## The capacity of transmitting atomic qubit with light

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## Abstract

The quantum information transfer between a single photon and a two-level atom is considered as a part of a quantum channel. The channel is a degradable channel even when there are decays of the atomic excited state and the single photon state, as far as the total excitation of the combined initial state does not exceed 1. The single letter formula for quantum capacity is obtained.

Recent experiment has realized the coherent transfer of coherent state of light to and from the hyperfine states of an atom trapped within the modes of high finesse optical cavity [1]. This may provide the basic nodes of quantum networks. In the proposal for the implementation of quantum networks[2], each node is a quantum system that stores and locally processes quantum information in quantum bits, atomic internal states with long coherent times serve as these 'stationary' qubits, exchange of information between the nodes of the network is realized by the transmission of photons ('flying' qubits) over optical fiber. In the the experimental setting, a ' $\Lambda$ ' type three-level atom is used. A three-level system can be reduced to a twolevel system adiabatically for far detuning system [2]. Thus a problem of basic interests is the interface of light and two-level atom. We will treat the quantum information transfer process as a quantum channel (or a part of quantum channel) for the first time. The whole process involves transfer of atomic state to field, the optical transmission and transfer quantum state back to atom. The field is restricted to single photon state and in each step of transfer the part that receives quantum state is prepared as ground state or vacuum field state. Field decay and atomic excited state decay are included in our model.

The state transfer: The qubit is a two-level atom with ground state  $|\downarrow\rangle$  and excited state  $|\uparrow\rangle$ . It interacts with a single-mode near-resonant cavity field prepared in vacuum state  $|0\rangle$ . The dynamics of the system is given by the Jaynes-Cumming interaction Hamiltonian

[3] 
$$(\hbar = 1)$$
  
 $\widehat{H} = \nu(a^{\dagger}a + \frac{1}{2}) + \frac{\omega}{2}\sigma_z + g(a^{\dagger}\sigma_- + a\sigma_+),$  (1)

where  $\nu$  is the frequency of the field,  $\omega$  is the atomic transition frequency between the two levels, the coupling constant between the atom and field is g, a and  $a^{\dagger}$  are the annihilation and creation operators of the field,  $\sigma_{-}$  and  $\sigma_{+}$  are the atomic state flip operators defined as  $\sigma_{-} |\uparrow\rangle = |\downarrow\rangle$ ,  $\sigma_{-} |\downarrow\rangle = 0$ ,  $\sigma_{+} |\downarrow\rangle = |\uparrow\rangle$ ,  $\sigma_{+} |\uparrow\rangle = 0$ ,  $\sigma_{z} = \sigma_{+}\sigma_{-} - \sigma_{-}\sigma_{+}$  and  $\sigma_{z} |\uparrow\rangle = |\uparrow\rangle$ ,  $\sigma_{z} |\downarrow\rangle = - |\downarrow\rangle$ . The evolution operator of the system is

$$\widehat{U} = \exp(-i\widehat{H}t) = e^{-i\nu\widehat{\xi}t} \{\cos(\widehat{\Omega}t)$$
(2)

$$-\frac{\iota}{\widehat{\Omega}}\sin(\widehat{\Omega}t)[\frac{\Delta}{2}\sigma_z + g(a^{\dagger}\sigma_- + a\sigma_+)]\}, \quad (3)$$

