Continuous-time quantum walks on semi-regular spidernet graphs via quantum probability theory

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Abstract

We analyze continuous-time quantum and classical random walk on spidernet lattices. In the framework of Stieltjes transform, we obtain density of states, which is an efficiency measure for the performance of classical and quantum mechanical transport processes on graphs, and calculate the spacetime transition probabilities between two vertices of the lattice. Then we analytically show that there are two power law decays $\sim t^{-3}$ and $\sim t^{-1.5}$ at the beginning of the transport for transition probability in the continuous-time quantum and classical random walk respectively. This results illustrate the decay of quantum mechanical transport processes is quicker than that of the

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classical one. Due to the result, the characteristic time t_c , which is the time when the first maximum of the probabilities occur on an infinite graph, for the quantum walk is shorter than that of the classical walk. Therefore, we can interpret that the quantum transport speed on spidernet is faster than that of the classical one. In the end, we investigate the results by numerical analysis for two examples.

Keywords: Continuous-time quantum walk, Spidernet graphs, Spectral distribution.

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1 Introduction

Quantum walks were introduced in the early 1990s by Aharonovich, Davidovich and Zaggury [1]. Since then the topic has attracted considerable interest. The continuing attraction can be traced back to at least two reasons. First, the quantum walk is of sufficient interest in its own right because there are fundamental differences compared to the classical random walk. Next, quantum walks offer quite a number of possible applications. One of the best known is the link between quantum walks and quantum search algorithms which are superior to their classical counterparts [2, 3]. Similar to classical random walk there are two types of quantum walks, discrete and continuous time [1, 4]. A study of quantum walks on a simple graph is well known in physics (for more details see [5]). Recent studies of quantum walks on more general graphs were described in [2, 6, 7, 8, 9, 10, 11, 12, 13]. Some of these works study the problem in the important context of algorithmic problems on graphs and suggest that quantum walk is a promising algorithmic technique for designing future quantum algorithms. One approach for investigation of continuous-time quantum walk (CTQW) on graphs is using the spectral distribution associated with the adjacency matrix of graphs [14, 15, 16, 17, 18, 19]. Authors in Refs. [14, 15] have introduced a new method for calculating the probability amplitudes of quantum walk based on spectral distribution. In this method a canonical relation between the Fock space of stratification graph and set of orthogonal polynomials has been established which leads to obtain the probability measure (spectral distribution) of adjacency matrix graph. The method of spectral distribution only requires simple structural data of graph and allows us to avoid a heavy combinational argument often necessary to obtain full description of spectrum of the adjacency matrix.

In fact the dendrimers play an important role in recent researches, both experimentally and theoretically (e.g. see Ref.[20]). Dendrimers are hyperbranched macromolecules with very regular structure, and are important in drug delivery.

4

Therefore, that part of theoretical researches depending on probe transport process is interesting. Since dendrimers are synthesized in a self-similar fashion by hierarchically growing dendrimer from a core, they can be described by spidernet graphs. In recent years, the Mulken, Bierbaum and Blumen studied the coherent transport on dendrimers by CTQW [21]. Dendrimers may have the spidernet lattices structure. As an example, we can mention the dendrimers that the authors of Ref.[21] considered are spidernet lattices with parameters a = b = 3; c = 2, i.e. S(3,3,2). Therefore, in this paper we study CTQW and continuous-time random walk (CTRW) on spidernet lattices. For this work, by using the Stieltjes transform, we obtain spectral distribution (density of states) that is an efficiency measure for the performance of the classical and quantum mechanical transport processes on graphs [22]. Considering the results of Ref. [21, 22, 23], one way of quantifying the global efficiency of classical and quantum walk is the average probability of a walker to return to or stay at the starting point. In the classical case this quantity is equal to $p_0(t)$ (transition probability at the starting point at the time t) and in the quantum mechanical case $|q_0(t)|^2$ is the lower bound for the average probability. Then we analytically obtain these quantities for CTQW and CTRW on spidernet graphs as $\sim t^{-3}$ and $\sim t^{-3/2}$, respectively. Since the decay of the quantum mechanical average probability is quicker than that of the classical one, the quantum walk on the spidernet graphs is more efficient than the classical random walk. In the end, by numerical analysis we confirm these results for CTQW and CTRW on two spidernet graphs S(4,6,3) and S(a, a, a-1). As one can see from the figures, there are power law decay probabilities ($\sim t^{-1.5})$ and $\sim t^{-3}$ at the beginning of the transport for transition probability for CTRW and CTQW, respectively. Due to results of this paper, since the characteristic time t_c depends on the decay of the average probability (t_c is the time when the first maximum of the probabilities occurs on an infinite graph,

such that this definition is held both for the classical and quantum transport. For the classical transport, there is only one maximal value and the characteristic time corresponds to the time when the equipartitioned probability 1/N is reached on finite graphs [24].) the t_c for the quantum walk is shorter than the classical counterpart. Therefore, one can interpret that the quantum transport speed on spidernet is faster than that of the classical one.

