the identity*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

holds in it, then each partial sum of the series is also a distributive lattice.

Proof. A lattice is distributive if and only if the identity

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$$

holds in it (see e.g. [2]). Since this identity is symmetric in x, y, z , to establish it in the partial sum of a lattice series it is sufficient to prove it for $(x, j), (y, k), (z, l)$ where $j \leq k \leq l$. This can be done as follows:

$$\begin{aligned} & ((x, j) \vee (y, k)) \wedge ((y, k) \vee (z, l)) \wedge ((z, l) \vee (x, j)) \\ &= (x^k \vee y, k) \wedge (y^l \vee z, l) \wedge (z \vee x^l, l) \\ &= (x^k \vee y, k) \wedge ((y^l \vee z) \wedge (z \vee x^l), l) \\ &= (((x^k \vee y)^l \wedge (y^l \vee z) \wedge (z \vee x^l))^{-k}, k) \\ &= (((x^l \vee y^l) \wedge (y^l \vee z) \wedge (z \vee x^l))^{-k}, k) \\ &= (((x^l \wedge y^l) \vee (y^l \wedge z) \vee (z \wedge x^l))^{-k}, k) \\ &= ((x^l \wedge y^l)^{-k} \vee (z \wedge x^l)^{-k} \vee (y^l \wedge z)^{-k}, k) \\ &= (((x^l \wedge y^l)^{-j} \vee (z \wedge x^l)^{-j})^k \vee (y^l \wedge z)^{-k}, k) \\ &= (((x^k \wedge y)^{-j} \vee (z \wedge x^l)^{-j})^k \vee (y^l \wedge z)^{-k}, k) \\ &= ((x^k \wedge y)^{-j} \vee (z \wedge x^l)^{-j}, j) \vee ((y^l \wedge z)^{-k}, k) \\ &= ((x^k \wedge y)^{-j}, j) \vee ((y^l \wedge z)^{-k}, k) \vee ((z \wedge x^l)^{-j}, j) \\ &= ((x, j) \wedge (y, k)) \vee ((y, k) \wedge (z, l)) \vee ((z, l) \wedge (x, j)). \end{aligned}$$

□

All lattices in Figure 1 are distributive. Moreover, all maps there are join-preserving and order-reflecting, with down-closed images. So Theorems 5 and 6 allow us to conclude that iterated stacking of these lattices along rows or columns (or in fact, along any path of consecutive arrows, for that matter), will always give rise to distributive lattices. Note that the fact that the lattices in Figure 1 are distributive itself follows from the theorems above: C_m can be obtained by staking C_0 along identity morphisms; then, C_m^n is a cartesian power of a distributive lattice, and so is distributive.

Remark 7. *The link with noetherian forms briefly mentioned above suggest to look for lattice series that arise as series of lattices of substructures of group-like structures, given by sequences of group homomorphisms. A combinatorial investigation of such lattice series, similar to what we did in the present paper for the columns of Figure 1, would be interesting. These lattices would be far from being distributive, unlike the ones considered in the present paper. Lattice series of the present paper all live inside the ‘noetherian form of distributive lattices’, studied in depth in [6]. The series C_1^∞ can be realized as a series of substructures in another noetherian form — that of sets and partial bijections: the lattice C_1^n is of course nothing other than the lattice (Boolean algebra) of all subsets of an n -element set.*

4. PROOF OF THEOREM 3

The proof of Theorem 3 that we present below relies on further analysis of the concept of lax sum. We will show first that each of the squares in Figure 6 is a ‘co-comma’ diagram of posets seen as categories, which we simply call a ‘lax pushout’; note however that it is not an instance of a lax colimit — rather, it is an instance of (another) type of ‘indexed colimit’ (also called a ‘weighted colimit’) in a 2-category [8]. This will provide another way of constructing the posets $\Sigma_n^k M_n$: the morphisms ι_0^0 are identity morphisms and so we can start by forming the top left square, the one next to it, and so on in each row successively.

A diagram

$$\begin{array}{ccc}
 K & \xrightarrow{f} & M \\
 g \downarrow & & \downarrow g' \\
 N & \xrightarrow{f'} & L
 \end{array}$$

in **Pos** is a *lax pushout* if and only if:

- f' and g' are order-reflecting morphisms of posets (hence injective) whose images are disjoint, with the union of images giving the entire L ,
- the image of f' is down-closed and for any $a \in N$ and $b \in M$, we have $f'(a) \leq g'(b)$ if and only if there exists $c \in K$ such that $a \leq g(c)$ and $f(c) \leq b$.

