

Tractable measure of nonclassical correlation using density matrix truncations

Akira SaiToh · Robabeh Rahimi · Mikio Nakahara

Received: date / Accepted: date

Abstract In the context of the Oppenheim-Horodecki paradigm of nonclassical correlation, a bipartite quantum state is (properly) classically correlated if and only if it is represented by a density matrix having a product eigenbasis. On the basis of this paradigm, we propose a measure of nonclassical correlation by using truncations of a density matrix down to individual eigenspaces. It is computable within polynomial time in the dimension of the Hilbert space albeit imperfect in the detection range. This is in contrast to the measures conventionally used for the paradigm. The computational complexity and mathematical properties of the proposed measure are investigated in detail and the physical picture of its definition is discussed.

Keywords Nonclassical correlation · Quantumness · Computational tractability · Informational entropy · Matrix truncation

PACS 03.65.Ud · 03.67.Mn

1 Introduction

Classical/nonclassical separation of correlations between subsystems of a bipartite quantum system has been an essential and insightful subject in quantum information theory. The entanglement paradigm [27, 19] is based on the state preparation stage: any quantum state that cannot be prepared by local operations and classical communications (LOCC) [22] is entangled. There are paradigms [3, 16, 17, 9] based on post-preparation stages, which use different definitions of classical and nonclassical correlations. On the basis of the Oppenheim-Horodecki definition [17, 12, 18], a quantum bipartite system consisting of subsystems A and B is (properly) classically correlated if and only if it is described by a density matrix having a product eigenbasis (PE), $\rho_{PE}^{AB} = \sum_{j,k=1,1}^{d^A, d^B} e_{jk} |v_j^A\rangle\langle v_j^A| \otimes |v_k^B\rangle\langle v_k^B|$, where d^A (d^B) is the dimension of the Hilbert space of A (B), e_{jk} is the eigenvalue of ρ_{PE}^{AB} corresponding to an eigenvector $|v_j^A\rangle \otimes |v_k^B\rangle$. Thus, a quantum bipartite system consisting of subsystems A and B is nonclassically correlated if and only if it is described by a density matrix having no product eigenbasis. In this Paper, we employ this classical/nonclassical separation.

This definition was introduced in the discussions by Oppenheim *et al.* [17, 12] on information that can be localized by applying closed LOCC (CLOCC, a branch family of LOCC) operations. The CLOCC protocol allows only local unitary operations and the operations of sending subsystems through a complete dephasing

A. SaiToh

Research Center for Quantum Computing, Interdisciplinary Graduate School of Science and Engineering, Kinki University, 3-4-1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan
Tel.: +81-6-6721-2332
Fax: +81-6-6727-4301
E-mail: saitoh@alice.math.kindai.ac.jp

R. Rahimi

Departments of Chemistry and Materials Science, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
Note: She belonged to Department of Physics, Kinki University when this work was mostly conducted.
E-mail: rahimi@sci.osaka-cu.ac.jp

M. Nakahara

Research Center for Quantum Computing, Interdisciplinary Graduate School of Science and Engineering, Kinki University, 3-4-1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan; Department of Physics, Kinki University, 3-4-1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan
E-mail: nakahara@math.kindai.ac.jp

arXiv:0906.4187v6 [quant-ph] 14 Oct 2010

channel. The classical/nonclassical separation is linked to a localizable information under the zero-way CLOCC protocol in which coherent terms are deleted completely by local players before communicating under CLOCC. A bipartite state with a product eigenbasis carries information completely localizable under zero-way CLOCC. The nonlocalizable information under zero-way CLOCC is a measure of nonclassical correlation. To bring physical insights to the paradigm, the operational interpretations of nonclassical correlation have been investigated in view of impossibility of local broadcasting [20] and in view of nonclassicality in local measurements [12, 15]. Furthermore, a correspondence between a separable state and a classically correlated state in the context of a system extension by using local ancillary systems was reported [13]. The paradigm is thus physically meaningful, for which it is of growing interest to introduce measures [16, 17, 12, 8, 26, 24, 20] and detection methods [7, 23, 4] (a measure can, of course, be regarded as a detection tool).

