THE SATURATION SPECTRUM FOR ANTICHAINS OF SUBSETS

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Abstract. Extending a classical theorem of Sperner, we characterize the integers m such that there exists a maximal antichain of size m in the Boolean lattice B_n, that is, the power set of $[n] := \{1, 2, \dots, n\}$, ordered by inclusion. As an important ingredient in the proof, we initiate the study of an extension of the Kruskal-Katona theorem which is of independent interest. For given positive integers t and k, we ask which integers s have the property that there exists a family $\mathcal{F}$ of k-sets with $|\mathcal{F}| = t$ such that the shadow of $\mathcal{F}$ has size s, where the shadow of $\mathcal{F}$ is the collection of $(k-1)$-sets that are contained in at least one member of $\mathcal{F}$. We provide a complete answer for $t \leqslant k+1$. Moreover, we prove that the largest integer which is not the shadow size of any family of k-sets is $\sqrt{2}k^{3/2} + \sqrt[4]{8}k^{5/4} + O(k)$.

1. Introduction

In the Boolean lattice B_n of all subsets of $[n] := \{1, 2, \dots, n\}$, ordered by inclusion, an *antichain* is a family of subsets such that no one contains any other one. By a classical theorem of Sperner [19], the maximum size of an antichain in B_n is $\binom{n}{\lfloor n/2 \rfloor}$, and it is obtained only by the collections $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$, where for any set M we write $\binom{M}{k}$ for the collection of all k-element subsets of M.

Of course there are antichains in B_n of every size s up to $\binom{n}{\lfloor n/2 \rfloor}$, since any s subsets of size $\lfloor n/2 \rfloor$ will do. Trotter (pers. comm. 2016) wondered what is the second largest size of a *maximal* antichain of subsets in B_n? Here, an antichain $\mathcal{A}$ is maximal if no subset can be added to it without destroying the antichain property. Looking at examples, a natural candidate for $\mathcal{A}$ is obtained by taking the collection $\binom{[n]}{\lceil n/2 \rceil}$, add in the set $\{1, \dots, \lceil n/2 \rceil - 1\}$, and then remove the sets above it. Using methods in the field, one can show that this is indeed a maximal antichain with the second largest size, which is $\binom{n}{\lfloor n/2 \rfloor} - \lfloor n/2 \rfloor$. Notice the gap between the first and second largest sizes of maximal antichains in B_n. What can be said about, say, the third largest size? Indeed, what can we say in general about the possible sizes of maximal antichains in B_n? Our group set out to investigate this natural question, and we found that things get increasingly complicated the closer we look. We can now describe where the gaps are (intervals of values for which there is no maximal antichain), and construct antichains for all sizes below the gaps, thus answering the following question.

Question 1. For which integers m with $1 \leqslant m \leqslant \binom{n}{\lfloor n/2 \rfloor}$ does there exist a maximal antichain of size m?

In other words, we determine the set

$$S(n) = \{|\mathcal{A}| : \mathcal{A} \text{ is a maximal antichain in } B_n\}.$$

If $\mathcal{A}$ contains only sets of a single size k, then we must take all of them to get a maximal antichain, and this gives us $\binom{n}{k} \in S(n)$ for $k = 1, 2, \dots, \lfloor n/2 \rfloor$. As a next step we consider maximal antichain sizes obtained from antichains $\mathcal{A}$ that are contained in *two* consecutive levels of B_n, that is, $\mathcal{A} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$, for some k. Such antichains are called *flat*, and we are also interested in the following more restricted version of Question 1.

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Date: October 21, 2022.
The research of Jerrold Griggs is supported in part by grant #282896 from the Simons Foundation.

Question 2. For which integers m with $1 \leqslant m \leqslant \binom{n}{\lfloor n/2 \rfloor}$ does there exist a maximal flat antichain of size m?

In the present paper we make a first step towards answering this question by establishing that the sizes close to the maximum $\binom{n}{\lfloor n/2 \rfloor}$ are achieved by flat maximal antichains. This is complemented in [9] by showing that all the sizes, with some exceptions below $\binom{n}{2}$, can be obtained by flat antichains, thus completing the answer to Question 2.

These investigations are analogous to the study of the *saturation spectrum* in graphs: For a family $\mathcal{H}$ of graphs, one is interested in the possible edge numbers of saturated $\mathcal{H}$-free graphs on n vertices, where saturation means that adding any edge creates a copy of a member of $\mathcal{H}$. Starting with [2], this natural extension of Turán type problems (asking for the maximal number of edges in an $\mathcal{H}$-free graph) has been studied for various families $\mathcal{H}$, see [4, 7] for overviews. More recently, similar saturation problems have been studied for posets in B_n: For a given poset P, what are the possible sizes of saturated P-free families $\mathcal{F} \subseteq 2^{[n]}$, that is, such that no subset can be added to $\mathcal{F}$ without creating a copy of P? In [6, 13, 17] the focus is on the lower end of the saturation spectrum, that is, the minimum size of a saturated P-free family, while the classical Sperner theory looks at the upper end of the spectrum, that is, the maximum size of a P-free family. A natural next step is to extend this by asking for a characterization of all possible sizes of saturated P-free families. Our work on maximal antichains can be seen as a first step in this direction by looking at the special case where P is a chain of length 2. Another related direction is the study of stability questions: What can be said about the structure of a P-free family whose size is close to the maximum possible? For instance, in [18] a stability result for P being a chain of length 3 has been used in connection with supersaturation results for the butterfly poset. Stability results are related to our problem as these structural statements can imply gaps at the upper end of the spectrum, and our arguments in Section 3 have this flavor.

As an intermediate step towards characterizing the maximal antichain sizes very close to $\binom{n}{\lfloor n/2 \rfloor}$, we investigate a problem related to another classical result, known as the Kruskal-Katona Theorem [12, 14]. This theorem answers the following question: Given positive integers t and k, how can one select a family $\mathcal{F}$ of t k-sets in order to minimize the size of the family $\Delta\mathcal{F}$ of $(k-1)$-subsets that are each contained in some set in $\mathcal{F}$? The family $\Delta\mathcal{F}$ is called the *shadow* of $\mathcal{F}$. It is then natural to ask the following question.

Question 3. Given t and k, what is the *shadow spectrum*

$$\sigma(t,k) := \{|\Delta\mathcal{F}| \,:\, \mathcal{F} \text{ is a family of } k\text{-sets with } |\mathcal{F}| = t\}\ ?$$

The Kruskal-Katona Theorem characterizes the smallest element of $\sigma(t,k)$, and the largest element is obviously tk. Question 2 and Question 3 are closely related, since a maximal flat antichain $\mathcal{A}$ which consists of k-sets and $(k-1)$-sets is determined by its collection of k-sets, $\mathcal{F} := \mathcal{A} \cap \binom{[n]}{k}$, namely $\mathcal{A} = \mathcal{F} \cup \binom{[n]}{k-1} \setminus \Delta\mathcal{F}$. However, it is important to note that an antichain of this form is not necessarily maximal, as there can be k-sets not in $\mathcal{F}$ which are not supersets of any $(k-1)$-set outside $\Delta\mathcal{F}$. To be more precise, we introduce the notion of the *shade* (or *upper shadow*) of a family $\mathcal{G} \subseteq \binom{[n]}{\ell}$ which is defined to be the family $\nabla\mathcal{G} = \{G \cup \{x\} \,:\, G \in \mathcal{G},\ x \in [n] \setminus G\}$. Then $\mathcal{A} = \mathcal{F} \cup \mathcal{G}$ with $\mathcal{F} \subseteq \binom{[n]}{k}$ and $\mathcal{G} \subseteq \binom{[n]}{k-1}$ is a maximal antichain if and only if $\mathcal{G} = \binom{[n]}{k-1} \setminus \Delta\mathcal{F}$ and $\mathcal{F} = \binom{[n]}{k} \setminus \nabla\mathcal{G}$.

Related work. The paper [10] asks for the minimum size of a maximal flat antichain for given n and k. The case $k = 3$ is settled by showing that the minimum size of a maximal flat antichain consisting of 2-sets and 3-sets is $\binom{n}{2} - \lfloor (n+1)^2/8 \rfloor$, and all extremal antichains are determined. A stability version of this result is proved in [6], together with bounds for general k which depend on the Turán densities for complete k-uniform hypergraphs. For $k \geqslant 4$ these bounds are very far away from the best known constructions, and a lot of work remains to be done. Further results are given in [11] for the general setting of maximal antichains of subsets, where set sizes belong to some given set K. The flat case corresponds to $K = \{k, k-1\}$. In [8] the focus is on (not necessarily maximal) antichains of the form $\mathcal{F} \cup \binom{[n]}{k-1} \setminus \Delta\mathcal{F}$, where $\mathcal{F}$ is a family that minimizes the shadow size among all families of $|\mathcal{F}|$ k-sets. It has been observed several times, for instance in [5], that the size $\left|\mathcal{F} \cup \binom{[n]}{k-1} \setminus \Delta\mathcal{F}\right|$ is a rather complicated function of $|\mathcal{F}|$, and [8] studies the structure of (weighted variants of) this function. The sets $\sigma(t,k)$ were first considered in [15], where certain classes of triples (s,t,k) with $s \notin \sigma(t,k)$ were characterized.

Main Results.

Theorem 1. *Let n and m be positive integers with $m \leqslant \binom{n}{\lceil n/2\rceil}$.*

(i) For $m > \binom{n}{\lceil n/2\rceil} - \left\lceil \frac{n}{2}\right\rceil^2$, there exists a maximal antichain of size m if and only if

$$m = \binom{n}{\lceil n/2\rceil} - tl + \binom{a}{2} + \binom{b}{2} + c$$

for some integers $l \in \{\lceil n/2\rceil, \lfloor n/2\rfloor\}$, $t \in \{0, \dots, \lceil n/2\rceil\}$, $a \geqslant b \geqslant c \geqslant 0$, $1 \leqslant a+b \leqslant t$. Moreover, every such m is obtained as the size of a maximal flat antichain.

(ii) If $m \leqslant \binom{n}{\lceil n/2\rceil} - \left\lceil \frac{n}{2}\right\rceil \left\lceil \frac{n+2}{4}\right\rceil$, then there is a maximal antichain of size m in B_n.

The last statement in part (i) of this theorem is an initial step towards an answer for Question 2: The sizes close to the maximum are already obtained by flat maximal antichains, and in [9] we can focus on the case $m \leqslant \binom{n}{\lceil n/2\rceil} - \lceil \frac{n}{2}\rceil^2$.

One direction of the equivalence in Theorem 1(i) will be proved by showing that, for any n, l, t, a, b, c as in the theorem and $k = \lceil n/2\rceil$, we can obtain a maximal antichain $\mathcal{A} \subseteq \binom{[n]}{l} \cup \binom{[n]}{l+1}$ of size $|\mathcal{A}| = \binom{n}{k} - tl + \binom{a}{2} + \binom{b}{2} + c$ by taking a suitable family $\mathcal{F} \subseteq \binom{[n]}{l+1}$ with $|\mathcal{F}| = t$ and setting $\mathcal{A} = \mathcal{F} \cup \left(\binom{[n]}{l} \setminus \Delta\mathcal{F}\right)$. To establish this result, we study the sets $\sigma(t,k)$. Our proof of part (i) of Theorem 1 in Section 3 is based on the following characterization of the sets $\sigma(t,k)$ for $t \leqslant k+1$, which will be established in Section 2.

Theorem 2. *For integers k and t with $k \geqslant 2$ and $1 \leqslant t \leqslant k+1$,*

$$\sigma(t,k) = \left\{ tk - \binom{a}{2} - \binom{b}{2} - c \,:\, a \geqslant b \geqslant c \geqslant 0,\ 1 \leqslant a+b \leqslant t \right\}. \tag{1}$$

Moreover, for every $s \in \sigma(t,k)$, there exists a family $\mathcal{F} \subseteq \binom{[k+4]}{k}$ with $|\mathcal{F}| = t$ and $|\Delta\mathcal{F}| = s$.

To get a better idea of the structure of the sets $\sigma(t,k)$, and also as a useful tool in the proof of Theorem 2, we write the right-hand side of (1) as a union of pairwise disjoint intervals. To state this precisely, set

$$I(t) = \left\{ \binom{a}{2} + \binom{b}{2} + c \,:\, a \geqslant b \geqslant c \geqslant 0,\ 1 \leqslant a+b \leqslant t \right\}, \tag{2}$$

so that Theorem 2 says $\sigma(t,k) = \{tk - x \,:\, x \in I(t)\}$. For a positive integer t, we define $j^*(t)$ to be the smallest non-negative integer j with $\binom{j+3}{2} \geqslant t$. An easy calculation (see Lemma 3.3 below) shows that $j^*(t) = \lceil \sqrt{2t} - 5/2\rceil$. We define intervals

$$I_j(t) = \left[\binom{t-j}{2}, \binom{t-j}{2} + \binom{j+1}{2} \right] \qquad \text{for } j = 0, 1, \dots, j^*(t) - 1,$$

$$I_j(t) = \left[0, \binom{t-j}{2} + \binom{j+1}{2} \right] \qquad \text{for } j = j^*(t).$$

The set $I(t)$ is the union of the intervals $I_j(t)$.

Proposition 1. *For every positive integer t, $I(t) = \bigcup_{j=0}^{j^*(t)} I_j(t)$.*

As a consequence, we obtain $\sigma(t,k)$ as a union of intervals, the rightmost of them ending at tk. This is illustrated in the following example.

Example 1. Figure 1 shows the sets $\sigma(t, 50)$ for $t = 1, \dots, 14$.

In our last result, we investigate the largest integer which is not the shadow size of a k-uniform set family, that is, the quantity $\psi(k) = \max \mathbb{N} \setminus \Sigma(k)$ where $\Sigma(k) = \bigcup_{t=1}^{\infty} \sigma(t,k)$. For instance, Figure 1 suggests $\psi(50) = 12 \times 50 - \binom{12-3}{2} - \binom{4}{2} - 1 = 557$. In the next theorem, we provide an asymptotic formula for $\psi(k)$,

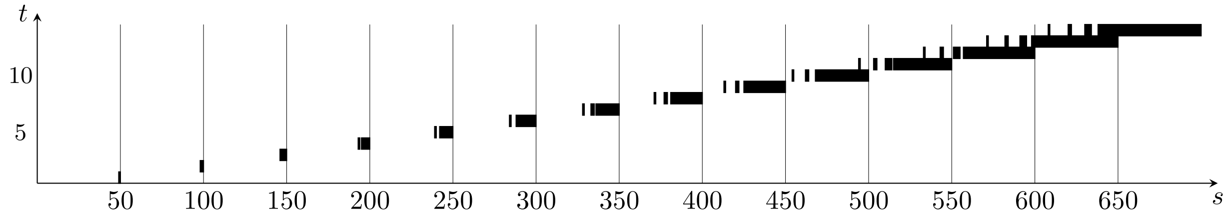

FIGURE 1. The sets $\sigma(1,50), \sigma(2,50), \dots, \sigma(14,50)$.

and combining this with Theorem 1, we obtain an asymptotic formula for the smallest positive integer which is not the size of a maximal antichain, that is, for $\varphi(n) = \min\{m \in \mathbb{N} : m \notin S(n)\}$.

Theorem 3.

(i) For $k \to \infty$, $\psi(k) = \sqrt{2}k^{3/2} + \sqrt[4]{8}k^{5/4} + O(k)$.

(ii) For $n \to \infty$, $\varphi(n) = \binom{n}{\lfloor n/2 \rfloor} - \left(\frac{1}{2} + o(1)\right) n^{3/2}$.

Outline of the paper.

(1) In Section 2, we prove Theorem 2 and Proposition 1. Using a construction based on certain graphs we show $\{tk - x : x \in I(t)\} \subseteq \sigma(t,k)$ (Lemma 2.1), then we prove Proposition 1, and use it to establish $\sigma(t,k) \subseteq \{tk - x : x \in I(t)\}$ (Lemma 2.3).
(2) The proof of Theorem 1(i) is the content of Section 3. The existence of maximal antichains with the claimed sizes is derived from Theorem 2 using the construction described just after the statement of Theorem 1, and this is complemented by showing that all antichains of the relevant sizes necessarily come from this construction.
(3) Section 4 contains the proof of Theorem 1(ii).
(4) The asymptotic results in Theorem 3 are derived in Section 5, and we conclude by listing a few open problems in Section 6.

A common theme in Sections 3 and 4 is the construction of families of maximal antichains whose sizes form intervals, and then establishing that these intervals overlap to yield all the required sizes. Both steps tend to get easier as n grows: For small n, the general constructions leave a few gaps, and even when there are no gaps it turns out that the straightforward arguments for the inequalities, which verify the overlaps, sometimes only work for, say, $n \geqslant 200$. In both situations one can try to optimize the constructions and the proofs to work for smaller n. As this would lead to a further blow-up of various case distinctions and computations (on which the present version is already rather heavy), we decided to take the easy way out, and treat the small cases by ad-hoc constructions and inequality verifications (either by hand or using a computer).

2. The shadow spectrum

The aim of this section is to prove Theorem 2. To this end we will construct families of k-sets by taking the complements of the edges of certain graphs whose vertex sets are contained in $[k+2]$. This class of graphs is defined as follows.

Definition 1. Let a, b, c be integers with $a \geqslant b \geqslant c \geqslant 0$ and $a \geqslant 1$. For $b = 0$ let $G(a,b,c)$ be an a-star, that is, a complete bipartite graph $K_{1,a}$, on the vertex set $[a+1]$. For $b \geqslant 1$ let $G(a,b,c)$ be the graph on the ground set $[a+b-c+2]$ that consists of one a-star and one b-star that share exactly c pendant vertices (see Figure 2).

The graph $G(a,b,c)$ has $a+b$ edges and $\binom{a}{2} + \binom{b}{2} + c$ pairs of adjacent edges.

Definition 2. For integers k and a, b, c with $a \geqslant b \geqslant c \geqslant 0$, let $\mathcal{F}_k(a,b,c)$ be the family

$$\mathcal{F}_k(a,b,c) = \{[k+2] \setminus e : e \text{ is an edge of } G(a,b,c)\}.$$

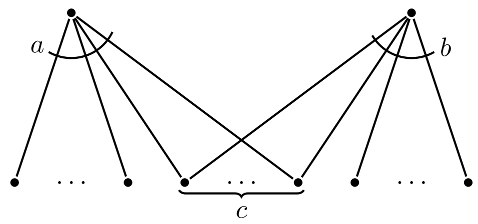

FIGURE 2. The graph $G(a,b,c)$.