It is easy to see that such a diagram has the following universal property (that is a specialization of the construction of a co-comma object in a general 2-category):

- the square lax commutes, i.e., there is a 2-cell from $f' \circ g$ to $g' \circ f$ (thus, $f' \circ g \leq g' \circ f$),
- for any other lax commuting square over f, g ,

$$\begin{array}{ccc}
 K & \xrightarrow{f} & M \\
 g \downarrow & \llcorner & \downarrow g'' \\
 N & \xrightarrow{f''} & L'
 \end{array}$$

there is a unique morphism u such that $u \circ f' = f''$ and $u \circ g' = g''$, as shown on the picture:

$$\begin{array}{ccc}
 K & \xrightarrow{f} & M \\
 g \downarrow & \llcorner & \downarrow g' \\
 N & \xrightarrow{f'} & L \\
 & & \downarrow u \\
 & & L'
 \end{array}$$

\curvearrowright g'' (curved arrow from M to L')
 \curvearrowright f'' (curved arrow from N to L')

Just as any other (weighted) colimit, a lax pushout is unique up to a canonical isomorphism. It exists for any given f and g as we can construct one *concretely* as follows:

- Let the elements of L be pairs of the form $(a, 1)$, where $a \in N$, as well as pairs of the form $(b, 2)$, where $b \in M$.
- Define $(a, 1) \leq (a', 1)$ when $a \leq a'$ in N .
- Define $(b, 2) \leq (b', 2)$ when $b \leq b'$ in N .
- Define $(a, 1) \leq (b, 2)$ when there exists $c \in K$ such that $a \leq g(c)$ and $f(c) \leq b$.
- Define f' and g' by the identities $f'(a) = (a, 1)$ and $g'(b) = (b, 2)$.

Lemma 8. *Each square in Figure 6 is a lax pushout.*

Proof. It suffices to prove this for the squares in the first row, since the rest of the rows are obtained by iteration. So we prove that for each n , the following square is a lax pushout:

$$\begin{array}{ccc}
 M_n & \xrightarrow{f_n} & M_{n+1} \\
 \downarrow \iota_n^n & \llcorner & \downarrow \iota_{n+1}^{n+1} \\
 \Sigma_n M_n & \xrightarrow{f'_n} & \Sigma_{n+1} M_{n+1}
 \end{array}$$

For this, we need to check the following:

- f'_n an order-preserving morphism of posets (and hence is injective), having down-closed image.
- ι_{n+1}^{n+1} is an order-reflecting morphism of posets (and hence injective).
- The images of f'_n and ι_{n+1}^{n+1} are disjoint, and their union is the entire $\Sigma_{n+1} M_{n+1}$.
- For any $a \in \Sigma_n M_n$ and $b \in M_{n+1}$, we have $f'_n(a) \leq \iota_{n+1}^{n+1}(b)$ if and only if there exists $c \in K$ such that $a \leq \iota_n^n(c)$ and $f_n(c) \leq b$.

We know (i) and (ii) already from Theorem 5. Since f'_n is defined by $f'_n(x, j) = (x, j)$, where $j \leq n$ and $x \in M_j$, and $\iota_{n+1}^{n+1}(x) = (x, n+1)$ where $x \in M_{n+1}$, the images of these two functions are clearly disjoint. Their union is the entire $\Sigma_{n+1} M_{n+1}$ since the latter only consists of elements of the form (x, j) , where $x \in M_j$ with $j \leq n+1$. So we have (iii). For any $(x, j) \in \Sigma_n M_n$ and $y \in M_{n+1}$, we have $f'_n(x, j) \leq \iota_{n+1}^{n+1}(y)$ if and only if $(x, j) \leq (y, n+1)$. This is the case if and only if $x^{n+1} \leq y$ (see the previous section for this notation). If $x^{n+1} \leq y$, then define $c = x^n \in M_n$. We get $(x, j) \leq (c, n) = \iota_n^n(c)$ and $f_n(c) = x^{n+1} \leq y$. Conversely, if there exists $c \in M_n$ such that $(x, j) \leq \iota_n^n(c) = (c, n)$ and $f_n(c) \leq y$, then $x^n \leq c$ and so $x^{n+1} = f_n(x^n) \leq f_n(c) \leq y$, which gives $(x, j) \leq (y, n+1)$. The proof is now complete. \square

We are now ready to prove Theorem 3.