To define measures of nonclassical correlation, there must be certain axioms or requirements to satisfy. To find proper axioms for them, let us revisit some axioms [22] for entanglement measures. (Although we do not revisit, one may also be interested in axioms for classical correlation measures [10].) The commonly accepted axioms for entanglement measures are (i) a measure should vanish for any separable state, and (ii) entanglement and its measure should not increase under LOCC. These are regarded as the most important axioms. In addition, one must have certain maximally entangled states for which a measure takes its maximum value as a consequence of (ii). Similar axioms are naturally to be imposed to measures of nonclassical correlation. The following axioms are suggested in the present context where the basic protocol is the zero-way CLOCC. (i') a measure should vanish for any state having a product eigenbasis, and (ii') a measure should be invariant under local unitary transformations. It should be noted that one cannot require a measure to be nonincreasing under general local operations.¹ One may also need a measure to take the maximum value for certain maximally entangled states since they should possess quite large nonclassical correlation.

¹ There are local operations that increase nonclassical correlation. For example, consider a 3-qubit state of the system AB, $(|01\rangle^A |01\rangle \otimes |0\rangle^B |0\rangle + |1\rangle^A |1\rangle \otimes |1\rangle^B |1\rangle)/2$, with $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. This density matrix has a product eigenbasis and hence the state possesses only classical correlation. Let us apply a complete projection with the projectors $|+\rangle\langle +|$ and $|-\rangle\langle -|$ to the second qubit. The resultant state is represented as a density matrix that has the eigenvectors $|0+\rangle^A |0\rangle^B$, $|0-\rangle^A |0\rangle^B$, $|1+\rangle^A |1\rangle^B$, and $|1-\rangle^A |1\rangle^B$ with the equal populations. This has no product eigenbasis and hence the state possesses nonclassical correlation.

The above axioms (i') and (ii') are satisfied by the nonlocalizable information named zero-way quantum deficit [17, 12] and another well-known measure of nonclassical correlation, the minimized quantum discord [16]. They take the maximum values (dependent on the system size) for the generalized Bell state $|\Psi_{\text{Bell}}\rangle \langle \Psi_{\text{Bell}}|$ with $|\Psi_{\text{Bell}}\rangle = \sum_{i=1}^N |i\rangle |i\rangle / \sqrt{N}$ for an $N \times N$ -dimensional bipartite system. The quantum discord is a discrepancy between two different expressions of mutual informations that are identical in the classical probability theory. Its minimization over the choices of a local basis is local-unitary invariant and can be used as a measure of nonclassical correlation in the present context. Recently, the minimized one is simply referred to as quantum discord. We follow this custom hereafter. Quantum discord for one side, say, the subsystem A, as an ‘‘apparatus’’ vanishes if and only if [16] the bipartite density matrix is written as $\sum_k |k\rangle^A \langle k| \otimes \xi_k^B$ with ξ_k^B positive Hermitian operators acting on the other side B. Thus the average of the quantum discords for both sides is a measure with the full detection range of nonclassical correlation.

There are other measures [8, 26, 24, 20] that were later proposed on the basis of the same definition of classical/nonclassical correlations. In particular, Piani *et al.* [20] designed a measure which vanishes if and only if a state has a product eigenbasis. It is in a similar form as quantum discord and defined as a distance of two different quantum mutual informations that is minimized over local maps associated with local positive operator-valued measurements [6]. A problem in a practical point of view is that the original nonlocalizable information, quantum discord, and the Piani *et al.*'s measure all require expensive computational tasks to find minima over all possible local operations in their contexts. A similar difficulty exists in the measurement-induced disturbance [8, 15, 5] for which a minimization is actually required to find a proper Schmidt basis (or local dephasing basis) used to compute the measure in degenerate cases (namely, the cases where the eigenbases of one or both of the reduced density matrices of the state are not unique). Note that detecting nonclassical correlation can be easier than quantification; apart from measures, there are easy cases for detection as we later discuss in Section 4. Our interest is, however, not only detecting nonclassical correlation but quantification.

In our previous work [26], an entropic measure G based on a sort of a game to find the eigenvalues of a reduced density matrix from the eigenvalues of an original density matrix was proposed. This measure can be computed within a finite time although it does not have a perfect detection range. Its computational cost is exponential in the dimension of the Hilbert space.

One way [25] to achieve a polynomial cost is to introduce carefully-chosen maps similar to positive-but-not-completely-positive maps [19, 11]. We pursue a different way in this paper.