Remark 1. If $b=0$ and $k \geqslant a-1$ or $b \geqslant 1$ and $k \geqslant a+b-c$ then the vertex set of $G(a,b,c)$ is contained in $[k+2]$, and consequently, $\mathcal{F}_k(a,b,c)$ is a family of k-sets with $|\mathcal{F}_k(a,b,c)| = a+b$. Moreover, in this situation, the size of its shadow equals its own size multiplied by k reduced by the number of pairs of sets that have a shadow element in common. As the latter equals the number of pairs of adjacent edges of $G(a,b,c)$, we obtain

$$|\Delta \mathcal{F}_k(a,b,c)| = (a+b)k - \binom{a}{2} - \binom{b}{2} - c.$$

Before we present rigorous proofs for every detail that is needed to obtain Theorem 2, we sketch how the different expressions in (1) arise and how they are related. A t-family $\mathcal{F} = \{A_1, \dots, A_t\}$ of k-sets can be regarded as built by successively adding its elements $A_1, \dots, A_t$ in this order. Its shadow size $s = |\Delta\mathcal{F}|$ is the sum of the *marginal shadow sizes* (or *new-shadow sizes*)

$$s_i = |\Delta\{A_1, \dots, A_i\}| - |\Delta\{A_1, \dots, A_{i-1}\}| \qquad \text{for } i = 1,2,3,\dots,t.$$

We let $\boldsymbol{s} = (s_1, \dots, s_t)$ be the so called *marginal shadow vector* (or *new-shadow vector*). By the Kruskal-Katona Theorem, the shadow size $s = \sum_{i=1}^{t} s_i$ is minimal if $\mathcal{F}$ consists of the first t elements of $\binom{\mathbb{N}}{k}$ in *squashed order* $<_S$, which is defined by

$$A <_S B \iff \max\left[(A \setminus B) \cup (B \setminus A)\right] \in B.$$

For more on squashed order and the Kruskal-Katona Theorem see e.g. [1, pp. 112-124]. For $2 \leqslant t \leqslant k+1$ taking the first t elements in squashed order means choosing only subsets of $[k+1]$. The marginal shadow vector is then $\boldsymbol{s} = (k, k-1, \dots, k-(t-1))$ and the shadow size is given by $s = tk - \sum_{i=1}^{t-1} i = tk - \binom{t}{2}$.

How can one increase this minimal shadow size by the least possible amount? Clearly, by introducing exactly one new element, say $k+2$, within the last set A_t. Then the marginal shadow vector becomes $\boldsymbol{s} = (k, k-1, \dots, k-(t-2); k-1)$, where the semicolon indicates at which point we first used a set that is not a subset of $[k+1]$. The resulting shadow size is $s = tk - \binom{t-1}{2} - \binom{2}{2}$. By introducing two new elements instead of one within A_t we can get the marginal shadow vector $\boldsymbol{s} = (k, k-1, \dots, k-(t-2); k)$ and the shadow size $s = tk - \binom{t-1}{2}$.

In order to increase the shadow size again by the least possible amount, we have to introduce a new element one step earlier, that is within A_{t-1}. From there we proceed in squashed order and obtain the marginal shadow vector $\boldsymbol{s} = (k, k-1, \dots, k-(t-3); k-1, k-2)$, and the shadow size $s = tk - \binom{t-2}{2} - \binom{3}{2}$. It is easy to see in this special case and will be shown in general later that we can increase the shadow size from this value in single steps up to $s = tk - \binom{t-2}{2}$, which corresponds to the marginal shadow vector $\boldsymbol{s} = (k, k-1, \dots, k-(t-3); k, k)$.

The next increase by the least possible amount is achieved by introducing a new element already within A_{t-2} and results in $\boldsymbol{s} = (k, k-1, \dots, k-(t-4); k-1, k-2, k-3)$, having the sum $s = tk - \binom{t-3}{2} - \binom{4}{2}$. Again, we can increase the shadow size from this value in single steps up to $s = tk - \binom{t-3}{2}$, corresponding to the marginal shadow vector $\boldsymbol{s} = (k, k-1, \dots, k-(t-4); k, k, k)$, and so on.

The general scheme is having a marginal shadow vector of length t, that is split up into one (longer) part of length $t-j$ and one (shorter) part of length j, and has the form

$$\boldsymbol{s} = (k, k-1, \dots, k-(t-j-1); k-1, k-2, \dots, k-j),$$

and the sum $s = tk - \binom{t-j}{2} - \binom{j+1}{2}$. From here we can increase the shadow size in single steps up to the marginal shadow vector $\boldsymbol{s} = (k, k-1, \dots, k-(t-j-1); k, \dots, k)$, and thereby we obtain all shadow sizes in the interval

$$tk - I_j(t) = \left\{tk - x \,:\, \binom{t-j}{2} \leqslant x \leqslant \binom{t-j}{2} + \binom{j+1}{2}\right\}.$$

Then comes a gap before we get all shadow sizes in $tk - I_{j+1}(t)$, and so on. It is easily seen that the gaps between the intervals $I_j(t)$ and $I_{j+1}(t)$ close at some point for relatively small j. This happens exactly when

$$\binom{t-j-1}{2} + \binom{j+2}{2} + 1 \geqslant \binom{t-j}{2},$$

and a simple computation yields that this is the case if and only if $\binom{j+3}{2} \geqslant t$. Therefore, we chose $j^*(t)$ to be the smallest non-negative integer with this property.

How is this related to the graphs $G(a,b,c)$? We have seen that it is crucial to increase the shadow size from $s = tk - \binom{t-j}{2} - \binom{j+1}{2}$ up to $s = tk - \binom{t-j}{2}$ in single steps. Therefore, we rewrite

$$s = tk - \binom{t-j}{2} - \binom{j+1}{2} = tk - \binom{t-j}{2} - \binom{j}{2} - j,$$

and this has the form $tk - \binom{a}{2} - \binom{b}{2} - b$ with $a \geqslant b \geqslant 0$. This shadow size corresponds to $G(a,b,b)$, or more precisely to $\mathcal{F}_k(a,b,b)$ extended by $t-(a+b)$ sets with marginal shadow k, because this graph has exactly $\binom{a}{2} + \binom{b}{2} + b$ pairs of adjacent edges. To increase the shadow size in single steps we use the graphs $G(a,b,b-1), G(a,b,b-2), \dots, G(a,b,0)$ with $\binom{a}{2}+\binom{b}{2}+b-1$, $\binom{a}{2}+\binom{b}{2}+b-2, \dots, \binom{a}{2}+\binom{b}{2}$ pairs of adjacent edges. These are only $b = j$ single steps, but we can use an inductive argument for the missing steps later.

Now we turn to the details. We start by using the families $\mathcal{F}_k(a,b,c)$ to prove that the right-hand side of (1) is contained in the left-hand side.

Lemma 2.1. *For integers k and t with $k \geqslant 2$ and $1 \leqslant t \leqslant k+1$, $\{tk - x \,:\, x \in I(t)\} \subseteq \sigma(t,k)$.*

Proof. We fix $s = tk - \binom{a}{2} - \binom{b}{2} - c$ with $a \geqslant b \geqslant c \geqslant 0$ with $1 \leqslant a+b \leqslant t$, and show that there is a t-family of k-sets with shadow size s.

Case 1: $a+b = t = k+1$ and $c = 0$.

Case 1.1: $b \geqslant 2$. Then we let $\mathcal{F} = \mathcal{F}_1 \cup \{A\}$, where $\mathcal{F}_1 = \mathcal{F}_k(a, b-1, b-1)$, and is A is a k-set that is shadow-disjoint to all sets in $\mathcal{F}_1$. As the ground set of $\mathcal{F}_1$ is $[a+2]$, which is a subset of $[k+2]$, $\mathcal{F}_1$ is a family of k-sets of size $a+b-1$. Using Remark 1 we obtain

$$|\Delta\mathcal{F}| = |\Delta\mathcal{F}_1| + k = (a+b-1)k - \binom{a}{2} - \binom{b-1}{2} - (b-1) + k = tk - \binom{a}{2} - \binom{b}{2}.$$

Case 1.2: $b \in \{0,1\}$. Note that as $c = 0$ we are then looking for a set system with shadow size $tk - \binom{a}{2}$. If $b = 0$, we have $a = t$ and use $\mathcal{F} = \mathcal{F}_k(a,0,0)$ with shadow size $ak - \binom{a}{2} = tk - \binom{a}{2}$. If $b = 1$, we use $\mathcal{F} = \mathcal{F}_1 \cup \{A\}$, where $\mathcal{F}_1 = \mathcal{F}_k(a,0,0)$ and A is shadow disjoint to all members of $\mathcal{F}_1$, and obtain

$$|\Delta\mathcal{F}| = ak - \binom{a}{2} + k = tk - \binom{a}{2}.$$

Case 2: $a+b < t$ or $t \leqslant k$ or $c \geqslant 1$. Then, we have $a+b-c \leqslant k$ and we can use $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1 = \mathcal{F}_k(a,b,c)$, and $\mathcal{F}_2$ consists of $t-(a+b)$ pairwise shadow-disjoint sets such that $\Delta\mathcal{F}_1 \cap \Delta\mathcal{F}_2 = \emptyset$. By Remark 1 we obtain

$$|\Delta\mathcal{F}| = (a+b)k - \binom{a}{2} - \binom{b}{2} - c + (t-a-b)k = tk - \binom{a}{2} - \binom{b}{2} - c. \qquad \square$$

In the proof of the other inclusion in (1) we will use Proposition 1, so we first prove this result.

Proof of Proposition 1. For every $a \in \{1, 2, \ldots, t\}$,

$$\bigcup_{b=0}^{\min\{a,t-a\}} \left\{ \binom{a}{2} + \binom{b}{2} + c \,:\, 0 \leqslant c \leqslant b \right\} = \bigcup_{b=0}^{\min\{a,t-a\}} \left[\binom{a}{2} + \binom{b}{2}, \binom{a}{2} + \binom{b+1}{2} \right]$$
$$= \left[\binom{a}{2}, \binom{a}{2} + \binom{\min\{a, t-a\}+1}{2} \right],$$

and as a consequence,

$$I(t) = \bigcup_{a=1}^{t} \left[\binom{a}{2}, \binom{a}{2} + \binom{\min\{a, t-a\}+1}{2} \right]. \tag{3}$$

We split this union into the intervals for $a \geqslant t - j^*(t) + 1$ and the ones for $a \leqslant t - j^*(t)$. For $a \geqslant t - j^*(t) + 1$, it follows from $2j^*(t) \leqslant t + 2$ that $a \geqslant t - a$. Hence

$$\left[\binom{a}{2}, \binom{a}{2} + \binom{\min\{a, t-a\}+1}{2} \right] = \left[\binom{a}{2}, \binom{a}{2} + \binom{t-a+1}{2} \right] = I_{t-a}(t). \tag{4}$$

For $a \leqslant t - j^*(t) - 1$, it follows from $j^*(t) = \min\{j : \binom{j+3}{2} \geqslant t\}$ that

$$\binom{t-a+1}{2} \geqslant \binom{j^*(t)+2}{2} = \binom{j^*(t)+3}{2} - (j^*(t)+2) \geqslant t - j^*(t) - 2 \geqslant a - 1.$$

Together with $\binom{a+1}{2} \geqslant a - 1$, we obtain that there is no gap between the interval for a and the interval for $a + 1$:

$$\binom{a}{2} + \binom{\min\{a, t-a\}+1}{2} \geqslant \binom{a}{2} + (a-1) = \binom{a+1}{2} - 1.$$

Since the function $a \mapsto \binom{a}{2} + \binom{\min\{a,t-a\}+1}{2}$ is non-decreasing, $\binom{a}{2} + \binom{\min\{a,t-a\}+1}{2} \leqslant \binom{t-j^*(t)}{2} + \binom{j^*(t)+1}{2}$ for all a, and we obtain

$$\bigcup_{a=0}^{t-j^*(t)} \left[\binom{a}{2}, \binom{a}{2} + \binom{\min\{a, t-a\}+1}{2} \right] = \left[0, \binom{t-j^*(t)}{2} + \binom{j^*(t)+1}{2} \right] = I_{j^*(t)}(t). \tag{5}$$

Combining the ingredients, we conclude the proof as follows:

$$\bigcup_{j=0}^{j^*(t)} I_j(t) = \left(\bigcup_{j=0}^{j^*(t)-1} I_j(t) \right) \cup I_{j^*(t)}(t) = \left(\bigcup_{a=t-j^*(t)+1}^{t} I_{t-a}(t) \right) \cup I_{j^*(t)}(t)$$
$$\overset{(4),(5)}{=} \bigcup_{a=t-j^*(t)+1}^{t} \left[\binom{a}{2}, \binom{a}{2} + \binom{\min\{a, t-a\}+1}{2} \right] \cup \bigcup_{a=0}^{t-j^*(t)} \left[\binom{a}{2}, \binom{a}{2} + \binom{\min\{a, t-a\}+1}{2} \right]$$
$$\overset{(3)}{=} I(t). \qquad \square$$

To complete the proof of Theorem 2, we need the following bound.

Lemma 2.2. *For positive integers t, l and l' with $t/2 > l \geqslant l' \geqslant \lceil \frac{t-l}{2} \rceil$ and $l > 1$,*

$$\binom{l}{2} + \binom{l'}{2} + \binom{t-l-l'+1}{2} + (t-l) \leqslant \binom{\lceil t/2 \rceil}{2} + \binom{\lfloor t/2 \rfloor + 1}{2}.$$

Proof. Note that the assumptions imply $t \geqslant 5$. Fix $t \geqslant 5$ and set $f(l, l') = \binom{l}{2} + \binom{l'}{2} + \binom{t-l-l'+1}{2} + (t-l)$. If $l > l' \geqslant \lceil \frac{t-l}{2} \rceil$ then

$$f(l, l'+1) - f(l, l') = l' - (t - l - l') = 2l' - (t-l) \geqslant 0.$$

Hence, we can assume $l' = l$, and our task is reduced to showing that

$$2\binom{l}{2} + \binom{t-2l+1}{2} + (t-l) \leqslant \binom{\lceil t/2 \rceil}{2} + \binom{\lfloor t/2 \rfloor + 1}{2}$$

whenever $t/2 > l \geqslant \lceil \frac{t-l}{2} \rceil$. Now

$$f(l+1,l+1) - f(l,l) = 2l - (t-2l) - (t-2l-1) - 1 = 6l - 2t \geqslant 0,$$

hence, we can assume $l = \lceil t/2 \rceil - 1$, and our task is reduced to verifying

$$2\binom{\lceil t/2 \rceil - 1}{2} + \binom{t - 2\lceil t/2 \rceil + 3}{2} + \left\lfloor \frac{t}{2} \right\rfloor + 1 \leqslant \binom{\lceil t/2 \rceil}{2} + \binom{\lfloor t/2 \rfloor + 1}{2}.$$

For even t, this follows from

$$2\binom{t/2-1}{2} + 3 + \frac{t}{2} + 1 = \binom{t/2}{2} + \binom{t/2-1}{2} + 5 \leqslant \binom{t/2}{2} + \binom{t/2+1}{2},$$

where we used $t \geqslant 6$. For odd t, we conclude by noticing

$$2\binom{\frac{t-1}{2}}{2} + 1 + \frac{t-1}{2} + 1 = \binom{\frac{t+1}{2}}{2} + \binom{\frac{t-1}{2}}{2} + 2 \leqslant 2\binom{\frac{t+1}{2}}{2}.$$ □

Lemma 2.3. *For integers k and t with $k \geqslant 2$ and $1 \leqslant t \leqslant k+1$, $\sigma(t,k) \subseteq \{tk - x : x \in I(t)\}$.*

Proof. The claim is trivial for $t = 1$, and we assume $t > 1$. Let $\mathcal{F}$ be a t-family of k-sets. By Proposition 1, it is sufficient to show that $|\Delta\mathcal{F}| \in \{tk - x : x \in I_j(t)\}$ for some j. Let $\mathcal{A} \subseteq \mathcal{F}$ be a subset of maximum cardinality subject to $\left|\bigcup_{X \in \mathcal{A}} X\right| \leqslant k+1$. Further, let $l = |\mathcal{A}|$ and $\mathcal{B} = \mathcal{F} \setminus \mathcal{A}$, so that $|\mathcal{B}| = t - l$. If $l = 1$, then $|\Delta\mathcal{F}| = tk$ is in $\{tk - x : x \in I(t)\}$ because $0 \in I_{j^*(t)}(t) \subseteq I(t)$. We now assume $l > 1$. For the shadow sizes we have the bounds

$$|\Delta\mathcal{A}| = lk - \binom{l}{2}, \tag{6}$$

$$(t-l)k - \binom{t-l}{2} \leqslant |\Delta\mathcal{B}| \leqslant (t-l)k, \tag{7}$$

$$|\Delta\mathcal{A} \cap \Delta\mathcal{B}| \leqslant t - l. \tag{8}$$

We use induction on t to show the following claim.

Claim 1. *If $l \geqslant t/2$, then $tk - \binom{l}{2} - \binom{t-l+1}{2} \leqslant |\Delta\mathcal{F}| \leqslant tk - \binom{l}{2}$, and if $l < t/2$, then $|\Delta\mathcal{F}| \geqslant tk - \binom{\lceil t/2 \rceil}{2} - \binom{\lfloor t/2 \rfloor + 1}{2}$.*

Before proving this claim, we show how it can be used to deduce the statement of the lemma. For $l \geqslant t - j^*(t) + 1$, the claim implies $|\Delta\mathcal{F}| \in \{tk - x : x \in I_{t-l}(t)\}$. For $l \leqslant t - j^*(t)$, we obtain $|\Delta\mathcal{F}| \in \{tk - x : x \in I_{j^*(t)}(t)\}$ as follows. We have $I_{j^*(t)}(t) = [0, f(j^*(t))]$ for the function $f : j \mapsto \binom{t-j}{2} + \binom{j+1}{2}$ which is non-increasing for $j \in \{0, 1, \dots, \lfloor t/2 \rfloor\}$. For $t/2 \leqslant l \leqslant t - j^*(t)$,

$$|\Delta\mathcal{F}| \geqslant tk - \binom{l}{2} - \binom{t-l+1}{2} = tk - f(t-l) \geqslant tk - f(\lfloor t/2 \rfloor).$$

For $l < t/2$, $|\Delta\mathcal{F}| \geqslant tk - f(\lfloor t/2 \rfloor)$ and $|\Delta\mathcal{F}| \geqslant tk - f(j^*(t))$ follows from $j^*(t) \leqslant \lfloor t/2 \rfloor$.

It remains to establish Claim 1. The base case $t = 1$ is trivial, so we assume $t \geqslant 2$. Combining (6), (7) and (8), we obtain

$$tk - \binom{l}{2} - \binom{t-l}{2} - (t-l) \leqslant |\Delta\mathcal{F}| \leqslant tk - \binom{l}{2}.$$

As the left-hand side equals $tk - \binom{l}{2} - \binom{t-l+1}{2}$ this concludes the argument for $l \geqslant t/2$. For $l < t/2$, let $\mathcal{C} \subseteq \mathcal{B}$ be a subset of maximum cardinality subject to $\left|\bigcup_{X \in \mathcal{C}} X\right| \leqslant k+1$. Further, let $l' = |\mathcal{C}|$ and $\mathcal{D} := \mathcal{B} \setminus \mathcal{C}$, so that $|\mathcal{D}| = t - l - l'$. The maximality of $\mathcal{A}$ implies $l' \leqslant l$.