The organization of the paper is as follows: we give a brief review of stratification, quantum decomposition and spidernet graph in Section 2. Section 3 is devoted to study CTQW and CTRW on spidernet graphs via quantum probability theory and try to investigate CTQW and CTRW on their graphs. In the conclusion we summarize the obtained results and discuss possible development. Finally, in the appendix the determination of spectral distribution associated with adjacency matrix by Stieltjes transform is derived.

2 Stratification, Quantum decomposition and Spidernet graph

Let V be a non-empty set and E be a subset of $\{\{\alpha,\beta\} | \alpha,\beta \in V \text{ for } \alpha \neq \beta\}$. The pair G = (V, E) is called a graph, where elements of V and E are vertices and edges of graph, respectively. We say that two vertices of α and β are adjacent if $\{\alpha,\beta\} \in E$ and write $\alpha \sim \beta$. A finite sequence $\alpha_0, \alpha_1, ..., \alpha_n$ is said a walk of length n if $\alpha_k \sim \alpha_{k+1}$ for k = 0, 1, ..., n - 1. A graph is called connected if any pair of distinct vertices is connected by a walk. The degree or valency of a vertex $\alpha \in V$ is defined by

$$\kappa(\alpha) = |\{\beta \in V | \ \beta \sim \alpha\}| \tag{2-1}$$

where |.| denote the cardinality. For a graph G the adjacency matrix A is given by

$$A_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha \sim \beta \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, (i) A is a symmetric (ii) elements of A take a value in $\{0, 1\}$ (iii) diagonal elements of A are 0. Conversely, for a non-empty set V, a structure graph is uniquely determined by a such matrix which indexed by V. On the other hand, A is considered as an operator acting on the Hilbert space $l^2(V)$ in such a way that

$$A|\alpha\rangle = \sum_{\alpha\sim\beta} |\beta\rangle, \quad \alpha \in V,$$

where $\{|\alpha\rangle| \ \alpha \in V\}$ forms a complete orthogonal basis of $l^2(V)$.

The stratification is introduced [14, 26, 27] by taking o as the origin (initial site) and have

$$V = \bigcup_{k=0}^{\infty} V_k, \qquad V_i = \{ \alpha \in V | \ \partial(o, \alpha) = k \},$$
(2-2)

where $\partial(\alpha, \beta)$ stands for the length of the shortest walk connecting α and β . According to the stratification (2-2), we define a unit vector by

$$|\phi_k\rangle = \frac{1}{\sqrt{|V_k|}} \sum_{\alpha \in V_k} |k, \alpha\rangle, \tag{2-3}$$

where $|k, \alpha\rangle$ denotes the eigenket of the α -th vertex at the stratum k and let $\Gamma(G)$ the closed subspace of $l^2(V)$ be spanned by $\{|\phi_k\rangle\}$. Moreover, the stratification (2-2) give rise to define three matrices A^{ε} , $\varepsilon \in \{+, -, 0\}$, as follows [27]: for $\alpha \in V_k$, k = 0, 1, 2, ...

$$(A^{\varepsilon})_{\beta\alpha} = \begin{cases} A_{\beta\alpha} & \text{if } \beta \in V_{k+\varepsilon} \\ 0 & \text{otherwise.} \end{cases}$$

where $k + \varepsilon = n + 1, n - 1, n$ according as $\varepsilon \in +, -, 0$. Therefore, the adjacency matrix A is decomposed as

$$A = A^{+} + A^{-} + A^{0}. (2-4)$$

This is called quantum decomposition of A associated with the stratification (2-2). Now, according to the stratification (2-2), for $\alpha \in V_k$ we set

$$\omega_{\varepsilon}(\alpha) = |\{\beta \in V_{k+\varepsilon}, \alpha \sim \beta\}|, \qquad \varepsilon \in \{+, -, 0\}$$
(2-5)

where $k + \varepsilon = n + 1, n - 1, n$ according as $\varepsilon \in +, -, 0$. The degree or valency of $\alpha \in V$ is

$$\kappa(\alpha) = \omega_{+}(\alpha) + \omega_{0}(\alpha) + \omega_{-}(\alpha) = \begin{cases} a & \alpha = o \\ b & \alpha \neq o. \end{cases}$$

We consider the integers a, b, c such that $a \ge 1$, $b \ge 2$ and $1 \le c \le b - 1$. A spidernet is a graph which satisfies the following conditions:

$$\omega_{+}(o) = a \quad \omega_{-}(o) = 0 \quad \omega_{0}(o) = 0$$
$$\omega_{+}(\alpha) = c \quad \omega_{-}(\alpha) = 1 \quad \omega_{0}(\alpha) = b - 1 - c \quad \text{for } \alpha \neq o,$$
(2-6)

where it is denoted by S(a, b, c), for example see Fig.1 (for more details see Refs.[27]). Spidernet is not necessarily a regular graph so it is called a semi-regular graph (for more details see Refs.[27, 28]). It is easy to show, by using (2-6), the number of vertices in strata as

$$|V_0| = 1, \quad |V_k| = ac^{k-1}, \quad k = 1, 2, 3, \dots$$
 (2-7)