where  $\hat{\xi} = a^{\dagger}a + (1 + \sigma_z)/2$  is the operator of the total number of excitation of the system.  $\Delta = \omega - \nu$  is the detuning, and  $\hat{\Omega} = \sqrt{g^2 \hat{\xi} + \frac{\Delta^2}{4}}$ . For initial states  $|\downarrow 0\rangle$  or  $|\uparrow 0\rangle$ , the evolved states are  $\hat{U} |\downarrow 0\rangle = e^{i\Delta t/2} |\downarrow 0\rangle$ ,  $\hat{U} |\uparrow 0\rangle = e^{-i\nu t} \{ [\cos(\Omega t) - i\sin(\Omega t)\frac{\Delta}{2\Omega}] |\uparrow 0\rangle - i\sin(\Omega t)\frac{g}{\Omega} |\downarrow 1\rangle \}$ , respectively, where  $\Omega = \sqrt{g^2 + \frac{\Delta^2}{4}}$ . For a general atomic density matrix  $\rho_A = (1 - p) |\downarrow\rangle \langle\downarrow| + r |\downarrow\rangle \langle\uparrow| + r^* |\uparrow\rangle \langle\downarrow| + p |\uparrow\rangle \langle\uparrow|$ , the evolved system state is  $\hat{U}\rho_A \otimes |0\rangle \langle 0| \hat{U}^{\dagger}$ . At some proper time t, the atomic state is traced out, leaving the optical field state  $\rho_B = Tr_A(\hat{U}\rho_A \otimes |0\rangle \langle 0| \hat{U}^{\dagger})$ . In the photon number basis  $|0\rangle, |1\rangle$ , it reads

$$\rho_B = \begin{bmatrix} 1 - p |h_1(t)|^2], & rh_1(t) \\ r^* h_1^*(t), & p |h_1(t)|^2 \end{bmatrix}.$$
(4)

with  $h_1(t) = ie^{i(\Delta/2+\nu)t} \sin(\Omega t) \frac{g}{\Omega}$ .

The conversion as a quantum channel: The state of atomic system A now is transferred to the state of optical field system B. If we neglect the different natures of the atomic system and optical field for a while, the process of the state transfer can be viewed as a quantum channel. It maps a density matrix  $\rho_A$  to another density matrix  $\rho_B$ . Denote the conversion channel as  $\mathcal{E}$ , then

$$\rho_B = \mathcal{E}(\rho_A) = Tr_A(U\rho_A \otimes |0\rangle \langle 0| U^{\dagger}). \tag{5}$$

The channel  ${\mathcal E}$  can be expressed with Kraus operator sum representation with

$$A_{1} = e^{i\Delta t/2} |0\rangle \langle \downarrow | -ie^{-i\nu t} \sin(\Omega t) \frac{g}{\Omega} |1\rangle \langle \uparrow |, (6)$$
  

$$A_{2} = e^{-i\nu t} [\cos(\Omega t) - i\sin(\Omega t) \frac{\Delta}{2\Omega}] |0\rangle \langle \uparrow |, (7)$$

such that  $\rho_B = A_1 \rho_A A_1^{\dagger} + A_2 \rho_A A_2^{\dagger}$  and  $A_1^{\dagger} A_1 + A_2^{\dagger} A_2 = I$ .

The procedure of quantum information transfer from atomic system to field can be viewed as a quantum channel. Hence we can characterize the transfer capability of the Javnes-Cummings interaction with the quantum capacity of the conversion channel if we are concerned with the quantum information converted. The quantum capacity Q measures the maximum amount of quantum information that can be reliably transmitted (here transferred) though the map  $\mathcal{E}$  per channel use [4]. The quantum capacity can be computed by coherent information (CI)  $I_c(\sigma, \mathcal{E}) =$  $S(\mathcal{E}(\sigma)) - S(\sigma^{QR'})$ . Here  $S(\varrho) = -\text{Tr}\varrho \log_2 \varrho$  is the von Neumann entropy,  $\sigma$  is the input state, the application of the channel  $\mathcal{E}$  results the output state  $\mathcal{E}(\sigma)$ ;  $\sigma^{QR'} =$  $(\mathcal{E} \otimes \mathbf{I})(|\psi\rangle \langle \psi|)$ , with R referred to the 'reference' system (the system under process is Q system, we denote  $\sigma^Q$  as  $\sigma$  for simplicity),  $|\psi\rangle$  is the purification of the input state  $\sigma$ . The quantum channel capacity is

$$Q = \lim_{n \to \infty} \sup_{\sigma_n} \frac{1}{n} I_c(\sigma_n, \mathcal{E}^{\otimes n}).$$
(8)