Case 1: $l' \geqslant \frac{t-l}{2}$. Applying the induction hypothesis to $\mathcal{B}$, we obtain

$$(t-l)k - \binom{l'}{2} - \binom{t-l-l'+1}{2} \leqslant |\Delta\mathcal{B}| \leqslant (t-l)k - \binom{l'}{2}.$$

Combining this with (6) and (8),

$$|\Delta\mathcal{F}| \geqslant tk - \binom{l}{2} - \binom{l'}{2} - \binom{t-l-l'+1}{2} - (t-l).$$

Case 2: $l' < \frac{t-l}{2}$. By induction,

$$|\Delta\mathcal{B}| \geqslant (t-l)k - \binom{\lceil\frac{t-l}{2}\rceil}{2} - \binom{\lfloor\frac{t-l}{2}\rfloor+1}{2},$$

and together with (6) and (8),

$$|\Delta\mathcal{F}| \geqslant tk - \binom{l}{2} - \binom{\lceil\frac{t-l}{2}\rceil}{2} - \binom{\lfloor\frac{t-l}{2}\rfloor+1}{2} - (t-l).$$

In both cases, the claim follows with Lemma 2.2. □

Combining Lemmas 2.1 and 2.3, we have proved (1). The next lemma provides the final part of Theorem 2.

Lemma 2.4. *Let $k \geqslant 2$ and $1 \leqslant t \leqslant k+1$ be integers. Then for every $s \in \sigma(t,k)$ there exists a set system $\mathcal{F} \subseteq \binom{[k+4]}{k}$ such that $|\mathcal{F}| = t$ and $|\Delta\mathcal{F}| = s$.*

To prove this we need one more lemma which says that for a $(t-1)$-family of k-sets on the ground set $[k+4]$ we can always add a k-set that yields k new shadow elements.

Lemma 2.5. *Let $k \geqslant 2$ and t be integers with $1 \leqslant t \leqslant k+1$, and assume that $\mathcal{F} \subseteq \binom{[k+4]}{k}$ satisfies $|\mathcal{F}| = t-1$. Then there exists a set $A \in \binom{[k+4]}{k}$ such that $\mathcal{F}' = \mathcal{F} \cup \{A\}$ satisfies $|\Delta\mathcal{F}'| = |\Delta\mathcal{F}| + k$.*

Proof. We have to show, that for $\mathcal{F} \subseteq \binom{[k+4]}{k}$ with $|\mathcal{F}| \leqslant k$, there exists an $A \in \binom{[k+4]}{k} \setminus \nabla\Delta\mathcal{F}$ (taking the shade in the ground set $[k+4]$). For $k=2$, there are two elements $a, b \in [k+4] = [6]$, which do not appear in any $X \in \mathcal{F}$, and then $A = \{a,b\}$ does the job. For $k=3$, assume for the sake of contradiction, that $\nabla\Delta\mathcal{F} = \binom{[7]}{3}$. Then $E = \binom{[7]}{2} \setminus \Delta\mathcal{F}$ is the edge set of a triangle-free graph on the vertex set $[7]$ with $|E| = 21 - |\Delta\mathcal{F}| \geqslant 12$. By Mantel's theorem, it follows that $\Delta\mathcal{F} = \binom{A}{2} \cup \binom{B}{2}$ for some partition $[7] = A \cup B$ with $|A| = 4$ and $|B| = 3$. But then $|F| > 3$, which is the required contradiction. For $k \geqslant 4$, the claim is immediate from

$$|\nabla\Delta\mathcal{F}| \leqslant |\mathcal{F}| + 4\,|\Delta\mathcal{F}| \leqslant k + 4k^2 < \binom{k+4}{k},$$

where the last inequality holds because $k \geqslant 4$. □

Proof of Lemma 2.4. We proceed by induction on t. The base case $t=1$ is trivial, and we assume $t \geqslant 2$. Let $s \in \sigma(t,k)$. By (1), $s = tk - \binom{a}{2} - \binom{b}{2} - c$ for integers a, b, c satisfying $a \geqslant b \geqslant c \geqslant 0$ and $1 \leqslant a+b \leqslant t$. If $a+b \leqslant t-1$ then (1) implies that $s-k = s = (t-1)k - \binom{a}{2} - \binom{b}{2} - c \in \sigma(t-1,k)$. By induction, there exists $\mathcal{F}' \subseteq \binom{[k+4]}{k}$ with $|\mathcal{F}'| = t-1$ and $|\Delta\mathcal{F}'| = s-k$, and then, by Lemma 2.5, we can add one more k-set to obtain $\mathcal{F} \subseteq \binom{[k+4]}{k}$ with $|\mathcal{F}| = t$ and $|\Delta\mathcal{F}| = s$. Only the case $a+b=t$ remains. If $c \geqslant 1$, then $a+b-c \leqslant t-1 \leqslant k$, and Remark 1 ensures that $\mathcal{F}_k(a,b,c) \subseteq \binom{[k+2]}{k}$ does the job. If $c=0$ then we can assume that $b \geqslant 1$ because $\binom{1}{2} = \binom{0}{2}$. But then $s = tk - \binom{a}{2} - \binom{b}{2} - 0 = tk - \binom{a}{2} - \binom{b-1}{2} - (b-1)$, and we can take $\mathcal{F}_k(a, b-1, b-1)$. □

Now we put together the ingredients to prove Theorem 2.

Proof of Theorem 2. From Lemmas 2.1 and 2.3, we obtain (1). The final claim is Lemma 2.4. □

Remark 2. It follows easily from the proofs of Lemmas 2.4 and 2.5 that for $t \leqslant \frac{k+2}{2}$ even the ground set $[k+2]$ is sufficient. For the general statement, the bound $k+4$ cannot be improved, because for $k=2$ and $t=3$ the ground set must have at least 6 elements to allow a shadow of size $tk=6$.

3. Sizes of large maximal antichains

In this section, we establish Theorem 1(i) by proving the following result which, in view of Theorem 2, is an equivalent reformulation.

Theorem 4. *Let n be a positive integer, set $k=\lceil n/2\rceil$ and let m be a positive integer with $\binom{n}{k}-k^2\leqslant m\leqslant\binom{n}{k}$. There exists a maximal antichain of size m in B_n if and only if*

$$\binom{n}{k}-m\in\bigcup_{t=0}^{k}\left(\sigma(t,k)\cup\sigma(t,n-k)\right).$$

In order to obtain a maximal antichain $\mathcal{A}$ whose size is close to $\binom{n}{k}$, it is natural to take a small family $\mathcal{F}\subseteq\binom{[n]}{l+1}$, $l\in\{k,n-k\}$, say with $|\mathcal{F}|=t$ and set $\mathcal{A}=\mathcal{F}\cup\binom{[n]}{l}\setminus\Delta\mathcal{F}$. Then

$$m=|\mathcal{F}|+\binom{n}{k}-|\Delta\mathcal{F}|=\binom{n}{k}-(s-t)$$

for some $s\in\sigma(t,l+1)$, and $\binom{n}{k}-m\in\sigma(t,l)$ follows since (1) implies

$$\sigma(t,l)=\sigma(t,l-1)+t\quad\text{for all } t,l \text{ with } 1\leqslant t\leqslant l. \tag{9}$$

Recall that $S(n)$ denotes the set of sizes of maximal antichains in B_n. In the proof of Theorem 4 we will see that for $m\in S(n)$ with $m\geqslant\binom{n}{k}-k^2$ there is always an antichain of size m which has this form (except for some very small values of n which have to be treated separately). We start by proving the easier of the two directions in Theorem 4.

Lemma 3.1. *Let $1\leqslant k\leqslant n$. For every $\mathcal{F}\subseteq\binom{[n]}{k}$ with $|\mathcal{F}|<k$, the antichain $\mathcal{A}=\mathcal{F}\cup\binom{[n]}{k-1}\setminus\Delta\mathcal{F}$ is maximal.*

Proof. If $\mathcal{A}$ is not maximal, then there exists $X\in\binom{[n]}{k}\setminus\mathcal{F}$ with $\Delta X\subseteq\Delta\mathcal{F}$. But $|\Delta X\cap\Delta A|\leqslant 1$ for every $A\in\mathcal{F}$, hence $|\Delta X\cap\Delta\mathcal{F}|\leqslant|\mathcal{F}|<k=|\Delta X|$, a contradiction. □

Lemma 3.2. *Let n be a positive integer, set $k=\lceil n/2\rceil$ and let m be a positive integer with $\binom{n}{k}-k^2\leqslant m\leqslant\binom{n}{k}$. If $m=\binom{n}{k}-s$ with $s\in\sigma(t,k)\cup\sigma(t,n-k)$ for some $t\in\{0,\dots,k\}$ then $m\in S(n)$.*

In view of the discussion above, this lemma looks almost trivial, but a bit of work is still needed, because for $s\in\sigma(t,l)$, Theorem 2 guarantees the existence of $\mathcal{F}\subseteq\binom{[n]}{l+1}$ with $|\mathcal{F}|=t$ and $|\Delta\mathcal{F}|=s+t$ only if $n\geqslant(l+1)+4$.

Proof. (Lemma 3.2) If $n\geqslant 10$, then $n\geqslant(k+1)+4$ and, by Theorem 2, there exists a family $\mathcal{F}\subseteq\binom{[n]}{l+1}$, $l\in\{k,n-k\}$ with $|\mathcal{F}|=t$ and $|\Delta\mathcal{F}|=t+s$. The antichain $\mathcal{A}=\mathcal{F}\cup\binom{[n]}{l}\setminus\Delta\mathcal{F}$ has size $|\mathcal{A}|=m$ and is maximal by Lemma 3.1, and thus, $m\in S(n)$. We now assume $n\leqslant 9$. If $m\leqslant n$ then the antichain $\mathcal{A}=\{\{i\}\,:\,i=1,\dots,m-1\}\cup\{\{m,m+1,\dots,n\}\}$ has size $|\mathcal{A}|=m$. As the antichain $\mathcal{A}=\binom{[n]}{k}$ has size $\binom{n}{k}$, we can assume $n<m<\binom{n}{k}$. For $n\leqslant 3$, there is no such m, and for $n=4$, Theorem 2 implies that $m=5$ does not satisfy the assumption of the lemma. From now on, we specify a maximal flat antichain $\mathcal{A}\subseteq\binom{[n]}{l}\cup\binom{[n]}{l-1}$ by writing down its l-sets. For $n=5$, we need the sizes $m\in\{6,7,8\}$. These are obtained by the maximal antichain in $\binom{[5]}{3}\cup\binom{[5]}{2}$ with 3-sets $\{1,2,3\}$ and $\{1,4,5\}$, the maximal antichain in $\binom{[5]}{4}\cup\binom{[5]}{3}$ with 4-set $\{1,2,3,4\}$ and the maximal antichain in $\binom{[5]}{3}\cup\binom{[5]}{2}$ with 3-set $\{1,2,3\}$. For $n\geqslant 6$, we use graphs similar to $G(a,b,c)$, sometimes extended by some additional vertex disjoint edges. More precisely, if G is a triangle-free graph on the vertex set $[l+2]$ with t edges and c pairs of adjacent edges, then the maximal flat antichain on levels l and $l-1$ whose l-sets are the sets $[l+2]\setminus e$, $e\in E(G)$, has size $\binom{n}{l-1}-t(l-1)+c$. We will use the graphs with t edges and c pairs of adjacent edges with the following edge sets E_{tc}. For brevity, for an edge $\{a,b\}$ we just write ab.

- $E_{10}=\{12\}$,
- $E_{21}=\{12,13\}$, $E_{20}=\{12,34\}$,
- $E_{33}=\{12,13,14\}$, $E_{32}=\{12,23,34\}$ $E_{31}=\{12,13,45\}$, $E_{30}=\{12,34,56\}$
- $E_{44}=\{12,13,14,25\}$, $E_{43}=\{12,13,14,56\}$, $E_{42}=\{12,23,34,56\}$, $E_{41}=\{12,13,45,67\}$, $E_{40}=\{12,34,56,78\}$

- $E_{54} = \{12, 13, 14, 25, 67\}$, $E_{53} = \{12, 13, 14, 56, 78\}$, $E_{52} = \{12, 23, 34, 56, 78\}$, $E_{51} = \{12, 13, 45, 67, 89\}$
- $E_{65} = \{12, 13, 14, 25, 56\}$

$n = 6$**:** We need the sizes $m \in \{11, \dots, 15, 17\}$. These are obtained by the maximal flat antichains on levels 4 and 3, corresponding to the graphs with the edge sets E_{10}, E_{21}, E_{20}, E_{32}, E_{31}, E_{30}.

$n = 7$**:** We need the sizes $m \in \{19, \dots, 32\}$. The antichains on levels 5 and 4 corresponding to the graphs with edge sets E_{10}, E_{2c}, $c \in \{0, 1\}$, E_{3c}, $c \in \{0, \dots, 3\}$, E_{4c}, $c \in \{1, 2, 3\}$ and E_{54} have sizes 31, 28, 27, 26, 25, 24, 23, 22, 21, 20 and 19. The missing sizes 32, 30 and 29 are obtained by the antichains on levels 4 and 3 corresponding to the graphs with edge sets E_{10}, E_{21} and E_{20}.

$n = 8$**:** We need the sizes $m \in \{54, \dots, 63, 66\}$. These are obtained by the maximal antichains on levels 5 and 4 corresponding to the edge sets E_{10}, E_{21}, E_{20}, E_{3c}, $c \in \{0, \dots, 3\}$, and E_{4c}, $c \in \{0, \dots, 4\}$.

$n = 9$**:** We need the sizes $m \in \{101, \dots, 119, 121, 122\}$. The sizes of the maximal antichains on levels 6 and 5, and on levels 5 and 4, respectively, corresponding to the edge sets are indicated in Table 1, and this concludes the proof. □

TABLE 1. The sizes of the maximal flat antichains corresponding to edge sets for $n = 9$.

edge sets	levels 6 and 5	levels 5 and 4
E_{10}	121	122
E_{20}, E_{21}	116, 117	118, 119
$E_{30}, \dots, E_{33}$	$111, \dots, 114$	$114, \dots, 117$
$E_{40}, \dots, E_{44}$	$106, \dots, 110$	$110, \dots, 114$
$E_{51}, \dots, E_{54}$	$102, \dots, 105$	$107, \dots, 110$
E_{65}	101	107

The hard direction in Theorem 4 is the implication

$$\binom{n}{k} - m \notin \bigcup_{t=0}^{k} (\sigma(t, k) \cup \sigma(t, n-k)) \implies m \notin S(n) \tag{10}$$

for all m with $\binom{n}{k} - k^2 \leqslant m \leqslant \binom{n}{k}$. Lemma 3.4 below allows us to assume $m \geqslant \binom{n}{k} - \binom{k+1}{2} + 2$. In the proof of Lemma 3.4 we will make use of the following explicit formula for $j^*(t)$.

Lemma 3.3. $j^*(t) = \lceil \sqrt{2t} - 5/2 \rceil$.

Proof. By definition, $\binom{j+3}{2} \geqslant t$, hence $(j+3)(j+2) \geqslant 2t$. This implies $j^2 + 5j + 6 - 2t \geqslant 0$, and then

$$j \geqslant -\frac{5}{2} + \sqrt{\frac{25}{4} - 6 + 2t} = \frac{\sqrt{8t+1} - 5}{2}.$$

From this, we obtain

$$j^*(t) = \left\lceil \frac{\sqrt{8t+1} - 5}{2} \right\rceil \geqslant \frac{\sqrt{8t} - 5}{2} = \sqrt{2t} - \frac{5}{2}.$$

If $j^*(t) > \lceil \sqrt{2t} - 5/2 \rceil$ then

$$\sqrt{2t} - \frac{5}{2} \leqslant l < \frac{\sqrt{8t+1} - 5}{2}$$

for some integer l. But this implies $8t \leqslant (2l+5)^2 < 8t + 1$, which is impossible, because the number in the middle is an odd integer. □

Lemma 3.4. *Let n be a positive integer, set $k = \lceil n/2 \rceil$ and let m be a positive integer with $\binom{n}{k} - k^2 \leqslant m \leqslant \binom{n}{k}$. If $\binom{n}{k} - m \notin \bigcup_{t=0}^{k} \sigma(t, k)$ then $m \geqslant \binom{n}{k} - \binom{k+1}{2} + 2$.*

Proof. The claim follows from the inclusion

$$\left[\binom{k+1}{2}-1,\ k^2\right] \subseteq \bigcup_{t=0}^{k} \sigma(t,k). \tag{11}$$

For $1 \leqslant k \leqslant 4$, we verify this by hand: $\{0,1\} \subseteq \sigma(0,1) \cup \sigma(1,1)$, $[2,4] \subseteq \sigma(1,2) \cup \sigma(2,2)$, $[5,9] \subseteq \sigma(2,3) \cup \sigma(3,3)$, and $[9,16] \subseteq \sigma(3,4) \cup \sigma(4,4)$. In order to prove (11) for $k \geqslant 5$, we use $\sigma(t,k) \supseteq \{tk - x : x \in I_{j^*(t)}(t)\} = [tk - f(t), tk]$, where

$$f(t) = \binom{t - \lceil\sqrt{2t} - 5/2\rceil}{2} + \binom{\lceil\sqrt{2t} - 5/2\rceil + 1}{2},$$

and we take the union only over $t \geqslant \lceil (k+1)/2 \rceil$. In other words, we verify the following inequalities:

(i) For $t = \left\lceil \frac{k+1}{2} \right\rceil$, $tk - f(t) \leqslant \binom{k+1}{2} - 1$, and
(ii) for $\left\lceil \frac{k+1}{2} \right\rceil < t \leqslant k$, $tk - f(t) \leqslant (t-1)k + 1$.