Then by using Eqs.(2-6) and (2-7), one can obtain

$$A^+|\phi_0\rangle = \sqrt{a}|\phi_1\rangle, \quad A^+|\phi_k\rangle = \sqrt{c}|\phi_{k+1}\rangle, \quad k \ge 1$$
 (2-8)

$$A^{-}|\phi_{0}\rangle = 0, \quad A^{-}|\phi_{1}\rangle = \sqrt{a}|\phi_{0}\rangle, \quad A^{-}|\phi_{k}\rangle = \sqrt{c}|\phi_{k-1}\rangle, \quad k \ge 2$$
(2-9)

$$A^{0}|\phi_{k}\rangle = (b-1-c)|\phi_{k}\rangle, \quad k \ge 1.$$
(2-10)

Two Szegö- Jacobi sequences $\{\omega_k\}_{k=1}^{\infty}$ and $\{\alpha_k\}_{k=1}^{\infty}$ can obtain to take into consideration above equation as following [27]

$$\omega_1 = a, \ \omega_2 = \omega_3 = \dots = c, \ \alpha_1 = 0, \ \alpha_{k+1} = b - 1 - c \ k \ge 1,$$
 (2-11)

where $(\Gamma(G), A^+, A^-, A^o)$ is an interacting Fock space associated with Szegö- Jacobi sequences $\{\omega_k, \alpha_k\}.$

3 CTQW and CTRW on spidernets via quantum probability theory

We start our discussion by considering a walk on general graphs. Classically, the evolution of CTRW is governed by Kolmogorov's equation (master equation)[29, 30],

$$\frac{dP_{i,j}}{dt} = \sum_{k} H_{ik} P_{k,j} \tag{3-12}$$

where H is Hamiltonian of the walk and $P_{i,j}$ is the conditional probability to find the CTRW on vertex i at time t when the walk starting in the vertex j. It is natural to choose the Laplacian of the graph, defined as L = D - A, as Hamiltonian of walk where D is a diagonal matrix with entries as $D_{jj} = deg(\alpha_j)$. Then the solution of the above equation is

$$P_{k,j} = \langle k | e^{tH} | j \rangle. \tag{3-13}$$

Quantum mechanically, the evolution of CTQW is obtained by replacing Kolmogorov's equation with Schrödinger's equation

$$i\hbar \frac{d|\phi(t)\rangle}{dt} = H|\phi(t)\rangle.$$
(3-14)

where we assume $\hbar = 1$ and $|\phi_0\rangle$ is the initial amplitude wave function of the particle. The solution of the above equation is given by

$$|\phi(t)\rangle = e^{-itH}|\phi_0\rangle \tag{3-15}$$

On s-regular graphs D = sI, then we get

$$e^{-itH} = e^{-it(A-sI)} = e^{itsI}e^{-itA}.$$
(3-16)

This introduces an irrelevant phase factor in the wave evolution, hence we can consider H = A.

In the study of CTQW on graphs, the spectral distribution or density of states of the system, H contains essential information about the system [14] (in fact, the spectral distribution is an efficiency measure for the performance of classical and quantum mechanical transport processes on graphs [22]) and by definition the spectral distribution is a probability distribution μ uniquely specified by

$$\langle H^m \rangle = \langle \phi_0 | H^m | \phi_0 \rangle = \int x^m \mu(dx), \quad m = 0, 1, 2, ...,$$
 (3-17)

where, according to [14, 15, 26], $\langle H^m \rangle$ coincides with the number of *m*-step walks starting and terminating at the origin site *o*. For analyzing the spectral distribution μ of the adjacency matrix *A*, we use the method of quantum decomposition method which is a powerful tool. The spectral distribution μ is determined by applying the canonical isomorphism from the interacting Fock space onto the closed linear span of orthogonal polynomials determined by Szegö- Jacobi sequences { ω_k, α_k }. In fact the determination of μ is the main problem in the spectral theory of operators, where in the case is quite possible by using the Stieltjes method, as it is explained in appendix A. Then by using the quantum decomposition relations (2-4)-(2-10) and the recursion relation of polynomials $P_n(n)$ (A-1), the other matrix elements as

$$\langle \phi_k | H^m | \phi_0 \rangle = \frac{1}{\sqrt{\omega_1 \omega_2 \dots \omega_k}} \int x^m P_k(x) \mu(dx), \quad m = 0, 1, 2, \dots$$
 (3-18)