This general result of capacity involves multiple use of the channel, it is called the regulation of the channel. The coherent information is not a convex function of the input state, hence it is very difficult to carry out the capacity for a general channel map. Fortunately, if the channel map is degradable, the capacity can be calculated with single letter formula [5]

$$Q = \sup I_c(\sigma, \mathcal{E}).$$
(9)

The degradability of a quantum channel is defined as follows: The sender A prepares quantum state  $\rho_A$ , the receiver B obtains state  $\rho_B = \mathcal{E}(\rho_A)$ , where  $\mathcal{E}$  is the channel map, the environment E obtains the state  $\rho_E = \widetilde{\mathcal{E}}(\rho_A)$  in the transmission process, where  $\widetilde{\mathcal{E}}$  is called complementary channel. If all input state  $\rho_A$ , there exist a quantum channel  $\mathcal{N}$  such that

$$\mathcal{N}(\mathcal{E}(\rho_A)) = \mathcal{E}(\rho_A), \tag{10}$$

then the channel  $\mathcal{E}$  is called degradable. The physical meaning of the degradable condition is: the quantum

state leaked to the environment can be reconstructed by the receiver with some quantum map  $\mathcal{T}$ .

In the following, we will prove that the conversion due to Jaynes-Cummings model with proper interaction time is a degradable quantum channel for vacuum initial field. The complementary channel  $\widetilde{\mathcal{E}}$  of  $\mathcal{E}$  is defined by

$$\widetilde{\mathcal{E}}(\rho_A) = Tr_B(\widehat{U}\rho_A \otimes |0\rangle \langle 0|\, \widehat{U}^{\dagger}), \qquad (11)$$

which is

$$\widetilde{\mathcal{E}}(\rho_A) = \begin{bmatrix} 1 - p |h_2(t)|^2, & rh_2(t) \\ r^* h_2^*(t), & p |h_2(t)|^2 \end{bmatrix}$$
(12)

in the basis  $|\downarrow\rangle,|\uparrow\rangle$ , with  $h_2(t) = e^{i(\Delta/2+\nu)t} [\cos(\Omega t) + i\sin(\Omega t)\frac{\Delta}{2\Omega}]$ . Note that  $|h_1(t)|^2 + |h_2(t)|^2 = 1$ . The quantum channel  $\mathcal{N}$  then should convert field to atomic system. It is implemented also by Jaynes-Cummings interaction with Hamiltonian

$$\widehat{H}' = \nu'(a^{\dagger}a + \frac{1}{2}) + \frac{\omega'}{2}\sigma_z + g'(a^{\dagger}\sigma_- + a\sigma_+), \quad (13)$$

 $\begin{array}{ll} \mathrm{then} \ \widehat{U'} &= \exp(-i\widehat{H'}t') \ \mathrm{and} \ \widehat{U'} \mid \downarrow 0 \rangle = \\ e^{i\Delta't/'2} \mid \downarrow 0 \rangle, \widehat{U'} \mid \downarrow 1 \rangle &= e^{-i\nu't'} \{ [\cos\left(\Omega't'\right) + \\ i\sin\left(\Omega't'\right)\frac{\Delta'}{2\Omega'} ] \mid \downarrow 1 \rangle &- i\sin\left(\Omega't'\right)\frac{g'}{\Omega'} \mid \uparrow 0 \rangle \} \ \text{ with } \\ \Delta' &= \omega' - \nu' \ \mathrm{and} \ \Omega' = \sqrt{g'^2 + \frac{\Delta'^2}{4}}. \ \mathrm{We \ have} \end{array}$ 

$$\mathcal{N}(\rho_B) = Tr_B(\widehat{U}' |\downarrow\rangle \langle\downarrow| \otimes \rho_B \widehat{U}'^{\dagger}).$$
(14)