Here, we introduce a measure of nonclassical correlation for a bipartite state using the eigenvalues of reduced matrices obtained by tracing out a subspace after certain truncations of a density matrix. Its construction is rather simple as we see in Definition 4 of Section 3. The computational cost is shown to be polynomial in the dimension of the Hilbert space. Although the measure is imperfect in the detection range and possesses no additivity property, it is practically useful as an economical measure invariant under local unitary operations. It takes the maximum value for the generalized Bell states. In addition, it reduces to the entropy of entanglement (see, e.g., [2] for the definition) for pure states with the Schmidt coefficients $\leq 1/\sqrt{2}$. (Here, a Schmidt coefficient is a square root of a non-zero eigenvalue of a reduced density matrix of a subsystem for a given pure state.)

This paper is organized as follows. We begin with a brief overview of the measure G in Section 2. The measure M is introduced and its properties are investigated in Section 3. The validity of M as a measure and a physical interpretation of its definition are discussed in Section 4. Section 5 summarizes this work.

2 Brief overview of the measure by partitioning eigenvalues

We first make a brief overview of the measure G , an existing measure computable in finite time. In the context of bipartite splitting, it is defined as the minimized discrepancy between the set of the mimicked eigenvalues of a local system (say, subsystem A), $\{\tilde{e}_i\}_{i=1}^{d^A}$, and the set of the genuine eigenvalues of the local system, $\{e_i\}_{i=1}^{d^A}$. Here, \tilde{e}_i 's are calculated by (i) partitioning the $d^A \times d^B$ eigenvalues of the original bipartite state ρ^{AB} into d^A sets; and (ii) calculating the sum of the d^B elements in each set. The discrepancy in view from one side (from Alice's side in this context) is defined as

$$F^A(\rho^{AB}) = \min_{\text{partitionings}} \left| \sum_i (\tilde{e}_i \log_2 \tilde{e}_i - e_i \log_2 e_i) \right|.$$

Similarly $F^B(\rho^{AB})$ is defined. The measure is defined as

$$G(\rho^{AB}) = \max[F^A(\rho^{AB}), F^B(\rho^{AB})].$$

A drawback of the measure is that the number of combinations of eigenvalues that should be tried in the

minimization is $d^A d^B C_{d^B} \times (d^A - 1) d^B C_{d^B} \times \dots \times [d^A - (d^A - 1)]_{d^B} C_{d^B} = (d^A d^B)! / (d^B!)^{d^A} \simeq 2^{d^A d^B \log_2 d^A}$ when the subsystem of concern is A [$(d^A d^B)! / (d^A!)^{d^B} \simeq 2^{d^A d^B \log_2 d^B}$ when it is B]. Indeed, this complexity is better in practice than that for minimization over all certain local operations required for calculating zero-way quantum deficit, quantum discord, and Piani *et al.*'s measure. The complexity of a minimization over all local operations for a subsystem, say A, is $O[\text{poly}(d^A, d^B) \times 2^{(d^A)^2 \log_2 c}]$ with c the number of values tried for each parameter of a local operation. The complexity for computing G is smaller in the range $d^A, d^B \lesssim c$. Computing G is, however, still very expensive.

3 Measure based on partial traces of truncated density matrices

A measure that is computable within realistic time is desired for practical use. We introduce in the following a measure that achieves a realistic computational time, namely, polynomial time in the dimension of the Hilbert space.

3.1 Introduction of the measure

Let us begin with a basic definition.

Definition 1 Let us write the eigenspace corresponding to the eigenvalue η of a bipartite density matrix ρ^{AB} as $\text{span}\{|v_k^\eta\rangle\}_{k=1}^{d^\eta}$ where d^η is the dimension of the eigenspace and $|v_k^\eta\rangle$'s are the eigenvectors. Let us define a "truncated" density matrix down to the η eigenspace as

$$\tilde{\rho}^\eta = \eta \sum_{k=1}^{d^\eta} |v_k^\eta\rangle\langle v_k^\eta|.$$

The following proposition holds for $\tilde{\rho}^\eta$.

Proposition 1 Consider $\tilde{\rho}^\eta$ introduced above. The eigenvalues of the reduced matrix $\text{Tr}_B \tilde{\rho}^\eta$ ($\text{Tr}_A \tilde{\rho}^\eta$) of the system A (B) are integer multiples of η if ρ^{AB} has a product eigenbasis.