First, we show the following (which, actually, does not require any of what we have done above):

Lemma 9. *The elements of C_{k+m}^n satisfying (*) are closed under meets and joins in C_{k+m}^n .*

Proof. Suppose both (x_1, \dots, x_n) and (x'_1, \dots, x'_n) satisfy (*). Their meet in C_{k+m}^n is given by

$$(x_1, \dots, x_n) \wedge (x'_1, \dots, x'_n) = (\min(x_1, x'_1), \dots, \min(x_n, x'_n)).$$

Suppose $\min(x_{i+1}, x'_{i+1}) \in \{1, \dots, k-1\}$. Without loss of generality, assume $x_{i+1} \leq x'_{i+1}$. Then $x_i \leq x_{i+1}$ and so

$$\min(x_i, x'_i) \leq x_{i+1} = \min(x_{i+1}, x'_{i+1}).$$

This proves that tuples satisfying $(*)$ are closed under meets. The join in C_{k+m}^n is given by

$$(x_1, \dots, x_n) \vee (x'_1, \dots, x'_n) = (\max(x_1, x'_1), \dots, \max(x_n, x'_n)).$$

Suppose $\max(x_{i+1}, x'_{i+1}) \in \{1, \dots, k-1\}$. Then both x_{i+1} and x'_{i+1} belong to $\{1, \dots, k-1\}$ and the conclusion

$$\max(x_i, x'_i) \leq \max(x_{i+1}, x'_{i+1})$$

is immediate by the fact that both tuples satisfy $(*)$. \square

To prove Theorem 3, it remains to show that each poset $\Sigma_n^k C_m^n$ is isomorphic to the sublattice of C_{k+m}^n consisting of n -tuples satisfying $(*)$. Consider the diagram

$$\begin{array}{ccccccc}
 C_{0+m}^0 & \longrightarrow & C_{0+m}^1 & \longrightarrow & C_{0+m}^2 & \longrightarrow & C_{0+m}^3 \longrightarrow \dots \\
 \downarrow & \leq & \downarrow & \leq & \downarrow & \leq & \downarrow \\
 C_{1+m}^0 & \longrightarrow & C_{1+m}^1 & \longrightarrow & C_{1+m}^2 & \longrightarrow & C_{1+m}^3 \longrightarrow \dots \\
 \downarrow & \leq & \downarrow & \leq & \downarrow & \leq & \downarrow \\
 C_{2+m}^0 & \longrightarrow & C_{2+m}^1 & \longrightarrow & C_{2+m}^2 & \longrightarrow & C_{2+m}^3 \longrightarrow \dots \\
 \downarrow & \leq & \downarrow & \leq & \downarrow & \leq & \downarrow \\
 C_{3+m}^0 & \longrightarrow & C_{3+m}^1 & \longrightarrow & C_{3+m}^2 & \longrightarrow & C_{3+m}^3 \longrightarrow \dots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \vdots & & \vdots & & \vdots & & \vdots
 \end{array}$$

of lattices C_{k+m}^n , where each vertical arrow $C_{k+m}^n \rightarrow C_{k+1+m}^n$ maps (x_1, \dots, x_n) to $(x_1 + 1, \dots, x_n + 1)$ and each horizontal arrow $C_{k+m}^n \rightarrow C_{k+m}^{n+1}$ maps (x_1, \dots, x_n) to $(0, x_1, \dots, x_n)$. Each square in this diagram lax commutes: left-bottom composite is below the top-right composite. Indeed:

$$(0, x_1 + 1, \dots, x_n + 1) \leq (1, x_1 + 1, \dots, x_n + 1).$$

It is easy to see that both the horizontal and the vertical morphisms preserve the property $(*)$. So the diagram above restricts to the following diagram, where C_{k+m}^{n*} denotes the sublattice of C_{k+m}^n determined by the property $(*)$, while the morphisms are defined in the same way as above:

$$\begin{array}{ccccccc}
 C_{0+m}^{0*} & \longrightarrow & C_{0+m}^{1*} & \longrightarrow & C_{0+m}^{2*} & \longrightarrow & C_{0+m}^{3*} & \longrightarrow & \dots \\
 \downarrow & \llcorner & \downarrow & \llcorner & \downarrow & \llcorner & \downarrow & & \\
 C_{1+m}^{0*} & \longrightarrow & C_{1+m}^{1*} & \longrightarrow & C_{1+m}^{2*} & \longrightarrow & C_{1+m}^{3*} & \longrightarrow & \dots \\
 \downarrow & \llcorner & \downarrow & \llcorner & \downarrow & \llcorner & \downarrow & & \\
 C_{2+m}^{0*} & \longrightarrow & C_{2+m}^{1*} & \longrightarrow & C_{2+m}^{2*} & \longrightarrow & C_{2+m}^{3*} & \longrightarrow & \dots \\
 \downarrow & \llcorner & \downarrow & \llcorner & \downarrow & \llcorner & \downarrow & & \\
 C_{3+m}^{0*} & \longrightarrow & C_{3+m}^{1*} & \longrightarrow & C_{3+m}^{2*} & \longrightarrow & C_{3+m}^{3*} & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \vdots & & \vdots & & \vdots & & \vdots & &
 \end{array}$$

Lemma 8 and the fact that lax pushouts are unique up to a canonical isomorphism, reduces the proof of Theorem 3 to showing that each square in the above diagram is a lax pushout. Indeed, for if it is so, we will be able to recursively create isomorphisms $d_{k,n,m} : \Sigma_n^k C_m^n \rightarrow C_{k+m}^{n*}$, the diagonal arrows in the following diagram,

$$\begin{array}{ccccccc}
 & & C_{0+m}^{0*} & \longrightarrow & C_{0+m}^{1*} & \longrightarrow & C_{0+m}^{2*} & \longrightarrow & C_{0+m}^{3*} & \longrightarrow & \dots \\
 & \nearrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \Sigma_0^0 C_m^0 & \longrightarrow & \Sigma_1^0 C_m^1 & \longrightarrow & \Sigma_2^0 C_m^2 & \longrightarrow & \Sigma_3^0 C_m^3 & \longrightarrow & \dots \\
 \downarrow & & \\
 & \nearrow & C_{1+m}^{0*} & \longrightarrow & C_{1+m}^{1*} & \longrightarrow & C_{1+m}^{2*} & \longrightarrow & C_{1+m}^{3*} & \longrightarrow & \dots \\
 \Sigma_0^1 C_m^0 & \longrightarrow & \Sigma_1^1 C_m^1 & \longrightarrow & \Sigma_2^1 C_m^2 & \longrightarrow & \Sigma_3^1 C_m^3 & \longrightarrow & \dots \\
 \downarrow & & \\
 & \nearrow & C_{2+m}^{0*} & \longrightarrow & C_{2+m}^{1*} & \longrightarrow & C_{2+m}^{2*} & \longrightarrow & C_{2+m}^{3*} & \longrightarrow & \dots \\
 \Sigma_0^2 C_m^0 & \longrightarrow & \Sigma_1^2 C_m^1 & \longrightarrow & \Sigma_2^2 C_m^2 & \longrightarrow & \Sigma_3^2 C_m^3 & \longrightarrow & \dots \\
 \downarrow & & \\
 & \nearrow & C_{3+m}^{0*} & \longrightarrow & C_{3+m}^{1*} & \longrightarrow & C_{3+m}^{2*} & \longrightarrow & C_{3+m}^{3*} & \longrightarrow & \dots \\
 \Sigma_0^3 C_m^0 & \longrightarrow & \Sigma_1^3 C_m^1 & \longrightarrow & \Sigma_2^3 C_m^2 & \longrightarrow & \Sigma_3^3 C_m^3 & \longrightarrow & \dots \\
 \downarrow & & \\
 \vdots & &
 \end{array}$$

starting with identity morphisms along the top and the left side of the picture: note that when $n = 0$ or $k = 0$, we have $\Sigma_n^k C_m^n = C_{k+m}^{n*}$, and so we can set the $d_{0,n,m}$ and $d_{k,0,m}$ diagonal morphisms to be the identity morphisms.