Rearranging these inequalities, we want to verify that

$$f\left(\left\lceil \frac{k+1}{2} \right\rceil\right) \geqslant \begin{cases} 1 & \text{if } k \text{ is odd,} \\ \frac{k}{2} + 1 & \text{if } k \text{ is even.} \end{cases}$$

and $f(t) \geqslant k-1$ for $\left\lceil \frac{k+1}{2} \right\rceil < t \leqslant k$. For $5 \leqslant k \leqslant 26$, this follows from Table 2 and the fact that f is increasing. For $k \geqslant 27$ we show that $f(\lceil k/2 \rceil) \geqslant k$, and this is sufficient by the monotonicity of f. We start with the

TABLE 2. The values $f(t)$ for $3 \leqslant t \leqslant 10$.

t	3	4	5	6	7	8	9	10
$f(t)$	3	4	7	11	13	18	24	31

bound

$$f(\lceil k/2 \rceil) \geqslant \binom{\lceil k/2\rceil - \lceil\sqrt{2\lceil k/2\rceil} - 5/2\rceil}{2} \geqslant \frac{1}{2}\left(\frac{k}{2} - \sqrt{k}\right)\left(\frac{k}{2} - \sqrt{k} - 1\right).$$

Simplifying the inequality $\frac{1}{2}\left(\frac{k}{2} - \sqrt{k}\right)\left(\frac{k}{2} - \sqrt{k} - 1\right) \geqslant k$ we find that it is equivalent to

$$0 \leqslant k\sqrt{k} - 4k - 6\sqrt{k} + 4 = \sqrt{k}\left(\sqrt{k} - 2 - \sqrt{10}\right)\left(\sqrt{k} - 2 + \sqrt{10}\right) + 4,$$

which is true for every $k \geqslant \lceil (2+\sqrt{10})^2 \rceil = 27$. □

From Lemma 3.4 it follows that in order to complete the proof of Theorem 4 by establishing (10), we need to prove the following lemma.

Lemma 3.5. *Let n be a positive integer, set $k = \lceil n/2 \rceil$ and let m be a positive integer with $\binom{n}{k} - \binom{k+1}{2} + 2 \leqslant m \leqslant \binom{n}{k}$. If $m \in S(n)$ then $\binom{n}{k} - m \in \bigcup_{t=0}^{k} (\sigma(t,k) \cup \sigma(t, n-k))$.*

Before proving Lemma 3.5, we briefly summarize the proof of Theorem 4, assuming the lemmas that have been stated so far.

Proof of Theorem 4. The "if"-direction of the claimed equivalence is Lemma 3.2. For the "only-if"-direction, we can assume, by Lemma 3.4, that $m \geqslant \binom{n}{k} - \binom{k+1}{2} + 2$, and then Lemma 3.5 concludes the argument. □

It remains to prove Lemma 3.5, and we obtain this as a consequence of the following result.

Lemma 3.6. *Let n be a positive integer, set $k = \lceil n/2 \rceil$ and let m be an integer with $m \in S(n)$ and $\binom{n}{k} - \binom{k+1}{2} + 2 \leqslant m \leqslant \binom{n}{k}$. Then there exists a family $\mathcal{F} \subseteq \binom{[n]}{l+1}$ with $l \in \{k, n-k\}$ and $|\mathcal{F}| \leqslant k+1$ such that $m = \binom{n}{k} + |\mathcal{F}| - |\Delta\mathcal{F}|$.*

Before proving Lemma 3.6 we explain how Lemma 3.5 follows from it.

Proof of Lemma 3.5. It follows from Lemma 3.6, that $m \in S(n)$ implies the existence of a family $\mathcal{F} \subseteq \binom{[n]}{l+1}$, $l \in \{k, n-k\}$ with $|\mathcal{F}| = t \leqslant k$ and

$$\binom{n}{k} - m = |\Delta\mathcal{F}| - |\mathcal{F}| \in \sigma(t, l+1) - t \stackrel{(9)}{=} \sigma(t, l). \qquad \square$$

The proof of Lemma 3.6 is a bit involved and uses some auxiliary lemmas. We start with an antichain $\mathcal{A}$ of the required large size, and first establish that this forces $\mathcal{A}$ to be contained in three central levels of B_n (Lemmas 3.8 and 3.9). In Lemma 3.10, we show that $\mathcal{A}$ is very close to being a complete level. Then we need a lemma which says that certain sums of shadow sizes are again shadow sizes (Lemma 3.11) to conclude that we can assume that $\mathcal{A}$ is flat, that is, it is contained in two consecutive levels (Lemma 3.12).

Let us recall a well known inequality, the *Normalized Matching Property (NMP)*. If $\mathcal{F} \subseteq \binom{[n]}{k+1}$ for some $k \in [n]$, then every $F \in \mathcal{F}$ contains exactly $k+1$ sets from $\Delta\mathcal{F}$ and every $X \in \Delta\mathcal{F}$ is contained in at most $n-k$ sets from $\mathcal{F}$. This implies the NMP, $(n-k)|\Delta\mathcal{F}| \geqslant (k+1)|\mathcal{F}|$. As a consequence, $|\Delta\mathcal{F}| \geqslant |\mathcal{F}|$ whenever $k \geqslant \frac{n-1}{2}$ with strict inequality for $k > \frac{n-1}{2}$. Similarly, one has $|\nabla\mathcal{F}| \geqslant |\mathcal{F}|$ for $\mathcal{F} \subseteq \binom{[n]}{k}$ and $k < \frac{n}{2}$. The latter inequalities were the essential tool in Sperner's original proof of his famous theorem.

We will also need the following strengthening for $k = \lceil n/2 \rceil$.

Lemma 3.7. *Let n be a positive integer and $k = \lceil n/2 \rceil$.*

(i) If $\mathcal{F} \subseteq \binom{[n]}{k+1}$ with $|\mathcal{F}| \geqslant k$, then $|\Delta\mathcal{F}| - |\mathcal{F}| \geqslant \binom{k+1}{2} - 1$.

(ii) If $\mathcal{F} \subseteq \binom{[n]}{k+2}$ is nonempty, then $|\Delta\mathcal{F}| - |\mathcal{F}| \geqslant k+1$.

Proof. *(i)* Let $\mathcal{F} \subseteq \binom{[n]}{k+1}$ with $|\mathcal{F}| \geqslant k$. If $|\mathcal{F}| = k$, then, by the Kruskal-Katona Theorem, we have $|\Delta\mathcal{F}| - |\mathcal{F}| \geqslant k(k+1) - \binom{k}{2} - k = \binom{k+1}{2}$. If $|\mathcal{F}| = k+1$, then we have $|\Delta\mathcal{F}| - |\mathcal{F}| \geqslant (k+1)^2 - \binom{k+1}{2} - (k+1) = \binom{k+1}{2}$.

Assume that $|\mathcal{F}| \geqslant k+2$. Then $|\mathcal{F}| = \binom{x}{k+1}$ for some real number $x \geqslant k+2$ and, by Lovasz' [16] continuous version of the Kruskal-Katona Theorem, $|\Delta\mathcal{F}| \geqslant \binom{x}{k}$. Hence,

$$|\Delta\mathcal{F}| - |\mathcal{F}| \;\geqslant\; \binom{x}{k} - \binom{x}{k+1} \;=\; \frac{2k+1-x}{(k+1)!} \prod_{j=0}^{k-1}(x-j) \;=:\; P_k(x).$$

It is sufficient to show that $P_k(x) \geqslant \binom{k+1}{2} - 1$ for $k+2 \leqslant x \leqslant n$. The two largest zeros of the polynomial $P_k(x)$ are $k-1$ and $2k+1$ and $P_k(x) > 0$ for $k-1 < x < 2k+1$. As $n \in \{2k-1, 2k\}$, the minimum of P_k on the interval $[k+2, n]$ is attained at one of the endpoints. We have $P_k(k+2) = \binom{k+2}{k} - (k+2) = \binom{k+1}{2} - 1$. Observe that

$$P_k(2k-1) \;= P_k(2k) \;=\; \frac{1}{(k+1)!} \prod_{j=k+1}^{2k} j.$$

It follows that the unique maximum of P_k on $[k-1, 2k+1]$ is attained between $2k-1$ and $2k$. Consequently, $P_k(x)$ is increasing on $[k-1, 2k-1]$ and attains its minimum on $[k+1, n]$ at $x = k+1$. This implies the claim.

(ii) Here we consider P_{k+1} on the interval $[k+2, n]$. The two largest zeros of P_{k+1} are k and $2k+3$. Moreover, $P_{k+1}(2k+1) = P_{k+1}(2k+2)$ and $n < 2k+1$. Therefore, P_{k+1} is increasing on $[k+2, n]$, and its minimum on $[k+2, n]$ is $P_{k+1}(k+2) = k+1$. $\square$

Lemma 3.8. *Let n be a positive integer, set $k = \lceil n/2 \rceil$, and let $\mathcal{A}$ be an antichain in B_n with size $|\mathcal{A}| \geqslant \binom{n}{k} - \binom{k+2}{2} + 2$.*

(i) If $n = 2k$ then $\mathcal{A} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k} \cup \binom{[n]}{k+1}$.

(ii) If $n = 2k-1$ then $\mathcal{A} \subseteq \binom{[n]}{k-2} \cup \binom{[n]}{k-1} \cup \binom{[n]}{k} \cup \binom{[n]}{k+1}$.

Proof. We show that all the sets in $\mathcal{A}$ have size at most $k+1$. As the family consisting of the complements of the members of $\mathcal{A}$ is also an antichain this implies that all members of $\mathcal{A}$ have size at least $k-1$ if n is even, and at least $k-2$ if n is odd, and this proves the statement.

For the sake of contradiction, assume that $\mathcal{A}$ contains a set of size at least $k+2$. As in Sperner's proof of his theorem, we can replace $\mathcal{A} \cap \binom{[n]}{i}$ by its shade if $i = \min\{j : \mathcal{A} \cap \binom{[n]}{j} \neq \emptyset\} < k$ and by its shadow if $i=\max\{j : \mathcal{A}\cap\binom{[n]}{j}\neq\emptyset\}>k+2$. Repeating this yields an antichain $\mathcal{A}'\subseteq\binom{[n]}{k}\cup\binom{[n]}{k+1}\cup\binom{[n]}{k+2}$ with $|\mathcal{A}'|\geqslant|\mathcal{A}|$. Let $\mathcal{B}=\mathcal{A}'\cap\binom{[n]}{k+2}$. By Lemma 3.7 (ii), $\mathcal{A}''=(\mathcal{A}'\setminus\mathcal{B})\cup\Delta\mathcal{B}$ is an antichain with $|\mathcal{A}''|\geqslant|\mathcal{A}'|+k+1$ and $|\mathcal{A}''\cap\binom{[n]}{k+1}|\geqslant k+2$. Let $\mathcal{C}=\mathcal{A}''\cap\binom{[n]}{k+1}$. Finally, $\mathcal{A}'''=(\mathcal{A}''\setminus\mathcal{C})\cup\Delta\mathcal{C}$ is an antichain with $|\mathcal{A}'''|=|\mathcal{A}''|+|\Delta\mathcal{C}|-|\mathcal{C}|\leqslant\binom{n}{k}$. By Lemma 3.7 (i), $|\Delta\mathcal{C}|-|\mathcal{C}|\geqslant\binom{k+1}{2}-1$. This implies $|\mathcal{A}''|\leqslant\binom{n}{k}-\binom{k+1}{2}+1$ and $|\mathcal{A}'|\leqslant\binom{n}{k}-\binom{k+2}{2}+1$, a contradiction. ☐

Lemma 3.9. *Let $n=2k-1$, and let $\mathcal{A}$ be an antichain in B_n with size $|\mathcal{A}|\geqslant\binom{n}{k}-\binom{k+1}{2}+2$. Then $\mathcal{A}\subseteq\binom{[n]}{k-2}\cup\binom{[n]}{k-1}\cup\binom{[n]}{k}$ or $\mathcal{A}\subseteq\binom{[n]}{k-1}\cup\binom{[n]}{k}\cup\binom{[n]}{k+1}$.*

Proof. By Lemma 3.8 we know that $\mathcal{A}\subseteq\binom{[n]}{k-2}\cup\binom{[n]}{k-1}\cup\binom{[n]}{k}\cup\binom{[n]}{k+1}$. Assume that $\mathcal{A}$ contains at least one $(k+1)$-set and at least one $(k-2)$-set. For $i=1,2,3,4$, let $\mathcal{A}_i=\mathcal{A}\cap\binom{[n]}{k+2-i}$. By the Kruskal-Katona Theorem, without loss of generality, we can assume that $\mathcal{A}$ is left-compressed, i.e., that $\mathcal{A}_1$, $\mathcal{A}_2'=\Delta\mathcal{A}_1\cup\mathcal{A}_2$, $\mathcal{A}_3'=\Delta\mathcal{A}_2'\cup\mathcal{A}_3$, $\mathcal{A}_4'=\Delta\mathcal{A}_3'\cup\mathcal{A}_4$ are initial segments of $\binom{[n]}{k+1}$, $\binom{[n]}{k}$, $\binom{[n]}{k-1}$, $\binom{[n]}{k-2}$, respectively, with respect to squashed order. Partition $\mathcal{A}$ into $\mathcal{A}^-=\{A\in\mathcal{A} : n\notin A\}$ and $\mathcal{A}^+=\{A\in\mathcal{A} : n\in A\}$. As $\mathcal{A}$ is left-compressed, we have $\mathcal{A}^-\subseteq\binom{[n]}{k+1}\cup\binom{[n]}{k}$ or $\mathcal{A}^+\subseteq\binom{[n]}{k-1}\cup\binom{[n]}{k-2}$. Assume that $\mathcal{A}^+\subseteq\binom{[n]}{k-1}\cup\binom{[n]}{k-2}$, the other case is analogous. Then $|\mathcal{A}^+|\leqslant\binom{n-1}{k-2}$ and, by Lemma 3.8 (i), $|\mathcal{A}^-|\leqslant\binom{n-1}{k-1}-\binom{k+1}{2}+1$. Hence, $|\mathcal{A}|=|\mathcal{A}^-|+|\mathcal{A}^+|\leqslant\binom{n}{k}-\binom{k+1}{2}+1$. ☐

At this point, without loss of generality, we can assume that $\mathcal{A}$ lives on the three levels $k-1$, k and $k+1$. Next we show that $\mathcal{A}$ must be almost a complete largest level.

Lemma 3.10. *Let n be a positive integer, set $k=\lceil n/2\rceil$, and let $\mathcal{A}\subseteq\binom{[n]}{k-1}\cup\binom{[n]}{k}\cup\binom{[n]}{k+1}$ be an antichain with size $|\mathcal{A}|\geqslant\binom{n}{k}-\binom{k+1}{2}+2$.*

(i) If $n=2k$, then $\left|\mathcal{A}\cap\binom{[n]}{k-1}\right|+\left|\mathcal{A}\cap\binom{[n]}{k+1}\right|\leqslant k+1$.

(ii) If $n=2k-1$, then $\left|\mathcal{A}\cap\binom{[n]}{k-1}\right|+\left|\mathcal{A}\cap\binom{[n]}{k+1}\right|\leqslant k+1$ or $\mathcal{A}\subseteq\binom{[n]}{k-1}\cup\binom{[n]}{k}$ and $\left|\mathcal{A}\cap\binom{[n]}{k}\right|\leqslant k+1$.

Proof. (i) Set $\mathcal{A}_1=\mathcal{A}\cap\binom{[n]}{k+1}$, $\mathcal{A}_2=\mathcal{A}\cap\binom{[n]}{k-1}$, $t_1=|\mathcal{A}_1|$ and $t_2=|\mathcal{A}_2|$. If $t_1\geqslant k+2$, then $|\Delta\mathcal{A}_1|-|\mathcal{A}_1|\geqslant\binom{k+1}{2}-1$ by Lemma 3.7 (i), hence

$$|\mathcal{A}|\leqslant\binom{n}{k}+(|\mathcal{A}_1|-|\Delta\mathcal{A}_1|)+(|\mathcal{A}_2|-|\nabla\mathcal{A}_2|)\leqslant\binom{n}{k}+(|\mathcal{A}_1|-|\Delta\mathcal{A}_1|)\leqslant\binom{n}{k}-\binom{k+1}{2}+1.$$

Consequently, $t_1\leqslant k+1$ and similarly, $t_2\leqslant k+1$. Now, by Kruskal-Katona, we have $|\Delta\mathcal{A}_1|\geqslant(k+1)t_1-\binom{t_1}{2}$ and $|\nabla\mathcal{A}_2|\geqslant(k+1)t_2-\binom{t_2}{2}$. This implies

$$|\mathcal{A}|\leqslant\binom{n}{k}-k(t_1+t_2)+\binom{t_1}{2}+\binom{t_2}{2}.$$

For fixed $t=t_1+t_2$ with $k+2\leqslant t\leqslant 2(k+1)$, the expression on right-hand side of the above inequality attains its maximum if $\max\{t_1,t_2\}=k+1$ and $\min\{t_1,t_2\}=t-k-1$. Hence, for $k+2\leqslant t\leqslant 2(k+1)$ we have

$$|\mathcal{A}|\leqslant\binom{n}{k}-kt+\binom{k+1}{2}+\binom{t-k-1}{2}=\binom{n}{k}-\binom{k+1}{2}-\frac{1}{2}(t-k-1)(3k+2-t)<\binom{n}{k}-\binom{k+1}{2},$$

and this concludes the proof.

(ii) Set $\mathcal{A}_i=\mathcal{A}\cap\binom{[n]}{k+2-i}$ and $t_i=|\mathcal{A}_i|$ for $i=1,2,3$. Like in the proof of Lemma 3.9, without loss of generality, we assume that $\mathcal{A}$ is left-compressed, i.e., $\mathcal{A}_1$, $\mathcal{A}_2'=\Delta\mathcal{A}_1\cup\mathcal{A}_2$, $\mathcal{A}_3'=\Delta\mathcal{A}_2'\cup\mathcal{A}_3$ are initial segments of $\binom{[n]}{k+1}$, $\binom{[n]}{k}$, $\binom{[n]}{k-1}$, respectively, with respect to squashed order. Furthermore, let $\mathcal{A}^-=\{A\in\mathcal{A} : n\notin A\}$ and $\mathcal{A}^+=\{A\in\mathcal{A} : n\in A\}$.

Case 1: Assume that $t_1=0$. We have to show that $\min\{t_2,t_3\}\leqslant k+1$. Assume for a contradiction that $t_2\geqslant k+2$ and $t_3\geqslant k+2$. We have $t_2\leqslant\binom{n-1}{k}$ or $t_3\leqslant\binom{n-1}{k}$ because $|\mathcal{A}_2|>\binom{n-1}{k}$ implies $|\mathcal{A}_3|\leqslant\binom{n}{k-1}-|\Delta\mathcal{A}_2|<\binom{n}{k-1}-\binom{n}{k-1}=\binom{n-1}{k-1}=\binom{n-1}{k-1}$. Without loss of generality, we assume that $t_2\leqslant\binom{n-1}{k}$. Then

$\mathcal{A}_2 \subseteq \mathcal{A}^-$, and by Lemma 3.7 (i), we have $|\Delta\mathcal{A}_2| - |\mathcal{A}_2| \geqslant \binom{k+1}{2} - 1$. This implies $|\mathcal{A}| \leqslant \binom{n}{k-1} - |\Delta\mathcal{A}_2| + |\mathcal{A}_2| \leqslant \binom{n}{k} - \binom{k+1}{2} + 1$, a contradiction.