Then by using Eq.(3-18), the classical transition probability and quantum mechanical transition amplitude to go from the initial site o at time 0 to the stratum $|\phi_k\rangle$ in time t are given by

$$p_0(t) = \langle \phi_0 | e^{tH} | \phi_0 \rangle = \int e^{tx} \mu(dx),$$

$$p_k(t) = \langle \phi_k | e^{tH} | \phi_0 \rangle = \frac{1}{\sqrt{\omega_1 \omega_2 \dots \omega_k}} \int e^{tx} P_k(x) \mu(dx) \quad \text{for } k \ge 1,$$
 (3-19)

and

$$q_0(t) = \langle \phi_0 | e^{-itH} | \phi_0 \rangle = \int e^{-itx} \mu(dx)$$
$$q_k(t) = \langle \phi_k | e^{-itH} | \phi_0 \rangle = \frac{1}{\sqrt{\omega_1 \omega_2 \dots \omega_k}} \int e^{-itx} P_k(x) \mu(dx) \quad \text{for } k \ge 1,$$
(3-20)

respectively, where $|q_k(t)|^2$ is the transition probability of the quantum walk at the stratum k at time t. The conservation of probability $\sum_k |q_k(t)|^2 = 1$ ($\sum_k p_k(t) = 1$) follows immediately

from Eq. (3-20) by using the completeness relation of orthogonal polynomials $P_n(x)$. In the appendix A reference [14] is provided the walker has the same transition probability at the all sites belonging to the same stratum, i.e., we have $|q_{ik}(t)|^2 = \frac{|q_k(t)|^2}{|V_k|}$, for $i \in V_k$ ($p_{ik}(t) = \frac{p_k(t)}{|V_k|}$, for $i \in V_k$), where $|q_{ik}(t)|^2$ ($p_{ik}(t)$) denotes the transition probability of the quantum walk (classical walk) at the *i*-th vertex of *k*-th stratum V_k . Investigation of CTQW via spectral distribution method, which is introduced as a new method for calculating the probability amplitudes quantum walk (for more details see [14]), allows us to avoid a heavy combinational argument often necessary to obtain full description of spectrum of the Hamiltonian.

We can now investigate CTQW on spidernets. For calculating CTQW we need the Stieltjes transform $G_{\mu}(z)$ which takes the following form

$$G_{\mu}(z) = \frac{1}{z - \frac{a}{z - (b - 1 - c) - \frac{a}{z - (b - 1 - c) - \frac{c}{z - (b - 1 - c) -$$

In order to evaluate the continued fraction, we need firstly to evaluate the following infinite continued fraction defined as

$$\tilde{G}_{\mu}(z) = \frac{c}{z - (b - 1 - c) - \frac{c}{z$$

where by solving the above equation, we have

$$\tilde{G}_{\mu}(z) = \frac{1}{2}(z - (b - 1 - c)) - \sqrt{(z - (b - 1 - c))^2 - 4c}).$$
(3-23)

By substituting (3-23) into (3-21), we obtain the following expression for the Stieltjes transform of μ

$$G_{\mu}(z) = \frac{1}{z - \frac{a}{z - (b - 1 - c) - \tilde{G}_{\mu}(z)}} = \frac{1}{2} \frac{a(b - 1 - c) + (2c - a)z - a\sqrt{(z - (b - 1 - c))^2 - 4c}}{a^2 + ((c - a)z + a(b - 1 - c))z}.$$
(3-24)

Finally, by applying Stieltjes inversion formula, we acquire the absolutely continuous part of

spectral distribution μ as follows

$$\mu(x) = \frac{1}{2\pi} \frac{a\sqrt{4c - (x - (b - 1 - c))^2}}{a^2 + ((c - a)x + a(b - 1 - c))x}; \qquad -2\sqrt{c} + (b - 1 - c) \le x \le 2\sqrt{c} + (b - 1 - c).$$
(3-25)

Referring to the results of Refs. [21, 22, 23], $|q_0(t)|^2$ (return probability at the initial vertex) is the lower bound of the average probability to be still or again at the initially excited vertex for quantum walker and $p_0(t)$ is the average probability to return the initial vertex for classical walker. Since the lower bound in the CTQW oscillates, due to unitary time evolution, one can use the envelope of this oscillations as a measure for quantifying the efficiency (for more detail, see Ref. [23]). Now, by considering Refs. [21, 22, 23], we are in the position to quantify the efficiency of classical and quantum mechanical transport processes on these graphs. For doing this work, we study the asymptotic expansion of integral $p_o(t)$ in the Eq.(3-19) for small x and large t. The spectral distribution of Eq.(3-24) for small x is $\mu(x) \sim x^{-1/2}(d+x)^{-1/2} \sim x^{-1/2}(1-\frac{1}{2}x)$, then we have

$$p_0(t) \sim t^{-3/2}$$
. (3-26)

To obtain this result we use the Ref.[25] (for more details see chapter 6). Therefore, one can obtain the lower bound of quantum mechanical for transport processes as

$$|q_0(t)|^2 \sim t^{-3},\tag{3-27}$$

(for more details see Ref. [22]). Since the decay of $|q_0(t)|^2$ is much quicker than that of $p_0(t)$, the quantum walk on this graph is more efficient than the classical random walk [21, 23]. Now, by numerical analysis, we confirm this results for CTQW and CTRW on two infinite graphs which can be obtained from spidernet graph by an appropriate choice of a, b, c.

Example A.