So the final step in the proof is given by the following:

Lemma 10. *For any k, n, m , the following square is a lax pushout:*

$$\begin{array}{ccccc}
 (x_1, \dots, x_n) & \xrightarrow{\quad} & (0, x_1, \dots, x_n) & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & C_{k+m}^{n*} & \xrightarrow{\quad} & C_{k+m}^{n+1*} & (y_1, \dots, y_{n+1}) \\
 & \downarrow & & \downarrow & \downarrow \\
 (x_1 + 1, \dots, x_n + 1) & C_{k+1+m}^{n*} & \xrightarrow{\quad} & C_{k+1+m}^{n+1*} & (y_1 + 1, \dots, y_{n+1} + 1) \\
 & & & & \downarrow \\
 (z_1, \dots, z_n) & \xrightarrow{\quad} & (0, z_1, \dots, z_n) & &
 \end{array}$$

Proof. The bottom map is clearly an order-reflecting morphism of posets, and it is easy to see that its image is down-closed. The right map is also obviously an order-reflecting morphism of posets. The images of these two maps certainly do not intersect. Consider any tuple (x_1, \dots, x_{n+1}) in C_{k+1+m}^{n+1*} . If $x_1 = 0$, then it belongs to the image of the bottom map. If $x_1 \neq 0$, then by the property (*), we can never have $x_j = 0$ for any other coordinate and so $(x_1, \dots, x_{n+1}) = (y_1 + 1, \dots, y_{n+1} + 1)$ for some tuple (y_1, \dots, y_{n+1}) in C_{k+m}^{n+1} . We need to check that (y_1, \dots, y_{n+1}) satisfies (*). Let $y_{i+1} \in \{0, \dots, k-1\}$. Then $y_{i+1} + 1 \in \{0, \dots, k\}$. Since $(y_1 + 1, \dots, y_{n+1} + 1)$ satisfies (*), we have $y_i + 1 \leq y_{i+1} + 1$, and so $y_i \leq y_{i+1}$. This proves $(y_1, \dots, y_{n+1}) \in C_{k+m}^{n+1}$, showing that every element in the bottom right lattice falls in the image of one of the maps going into it. Complete the proof that the square is a lax pushout, it remains to show that $(0, z_1, \dots, z_n) \leq (y_1 + 1, \dots, y_{n+1} + 1)$ for two $(n+1)$ -tuples in C_{k+1+m}^{n+1*} if and only if

$$(z_1, \dots, z_n) \leq (x_1 + 1, \dots, x_n + 1) \text{ and } (0, x_1, \dots, x_n) \leq (y_1, \dots, y_{n+1})$$

for some n -tuple (x_1, \dots, x_n) in C_{k+m}^{n*} . The ‘if’ part is easy to see (it follows from the fact that the square is lax commuting). To show the ‘only if’ part, assume $(0, z_1, \dots, z_n) \leq (y_1 + 1, \dots, y_{n+1} + 1)$ in C_{k+1+m}^{n+1*} . Then the $(n+1)$ -tuple $(y_1, y_2, \dots, y_{n+1})$ belongs to C_{k+m}^{n+1*} , which easily implies that the n -tuple (y_2, \dots, y_n) belongs to C_{k+m}^{n*} . This is the desired n -tuple, since

$$(z_1, \dots, z_n) \leq (y_2 + 1, \dots, y_n + 1) \text{ and } (0, y_2, \dots, y_n) \leq (y_1, \dots, y_{n+1}).$$

This completes the proof. □

5. THE ROWS OF FIGURE 1

In this section we briefly consider the ‘orthogonal’ situation, i.e., iterated stacking of rows of Figure 1 instead of columns. Theorem 3 has the following analogue in this case.

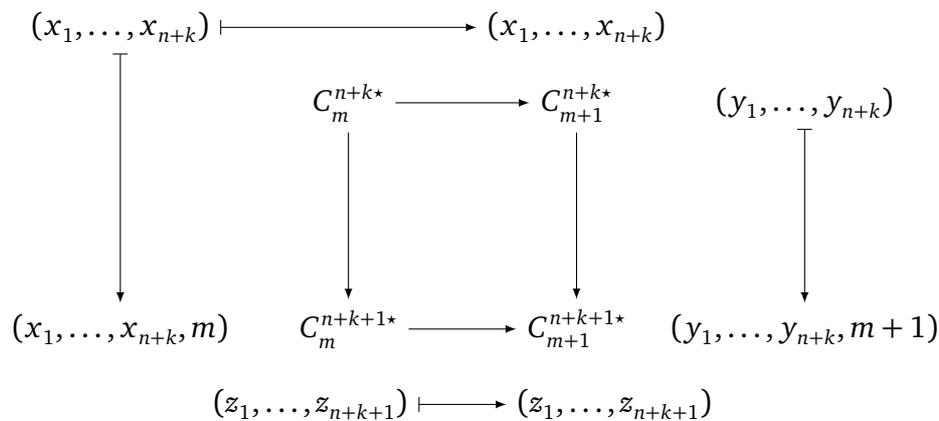
Theorem 11. *The poset $\Sigma_m^k C_m^n$ is isomorphic to the sublattice C_m^{n+k*} of C_m^{n+k} consisting of those elements that satisfy*