Case 2: Assume that $t_1 \geqslant 1$. For a contradiction, assume further that $t = t_1 + t_3 \geqslant k+2$. If $t_1 \geqslant k$, then by Lemma 3.7 (i) and $|\nabla\mathcal{A}_3| \geqslant |\mathcal{A}_3|$, we have

$$|\mathcal{A}| \;\leqslant\; \binom{n}{k} - (|\Delta\mathcal{A}_1| - |\mathcal{A}_1|) - (|\nabla\mathcal{A}_3| - |\mathcal{A}_3|) \;\leqslant\; \binom{n}{k} - \binom{k+1}{2} + 1,$$

a contradiction. Therefore, $t_1 < k$ which implies

$$|\Delta\mathcal{A}_1| - |\mathcal{A}_1| \;\geqslant\; kt_1 - \tbinom{t_1}{2} \;=\; k + (t_1 - 1)(k - \tfrac{t_1}{2}) \;\geqslant\; k. \tag{12}$$

Case 2.1: Assume that $\mathcal{A}^- \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$. Then $\mathcal{A}_3 \subseteq \mathcal{A}^+$. If $t_3 \geqslant k$, then by Lemma 3.7 (i), we have $|\nabla\mathcal{A}_3| - |\mathcal{A}_3| \geqslant \binom{k}{2} - 1$ which, together with (12), yields $|\mathcal{A}| \leqslant \binom{n}{k} - k - (\binom{k}{2} - 1) = \binom{n}{k} - \binom{k+1}{2} + 1$, a contradiction. Hence, $t_3 < k$, and a contradiction follows by

$$\begin{aligned} |\mathcal{A}| \;&\leqslant\; \tbinom{n}{k} - kt_1 + \tbinom{t_1}{2} - (k-1)t_3 + \tbinom{t_3}{2} \;=\; \tbinom{n}{k} - k(t_1 + t_3) + \tbinom{t_1}{2} + \tbinom{t_3+1}{2} \\ &\leqslant\; \tbinom{n}{k} - tk + \tbinom{t-k+1}{2} + \tbinom{k}{2} \;=\; \tbinom{n}{k} - \tbinom{k+1}{2} - \tfrac{1}{2}(3k - 1 - t)(t - k) \;<\; \tbinom{n}{k} - \tbinom{k+1}{2}, \end{aligned}$$

where the last equality comes from substituting $\binom{k+1}{2} + \frac{1}{2}k(2t - k - 1)$ for tk, and then rearranging the terms.

Case 2.2: Assume that $\mathcal{A}^- \cap \mathcal{A}_3 \neq \emptyset$. Then $\mathcal{A}^+ \subseteq \mathcal{A}_3$, and hence, $|\mathcal{A}^+| \leqslant \binom{n-1}{k-2}$. On the other hand, by Lemma 3.8 (i), $|\mathcal{A}^-| \leqslant \binom{n-1}{k-1} - \binom{k+1}{2} + 1$. Finally, we obtain $|\mathcal{A}| = |\mathcal{A}^-| + |\mathcal{A}^+| \leqslant \binom{n}{k} - \binom{k+1}{2} + 1$, a contradiction. □

In the final step we want to replace a maximal antichain $\mathcal{A}$ on three consecutive levels by another maximal antichain on two consecutive levels which has the same size as $\mathcal{A}$. To prove that this is always possible we will use the fact that sums of certain shadow sizes are themselves shadow sizes. Specifically, we need the following result.

Lemma 3.11.

(i) If $t \leqslant k$ then $\sigma(1,k) + \sigma(t,k-1) \subseteq \sigma(1+t,k) \cup \sigma(1+t,k-1)$.

(ii) If $t_1 \geqslant 2$ and $t_1 + t_2 \leqslant k+1$ then $\sigma(t_1,k) + \sigma(t_2,k-1) \subseteq \sigma(t_1+t_2,k)$.

In the proof of Lemma 3.11 we will use graphs obtained by gluing $G(a,b,c)$ and $G(a',b',c')$, as defined in Section 2, in the following way.

Definition 3. Let a,b,c,a',b',c' be integers with $a \geqslant b \geqslant c \geqslant 0, a' \geqslant b' \geqslant c' \geqslant 0$, and $a,a' \geqslant 1$. Set $G = G(a,b,c)$ and $G' = G(a',b',c')$. Moreover, assume that G has at least one pendant vertex if $b' = 0$ and at least two pendant vertices if $b' \geqslant 1$.

- For $b' = 0$, let $G \oplus G'$ denote a graph that is obtained from G and G' by identifying one pendant vertex of G with the center of the a'-star G'.
- For $b' \geqslant 1$, let $G \oplus G'$ denote a graph that is obtained from G and G' by identifying two pendant vertices of G with the centers of the two stars forming G'.

The assumption on the number of pendant vertices in G is satisfied whenever any of the following statements is true:

(i) $a + b - 2c \geqslant 2$, or (ii) $b' = 0$, $a + b - 2c \geqslant 1$, or (iii) $b = 1$, $a - 2c \geqslant 0$.

The construction for $b' \geqslant 1$ and $a + b - 2c \geqslant 2$ is illustrated in Figure 3. The given description does not characterize the graph $G \oplus G'$ up to isomorphism, as we do not specify which of the pendant vertices of G are used for gluing G'. For our purpose this is not important as we will only use the number of edges and the number of pairs of adjacent edges. The graph $G(a,b,c) \oplus G(a',b',c')$ has $a + a' + b' - c' + 1$ vertices if $b = 0$ and $a + b - c + a' + b' - c' + 2$ vertices if $b \geqslant 1$. The number of edges is always $a + b + a' + b'$ and there are

$$\binom{a}{2} + \binom{b}{2} + c + \binom{a'}{2} + \binom{b'}{2} + c' + a' + b'$$

pairs of adjacent edges.

Definition 4. For integers k and a, b, c, a', b', c' with $a \geqslant b \geqslant c \geqslant 0$ and $a' \geqslant b' \geqslant c' \geqslant 0$, let
$$\mathcal{G}_k(a, b, c, a', b', c') = \{[k+2] \setminus e \,:\, e \text{ is an edge of } G(a, b, c) \oplus G(a', b', c')\}\,.$$

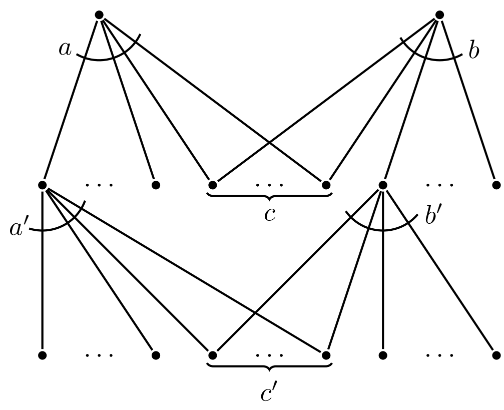

FIGURE 3. A graph $G(a, b, c) \oplus G(a', b', c')$.

Remark 3. If $b = 0$ and $k \geqslant a + a' + b' - c' - 1$ or $b \geqslant 1$ and $k \geqslant a + b - c + a' + b' - c'$ then the vertex set of $G(a, b, c) \oplus G(a', b', c')$ is contained in $[k + 2]$, and consequently, $\mathcal{G}_k(a, b, c, a', b', c')$ is a family of k-sets with $|\mathcal{G}_k(a, b, c, a', b', c')| = a + b + a' + b'$. Moreover, in this situation, the size of its shadow equals its size multiplied by k reduced by the number of pairs of sets that have a shadow element in common. As the latter equals the number of pairs of adjacent edges of $G(a, b, c) \oplus G(a', b', c')$, we obtain
$$|\Delta\mathcal{G}_k(a, b, c, a', b', c')| = (a + b + a' + b')k - \binom{a}{2} - \binom{b}{2} - c - \binom{a'}{2} - \binom{b'}{2} - c' - a' - b'.$$
We also note that $|\Delta\mathcal{G}_k(a, b, c, a', b', c')| = |\Delta\mathcal{F}_k(a, b, c)| + |\Delta\mathcal{F}_{k-1}(a', b', c')|$.

Proof of Lemma 3.11. For (i), assume $t \leqslant k$ and let $s \in \sigma(t, k - 1)$. By (1), there are integers $a \geqslant b \geqslant c \geqslant 0$ with $1 \leqslant a + b \leqslant t$ and $s = t(k - 1) - \binom{a}{2} - \binom{b}{2} - c$. If $c \geqslant 1$, then
$$s + k = (t + 1)(k - 1) - \binom{a}{2} - \binom{b}{2} - (c - 1) \in \sigma(t + 1, k - 1).$$
If $c = 0$ and $b \geqslant 2$, then
$$s + k = (t + 1)(k - 1) - \binom{a}{2} - \binom{b}{2} + 1 = (t + 1)(k - 1) - \binom{a}{2} - \binom{b-1}{2} - (b - 2) \in \sigma(t + 1, k - 1).$$
If $c = 0$ and $b \leqslant 1$, then
$$s + k = t(k - 1) - \binom{a}{2} + k = (t + 1)k - \binom{a}{2} - t = (t + 1)k - \binom{a+1}{2} - (t - a),$$
and for
$$\mathcal{F} = \{[k + 1] \setminus \{i\} \,:\, i \in \{1, 2, \dots, a + 1\}\} \cup \{\{2, \dots, k\} \cup \{k + 1 + i\} \,:\, i \in [t - a]\}$$
we obtain
$$|\Delta\mathcal{F}| = (a + 1)k - \binom{a+1}{2} + (t - a)(k - 1) = (t + 1)k - \binom{a+1}{2} - (t - a) = s,$$
and this concludes the proof of (i).

For (ii), let $s_1 \in \sigma(t_1, k)$ and $s_2 \in \sigma(t_2, k-1)$. By (1), there are integers $a \geqslant b \geqslant c \geqslant 0$ and $a' \geqslant b' \geqslant c' \geqslant 0$ with $1 \leqslant a + b \leqslant t_1$, $1 \leqslant a' + b' \leqslant t_2$,
$$s_1 = t_1k - \binom{a}{2} - \binom{b}{2} - c \qquad \text{and} \qquad s_2 = t_2(k - 1) - \binom{a'}{2} - \binom{b'}{2} - c'.$$

First, we assume $a \geqslant 3$, and argue that without loss of generality $a+b-2c \geqslant 2$. If $a+b-2c=0$, then $a=b=c$, and from

$$\binom{a}{2}+\binom{a}{2}+a=\binom{a+1}{2}+\binom{a-1}{2}+(a-1),$$

it follows that we can use $(a+1,a-1,a-1)$ instead of (a,a,a). Similarly, if $a+b-2c=1$, then $b=c=a-1$, and from

$$\binom{a}{2}+\binom{a-1}{2}+a-1=\binom{a+1}{2}+\binom{a-2}{2}+a-3,$$

it follows that we can use $(a+1,a-2,a-3)$ instead of $(a,a-1,a-1)$.

So we assume $a+b-2c \geqslant 2$ and show that there is a (t_1+t_2)-family $\mathcal{F}$ of k-sets such that $|\Delta\mathcal{F}|=s_1+s_2$. Without loss of generality, we assume that $b=0$ or $c \geqslant 1$. This is possible, because if $b \geqslant 1$ and $c=0$ then $\binom{b}{2}+c=\binom{b-1}{2}+(b-1)$, and we can use $(a,b-1,b-1)$ instead of $(a,b,0)$. If $b=0$ then $a+a'+b'-c' \leqslant t_1+t_2 \leqslant k+1$, and if $c \geqslant 1$ then $(a+b-c)+(a'+b'-c') \leqslant (t_1-1)+t_2 \leqslant k$. In both cases, by Remark 3 there is a family $\mathcal{F}'=\mathcal{G}_k(a,b,c,a',b',c') \subseteq \binom{[k+2]}{k}$ with $|\mathcal{F}'|=a+b+a'+b'$ and

$$|\Delta\mathcal{F}'|=(a+b+a'+b')k-\binom{a}{2}-\binom{b}{2}-c-\binom{a'}{2}-\binom{b'}{2}-c'-a'-b'.$$

Adding to $\mathcal{F}'$ a (t_1-a-b)-family of k-sets with marginal shadow k, and a $(t_2-a'-b')$-family of k-sets with marginal shadow $k-1$ we obtain a (t_1+t_2)-family $\mathcal{F}$ of k-sets with

$$|\Delta\mathcal{F}|=|\Delta\mathcal{F}'|+(t_1-a-b)k+(t_2-a'-b')(k-1)=s_1+s_2.$$

It remains to consider the case $a \leqslant 2$. Then $t_1k-4 \leqslant s_1 \leqslant t_1k$. For the values t_1k-l, $l=4,3,2,1$, we use the same construction as above with $\mathcal{F}'=\mathcal{G}_k(3,1,1,a',b',c')$, $\mathcal{G}_k(3,0,0,a',b',c')$, $\mathcal{G}_k(2,1,1,a',b',c')$, and $\mathcal{G}_k(2,0,0,a',b',c')$, respectively. For $s_1=t_1k$ we use $\mathcal{F}'=\mathcal{G}_k(1,1,0,a',b',c')$ if $a'+b' \leqslant t_2-1$ or $c' \geqslant 1$ (which ensures $2+a'+b'-c' \leqslant t_1+t_2-1 \leqslant k$). If $a'+b'=t_2$ and $c'=0$, then

$$s_1+s_2=(t_1+t_2)k-\binom{a'}{2}-\binom{b'}{2}-(a'+b')=(t_1+t_2)k-\binom{a'+1}{2}-\binom{b'}{2}-b' \in \sigma(t_1+t_2,k),$$

where we used $a'+1+b'=t_2+1 \leqslant t_1+t_2$. $\square$

We use Lemma 3.11 to show that the antichains on three levels, which appear in Lemma 3.10, can be replaced by flat antichains.

Lemma 3.12. *Let n be a positive integer, $k=\lceil n/2\rceil$, let $\mathcal{A} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k} \cup \binom{[n]}{k+1}$ be a maximal antichain, and set $t_1=\left|\mathcal{A}\cap\binom{[n]}{k+1}\right|$, $t_2=\left|\mathcal{A}\cap\binom{[n]}{k-1}\right|$. If $t_1+t_2 \leqslant k+2$ then there exists $\mathcal{F} \subseteq \binom{[n]}{k+1}$ or $\mathcal{F} \subseteq \binom{[n]}{n-k+1}$ with*

$$|\mathcal{A}|=|\mathcal{F}|+\binom{n}{k}-|\Delta\mathcal{F}|.$$

Proof. Let $\mathcal{A}_1=\mathcal{A}\cap\binom{[n]}{k+1}$ and $\mathcal{A}_2=\mathcal{A}\cap\binom{[n]}{k-1}$. If $t_2=0$ then $\mathcal{A}=\mathcal{F}\cup\binom{[n]}{k}\setminus\Delta\mathcal{F}$ for $\mathcal{F}=\mathcal{A}_1$, and if $t_1=0$ then $\mathcal{A}=\mathcal{F}\cup\binom{[n]}{k}\setminus\Delta\mathcal{F}$ for $\mathcal{F}=\mathcal{A}\cap\binom{[n]}{k}$. So we assume $t_1 \geqslant 1$ and $t_2 \geqslant 1$, and then

$$|\mathcal{A}|=t_1+t_2+\left|\binom{[n]}{k}\setminus(\Delta\mathcal{A}_1\cup\nabla\mathcal{A}_2)\right|=\binom{n}{k}+t_1+t_2-|\Delta\mathcal{A}_1|-|\nabla\mathcal{A}_2|.$$

Since $|\Delta\mathcal{A}_1| \in \sigma(t_1,k+1)$ and $|\nabla\mathcal{A}_2| \in \sigma(t_2,n-k+1)$, it follows that $|\Delta\mathcal{A}_1|+|\nabla\mathcal{A}_2| \in \sigma(t_1+t_2,k+1)$ if $n=2k$, and with Lemma 3.11,

$$|\Delta\mathcal{A}_1|+|\nabla\mathcal{A}_2| \in \sigma(t_1+t_2,k)\cup\sigma(t_1+t_2,k+1)$$

if $n=2k-1$. As a consequence, there exists a (t_1+t_2)-family $\mathcal{F}$ with $\mathcal{F} \subseteq \binom{\mathbb{N}}{k+1}$ or $\mathcal{F} \subseteq \binom{\mathbb{N}}{k}$ and $|\Delta\mathcal{F}|=|\Delta\mathcal{A}_1|+|\nabla\mathcal{A}_2|$. Using the second part of Theorem 2, we can assume that the members of $\mathcal{F}$ are subsets of $[k+4] \subseteq [n]$, as required. $\square$

Now we have all the ingredients to prove Lemma 3.6.

Proof of Lemma 3.6. As $m \in S(n)$, there exists a maximal antichain $\mathcal{A}$ in B_n with $|\mathcal{A}| = m$. By Lemmas 3.8 and 3.9, $\mathcal{A} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k} \cup \binom{[n]}{k+1}$ or $n = 2k-1$ and $\mathcal{A} \subseteq \binom{[n]}{k-2} \cup \binom{[n]}{k-1} \cup \binom{[n]}{k}$. In the latter case, the family consisting of the complements of the members of $\mathcal{A}$ is a maximal antichain in $\binom{[n]}{k-1} \cup \binom{[n]}{k} \cup \binom{[n]}{k+1}$, i.e., without loss of generality, we can assume that $\mathcal{A} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k} \cup \binom{[n]}{k+1}$. By Lemma 3.10, at most $k+1$ sets in $\mathcal{A}$ are not in $\binom{[n]}{k}$. Now, by Lemma 3.12, there exists a family $\mathcal{F} \subseteq \binom{[n]}{l+1}$ with $l \in \{k, n-k\}$ such that $m = \binom{n}{k} + |\mathcal{F}| - |\Delta\mathcal{F}|$. Finally, we have $|\mathcal{F}| \leqslant k+1$ by Lemma 3.10. □

4. The lower part of the spectrum

In this section we prove Theorem 1(ii). It will suffice by induction to construct antichains of large sizes m. More specifically, we set

$$w(n) = \binom{n}{\lceil n/2 \rceil} - \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n+1}{2} \right\rceil$$

and establish the following claim.

Claim 2. *If $n \geqslant 7$, then $[w(n-1)+2, w(n)] \subseteq S(n)$.*

Assuming this claim, the proof of Theorem 1(ii) is easy.