In this example we consider spidernet in Fig.1, i.e., S(4, 6, 3). Since this graph is not regular

it is sufficient to place $\dot{\alpha}_k = D_{kk} - \alpha_k$ (i.e., $\dot{\alpha}_1 = a = 4, \dot{\alpha}_2 = \dot{\alpha}_3 = \cdots = c + 1 = 4$) in the relation(3-24). Then the spectral distribution takes the following form

$$\mu(x) = \frac{1}{2\pi} \frac{4\sqrt{12 - (x+4)^2}}{-8x - x^2}, \qquad -4 - 2\sqrt{3} \le x \le -4 + 2\sqrt{3}. \tag{3-28}$$

The transition probability and amplitudes of the walker at the stratum k at the time t, for classical and quantum, are given by

$$p_{0}(t) = -\frac{2}{\pi} \int_{-4-2\sqrt{3}}^{-4+2\sqrt{3}} e^{tx} \frac{\sqrt{12 - (x+4)^{2}}}{8x + x^{2}} dx,$$

$$p_{k}(t) = -\frac{1}{\pi} \frac{1}{\sqrt{3^{k-1}}} \int_{-4-2\sqrt{3}}^{-4+2\sqrt{3}} e^{tx} P_{k}(x) \frac{\sqrt{12 - (x+4)^{2}}}{8x + x^{2}} dx, \quad \text{for } k \ge 1,$$

$$q_{0}(t) = -\frac{2}{\pi} \int_{-4-2\sqrt{3}}^{-4+2\sqrt{3}} e^{-itx} \frac{\sqrt{12 - (x+4)^{2}}}{8x + x^{2}} dx,$$

$$q_{k}(t) = -\frac{1}{\pi} \frac{1}{\sqrt{3^{k-1}}} \int_{-4-2\sqrt{3}}^{-4+2\sqrt{3}} e^{-itx} P_{k}(x) \frac{\sqrt{12 - (x+4)^{2}}}{8x + x^{2}} dx, \quad \text{for } k \ge 1,$$

$$(3-29)$$

respectively. Then, for analyzing the time dependent of the transition probability, we calculate numerically $p_0(t), |q_0(t)|^2, p_1(t)$ and $|q_1(t)|^2$ of Eq.(3-29). Figs.2 and 3 show the return probability for CTRW and CTQW on the initial site o of the graph. For CTRW, there is a power law decay $\sim t^{-1.5}$ at the beginning of the transport, but after some time, p_0 reaches a constant value $\frac{1}{N}$ (N is total number of the vertices of the graph i.e, N = |V|). Since the size of the graph is infinite this constant goes to the zero, as shown in Fig.2.

Also in Fig.3, the dashed curve indicates the quantum mechanical return probability on the initial site o of the graph. The dashed line shows the scaling behavior as $\sim t^{-3}$, but at large times, the return probability oscillates frequently and approaches to zero. This property indicates that the walk escapes from the starting site o.

In order to compare the transport speed on this graph, we define the characteristic time t_c as the time when the first maximum of the probabilities occurs on an infinite graph. Such definition is held for both the classical and quantum mechanical transport. For the classical

transport, there is one maximal value and the characteristic time corresponds to the time when the probability is reached uniform probability on finite graph. Figure 3 indicates that the characteristic time t_c for the quantum walk is shorter than that of the classical one. Therefore, we can interpret that the quantum transport speed on spidernet S(4, 6, 3) is faster than that of the classical one. The different behavior of the transport speeds between the quantum and classical random walk is striking characteristic that distinguishes the two transport processes.

Example B.

First we consider b = a and c = a - 1. With this choice the spidernet S(a, a, a - 1) is graph with degree $\kappa = a$ and μ obtains from Eq.(3-24) as

$$\mu(x) = \frac{1}{2\pi} \frac{a\sqrt{4(a-1) - x^2}}{a^2 - x^2}, \qquad -2\sqrt{a-1} \le x \le 2\sqrt{a-1}, \tag{3-30}$$

where this type of measure was first obtained by Kesten [31] in classical random walk with a different method. By using the Eq.(3-20), the transition probability and amplitudes for observing walker at the stratum k at the time t, for classical and quantum, are given by

$$p_{0}(t) = \frac{ae^{-at}}{2\pi} \int_{-2\sqrt{a-1}}^{2\sqrt{a-1}} e^{tx} \frac{\sqrt{4(a-1)-x^{2}}}{a^{2}-x^{2}} dx.$$

$$p_{k}(t) = \frac{ae^{-at}}{2\pi} \frac{1}{\sqrt{a(a-1)^{k-1}}} \int_{-2\sqrt{a-1}}^{2\sqrt{a-1}} e^{tx} P_{k}(x) \frac{\sqrt{4(a-1)-x^{2}}}{a^{2}-x^{2}} dx, \quad \text{for } k \ge 1,$$

$$q_{0}(t) = \frac{a}{2\pi} \int_{-2\sqrt{a-1}}^{2\sqrt{a-1}} e^{-itx} \frac{\sqrt{4(a-1)-x^{2}}}{a^{2}-x^{2}} dx.$$