$$(*) \text{ for each } i \in \{1, \dots, n\}, x_i \leq x_{n+1} \leq x_{n+2} \leq \dots \leq x_{n+k}.$$

$m = 0, 1, 2, 3, 4, 5, \dots$	
$n = 0 :$	
$k = 0$	1, 1, 1, 1, 1, 1, ... (A000012)
$k = 1$	1, 1, 1, 1, 1, 1, ... (A000012)
$k = 2$	1, 1, 2, 5, 14, 42, ... (A000108 – Catalan numbers)
$k = 3$	1, 1, 5, 42, 462, 6006, ... (A005789 – 3-dimensional Catalan numbers)
$k = 4$	1, 1, 14, 462, 24024, 1662804, ... (A005790 – 4-dimensional Catalan numbers)
$n = 1 :$	
$k = 0$	1, 1, 1, 1, 1, 1, ... (A000012)
$k = 1$	1, 1, 2, 15, 14, 42, ... (A000108 – Catalan numbers)
$k = 2$	1, 1, 5, 42, 462, 6006, ... (A005789 – 3-dimensional Catalan numbers)
$k = 3$	1, 1, 14, 462, 24024, 1662804, ... (A005790 – 4-dimensional Catalan numbers)
$k = 4$	1, 1, 42, 6006, 1662804, 701149020, ... (A005791 – 5-dimensional Catalan numbers)
$n = 2 :$	
$k = 0$	1, 2, 6, 20, 70, 252, ... (A000680 – $(2m)!/(m!)^2$)
$k = 1$	1, 2, 16, 192, 2816, 46592, ... (A006335 – Kreweras)
$k = 2$	1, 2, 46, 2240, 160504, 14594568, ... (no entry)
$k = 3$	1, 2, 140, 30108, 11721144, 6625780016, ... (no entry)
$k = 4$	1, 2, 444, 448272, 1024045836, 3936970992944, ... (no entry)
$n = 3 :$	
$k = 0$	1, 6, 90, 1680, 34650, 756756, ... (A006480 – $(3m)!/(m!)^3$)
$k = 1$	1, 6, 288, 24444, 2738592, 361998432, ... (no entry)
$k = 2$	1, 6, 918, 363984, 234506712, 203517798360, ... (no entry)
$k = 3$	1, 6, 2988, 5753484, 22547430432, 137927632096368, ... (no entry)
$k = 4$	1, 6, 9936, 96198840, 2404039625820, 109858268535649608, ... (no entry)

FIGURE 7. Maximal chain numbers $\#\Sigma_m^k C_m^n$ and the corresponding entry code (indicated in brackets) in the On-Line Encyclopedia of Integer Sequences (accessed 29/09/2020).

The proof of this theorem can be established along the lines of the proof of Theorem 3, where the lax pushout from Lemma 10 would now be replaced with the following lax pushout:



This representation recovers pictures from the Introduction in the case when $k = 1$ and $n = 1, 2$ — see Figures 8 and 9.

Maximal chains in the poset $\Sigma_m C_m^2$ correspond to ‘lattice walks’ on a cartesian plane of length $3m$ starting and ending at $(0, 0)$, remaining in the first quadrant and using only the NE, W, and S