We may set  $\Delta' = 0$  (resonant system), then

$$\mathcal{N}(\mathcal{E}(\rho_A)) = \begin{bmatrix} 1 - p |h_3(t)|^2], & ih_3(t)r \\ ih_3^*(t)r^*, & p |h_3(t)|^2 \end{bmatrix}$$
(15)

with  $h_3(t) = h_1(t) \sin(g't')$ . The condition  $\mathcal{N}(\mathcal{E}(\rho_A)) = \widetilde{\mathcal{E}}(\rho_A)$  requires  $1 - \sin^2(\Omega t) \frac{g^2}{\Omega^2} = \sin^2(\Omega t) \frac{g^2}{\Omega^2} \sin^2(g't')$ and  $\cos(\Omega t) + i \sin(\Omega t) \frac{\Delta}{2\Omega} = e^{i\nu't'} \sin(\Omega t) \frac{g}{\Omega} \sin(g't')$ . The amplitude part of the second equation is just the first equation, the phase factor can be adjusted by properly choosing  $\nu'$ . The condition reads

$$\left|\sin(\Omega t)\frac{g}{\Omega}\right| \ge \frac{1}{\sqrt{2}}.$$
(16)

With a similar calculation, it can be proved that the channel is anti-degradable  $(\mathcal{N}'(\tilde{\mathcal{E}}(\rho_A)) = \mathcal{E}(\rho_A))$  for  $|\sin(\Omega t)\frac{g}{\Omega}| \leq \frac{1}{\sqrt{2}}$ , and the channel capacity is 0 for anti-degradable channel by no-cloning theorem.

The channel capacity then is

$$Q = \max_{\rho_A} \{ S(\mathcal{E}(\rho_A)) - S[(\mathcal{E} \otimes \mathbf{I})(|\psi\rangle \langle \psi|)] \}$$
  
$$= \max_{p \in [0,1]} \{ H_2(\sin^2(\Omega t) \frac{g^2}{\Omega^2} p) - H_2((1 - \sin^2(\Omega t) \frac{g^2}{\Omega^2}) p) \}, \qquad (17)$$

with  $H_2(x) = -x \log_2 x - (1 - x) \log_2(x)$  the binary entropy function,  $|\psi\rangle$  is the purification of  $\rho_A$ .  $(\mathcal{E} \otimes \mathbf{I})(|\psi\rangle \langle \psi|)$  is a 4 × 4 matrix of rank 2. The two nonzero eigenvalues can easily be obtained. The explicit expression of the matrix for amplitude damping channel was given in the appendix of Ref. [6]. With a few modification, the matrix  $(\mathcal{E} \otimes \mathbf{I})(|\psi\rangle \langle \psi|)$  can be obtained.

Transmission of atomic qubit with light: The total procedure of quantum information conversion would not be ceased at the stage of the transfer of atomic to field system. The next two steps are optical field state transmission over fiber and convert the photonic state back to atomic state. Typically, optical fiber will damp the state with loss, and the inverse conversion may not be ideal. The whole process of transmitting atomic state with light consists at least three parts: (1) atom to field conversion, (2) transmission on fiber (or free space), (3) the inverse conversion. We may consider these three steps as quantum channel maps which are denoted with  $\mathcal{E}, \mathcal{T}, \mathcal{E}'$ , respectively. The optical transmission channel is characterized by the transmittance T. The inverse conversion  $\mathcal{E}'$  is just the channel map  $\mathcal{N}$ with the free choice of all its parameters. The whole process then is represented by a concatenate channel of  $\mathcal{C} = \mathcal{E}' \circ \mathcal{T} \circ \mathcal{E}$ . The quantum systems are denoted as A, B, C, D for the sender, the field before transmission, the field after transmission, the receiver, respectively. We have  $\rho_B = \mathcal{E}(\rho_A), \ \rho_C = \mathcal{T}(\rho_B), \ \rho_D = \mathcal{E}'(\rho_C)$ . The atomic state sent is  $\rho = \rho_A$ , the atomic state received is  $\rho' = \rho_D$ . Note that the lossy fiber  $\mathcal{T}$  is a degradable channel [7], and we have proved that the channel of atomic to field conversion  $\mathcal{E}$  is a degradable channel. It can be proved that the inverse conversion is also degradable. So we can anticipate that the concatenate channel  $\mathcal{C}$  is also degradable. In the following, we will prove that it is really the case.