Proof of Theorem 1(ii) (assuming Claim 2). We proceed by induction on n. As we want to use Claim 2 for the induction step, we have to establish the result for $n \leqslant 6$ as the base case. For $1 \leqslant m \leqslant n$, a maximal antichain of size m is given by $\{\{1\},\{2\},\dots,\{m-1\},\{m,\dots,n\}\}$. For $n \leqslant 5$, this is already sufficient, because then $w(n) \leqslant n$. As $w(6) = 14$, we need maximal antichains of sizes $m = 7, 8, \dots, 14$ for $n = 6$. Such are given by:

- $m = 7$: $\mathcal{A} = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{3,4\},\{3,5\},\{4,5\},\{6\}\}$,
- $m = 8$: $\mathcal{A} = \binom{[4]}{2} \cup \{\{5\},\{6\}\}$,
- $m = 9$: $\mathcal{A} = \{\{1,2,5\},\{1,2,6\},\{3,4,5\},\{3,4,6\},\{5,6\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$,
- $m = 10$: $\mathcal{A} = \{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,6\}\} \cup \binom{[6]}{3} \setminus \nabla\{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,6\}\}$,
- $m = 11$: $\mathcal{A} = \{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,5\}\} \cup \binom{[6]}{3} \setminus \nabla\{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,5\}\}$,
- $m = 12$: $\mathcal{A} = \{\{1,2\},\{2,3\},\{4,5\}\} \cup \binom{[6]}{3} \setminus \nabla\{\{1,2\},\{2,3\},\{4,5\}\}$,
- $m = 13$: $\mathcal{A} = \{\{1,2\},\{2,3\},\{3,4\}\} \cup \binom{[6]}{3} \setminus \nabla\{\{1,2\},\{2,3\},\{3,4\}\}$,
- $m = 14$: $\mathcal{A} = \{\{1,2\},\{3,4\}\} \cup \binom{[6]}{3} \setminus \nabla\{\{1,2\},\{3,4\}\}$.

For the induction step, we assume $n \geqslant 7$ and $[1, w(n-1)] \subseteq S(n-1)$. Noting that $1 \in S(n)$ for every n, and adding the singleton $\{n\}$ to each of the maximal antichains with sizes $m \in [1, w(n-1)]$ in B_{n-1}, we obtain $[1, w(n-1)+1] \subseteq S(n)$, and we use Claim 2 to conclude $[1, w(n)] \subseteq S(n)$ as required. □

The rest of the section is devoted to the proof of Claim 2. In Subsection 4.1 we describe a construction for maximal antichains on three consecutive levels $k-1, k, k+1$, we prove that this yields an interval $I(n,k)$ of maximal antichain sizes (Lemma 4.4), and we provide bounds for the endpoints of these intervals (Lemma 4.7). In Section 4.2 we show that for $n \geqslant 20$, the intervals $I(n,k)$ for varying k overlap and taking their union establishes Claim 2. Finally, in Section 4.3 we use a separate construction to fill the gaps between the intervals $I(n,k)$ for $7 \leqslant n \leqslant 19$, thus completing the proof of Claim 2.

4.1. **The main construction.** Before going into the details of the construction we outline the general idea. An antichain $\mathcal{A} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$ is a *maximal squashed flat antichain* if it has the form $\mathcal{A} = \mathcal{F} \cup \binom{n}{k-1} \setminus \Delta\mathcal{F}$ where $\mathcal{F} \subseteq \binom{[n]}{k}$ is an initial segment in squashed order. From a maximal squashed flat antichain $\mathcal{A}' \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$ with $\binom{[k+3]}{k} \subseteq \mathcal{A}'$, we can obtain a new maximal antichain $\mathcal{A}$ in the following way. We replace $\binom{[k+3]}{k}$ in $\mathcal{A}'$ by $\mathcal{F} \cup \binom{[k+3]}{k} \setminus \Delta\mathcal{F}$ where $\mathcal{F} \subseteq \binom{[k+3]}{k+1}$ contains only few sets (about $k/2$). Using Theorem 2, we can vary $\mathcal{F}$ without changing its cardinality to get k consecutive values for the size of $\Delta\mathcal{F}$. For a few small values of n we only get $k-1$ or $k-2$ consecutive shadow sizes, and we close the resulting gaps by extra constructions. Hence, from every maximal squashed $\mathcal{A}' \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$, we obtain an interval of k consecutive sizes of maximal antichains in B_n. We then vary $\mathcal{A}'$. We say that two distinct maximal squashed antichains

$\mathcal{A}_1, \mathcal{A}_2 \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$ are *consecutive* if there is no maximal squashed antichain $\mathcal{A}_3 \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$ with $|\mathcal{A}_1 \cap \binom{[n]}{k}| < |\mathcal{A}_3 \cap \binom{[n]}{k}| < |\mathcal{A}_2 \cap \binom{[n]}{k}|$. (Note that for consecutive $\mathcal{A}_1, \mathcal{A}_2$ it is still possible that there is a maximal squashed antichain $\mathcal{A}_3$ with size $|\mathcal{A}_3|$ between $|\mathcal{A}_1|$ and $|\mathcal{A}_2|$.) Observing that the sizes of consecutive maximal squashed flat antichains differ by at most $k+1$, we deduce that the maximal antichains on levels $k+1$, k and $k-1$ obtained from all $\mathcal{A}'$ yield an interval $I(n,k)$ of sizes of maximal antichains.

For an integer k with $\lfloor n/2 \rfloor \leqslant k \leqslant n-3$ we denote by $\mathcal{M}(n,k)$ the set of all maximal squashed flat antichains $\mathcal{A}$ with $\binom{[k+3]}{k} \subseteq \mathcal{A} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$. Moreover, we let $Y(n,k) = \{|\mathcal{A}| : \mathcal{A} \in \mathcal{M}(n,k)\}$ be the set of their sizes. The following result is proved using the construction outlined above.

Lemma 4.1. *Let n, k and t be integers with $n \geqslant 7$, $\frac{n-1}{2} \leqslant k \leqslant n-3$ and $1 \leqslant t \leqslant k$. Then for every $m \in Y(n,k)$ and $s \in \sigma(t,k)$ there is a maximal antichain $\mathcal{A} \subseteq \binom{[k+3]}{k+1} \cup \binom{[n]}{k} \cup \binom{[n]}{k-1}$ with $|\mathcal{A}| = m - s$.*

Proof. Fix $m \in Y(n,k)$, $s \in \sigma(t,k)$, and $\mathcal{A}' \in \mathcal{M}(n,k)$ with $|\mathcal{A}'| = m$. By Theorem 2 and Remark 2, $s + t \in \sigma(t, k+1)$, and there is a family $\mathcal{F} \subseteq \binom{[k+3]}{k+1}$ with $|\mathcal{F}| = t$ and $|\Delta\mathcal{F}| = s + t$. By Lemma 3.1 $\mathcal{F}' = \mathcal{F} \cup \binom{[k+3]}{k} \setminus \Delta\mathcal{F}$ is a maximal antichain in B_{k+3}. As a consequence, $\mathcal{A} = \left[\mathcal{A}' \setminus \binom{[k+3]}{k}\right] \cup \mathcal{F}'$ is a maximal antichain in B_n with

$$|\mathcal{A}| = |\mathcal{A}'| - \binom{k+3}{k} + |\mathcal{F}'| = m - \binom{k+3}{k} + |\mathcal{F}| - \binom{k+3}{3} + |\Delta\mathcal{F}| = m + t - (s+t) = m - s. \qquad \square$$

By Theorem 2 and Proposition 1, for $t \leqslant k+1$,

$$\left[tk - \binom{t - j^*(t)}{2} - \binom{j^*(t)+1}{2},\, tk\right] \subseteq \sigma(t,k),$$

where $j^*(t) = \lceil \sqrt{2t} - 5/2 \rceil$ is the smallest non-negative integer j with $\binom{j+3}{2} \geqslant t$. Thus we have the following consequence of Lemma 4.1.

Corollary 1. *Let $n \geqslant 7$ and k be integers with $\frac{n-1}{2} \leqslant k \leqslant n-3$. Let $m \in Y(n,k)$ and set $t = \lfloor \frac{k+3}{2} \rfloor$, $j = j^*(t)$. Then*

$$\left[m - tk,\, m - tk + \binom{t-j}{2} + \binom{j+1}{2}\right] \subseteq S(n).$$

In the next lemma we bound the lengths of the intervals in Corollary 1.

Lemma 4.2. *Let $k \geqslant 3$, $t = \lfloor \frac{k+3}{2} \rfloor$, $j = j^*(t)$, and set $L = \binom{t-j}{2} + \binom{j+1}{2}$.*

(i) If $k \geqslant 9$ or $k \in \{3, 7\}$, then $L \geqslant k$.
(ii) If $k \in \{4, 5, 8\}$, then $L = k - 1$.
(iii) If $k = 6$, then $L = 4$.

Proof. For $3 \leqslant k \leqslant 32$, the statement can be verified by calculating L. We now assume $k \geqslant 33$. Using $\frac{k+2}{2} \leqslant t \leqslant \frac{k+3}{2}$, we bound j by

$$j = \left\lceil \sqrt{2t} - \frac{5}{2} \right\rceil \leqslant \sqrt{2t} \leqslant \sqrt{k+3}.$$

Then

$$L \geqslant \binom{t-j}{2} \geqslant \frac{1}{2}\left(\frac{k+2}{2} - \sqrt{k+3}\right)\left(\frac{k}{2} - \sqrt{k+3}\right) \geqslant \frac{1}{2}\left(\frac{k}{2} - \sqrt{k+3}\right)^2 = \frac{k^2}{8} - \frac{1}{2}k\sqrt{k+3} + \frac{k+3}{2},$$

and

$$\begin{aligned} L - k \geqslant \frac{k^2}{8} - \frac{1}{2}k\sqrt{k+3} - \frac{k}{2} &= \frac{k}{8}\left(k - 4\sqrt{k+3} - 4\right) \geqslant \frac{k}{8}\left(k - 4\sqrt{k+3} - 9\right) \\ &= \frac{k}{8}\left((k+3) - 4\sqrt{k+3} - 12\right) = \frac{k}{8}\left(\sqrt{k+3} + 2\right)\left(\sqrt{k+3} - 6\right) \geqslant 0, \end{aligned}$$

where the last inequality follows from $k + 3 \geqslant 36$. $\square$

To show that the union of the intervals from Corollary 1 is an interval, we need an upper bound for the gaps between the elements of $Y(n,k)$. This is provided in the next lemma.

Lemma 4.3. *The sizes of any two consecutive maximal squashed flat antichains on levels k and $k-1$ in B_n differ by at most $\max\{k-1,n-k\}$. In particular, for $k \geqslant \frac{n+1}{2}$ any two consecutive elements of $Y(n,k)$ differ by at most $k-1$, for $k = \frac{n}{2}$ they differ by at most k, and for $k = \frac{n-1}{2}$ they differ by at most $k+1$.*

Proof. Let A_m be the m-th k-set in squashed order, and let $\mathcal{F}(k,m) = \{A_1,\dots,A_m\}$ be the collection of the first m k-sets in squashed order. Let $\Delta' A_m$ denote the marginal shadow of A_m, that is, $\Delta' A_m = \Delta A_m \setminus \Delta\mathcal{F}(k,m-1)$. For $m < \binom{n}{k}$, the collection $\mathcal{A}(k,m,n) = \mathcal{F}(k,m) \cup \binom{[n]}{k-1} \setminus \Delta\mathcal{F}(k,m)$ is a maximal antichain in B_n if and only if $\Delta' A_{m+1} \neq \emptyset$. Note that $\Delta' A_{m+1} \neq \emptyset$ if and only if $1 \in A_{m+1}$ because if $1 \in A_{m+1}$ then $A_{m+1} \setminus \{1\} \in \Delta' A_{m+1}$ and if $1 \notin A_{m+1}$ and $x \in A_{m+1}$, then $A_{m+1} \setminus \{x\} \in \Delta(A_{m+1} \setminus \{x\} \cup \{1\})$.

Suppose that $\mathcal{A} = \mathcal{A}(k,m,n)$ is a maximal squashed antichain and let $\mathcal{A}' = \mathcal{A}(k,m',n)$ be the next maximal squashed antichain. We will conclude the proof by verifying that $|\mathcal{A}| - (k-1) \leqslant |\mathcal{A}'| \leqslant |\mathcal{A}| + (n-k)$. If $\Delta' A_{m+2} \neq \emptyset$ then $m' = m+1$, and the claim follows from $|\mathcal{A}'| = |\mathcal{A}| + 1 - |\Delta' A_{m+1}|$. If $\Delta' A_{m+2} = \emptyset$, then $1 \in A_{m+1}$, $2 \notin A_{m+1}$ and $A_{m+2} = A_{m+1} \setminus \{1\} \cup \{2\}$. It follows that $\Delta' A_{m+1} = \{A_{m+1} \setminus \{1\}\}$. Moreover, with $y := \min A_{m'}$ we have $A_{m+i} = A_{m'} \setminus \{y\} \cup \{i\}$ for $i = 1,2,\dots,m'-m$. Hence, $2 \leqslant m - m' = y \leqslant n-k-1$, $|\Delta' A_{m+1}| = 1$ and $\Delta' A_{m+2} = \dots = \Delta' A_{m'} = \emptyset$. The claim follows with $|\mathcal{A}'| = |\mathcal{A}| + (m'-m) - |\Delta' A_{m+1} \cup \dots \cup \Delta' A_{m'}|$. □

Lemma 4.4. *Let $n \geqslant 7$, $\lfloor n/2 \rfloor \leqslant k \leqslant n-3$, $t = \lfloor \frac{k+3}{2} \rfloor$ and $j = j^*(t)$. Then $I(n,k) \subseteq S(n)$, where*

$$I(n,k) = \left[\min Y(n,k) - tk,\ \max Y(n,k) - tk + \binom{t-j}{2} + \binom{j+1}{2}\right].$$

Proof. If $k \geqslant 9$ or $k \in \{3,7\}$, then by Lemmas 4.2 and 4.3

$$I(n,k) = \bigcup_{m \in Y(n,k)} \left[m - tk,\ m - tk + \binom{t-j}{2} + \binom{j+1}{2}\right],$$

and the claim follows by Corollary 1. For $k \in \{4,5,8\}$, the claim follows if $n \leqslant 2k$, and for $n = 2k+1$ there is exactly one gap of length $k+1$ in $Y(n,k)$: the difference between the sizes of the last two maximal squashed flat antichains which have sizes $\binom{n}{k} - (k+1)$ and $\binom{n}{k}$, respectively. As a consequence, the only element of $I(n,k)$ which is not covered by Corollary 1 is $\binom{n}{k} - tk - 1$. For $k = 6$, we still need to do something for $n = 12$ and $n = 13$. For $n = 12$, we only need a maximal antichain of size $\binom{12}{6} - 25$ by the same argument as above. For $n = 13$, we need to look for the gaps in $Y(13,6)$ which have length 6 or 7. As before, there is only one gap of size 7. Writing down the elements of $Y(n,k)$ in the order in which they appear when we run through the antichains in $\mathcal{M}(n,k)$ ordered by the number of k-sets, then there are a few more differences of size 6. Fortunately, in most cases these gaps are filled by maximal squashed antichains that occur earlier or later. Using a computer, we verify that when we write down the elements of $Y(n,k)$ in increasing order, the only gaps of size at least six appear between the three largest values, that is, between $\binom{13}{6} - 13$, $\binom{13}{6} - 7$ and $\binom{13}{6}$. As a consequence, we only need the sizes $\binom{13}{6} - 25$, $\binom{13}{6} - 26$ and $\binom{13}{6} - 32$. Applying Lemma 4.1 with $\binom{n}{k} \in Y(n,k)$, it is sufficient to verify that the missing sizes can be written in the form $\binom{n}{k} - s$ for some $s \in \sigma(t',k)$ with $t' \leqslant k$. The proof is completed by observing that

- $\binom{9}{4} - 13 \in S(9)$ because $13 \in \sigma(4,4)$,
- $\binom{11}{5} - 21 \in S(11)$ because $21 \in \sigma(5,5)$,
- $\{\binom{13}{6} - 25,\ \binom{13}{6} - 26,\ \binom{13}{6} - 32\} \subseteq S(13)$ because $\{25,26\} \subseteq \sigma(5,6)$ and $32 \in \sigma(6,6)$,
- $\binom{12}{6} - 25 \in S(12)$ because $25 \in \sigma(5,6)$,
- $\binom{17}{8} - 41 \in S(17)$ because $41 \in \sigma(6,8)$. □

We use results from [8] to bound $\min Y(n,k)$. To state these results, let C_l be the sum of the first l Catalan numbers: $C_l = \sum_{i=1}^{l} \frac{1}{i+1}\binom{2i}{i}$.

Lemma 4.5 (Section 4.1 in [8])**.** *For $1 < k < n$, the minimum size of a maximal squashed flat antichain is*

$$\min\left\{\binom{n}{k-1}, \binom{n}{k}\right\} - C_{\min\{k-1,n-k\}}.$$

The largest number of k-sets in a maximal squashed flat antichain of minimum size is $\binom{a_k}{k}+\dots+\binom{a_2}{2}$ *where*

$$a_i=\begin{cases}2i-2 & \text{for } i\leqslant n-k+1,\\ n-k-1+i & \text{for } i\geqslant n-k+2.\end{cases}\tag{13}$$

From Lemma 4.5 we obtain the following values for $\min Y(n,k)$.

Lemma 4.6. *If* $5\leqslant k\leqslant n-4$ *then* $\min Y(n,k)=\min\left\{\binom{n}{k-1},\binom{n}{k}\right\}-C_{\min\{k-1,n-k\}}$.

Proof. To deduce this from Lemma 4.5, we verify $\binom{a_k}{k}+\dots+\binom{a_2}{2}\geqslant\binom{k+3}{k}$, where the a_i are given by (13). If $k\leqslant n-k+1$ then $\binom{a_k}{k}+\dots+\binom{a_2}{2}\geqslant\binom{a_k}{k}=\binom{2k-2}{k}\geqslant\binom{k+3}{k}$, where the inequality follows from $k\geqslant 5$. If $k\geqslant n-k+2$, then $\binom{a_k}{k}+\dots+\binom{a_2}{2}\geqslant\binom{a_k}{k}=\binom{n-1}{k}\geqslant\binom{k+3}{k}$, where the inequality follows from $n-1\geqslant k+3$. □

We are now ready to express the intervals $I(n,k)$ in terms of n and k.

Lemma 4.7. *Let* $k\geqslant 5$*, assume* $\frac{n-1}{2}\leqslant k\leqslant n-4$*, and set* $t=\left\lfloor\frac{k+3}{2}\right\rfloor$, $j=j^*(t)$.

(i) If $k>\frac{n+1}{2}$ *then*

$$I(n,k)\supseteq\left[\binom{n}{k}-C_{n-k}-tk,\ \binom{n}{k-1}+\binom{k+3}{k}-\binom{k+3}{k-1}-tk+\binom{t-j}{2}+\binom{j+1}{2}\right].$$

(ii) If $k\leqslant\frac{n+1}{2}$ *then* $I(n,k)=\left[\binom{n}{k-1}-C_{k-1}-tk,\ \binom{n}{k}-tk+\binom{t-j}{2}+\binom{j+1}{2}\right]$.

Moreover, $I(7,3)=[16,29]$, $I(8,4)=[41,61]$, *and* $I(9,4)=[69,117]$.