$$q_{k}(t) = \frac{a}{2\pi} \frac{1}{\sqrt{a(a-1)^{k-1}}} \int_{-2\sqrt{a-1}}^{2\sqrt{a-1}} e^{-itx} P_{k}(x) \frac{\sqrt{4(a-1)-x^{2}}}{a^{2}-x^{2}} dx, \quad \text{for } k \ge 1,$$
(3-31)

respectively. particularly, when a = 2 the network S(2, 2, 1) corresponds to a line graph which the transition amplitudes can be rewritten as $q_0(t) = J_0(2t)$ $(p_0(t) = e^{-2t}I_0(2t))$ and $q_k(t) = \sqrt{2}i^k J_k(2t)$ $(p_k(t) = e^{-2t}\sqrt{2}I_k(2t))$ for $k \ge 2$ where J_k and I_k are the Bessel and modified Bessel function of the first kind, respectively. This is consistent with the result in Ref.[14]. For analyzing the time dependent of probability the observing walk we calculate numerically $|q_k(t)|^2$, $p_k(t)$ of Eq.(3-31). The return probability at the initial vertex is a good measure to quantify the efficiency of the transport[22].

Having obtained transition probability for arbitrary spidernet S(a, a, a - 1) we investigate numerically CTRW and CTQW on its graph for a = 3, 4. Figs. 4, 5 show the transition probability for CTRW on spidernet with a = 3, 4, respectively. For CTRW, there is a power low decay ($\sim t^{-1.5}$) at the beginning of the transport, but after some time the transition probability reach a constant value. Also, the figures 6, 7 show the transition probability for CTQW this graph with with a = 3, 4, respectively. In this case the dashed line shows the scaling behavior $\sim t^{-3}$, and the return probability oscillate frequently and approach to zero which this property indicate the walk escape form a starting site o. This figures indicate that the characteristic time t_c for quantum walk is shorter than it classical. Therefore, we can interpret which the quantum transport speed on spidernet is faster than that of the classical one.

Finally, for large a (i.e., $a \to \infty$) we discuss this question as a quantum central limit theorem [19]. Since $q_k(t) = 0$ in the limit $a \to \infty$, then we have normalization Hamiltonian and probability amplitudes as

$$q_{k}(t) = \lim_{a \to \infty} \langle \phi_{k} | e^{-iAt/\sqrt{a}} | \phi_{0} \rangle = \lim_{a \to \infty} \frac{1}{\sqrt{a(a-1)^{k}}} \int_{-2\sqrt{a-1}}^{2\sqrt{a-1}} e^{-itx/\sqrt{a}} P_{k}(x) \frac{a}{2\pi} \frac{\sqrt{4(a-1) - x^{2}}}{a^{2} - x^{2}} dx$$

$$= \lim_{a \to \infty} \frac{1}{2\pi \sqrt{a(a-1)^{k}}} \int_{-2\sqrt{(a-1)/a}}^{2\sqrt{(a-1)/a}} e^{-itx} P_{k}(\sqrt{a}x) \frac{\sqrt{4(a-1)/a - x^{2}}}{1 - x^{2}/a} dx$$

$$= \frac{1}{2\pi} \int_{-2}^{2} e^{-itx} P_{k,\infty}(x) \sqrt{4 - x^{2}} dx$$

$$= \frac{2}{\pi} \int_{-1}^{1} e^{-i2xt} P_{k,\infty}(2x) \sqrt{1 - x^{2}} dx, \qquad (3-32)$$

where the polynomials $P_{k,\infty}(x)$ is defined as

$$P_{k,\infty}(x) = \lim_{a \to \infty} \frac{1}{\sqrt{a(a-1)^k}} P_k(x).$$
(3-33)

By comparing this recursion relation with Tchebichef polynomials, we have

$$P_{k,\infty}(x) = U_k(x/2), (3-34)$$

where $U_k(x)$ is the Tchebichef polynomials of second kind. Therefore, the probability amplitudes of Eq.(3-32) obtain

$$q_k(t) = \frac{2}{\pi} \int_{-1}^{1} e^{-i2xt} U_k(x) \sqrt{1 - x^2} dx = i^k (k+1) \frac{J_{k+1}(2t)}{t}, \qquad (3-35)$$

where in this case the return probability is $|q_0(t)|^2 = \left(\frac{J_1(2t)}{t}\right)^2$ which the results are obtained in agreement with Ref.[17]. In this case, we see that the spectral distribution obeys Wigner's semi-circle law (i.e, $\mu(x) = \frac{1}{\pi}(\sqrt{1-x^2})$) and one can obtain that the power law decay $\sim t^{-3}$ and $\sim t^{-3/2}$ for the quantum mechanical transport processes and classical counterpart (for more details see Ref. [22]).

4 Conclusion

In this paper we have studied CTQW and CTRW on spidernet lattices. For this work, by using the Stieltjes transform, we have obtained spectral distribution (density of states) that it is a efficiency measure for the performance of classical and quantum mechanical transport processes on graphs. The we have analytically obtained the power laws ~ t^{-3} and ~ $t^{-3/2}$ for CTQW and CTRW on spidernet graphs, respectively. By numerical analysis for two examples S(4, 6, 3) and S(a, a, a - 1), we have confirmed this results. Due to quick decrease of the quantum mechanical transport processes than that of the classical one, the quantum walk on spidernet graphs is more efficient than the classical counterpart. Also, this results have shown that the characteristic time t_c (the t_c is the time when the first maximum of the probabilities occur on infinite graph) for quantum walk is shorter than it

classical. Therefore, in this lattices the speed of quantum transport is faster than that of the classical one.