The action of the lossy fiber is equivalent to that of a beam splitter. The channel  $\mathcal{T}$  can be represented by beam splitter operator. Now the optical state is in the Hilbert space with basis  $|0\rangle$  and  $|1\rangle$ , the action of the channel can be simplified to an operator  $\hat{V}$  with

$$\widehat{V} |0\rangle_B |0\rangle_C = |0\rangle_B |0\rangle_C,$$

$$\widehat{V} |1\rangle_B |0\rangle_C = \sqrt{1-T} |1\rangle_B |0\rangle_C + \sqrt{T} |0\rangle_B |1\rangle_C$$

$$(18)$$

The concatenate channel  $\mathcal{C}$  maps the input state  $\rho$  to  $\rho' = \mathcal{C}(\rho)$ , and  $\mathcal{C}(\rho) = Tr_{ABC}[\hat{U}'\hat{V}\hat{U}\rho \otimes |0\rangle_B \langle 0| \otimes |0\rangle_C \langle 0|\otimes|\downarrow\rangle_D \langle \downarrow| \hat{U}^{\dagger}\hat{V}^{\dagger}\hat{U}'^{\dagger}]$ . The detail calculation gives

$$C(\rho) = \begin{bmatrix} 1 - p |h_4(t, t')|^2, & rh_4(t, t') \\ r^*h_4(t, t')^* & p |h_4(t, t')|^2 \end{bmatrix}$$
(20)

in the atomic basis  $|\downarrow\rangle$  and  $|\uparrow\rangle$ , where  $h_4(t,t') = -e^{i(\Delta/2+\nu)t+i(\Delta'/2+\nu't')}\sin(\Omega t)\frac{g}{\Omega}\sqrt{T}\sin(\Omega't')\frac{g'}{\Omega'}$ .

We have three kinds of complementary channels  $\widetilde{\mathcal{E}}$ ,  $\widetilde{\mathcal{T}} \circ \mathcal{E}$  and  $\widetilde{\mathcal{E}}' \circ \mathcal{T} \circ \mathcal{E}$ . They map the input state  $\rho$  to states that leak to environment at the three steps. The degradability of  $\mathcal{C}$  requires the existence of channel maps  $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$  such that  $\mathcal{N}_1 \circ \mathcal{C} = \widetilde{\mathcal{E}}$ ,  $\mathcal{N}_2 \circ \mathcal{C} = \widetilde{\mathcal{T}} \circ \mathcal{E}$ ,  $\mathcal{N}_3 \circ \mathcal{C} = \widetilde{\mathcal{E}}' \circ \mathcal{T} \circ \mathcal{E}$  for all input state  $\rho$ . The channels  $\mathcal{N}_i$  can be constructed as in the previous section. The conditions of degradability are  $1 - \sin^2(\Omega t) \frac{g^2}{\Omega^2} \leq |h_4(t,t')|^2, 1 - T \sin^2(\Omega t) \frac{g^2}{\Omega^2} \leq |h_4(t,t')|^2$ , respectively. The three conditions can be combined to the condition  $1 - |h_4(t,t')|^2 \leq |h_4(t,t')|^2$ , which is

$$\sin(\Omega t)\frac{g}{\Omega}\sqrt{T}\sin(\Omega' t')\frac{g'}{\Omega'} \ge \frac{1}{\sqrt{2}}.$$
 (21)

The channel capacity is

$$Q = \max_{p \in [0,1]} \{ H_2(p | h_4(t,t') |^2) - H_2(p(1 - | h_4(t,t') |^2)) \}$$
(22)

for  $|h_4(t,t')|^2 \ge \frac{1}{2}$  and Q = 0 for  $|h_4(t,t')|^2 < \frac{1}{2}$ . The effect of decay: To comply with experimental