Proof. For $k\geqslant 5$, by Lemma 4.6,

$$\min I(n,k)=\min\left\{\binom{n}{k-1},\binom{n}{k}\right\}-C_{\min\{k-1,n-k\}}-tk,$$

and this gives the left ends of the intervals in (i) and (ii). For $k>\frac{n+1}{2}$, the right end of the interval comes from $\binom{[k+3]}{k}\cup\binom{[n]}{k-1}\setminus\binom{[k+3]}{k-1}\in\mathcal{M}(n,k)$. For $k\leqslant\frac{n+1}{2}$, $\binom{[n]}{k}\in\mathcal{M}(n,k)$ implies

$$\max I(n,k)=\binom{n}{k}-tk+\binom{t-j}{2}+\binom{j+1}{2}.$$

This is also valid for $k\in\{3,4\}$, and gives the right ends of the intervals for $(n,k)\in\{(7,3),(8,4),(9,4)\}$. For the left ends of these intervals we do an exhaustive search over $\mathcal{M}(n,k)$ and find the following antichains. For $(n,k)=(7,3)$, the maximal squashed flat antichain with 21 3-sets has size 25. For $(n,k)=(8,4)$, the maximal squashed flat antichain with 37 4-sets has size 53. For $(n,k)=(9,4)$, the maximal squashed flat antichain with 37 4-sets has size 81. □

4.2. **Proof of Claim 2 for large n.** In this section, we prove Claim 2 for $n\geqslant 20$.

Lemma 4.8. *If* $n\geqslant 20$ *then* $[w(n-1)+2,w(n)]\subseteq S(n)$.

Proof. For $20\leqslant n\leqslant 199$, we use Lemmas 4.4 and 4.7 (and a computer) to verify the statement. For $n\geqslant 200$, we conclude using the following inequalities which are proved below:

- For $k=\left\lceil\frac{9n}{10}\right\rceil$, $\min I(n,k)<w(n-1)$ (Lemma 4.9).
- For $k=\lceil n/2\rceil$, $\max I(n,k)\geqslant w(n)$ (Lemma 4.10).
- For $\left\lceil\frac{n+2}{2}\right\rceil\leqslant k\leqslant\left\lceil\frac{9n}{10}\right\rceil$, $\max I(n,k)\geqslant\min I(n,k-1)$ (Lemma 4.11). □

Lemma 4.9. *For* $n\geqslant 200$ *and* $k=\left\lceil\frac{9n}{10}\right\rceil$, $\min I(n,k)<w(n-1)$.

Proof. By Lemma 4.7, we have to show

$$\binom{n}{k}-C_{n-k}-\left\lfloor\frac{k+3}{2}\right\rfloor k\leqslant w(n-1).$$

We prove the stronger inequality $\binom{n}{k} \leqslant \binom{n-1}{\lfloor\frac{n-1}{2}\rfloor} - n^2$. We bound the left-hand side by

$$\binom{n}{k} = \binom{n}{n-k} \leqslant \left(\frac{en}{n-k}\right)^{n-k} \leqslant \left(\frac{200e}{19}\right)^{n/10} < 1.4^n.$$

For the right-hand side, we start with

$$\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor} \geqslant 2^{\lfloor (n-1)/2\rfloor} \geqslant 2^{\frac{99n}{200}}.$$

Combining this with $n^2 \leqslant 2^{\frac{98n}{200}} \leqslant 2^{\frac{98n}{200}}\left(2^{\frac{n}{200}}-1\right)$, we obtain $\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor} - n^2 \geqslant 2^{\frac{49n}{100}} > 1.4^n$. □

Lemma 4.10. *For $n \geqslant 200$ and $k = \lceil n/2\rceil$, $\max I(n,k) \geqslant w(n)$.*

Proof. Set $t = \left\lfloor\frac{k+3}{2}\right\rfloor$ and $j = j^*(t)$. By Lemma 4.7, we have to show

$$\binom{n}{k} - tk + \binom{t-j}{2} + \binom{j+1}{2} \geqslant \binom{n}{k} - k\left\lceil\frac{k+1}{2}\right\rceil.$$

We prove the stronger inequality $\binom{n}{k} - tk + \binom{t-j}{2} \geqslant \binom{n}{k} - k\left\lceil\frac{k+1}{2}\right\rceil$, or, equivalently,

$$\binom{t-j}{2} \geqslant k\left(\left\lfloor\frac{k+3}{2}\right\rfloor - \left\lceil\frac{k+1}{2}\right\rceil\right).$$

If k is even then the right-hand side is 0, and there is nothing to do. For odd k, the right-hand side is equal to k, and we can show $\binom{t-j}{2} \geqslant k$ as in the proof of Lemma 4.2. The assumption $k \geqslant 33$ made in this proof follows from $n \geqslant 200$. □

Lemma 4.11. *For $n \geqslant 200$ and $\left\lceil\frac{n+2}{2}\right\rceil \leqslant k \leqslant \left\lceil\frac{9n}{10}\right\rceil$, $\max I(n,k) \geqslant \min I(n,k-1)$.*

Proof. Set $t = \left\lfloor\frac{k+3}{2}\right\rfloor$, $t' = \left\lfloor\frac{k+2}{2}\right\rfloor$ and $j = j^*(t)$. By Lemma 4.7, we have to show that

$$\binom{n}{k-1} + \binom{k+3}{k} - \binom{k+3}{k} - tk + \binom{t-j}{2} + \binom{j+1}{2} \geqslant \min I(n,k-1).$$

For $k = \frac{n+2}{2}$, this is

$$\binom{n}{k-1} + \binom{k+3}{k} - \binom{k+3}{k-1} - tk + \binom{t-j}{2} + \binom{j+1}{2} \geqslant \binom{n}{k-2} - C_{k-2} - t'(k-1).$$

From $n^5 \leqslant \binom{n}{k-1}$, we obtain

$$n^4 \leqslant \frac{1}{k}\binom{n}{k-1} = \binom{n}{k-1} - \binom{n}{k-2},$$

and this can be used to bound the left-hand side:

$$\binom{n}{k-1} + \binom{k+3}{k} - \binom{k+3}{k-1} - tk + \binom{t-j}{2} + \binom{j+1}{2} \geqslant \binom{n}{k-1} - n^4 \geqslant \binom{n}{k-2},$$

which is obviously larger than the right-hand side. Now we assume $k \geqslant \left\lfloor\frac{n+4}{2}\right\rfloor$, and the claim will follow from

$$\binom{n}{k-1} + \binom{k+3}{k} - \binom{k+3}{k-1} - tk + \binom{t-j}{2} + \binom{j+1}{2} \geqslant \binom{n}{k-1} - C(n-k+1) - t'(k-1).$$

The left-hand side is larger than $\binom{n}{k-1} - n^4$ and the right-hand side is smaller than

$$\binom{n}{k-1} - C_{n-k+1} \leqslant \binom{n}{k-1} - C_{\lfloor n/10\rfloor} \leqslant \binom{n}{k-1} - \frac{1}{\lfloor n/10\rfloor + 1}\binom{2\lfloor n/10\rfloor}{\lfloor n/10\rfloor}.$$

Therefore, it is sufficient to verify $\frac{1}{l+1}\binom{2l}{l} \geqslant n^4$ for $l = \lfloor n/10\rfloor$. We use the bound

$$\frac{1}{l+1}\binom{2l}{l} \geqslant \frac{4^l}{(l+1)\left(\pi l\frac{4l}{4l-1}\right)^{1/2}}$$

from [3]. From $n \geqslant 200$, it follows that $l \geqslant 20$, and this implies $4^l \geqslant 25000 l^{11/2}$. Bounding the denominator by

$$(l+1)\left(\pi l \frac{4l}{4l-1}\right)^{1/2} \leqslant \frac{23l}{22}\left(\pi l \frac{88}{87}\right)^{1/2} \leqslant 2l^{3/2},$$

we obtain

$$\frac{4^l}{(l+1)\left(\pi l \frac{4l}{4l-1}\right)^{1/2}} \geqslant 12500 l^4 \geqslant (11l)^4 \geqslant n^4,$$

which concludes the proof. $\square$

4.3. **Proof of Claim 2 for small n.** In this Section, we prove Claim 2 for $7 \leqslant n \leqslant 19$. We need a few auxiliary results. Let

$$S(n,k) = \left\{ |\mathcal{A}| \,:\, \mathcal{A} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k} \text{ is a maximal antichain in } B_n \right\}.$$

Lemma 4.12. *For $k \geqslant 2$ and $n \geqslant k+1$, $S(n,k) \supseteq S(n-1,k) + \binom{n-1}{k-2}$ and $S(n,k) \supseteq S(n-1,k-1) + \binom{n-1}{k}$.*

Proof. Let $\mathcal{A}' \subseteq \binom{[n-1]}{[k-1]} \cup \binom{[n-1]}{[k]}$ be a maximal antichain in B_{n-1}. Then

$$\mathcal{A} = \mathcal{A}' \cup \left\{ A \cup \{n\} \,:\, A \in \binom{[n-1]}{k-2} \right\} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k}$$

is a maximal antichain in B_n, and this implies the first inclusion. For the second one, let $\mathcal{A}' \subseteq \binom{[n-1]}{[k-2]} \cup \binom{[n-1]}{[k-1]}$ be a maximal antichain in B_{n-1}. Then

$$\mathcal{A} = \{A \cup \{n\} \,:\, A \in \mathcal{A}'\} \cup \binom{[n-1]}{k} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k}$$

is a maximal antichain in B_n. $\square$

Definition 5. A flat antichain $\mathcal{A} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k}$ is called *$\{1,2\}$-separated* if $\{1,2\} \subseteq A$ for every k-set $A \in \mathcal{A}$ $|A \cap \{1,2\}| \leqslant 1$ for every $(k-1)$-set $A \in \mathcal{A}$.

Observation 1. Let $\mathcal{A}' \subseteq \binom{[n-1]}{k-1} \cup \binom{[n-1]}{k}$ and $\mathcal{A}'' \subseteq \binom{[n-1]}{k-2} \cup \binom{[n-1]}{k-1}$ be $\{1,2\}$-separated maximal antichains in B_{n-1}. Then $\mathcal{A} = \mathcal{A}' \cup \{A \cup \{n\} \,:\, A \in \mathcal{A}''\} \subseteq \binom{[n-1]}{k-1} \cup \binom{[n-1]}{k}$ is a $\{1,2\}$-separated maximal antichain in B_n.

This can be used in an induction to establish the following result.

Lemma 4.13. *Let n, k and m be integers with $k \geqslant 2$, $n \geqslant k+1$, and*

$$\binom{n-1}{k-1} \leqslant m \leqslant \binom{n}{k-1} - 2\binom{n-3}{k-3} - \binom{n-4}{k-5}.$$

Then there exists a $\{1,2\}$-separated antichain $\mathcal{A} \subseteq \binom{[n]}{k-1} \cup \binom{[n]}{k}$ with $|\mathcal{A}| = m$.

Proof. For $k=2$, we have $m \in \{n-1, n\}$. Then $\binom{[n]}{1}$ is a $\{1,2\}$-separated maximal antichain of size n, and $\{\{1,2\}\} \cup \left(\binom{[n]}{1} \setminus \{\{1\},\{2\}\}\right)$ is a $\{1,2\}$-separated maximal antichain of size $n-1$. For $k \geqslant 3$, $n = k+1$, we have $k \leqslant m \leqslant 2k-2$. For $l \in \{3,4,\dots,n\}$,

$$\mathcal{A} = \{[n] \setminus \{j\} \,:\, 3 \leqslant j \leqslant l\} \ \cup\ \{\{3,4,\dots,n\}\} \ \cup\ \{\{i\} \cup ([3,n] \setminus \{j\}) \,:\, i \in \{1,2\},\, j \in \{l+1,\dots,n\}\}$$

is a maximal $\{1,2\}$-separated antichain with $|\mathcal{A}| = l - 2 + 2(n-l) + 1 = 2k - l + 1$. For $n \geqslant k+2$, we proceed by induction, and note that Observation 1 implies that we get all sizes m with

$$\binom{n-2}{k-1} + \binom{n-2}{k-2} \leqslant m \leqslant \binom{n-1}{k-1} - 2\binom{n-4}{k-3} - \binom{n-5}{k-5} + \binom{n-1}{k-2} - 2\binom{n-4}{k-4} - \binom{n-5}{k-6},$$

and this simplifies to the claimed range. $\square$

Lemma 4.14. *For $3 \leqslant k \leqslant 10$ and $2k \leqslant n \leqslant 20$, $\left[\binom{n-1}{k-1}, \binom{n}{k-1}\right] \subseteq S(n,k)$.*

Proof. We proceed by induction on k, and for fixed k by induction on n. For the base case $(k,n) = (3,6)$, we have to verify $[10,15] \subseteq S(6,3)$. For $m \in [10,14]$ we can use the maximal antichains on levels 2 and 3 in B_6 that are given in the proof of Theorem 1(ii) just before Section 4.1, and for $m = 15$ we use the maximal antichain $\binom{[6]}{2}$. Next, we look at the induction step from $(k, n-1)$ to (k,n) for $n \geqslant 2k+1$. From the induction hypothesis and Lemma 4.12,

$$S(n,k) \supseteq \left[\binom{n-2}{k-1}, \binom{n-1}{k-1}\right] + \binom{n-1}{k-2} = \left[\binom{n-2}{k-1} + \binom{n-1}{k-2}, \binom{n}{k-1}\right],$$

and the claim follows with Lemma 4.13 and

$$\binom{n}{k-1} - 2\binom{n-3}{k-3} - \binom{n-4}{k-5} \geqslant \binom{n-2}{k-1} + \binom{n-1}{k-2}$$

for all (k,n) with $4 \leqslant k \leqslant 10$ and $2k+1 \leqslant n \leqslant 20$. To complete the induction argument, we check that the claim for $k \geqslant 4$ and $n = 2k$ is implied by the statement for $k-1$. We need four ingredients:

- By Lemma 4.13,
$$\left[\binom{2k-1}{k-1}, \binom{2k}{k-1} - 2\binom{2k-3}{k-3} - \binom{2k-4}{k-5}\right] \subseteq S(2k,k).$$
- By Lemmas 4.12 and 4.13,
$$S(2k,k) \supseteq \left[\binom{2k-2}{k-1}, \binom{2k-1}{k-1} - 2\binom{2k-4}{k-3} - \binom{2k-5}{k-5}\right] + \binom{2k-1}{k-2}.$$
- By Lemma 4.12, $S(2k,k) \supseteq S(2k-1,k) + \binom{2k-1}{k-2} \supseteq S(2k-2,k-1) + \binom{2k-2}{k} + \binom{2k-1}{k-2}$, and with induction,
$$S(2k,k) \supseteq \left[\binom{2k-3}{k-2}, \binom{2k-2}{k-2}\right] + \binom{2k-2}{k} + \binom{2k-1}{k-2}.$$
- By Lemma 4.12 and induction,
$$S(2k,k) \supseteq \left[\binom{2k-2}{k-2}, \binom{2k-1}{k-2}\right] + \binom{2k-1}{k} = \left[\binom{2k-2}{k-2} + \binom{2k-1}{k}, \binom{2k}{k-1}\right].$$

We check that for $4 \leqslant k \leqslant 10$, the union of these four intervals is $[\binom{2k-1}{k-1}, \binom{2k}{k-1}]$, as required. □

We need one more simple construction before we can complete the argument for $n \leqslant 19$.

Lemma 4.15. *Let $n \geqslant 2$ be an integer. If $m \in S(n-1)$ and $m > \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor} + 1$, then $m + n - 1 \in S(n)$.*

Proof. Let $m \in S(n-1)$ with $m > \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor} + 1$, and $\mathcal{A}$ be a maximal antichain in B_{n-1} with size m. Then, by Sperner's Theorem, $\mathcal{A}$ cannot contain a singleton. Thus, $\mathcal{A} \cup \{\{i,n\} : i \in [n-1]\}$ is a maximal antichain in B_n of size $m+n-1$. □

Finally, we combine Lemmas 4.4, 4.7, 4.14 and 4.15 to prove Claim 2 for $7 \leqslant n \leqslant 19$.

Lemma 4.16. *For all $n \in \{7, \dots, 19\}$, $[w(n-1)+2, w(n)] \subseteq S(n)$.*

Proof. For each value of n we list the sizes coming from the various lemmas.

$n = 7$: We need the interval $[w(6)+2, w(7)] = [16,23]$, and Lemma 4.7 yields $I(7,3) = [16,29]$.

$n = 8$: We need $[w(7)+2, w(8)] = [25,58]$. From Lemma 4.7 we have $I(8,4) = [41,61]$, and from Lemma 4.14, $\left[\binom{7}{2}, \binom{8}{2}\right] \cup \left[\binom{7}{3}, \binom{8}{3}\right] = [21,28] \cup [35,56]$. Finally, Lemma 4.15 with $[22,27] \subseteq S(7)$ yields $[29,34] \subseteq S(8)$.

$n = 9$: We need $[w(8)+2, w(9)] = [60,111]$. From Lemma 4.7 we have $I(9,4) = [69,117]$, and from Lemma 4.14, $\left[\binom{8}{3}, \binom{9}{3}\right] = [56,84]$.

$n = 10$: We need $[w(9)+2, w(10)] = [113, 237]$. From Lemma 4.7 we have

$$I(10,5) \cup I(10,6) = [164, 236],$$

and from Lemma 4.14, $\left[\binom{9}{3}, \binom{10}{3}\right] \cup \left[\binom{9}{4}, \binom{10}{4}\right] = [84, 120] \cup [126, 210]$. From Lemma 4.15 and $[112, 116] \subseteq S(9)$, we deduce $[121, 125] \subseteq S(10)$, and finally, a maximal antichain of size 237 is given by $\mathcal{A} = \mathcal{F} \cup \binom{[10]}{5} \setminus \Delta\mathcal{F}$ where $\mathcal{F} = \{\{1,2,3,4,5,6\}, \{1,2,3,4,7,8\}, \{1,2,3,4,9,10\}\}$.

$n = 11$: We need $[w(10)+2, w(11)] = [239, 438]$. From Lemma 4.7 we have

$$I(11,5) \cup I(11,7) = [273, 446],$$

and from Lemma 4.14, $\left[\binom{10}{4}, \binom{11}{4}\right] = [210, 330]$.

$n = 12$: We need $[w(11)+2, w(12)] = [440, 900]$. From Lemma 4.7 we have

$$I(12,6) \cup I(12,7) = [693, 904],$$

and from Lemma 4.14, $\left[\binom{11}{4}, \binom{12}{4}\right] \cup \left[\binom{11}{5}, \binom{12}{5}\right] = [330, 792]$.

$n = 13$: We need $[w(12)+2, w(13)] = [902, 1688]$. From Lemma 4.7 we have

$$I(13,7) \cup I(13,8) = [1138, 1688],$$

and from Lemma 4.14, $\left[\binom{12}{5}, \binom{13}{5}\right] = [792, 1287]$.