Appendix A

Determination of spectral distribution by the Stieltjes transform

In this appendix we explain how we can determine spectral distribution $\mu(x)$ of the graphs, by using the Szegö-Jacobi sequences ($\{\omega_k\}, \{\alpha_k\}$). To this aim we may apply the canonical isomorphism from the interacting Fock space onto the closed linear span of the orthogonal polynomials determined by the Szegö-Jacobi sequences ($\{\omega_i\}, \{\alpha_i\}$). More precisely, the spectral distribution μ under question is characterized by the property of orthogonalizing the polynomials $\{P_n\}$ defined recurrently by

$$P_0(x) = 1, \qquad P_1(x) = x - \alpha_1,$$

$$xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \qquad (A-1)$$

for $n \ge 1$.

As it is shown in [32], the spectral distribution can be determined by the following identity:

$$G_{\mu}(z) = \int_{R} \frac{\mu(dx)}{z - x} = \frac{1}{z - \alpha_{1} - \frac{\omega_{1}}{z - \alpha_{2} - \frac{\omega_{1}}{z - \alpha_{3} - \frac{\omega_{3}}{z - \alpha_{4} - \cdots}}} = \frac{Q_{n-1}^{(1)}(z)}{P_{n}(z)} = \sum_{l=1}^{n} \frac{A_{l}}{z - x_{l}},$$
 (A-2)

,

where $G_{\mu}(z)$ is called the Stieltjes transform and A_l is the coefficient in the Gauss quadrature formula corresponding to the roots x_l of polynomial $P_n(x)$ and where the polynomials $\{Q_n^{(1)}\}$ are defined recurrently as

$$Q_0^{(1)}(x) = 1,$$

$$Q_1^{(1)}(x) = x - \alpha_2,$$

$$xQ_n^{(1)}(x) = Q_{n+1}^{(1)}(x) + \alpha_{n+2}Q_n^{(1)}(x) + \omega_{n+1}Q_{n-1}^{(1)}(x)$$

for $n \ge 1$.

Now if $G_{\mu}(z)$ is known, then the spectral distribution μ can be recovered from $G_{\mu}(z)$ by means of the Stieltjes inversion formula:

$$\mu(y) - \mu(x) = -\frac{1}{\pi} \lim_{v \to 0^+} \int_x^y Im\{G_\mu(u+iv)\}du.$$
 (A-3)

Substituting the right hand side of (A-2) in (A-3), the spectral distribution can be determined in terms of $x_l, l = 1, 2, ...$, the roots of the polynomial $P_n(x)$, and Guass quadrature constant $A_l, l = 1, 2, ...$ as

$$\mu = \sum_{l} A_l \delta(x - x_l) \tag{A-4}$$

(for more details see Refs. [14, 15, 32, 33].)

References

- Y. Aharonov, L. Davidovich, N. Zagury. Quantum random walks. Phys. Rev. A, 48: 1687-1690, 1993.
- [2] A. Childs, R. Cleve, E. Deotto, E. Farhi, S. Gutmann, D. Spielman. Exponential algorithmic speedup by quantum walk. Proc. 35th Ann. Symp. Theory of Computing (ACM Press), 59-68, 2003.
- [3] N. Shenvi, J. Kempe, K. B. Whaley. Quantum random-walk search algorithm. Phys. Rev. A, 67: 052307-052318, 2003.
- [4] J. Kempe. Quantum random walks an introductory overview. Contemporary Physics, 44: 307-327, 2003.
- [5] R. Feynman, R. Leighton, M. Sands. The Feynman Lectures on Physics, Volume 3, Addison-Wesley, 1965.
- [6] E. Farhi, S. Gutmann. Quantum computation and decision trees. Phys. Rev. A, 58: 915-928, 1998.

- [7] M. Childs, E. Farhi, S. Gutmann. An example of the difference between quantum and classical random walks. Quantum. Inf. Process, 1: 35-43, 2002.
- [8] A. Ambainis, E. Bach, A. Nayak, A. Viswanath, J. Watrous. One-dimensional quantum walks. Conference Proceedings of the Annual ACM Symposium on Theory of Computing, 37-49, 2001.
- [9] D. Aharonov, A. Ambainis, J. Kempe, U. Vazirani. Quantum walks on graphs. Conference Proceedings of the Annual ACM Symposium on Theory of Computing, 50-59, 2001.
- [10] C. Moore, A. Russell. Quantum walks on the hypercube. Proceedings of the 6th Int. Workshop on Randomization and Approximation in Computer Science (RANDOM'02), 164, 2002.
- [11] J. Kempe. Quantum Random Walks Hit Exponentially Faster. Proceedings of 7th International Workshop on Randomization and Approximation Techniques in Computer Science (RANDOM'03): 354-69, 2003.
- [12] H. Krovi, Todd A. Brun. Quantum walks on quotient graphs. Phys. Rev. A, 75: 062332-062345, 2007.
- [13] N. Konno. Continuous-time quantum walks on ultrametric spaces. Int. J. Quantum Information, 4: 1023-1035, 2006.
- [14] M. A. Jafarizadeh, S. Salimi. Investigation of continuous-time quantum walk via spectral distribution associated with adjacency matrix. Ann. Phys, 322: 1005-1033, 2007.
- [15] M. A. Jafarizadeh, S. Salimi. Investigation of continuous-time quantum walk via modules of BoseMesner and Terwilliger algebras. J. Phys. A, 39: 1329513323, 2006.