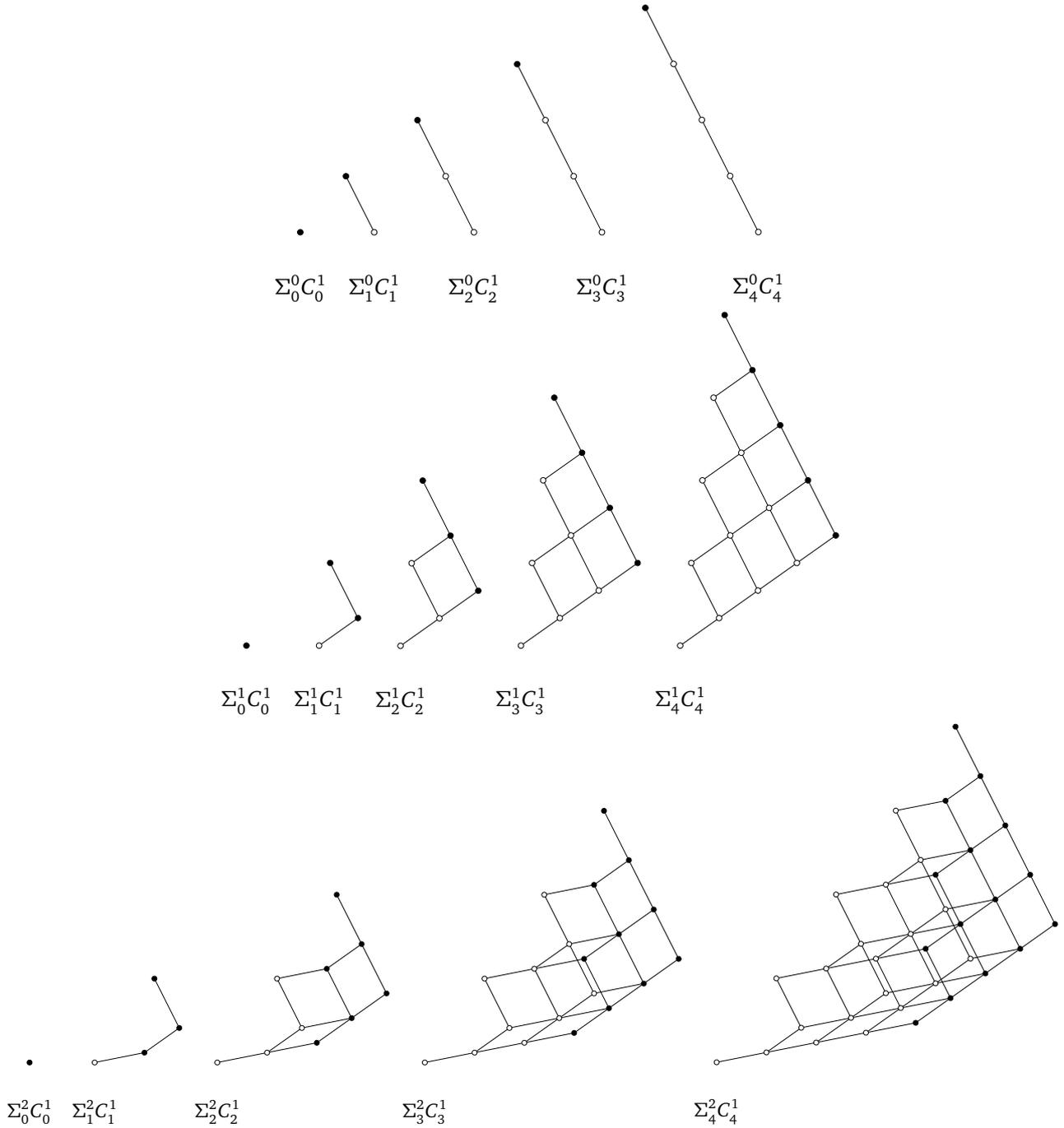


FIGURE 8. Lattices $\Sigma_m^k C_m^1$ for $k \in \{0, 1, 2\}$ and $m \in \{0, 1, 2, 3, 4\}$.

steps, that is, $(1, 1)$, $(-1, 0)$ and $(0, -1)$ steps — this is the Kreweras situation mentioned in the Introduction (see [3, 9]). More generally, we have:

Theorem 12. *There is a bijection between the set of maximal chains in the poset $\Sigma_m C_m^n$ and the set of walks of length $(n + 1)m$ in C_m^n , starting and ending at $(0, \dots, 0)$, and using only $(1, 1, \dots, 1)$, $(-1, 0, \dots, 0)$, $(0, -1, \dots, 0)$, \dots , and $(0, 0, \dots, -1)$ steps.*

Proof (sketch). Using the representation $\Sigma_m C_m^n \approx C_m^{n+1*}$ of Theorem 11, the desired bijection is given by:

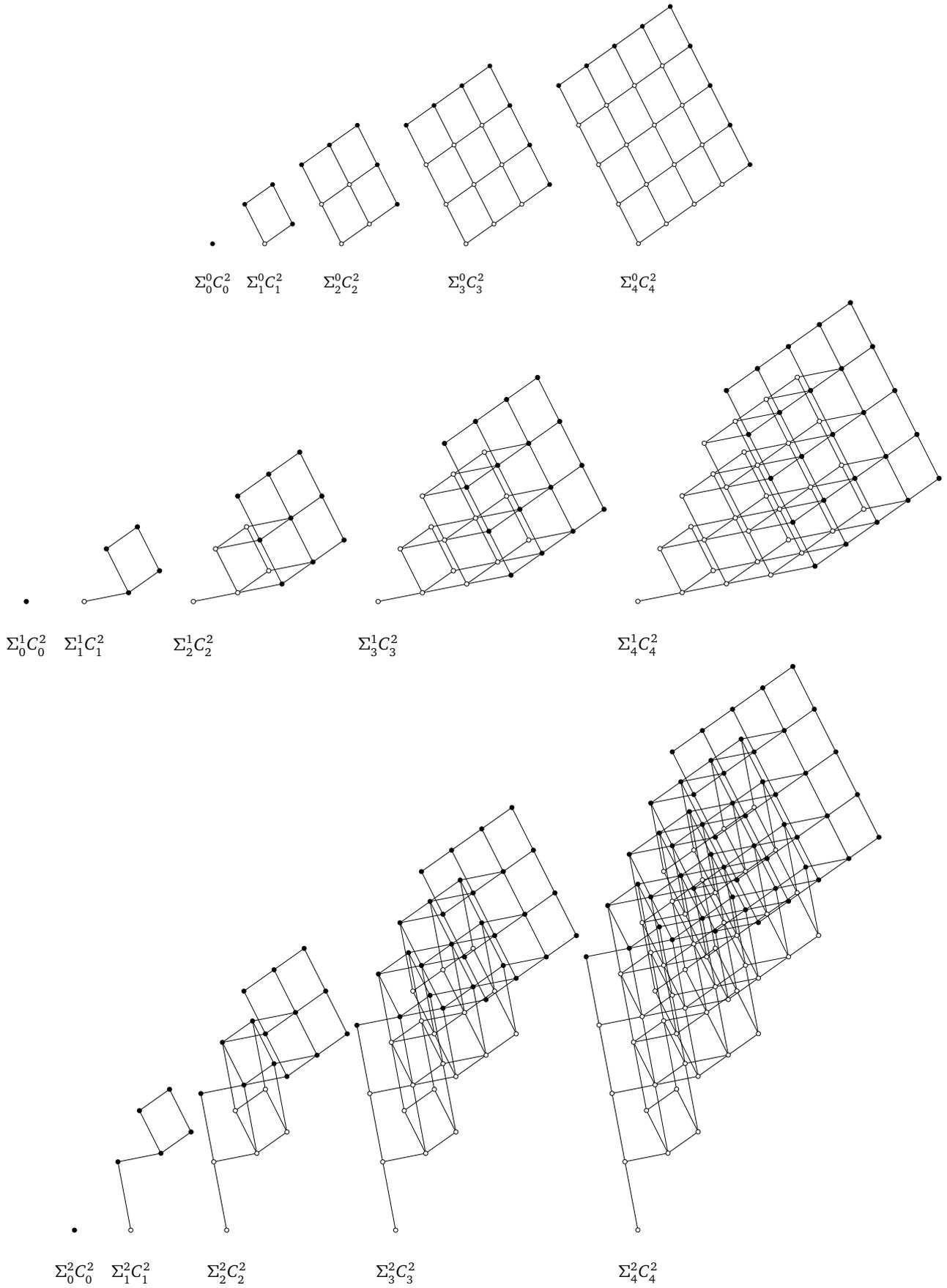


FIGURE 9. Lattices $\Sigma_m^k C_m^2$ for $k \in \{0, 1, 2\}$ and $m \in \{0, 1, 2, 3, 4\}$.

- for $1 \leq i \leq n$, associating an increase in the i -th coordinate of a maximal chain in C_m^{n+1} with the step in a walk given by -1 in the i -th position,
- and associating an increase in the $(n + 1)$ -th coordinate with the diagonal step $(1, 1, \dots, 1)$ in a walk.

□

The first few terms of the sequence of maximal chain numbers for the stacking of lattices along the rows of Figure 1 are shown in Figure 7.

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