The effect of aecay: To comply with experimental environment, we introduce photonic decay rate k and atomic excited decay rate  $\gamma$  into our system. For a twolevel system reduced from three-level  $\Lambda$  cavity trapped atom, the two levels are two hyperfine ground states. The decay rate  $\gamma$  can be omitted comparing with other parameters, while photonic decay rate k should be considered in order to read out the quantum state encoded in the atom. The master equation of the whole atom and field system is

$$\frac{d\rho}{dt} = -i[\widehat{H},\rho] + \mathcal{L}_1\rho + \mathcal{L}_2\rho.$$
(23)

with  $\mathcal{L}_1 \rho = \frac{k}{2} (2a\rho a^{\dagger} - a^{\dagger} a\rho - \rho a^{\dagger} a), \mathcal{L}_2 \rho = \frac{\gamma}{2} (2\sigma_- \rho \sigma_+ - \rho a^{\dagger} a)$  $\sigma_+\sigma_-\rho - \rho \sigma_+ \sigma_-$ ). The basis of the whole system density matrix  $\rho$  are  $|\downarrow 0\rangle$ ,  $|\downarrow 1\rangle$ ,  $|\uparrow 0\rangle$ ,  $|\uparrow 1\rangle$ . To simplify the notation of the entries of  $\rho$ , we abbreviate the basis as  $|m\rangle$  with m = 0, 1, 2, 3 corresponding to  $|\downarrow 0\rangle, |\downarrow 1\rangle, |\uparrow 0\rangle, |\uparrow 1\rangle$ , respectively. Then we have the equations for  $\rho_{mn}$ . In our system, the total excitation of the initial state is assumed to be 1. In the later evolution, this number of the total excitation can not exceed 1. H is a excitation number conserved Hamiltonian, the decay can only decrease the excitation number. Hence, in such an initial condition,  $\rho_{m3}(t) = \rho_{3m}(t) \equiv 0$  for all m. We here consider the transfer of single photonic state to atom (In the situation of total excitation limited to 1, there is the symmetry between the field and atom, thus the case of transfer of quantum information from atom to field can be obtained accordingly). Suppose the general initial state of  $|\downarrow\rangle \langle \downarrow| \otimes \rho_{photon}$ with  $\rho_{photon} = \overline{p} |0\rangle \langle 0| + r |0\rangle \langle 1| + r^* |1\rangle \langle 0| + p |1\rangle \langle 1|$ and  $\overline{p} = 1 - p$ , then  $\rho_{00}(0) = \overline{p}, \rho_{02}(0) = r, \rho_{20}(0) =$  $r^*, \rho_{22}(0) = p$ , and all other entries are 0 initially. The linear equations of  $\rho_{mn}$  (m, n = 0, 1, 2) can be solved with Laplacian transformations. The solutions are  $\rho_{00}(t) = 1 - \rho_{11}(t) - \rho_{22}(t)$ , and

$$\rho_{01}(t) = \frac{igr}{X+iY}e^{-\frac{1}{2}k_1t+i(\nu+\frac{\Delta}{2})t} \\ [e^{\frac{1}{2}(X+iY)} - e^{-\frac{1}{2}(X+iY)}], \qquad (24)$$

$$\rho_{02}(t) = \frac{r}{2(X+iY)} e^{-\frac{1}{2}k_1t + i(\nu + \frac{\Delta}{2})t} \\ \{(k_2+i\Delta)[e^{\frac{1}{2}(X+iY)} - e^{-\frac{1}{2}(X+iY)}] \\ + (X+iY)[e^{\frac{1}{2}(X+iY)} + e^{-\frac{1}{2}(X+iY)}]\} (25)$$

$$\rho_{11}(t) = 2pg^2\eta(\cosh Xt - \cos Yt), \qquad (26)$$

$$\rho_{12}(t) = pg\eta[(\Delta - ik_2)(\cosh Xt - \cos Yt) + (Y - iX)(\sinh Xt - i\sin Yt), \quad (27)$$

$$\rho_{22}(t) = p\eta[(X^{2} + \Delta^{2} + 2g^{2}) \cosh Xt + (k_{2}X + \Delta Y) \sinh Xt + (Y^{2} - \Delta^{2} - 2g^{2}) \cos Yt + (k_{2}Y - \Delta X) \sin Yt].$$
(28)