$n = 14$: We need $[w(13)+2, w(14)] = [1690, 3404]$. From Lemma 4.7 we have

$$I(14,7) \cup I(14,8) = [2767, 3404],$$

and from Lemma 4.14, $\left[\binom{13}{5}, \binom{14}{5}\right] \cup \left[\binom{13}{6}, \binom{14}{6}\right] = [1287, 3003]$.

$n = 15$: We need $[w(14)+2, w(15)] = [3406, 6395]$. From Lemma 4.7 we have

$$I(15,8) \cup I(15,9) = [4755, 6402],$$

and from Lemma 4.14, $\left[\binom{14}{6}, \binom{15}{6}\right] = [3003, 5005]$.

$n = 16$: We need $[w(15)+2, w(16)] = [6397, 12\,830]$. From Lemma 4.7 we have

$$I(16,8) \cup I(16,10) = [7752, 12\,837],$$

and from Lemma 4.14, $\left[\binom{15}{6}, \binom{16}{6}\right] = [5005, 8008]$.

$n = 17$: We need $[w(16)+2, w(17)] = [12\,832, 24\,265]$. From Lemma 4.7 we have

$$I(17,9) \cup I(17,10) = [18\,763, 24\,267],$$

and from Lemma 4.14, $\left[\binom{16}{7}, \binom{17}{7}\right] = [11\,440, 19\,448]$.

$n = 18$: We need $[w(17)+2, w(18)] = [24\,267, 48\,575]$. From Lemma 4.7 we have

$$I(18,9) \cup I(18,10) \cup I(18,11) = [31\,122, 48\,577],$$

and from Lemma 4.14, $\left[\binom{17}{7}, \binom{18}{7}\right] = [19\,448, 31\,824]$.

$n = 19$: We need $[w(18)+2, w(19)] = [48\,577, 92\,318]$. From Lemma 4.7 we have

$$I(19,10) \cup I(19,11) \cup I(19,12) = [49\,679, 92\,329],$$

and from Lemma 4.14, $\left[\binom{18}{7}, \binom{19}{7}\right] = [31\,824, 50\,388]$. □

5. The largest missing value in the shadow spectrum

In this section, we prove Theorem 3. Recall that $j^*(t)$ is defined as the smallest non-negative integer j with $\binom{j+3}{2} \geqslant t$ and that $j^*(t) = \left\lceil \sqrt{2t} - 5/2 \right\rceil$ by Lemma 3.3.

Next, we make the observation that every integer $s \geqslant k^2$ is the shadow size for some family of k-sets. Recall that $\Sigma(k) = \bigcup_{t=1}^{\infty} \sigma(t,k)$ is the set of all these shadow sizes.

Lemma 5.1. *If $k \geqslant 3$ and s are integers with $s \geqslant k^2$ then $s \in \Sigma(k)$.*

Proof. Let $\mathcal{F}$ be a family of k-sets with $|\mathcal{F}|=t$, and let A be a k-set that contains at least two elements not in any of the members of $\mathcal{F}$. Then $\mathcal{F}'=\mathcal{F}\cup\{A\}$ is a family of k-sets with $|\mathcal{F}'|=t+1$ and $|\Delta\mathcal{F}'|=|\Delta\mathcal{F}|+k$. As a consequence, $\sigma(t+1,k)\supseteq\sigma(t,k)+k$. If we can show that $[k^2-k,k^2]\subseteq\sigma(k,k)$, then it follows by induction on t that $[(t-1)k,tk]\subseteq\sigma(t,k)$ for all $t\geqslant k$, and this is clearly sufficient for our claim. By Theorem 2 and Proposition 1,

$$\left[k^2-\binom{k-j^*(k)}{2}-\binom{j^*(k)+1}{2},k^2\right]\subseteq\sigma(k,k),$$

and it suffices to verify $\binom{k-j^*(k)}{2}+\binom{j^*(k)+1}{2}\geqslant k$. For $k\leqslant 7$, this can be checked by hand, and for $k\geqslant 8$ we have $\sqrt{k}\geqslant 2\sqrt{2}$, and with Lemma 3.3,

$$\binom{k-j^*(k)}{2}-k\geqslant\frac12\left(k-\sqrt{2k}\right)\left(k-\sqrt{2k}-1\right)-k\geqslant\frac12k^2-\sqrt{2}k^{3/2}\geqslant\frac12k^{3/2}(k-2\sqrt{2})\geqslant 0.\qquad\square$$

Combining Theorem 2, Proposition 1 and Lemma 5.1, the value $\psi(k)=\max\mathbb{N}\setminus\Sigma(k)$ is the integer immediately before some set $\{tk-x\,:\,x\in I_{j^*(t)}(t)\}$, and the t must be chosen so that this number is too big to be the shadow size of a $(t-1)$-family of k-sets. In other words, taking t to be maximal with the property that $tk-\binom{t-j^*(t)}{2}-\binom{j^*(t)+1}{2}>(t-1)k+1$, we will argue that (for $k\geqslant 21$) $\psi(k)=tk-\binom{t-j^*(t)}{2}-\binom{j^*(t)+1}{2}-1$. Equivalently, we are looking for the smallest $t^*=t^*(k)$ such that there is no gap between $\{tk-x\,:\,x\in I_{j^*(t^*)}(t^*)\}$ and $\{(t+1)k-x\,:\,x\in I_{j^*(t^*+1)}(t^*+1)\}$. To be more precise, we define

$$f(t)=\binom{t-j^*(t)}{2}+\binom{j^*(t)+1}{2},$$

so that $I_{j^*(t)}(t)=[0,f(t)]$, and then

$$t^*=t^*(k)=\min\{t\,:\,(t+1)k-f(t+1)\leqslant tk+1\}=\max\{t\,:\,f(t)\leqslant k-2\}.$$

Example 2. For $k=50$, we have

$$\begin{aligned}\sigma(10,50)&=[455,455]\cup[463,464]\cup[469,500],\\ \sigma(11,50)&=[495,495]\cup[504,505]\cup[511,514]\cup[516,550],\\ \sigma(12,50)&=[534,534]\cup[544,545]\cup[552,555]\cup[558,600],\\ \sigma(13,50)&=[572,572]\cup[583,584]\cup[592,595]\cup[599,650],\\ \sigma(14,50)&=[609,609]\cup[621,622]\cup[631,634]\cup[639,700].\end{aligned}$$

We see that $t^*(50)=12$ and $\psi(50)=557$.

For $t=t^*(k)$ and $j=j^*(t)$, we want to show that $\psi(k)=tk-f(t)-1$. One part of the argument is the verification that $tk-f(t)-1\notin\sigma(t+1,k)$. For this step we will need the following bound.

Lemma 5.2. *For $k\geqslant 374$ and $t=t^*(k)$, $tk-f(t)\leqslant(t+1)k-\binom{t+1}{k}$.*

Proof. Rearranging terms, we need to show $k+f(t)\geqslant\binom{t+1}{2}$. Lemma 3.3 implies the bound

$$f(t)\geqslant\binom{t-j^*(t)}{2}\geqslant\frac12\left(t-j^*(t)\right)\left(t-j^*(t)-1\right)\geqslant\frac12t^2-\sqrt{2}t^{3/2}.\tag{14}$$

From $k\geqslant 374$ we obtain $t\geqslant 34$ and $\sqrt{t}\geqslant 2\sqrt{2}+3$, which implies

$$\frac12t^2-2\sqrt{2}t^{3/2}-\frac12t=\frac12t\left(t^{1/2}-2\sqrt{2}-3\right)\left(t^{1/2}-2\sqrt{2}+3\right)\geqslant 0.$$

We rearrange terms, use $f(t)\leqslant k$ and (14):

$$\sqrt{2}t^{3/2}+\frac12t\leqslant\frac12t^2-\sqrt{2}t^{3/2}\leqslant f(t)\leqslant k,$$

hence $k-\sqrt{2}t^{3/2}\geqslant t/2$. Finally, using (14) again,

$$k+f(t)\geqslant k+\frac12t^2-\sqrt{2}t^{3/2}\geqslant\frac12t^2+\frac12t=\binom{t+1}{2},$$

as required. $\square$

The next lemma is the precise version of the above description of $\psi(k)$ in terms of $t^* = t^*(k)$.

Lemma 5.3. *For $k \geqslant 374$ and $t^* = t^*(k)$, $\psi(k) = t^*k - f(t^*) - 1$.*

Proof. Let $s = t^*k - f(t^*) - 1$. From $f(t^*) \leqslant k-2$ it follows that $s > (t^*-1)k$, hence $s \notin \bigcup_{t=1}^{t^*-1} \sigma(t,k)$. Clearly, $s \notin \sigma(t^*,k)$, and moreover Lemma 5.2 implies $s < \min \sigma(t,k)$ for all $t \geqslant t^*+1$. We conclude $s \notin \Sigma(k)$. From $(t+1)k - f(t+1) \leqslant tk+1$ for all $t \geqslant t^*$ it follows that

$$\left[t^*k - \binom{t^* - j^*(t^*)}{2} - \binom{j^*(t^*)+1}{2}, k^2\right] \subseteq \Sigma(k). \qquad \square$$

To complete the proof of Theorem 3(i), we need to bound the expression for $\psi(k)$ provided in Lemma 5.3. We start with the function f.

Lemma 5.4. $f(t) = \frac{1}{2}t^2 - \sqrt{2}t^{3/2} + O(t)$.

Proof. By Lemma 3.3, $\sqrt{2t} - 5/2 < j^*(t) < \sqrt{2t} - 3/2$, hence

$$\binom{t-\sqrt{2t}+3/2}{2} + \binom{\sqrt{2t}-1/2}{2} < \binom{t-j^*(t)}{2} + \binom{j^*(t)+1}{2} < \binom{t-\sqrt{2t}+5/2}{2} + \binom{\sqrt{2t}-3/2}{2},$$

and this implies

$$\binom{t-j^*(t)}{2} + \binom{j^*(t)+1}{2} = \frac{1}{2}\left(t-\sqrt{2t}\right)^2 + O(t) = \frac{1}{2}t^2 - \sqrt{2}t^{3/2} + O(t). \qquad \square$$

We set $t^*(k) = \max\{t : f(t) \leqslant k-2\}$, and look at the asymptotics of this function.

Lemma 5.5. $t^*(k) = \sqrt{2k} + \sqrt[4]{8k} + O(1)$.

Proof. From Lemma 5.4 it follows that $f(t+1) = \frac{1}{2}t^2 - \sqrt{2}t^{3/2} + O(t)$, and since by definition

$$f\left(t^*(k)\right) \leqslant k-2 < f\left(t^*(k)+1\right),$$

we deduce

$$\frac{1}{2}t^*(k)^2 - \sqrt{2}\,t^*(k)^{3/2} + O\left(t^*(k)\right) = k.$$

This implies $t^*(k) = \sqrt{2k} + g(k)$ with $g(k) = o\left(\sqrt{k}\right)$, and

$$\frac{1}{2}t^*(k)^2 - \sqrt{2}\,t^*(k)^{3/2} = k + O\left(\sqrt{k}\right). \tag{15}$$

The left-hand side is

$$\begin{aligned}\frac{1}{2}t^*(k)^2 - \sqrt{2}\,t^*(k)^{3/2} &= \frac{1}{2}\left(\sqrt{2k}+g(k)\right)^2 - \sqrt{2}\left(\sqrt{2k}+g(k)\right)^{3/2} \\ &= k + \sqrt{2k}g(k) + \frac{1}{2}g(k)^2 - 2^{5/4}k^{3/4} + o\left(k^{3/4}\right).\end{aligned}$$

This implies first $g(k) = O\left(k^{1/4}\right)$, and then $g(k) = 2^{3/4}k^{1/4} + h(k)$ with $h(k) = o\left(k^{1/4}\right)$. Now the left-hand side of (15) is

$$\begin{aligned}\frac{1}{2}t^*(k)^2 - \sqrt{2}\,t^*(k)^{3/2} &= \frac{1}{2}\left(\sqrt{2k}+2^{3/4}k^{1/4}+h(k)\right)^2 - \sqrt{2}\left(\sqrt{2k}+O\left(k^{1/4}\right)\right)^{3/2} \\ &= k + 2\sqrt{2k}\,h(k) + O\left(k^{1/2}\right).\end{aligned}$$

Now $h(k) = O(1)$ by (15), and the result follows. $\square$

Theorem 3(i) will be proved by combining Lemmas 5.3, 5.4 and 5.5. For the second part, we observe that for even n, say $n = 2k$, Theorem 1 implies $\varphi(n) = \binom{n}{k} - \psi(k)$, and then with Theorem 3(i), $\varphi(n) = \binom{n}{k} - \frac{1}{2}n^{3/2} - \frac{1}{\sqrt{2}}n^{5/4} + O(n)$. For odd n, say $n = 2k-1$, we have to work a bit harder, because Theorem 1 implies only $\varphi(n) = \binom{n}{k} - s$, where s is the largest integer less than k^2 which is not in $\Sigma(k) \cup \Sigma(k-1)$.

Lemma 5.6. *If $t = j^*(k-1)+1$ and $s = t(k-1) - \binom{t}{2} - 1$, then $s \notin \Sigma(k) \cup \Sigma(k-1)$. Consequently, for $n \in \{2k, 2k-1\}$,*

$$\binom{n}{k} - t(k-1) + \binom{t}{2} + 1 \notin S(n).$$

Proof. We prove this by checking that s is in the gap between $\sigma(t-1,k)$ and $\sigma(t,k)$ and also in the gap between $\sigma(t-1,k-1)$ and $\sigma(t,k-1)$. It follows from $s < t(k-1) - \binom{t}{2}$ that all the elements of $\sigma(t,k) \cup \sigma(t,k-1)$ are larger than s. To conclude the proof it is sufficient to verify the inequality $(t-1)\,k < s$. From $t-1 = j^*(k-1)$ and the definition of j^*, we obtain $\binom{t+1}{2} < k-1$, hence

$$(t-1)\,k = t(k-1) + t - k < t(k-1) + t - \binom{t+1}{2} - 1 = s. \qquad \square$$

Proof of Theorem 3. To prove part (i), we combine Lemmas 5.3, 5.4 and 5.5:

$$\psi(k) = t^*k - f(t^*) - 1 = \left(\sqrt{2k} + \sqrt[4]{8k}\right) k + O(k) = \sqrt{2}k^{3/2} + \sqrt[4]{8}k^{5/4} + O(k).$$

For part (ii), let $k = \lceil n/2 \rceil$ and set $m_1 = \binom{n}{k} - \psi(k) - 1$. By part (i),

$$m_1 = \binom{n}{k} - (\sqrt{2} + o(1))k^{3/2} = \binom{n}{k} - \left(\frac{1}{2} + o(1)\right) n^{3/2},$$

and by Theorem 1 together with the definition of $\psi(k)$, $[1, m_1] \subseteq S(n)$. For the other direction, let $t = j^*(k-1)+1$ and $m_2 = \binom{n}{k} - t(k-1) + \binom{t}{2} + 1$. By Lemma 5.6, $m_2 \notin S(n)$, and by Lemma 3.3,

$$m_2 = \binom{n}{k} - \left(\sqrt{2k} + O(1)\right)(k-1) + O(k) = \binom{n}{k} - \left(\frac{1}{2} + o(1)\right) n^{3/2}. \qquad \square$$

6. Open problems

As mentioned in the introduction, our main result is a characterization of the possible sizes of families in B_n which are maximal with respect to the property of not containing a pair of sets, one of which is contained in the other. In [6], the saturation number $\mathrm{sat}(n,P)$ for a poset P has been introduced as the smallest size of a saturated P-free family in B_n, and these numbers have been studied for various posets P, see [13] for an overview. For $P = C_{k+1}$ (a chain of length $k+1$), it was shown in [6] that there is a value $\mathrm{sat}(k)$ such that $\mathrm{sat}(n, C_{k+1}) = \mathrm{sat}(k)$ for all sufficiently large n, and it is known [17, Theorem 4] that $\mathrm{sat}(k) = 2^{(1+o(1))ck}$ for some c with $\frac{1}{2} \leqslant c < 1$. The largest size of a C_{k+1}-free family, $\mathrm{ex}(n, C_{k+1})$ is the sum of the k largest binomial coefficients. In analogy to Question 1, it might be interesting to investigate the following.

Problem 1. For which integers m with $\mathrm{sat}(k) \leqslant m \leqslant \mathrm{ex}(n, C_{k+1})$ does there exist a saturated C_{k+1}-free family of size m?

In Theorem 2, we characterize the shadow spectrum $\sigma(t,k)$ for $t \leqslant k+1$. It is an interesting problem to study the set $\sigma(t,k)$ for general t. Some initial results in this direction have been obtained in [15]. Let $F(t,k)$ be the minimum shadow size of a t-family of k-sets, so that $\sigma(t,k) \subseteq [F(t,k), tk]$. In [15], the following sufficient conditions for $s \notin \sigma(s,t)$ are established:

- If there is an integer $a > k$ with $F(t-1,k) \leqslant \binom{a}{k-1} < s \leqslant F(t-1,k) + k - 2$ then $s \notin \sigma(t,k)$.
- For an integer s with $F(t,3) \leqslant s \leqslant 3t$, $s \notin \sigma(t,3)$ if and only if $s = \binom{a}{2} + 1$ for some integer $a \geqslant 4$ with $\binom{a}{3} - a + 4 \leqslant t \leqslant \binom{a}{3}$.
- For an integer s with $F(t,4) \leqslant s \leqslant 4t$, each of the following three conditions implies $s \notin \sigma(t,4)$:
  (i) $s = \binom{a}{3} + 1$ for some integer $a \geqslant 5$ with $\binom{a}{4} - 2a + 9 \leqslant t \leqslant \binom{a}{4} - a + 4$,
  (ii) $s \in \left\{\binom{a}{3} + 1,\ \binom{a}{3} + 2\right\}$ for some integer $a \geqslant 5$ with $\binom{a}{4} - a + 5 \leqslant t \leqslant \binom{a}{4}$,
  (iii) $s = \binom{a}{3} + \binom{b}{2} + 1$ for some integers $a > b \geqslant 4$ with $\binom{a}{4} + \binom{b}{3} - b + 4 \leqslant t \leqslant \binom{a}{4} + \binom{b}{3}$.

It would be interesting to prove the following generalization.

Conjecture 1 (Vermutung 2.3 in [15])**.** *Let s be an integer and suppose there are integers $a_k > a_{k-1} > \cdots > a_r > r \geqslant 3$ with*

$$F(t-1,k) \leqslant \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_r}{r-1} < s \leqslant F(t-1,k) + r - 2.$$

Then $s \notin \sigma(t,k)$.

Acknowledgment

We are grateful to two anonymous referees for their constructive comments which helped us to improve the presentation of the arguments, and for pointing out references which were useful for putting our results into the context of related work.

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