- [16] M. A. Jafarizadeh, R. Sufiani, S. Salimi, S. Jafarizadeh. Investigation of continuous-time quantum walk by using Krylov subspace-Lanczos algorithm. Eur. Phys. J. B, 59: 199216, 2007.
- [17] N. Konno. Continuous-time quantum walks on trees in quantum probability theory. Inf. Dim. Anal. Quantum Probab. Rel. Topics, 9: 287-297, 2006.
- [18] S. Salimi. Quantum central limit theorem for continuous-time quantum walks on odd graphs in quantum probability theory. Int. J. Theor. Phys, 47: 3298-3309, 2008.
- [19] S. Salimi. Study of continuous-time quantum walks on quotient graphs via quantum probability theory. Int. J. Quantum Information, 6: 945-957, 2008.
- [20] F. Vogtle, ed., Dendrimers. Springer, Berlin, 1998.
- [21] O. Mulken, V. Bierbaum and A. Blumen. Coherent exciton transport in dendrimers and continuous-time quantum walks. J. Chem. Phys, 124: 124905-, 2006.
- [22] O. Mülken and A. Blumen. Efficiency of quantum and classical transport on graphs. Phys. Rev. E, 73: 066117-066111, 2006.
- [23] O. Mülken. Inefficient quantum walks on networks: the role of the density of states. arXiv: 0710.3453, 2007.
- [24] Xin-Ping Xu. Continuous-time quantum walks on one-dimensional regular networks. Phys. Rev. E, 77: 061127-061135, 2008.
- [25] Carl M. Bender and Steven A. Orszag, Advanced mathem atical methods for Scientist and Engineers, McGraw-Hill, Inc., New York, 1978.
- [26] N. Obata. Quantum probabilistic approach to spectral analysis of star graphs. Interdiscip. Inform. Sci., 10: 41-52, 2004.

- [27] A. Hora and N. Obata. Quantum Probability and Spectral Analysis of Graphs. Springer, Berlin Heidelberg, 2007.
- [28] W. Urakawa, Monatsh. math. 138: 225, 2003.
- [29] G. H. Weiss. Aspects and Applications of the Random Walk, Amsterdam. North-Holland, 1994.
- [30] N. van Kampen. Stochastic Processes in Physics and Chemistry. Amsterdam. North-Holland, 1990.
- [31] H. Kesten. Symmetric random walks on groups. Trans. Amer. Math. Soc., 92: 336, 1959.
- [32] T. S. Chihara. An Introduction to Orthogonal Polynomials. London: Gordon and Breach, 1978.
- [33] J. A. Shohat, J. D. Tamarkin. The Problem of Moments, American Mathematical Society. Providence, RI, 1943.

Figure Captions

Figure-1: Spidernet lattice S(4, 6, 3) given by Hora et al. ([27], Fig. 4.3, p. 121).

Figure-2: The solid and dashed curves show the return probability on initial site o and transition probability from initial site to the first strata for CTRW on spidernet lattice S(4, 6, 3), respectively.

Figure-3: (a) and (b) show the return probability on initial site o and transition probability from initial site to the first strata for CTQW on spidernet lattice S(4, 6, 3), respectively, in which the dashed curves indicate the scaling behavior $|q_0(t)|^2$ and $|q_1(t)|^2 \sim t^{-3}$.

Figure-4: Fig.4 indicates the transition probabilities $p_0(t)$ and $p_1(t)$ of CTRQ on S(a, a, a - 1), for a = 3.

Figure-5: Fig.5 shows the transition probabilities $p_0(t)$ and $p_1(t)$ of CTRQ on S(a, a, a-1), for a = 4.

Figure-6: (a) and (b) indicate the transition probabilities $|q_0(t)|^2$ and $|q_1(t)|^2$ for CTQW on S(a, a, a - 1), for a = 3, in which the dashed curves show the scaling behavior $|q_0(t)|^2$ and $|q_1(t)|^2 \sim t^{-3}$.

Figure-7: (a) and (b) indicate the transition probabilities $|q_0(t)|^2$ and $|q_1(t)|^2$ for CTQW on S(a, a, a - 1), for a = 4, in which the dashed curves show the scaling behavior $|q_0(t)|^2$ and $|q_1(t)|^2 \sim t^{-3}$.