where  $k_{1,2} = (k \pm \gamma)/2, \eta = \frac{e^{-k_1 t}}{X^2 + Y^2}$ , and  $X = \sqrt{\frac{1}{2}(\sqrt{z^2 + 4k_2^2\Delta^2} - z)}, Y = \sqrt{\frac{1}{2}(\sqrt{z^2 + 4k_2^2\Delta^2} - z)}$ , with  $z = 4g^2 + \Delta^2 - k_2^2$ . We now consider the process of quantum state transfer from field to atom as a quantum channel map  $\mathcal{E}$ . Then by tracing out the field freedom of  $\rho(t)$ , we obtain  $\mathcal{E}(\rho_{photon}) = [1 - \rho_{11}(t)] |\downarrow\rangle \langle\downarrow| + \rho_{01}(t)] |\downarrow\rangle \langle\uparrow| + \rho_{10}(t)] |\uparrow\rangle \langle\downarrow| + \rho_{11}(t)] |\uparrow\rangle \langle\uparrow|$ . By tracing out the atomic freedom of  $\rho(t)$ , we obtain the complimentary channel map  $\widetilde{\mathcal{E}}(\rho_{photon}) = [1 - \rho_{22}(t)] |0\rangle \langle 0| + \rho_{02}(t)] |0\rangle \langle 1| + \rho_{20}(t)] |1\rangle \langle 0| + \rho_{22}(t)] |1\rangle \langle 1|$ . It is not difficult to prove that  $|\rho_{01}(t)|^2 p = \rho_{11} |\gamma|^2$ , and  $|\rho_{02}(t)|^2 p = \rho_{22} |\gamma|^2$ , thus, the maps of the channel and the complimentary channel are

$$\mathcal{E}(\rho_{photon}) = \begin{bmatrix} 1 - p |h_5(t)|^2 & \gamma h_5(t) \\ \gamma^* h_5^*(t) & p |h_5(t)|^2 \end{bmatrix}, (29)$$

$$\widetilde{\mathcal{E}}(\rho_{photon}) = \begin{bmatrix} 1-p |h_6(t)|^2 & \gamma h_6(t) \\ \gamma^* h_6^*(t) & p |h_6(t)|^2 \end{bmatrix} (30)$$

in their own basis. Where  $h_5(t) = \sqrt{2g^2\eta(\cosh Xt - \cos Yt)}e^{i\theta_1(t)}, h_6(t) = \sqrt{\eta}[(X^2 + \sqrt{\eta})]$ 

 $\begin{array}{ll} \Delta^2 &+& 2g^2)\cosh Xt &+(k_2X &+& \Delta Y)\sinh Xt \\ +(Y^2-\Delta^2-2g^2)\cos Yt &+(k_2Y-\Delta X)\sin Yt]^{\frac{1}{2}}e^{i\theta_2(t)} \\ \text{for some phase factors } \theta_1(t) \text{ and } \theta_2(t). \text{ The degradabil-} \\ \text{ity condition is } |h_5(t)| \geq |h_6(t)| \text{ , which can be written} \\ \text{as} \end{array}$ 

$$(X^{2} + \Delta^{2}) \cosh Xt + (k_{2}X + \Delta Y) \sinh Xt + (Y^{2} - \Delta^{2}) \cos Yt + (k_{2}Y - \Delta X) \sin Yt] \leq (31)$$

When the channel is degradable, the capacity is

$$Q = \max_{p \in [0,1]} \{ H_2(p |h_5(t)|^2) - H_2(p(1 - |h_5(t)|^2)) \}, \quad (32)$$

otherwise Q = 0.

*Conclusion:* The quantum capacity of the conversion channel of quantum information between single photon number state and two-level atomic state is obtained as a function of coupling rate, detuning, decay rates and the operation time. For an efficient quantum information transfer, a proper operation time should be chosen and the coupling rate should be high enough comparing with the detunging and decay rates.

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