Proof Suppose ρ^{AB} has a product eigenbasis $\{|a_i\rangle\}_{i=1}^{d^A} \times \{|b_j\rangle\}_{j=1}^{d^B}$. Then $\tilde{\rho}^\eta$ becomes $\eta \sum_{k=1}^{d^\eta} |a_k^\eta\rangle\langle a_k^\eta| \otimes |b_k^\eta\rangle\langle b_k^\eta|$ where $|a_k^\eta\rangle$'s ($|b_k^\eta\rangle$'s) are some d^η vectors, with possible multiplicities, found in $\{|a_i\rangle\}_{i=1}^{d^A}$ ($\{|b_j\rangle\}_{j=1}^{d^B}$). Since $|a_i\rangle$'s ($|b_j\rangle$'s) are orthogonal to each other, it is now easy to find that the proposition holds. \square

Remark 1 The sum of the eigenvalues of $\text{Tr}_B \tilde{\rho}^\eta$ is equal to $\text{Tr} \tilde{\rho}^\eta = \eta d^\eta$; similarly, that of $\text{Tr}_A \tilde{\rho}^\eta$ is equal to ηd^η . This property is tacitly used in the calculations hereafter.

Let us introduce useful functions.

Definition 2 For $x, y \geq 0$, let us introduce the function $\text{nim}_y(x)$ which is the nearest integer multiple of y for x . For the exceptional case that $x \pm y/2$ are integer multiples of y , let it take the value $x - y/2$. Thus, the strict definition is

$$\text{nim}_y(x) = \begin{cases} y\lfloor x/y \rfloor & (x - y\lfloor x/y \rfloor \leq y\lceil x/y \rceil - x, y \neq 0) \\ y\lceil x/y \rceil & (x - y\lfloor x/y \rfloor > y\lceil x/y \rceil - x, y \neq 0) \\ 0 & (y = 0) \end{cases}$$

Definition 3 Consider the two collections of m nonempty sets,

$$X = \left\{ \{x_i^1\}_{i=1}^{d_1}, \{x_i^2\}_{i=1}^{d_2}, \dots, \{x_i^m\}_{i=1}^{d_m} \right\}$$

and

$$Y = \left\{ \{y_i^1\}_{i=1}^{d_1}, \{y_i^2\}_{i=1}^{d_2}, \dots, \{y_i^m\}_{i=1}^{d_m} \right\}$$

with nonnegative real numbers x_i^j and y_i^j ($j = 1, \dots, m$) such that $\sum_{j=1}^m \sum_{i=1}^{d_j} x_i^j = \sum_{j=1}^m \sum_{i=1}^{d_j} y_i^j = 1$ where d_j is the size of the j th set (d_j is common for X and Y for the same j). Here, Y is assumed to be a prediction or an estimate of X .

Let us here introduce the quota

$$T_j = \sum_{i=1}^{d_j} x_i^j$$

for the j th set. The quotas satisfy $\sum_{j=1}^m T_j = 1$.

As a discrepancy between x_i^j and y_i^j , we may use the quantity

$$s(x_i^j, y_i^j) = - \left| x_i^j - y_i^j \right| \log_2(x_i^j/T_j).$$

This can be interpreted as the claim that we obtain $[(-\log_2 x_i^j) - (-\log_2 T_j)]$ -bit of information for the (i, j) th event with the weight $|x_i^j - y_i^j|$ of surprise when we have the estimate value y_i^j for x_i^j and know beforehand that an event in the j th group occurs. Because $x_i^j \leq T_j$, it is guaranteed that $s(x_i^j, y_i^j) \geq 0$.

With the quantities, we define the function

$$\tilde{S}(X, Y) = \sum_{j=1}^m \sum_{i=1}^{d_j} s(x_i^j, y_i^j).$$

This can be regarded as a (nonsymmetric) distance between X and its prediction Y .

Proposition 2 *The relation $\tilde{S}(X, Y) \leq \log_2(\max_j d_j)$ holds if $y_i^j \leq 2x_i^j$ for $\forall i, j$.*

Proof We have $s(x_i^j, y_i^j) \leq -x_i^j \log_2(x_i^j/T_j)$ if $y_i^j \leq 2x_i^j$. Then,

$$\begin{aligned} \tilde{S}(X, Y) &\leq \sum_{j=1}^m \sum_{i=1}^{d_j} -x_i^j \log_2(x_i^j/T_j) \\ &= \sum_{j=1}^m T_j \sum_{i=1}^{d_j} -(x_i^j/T_j) \log_2(x_i^j/T_j) \leq \sum_{j=1}^m T_j \log_2(d_j) \\ &\leq \log_2(\max_j d_j). \end{aligned}$$

In this transformation, we have used the fact that, for each j , one may regard (x_i^j/T_j) as probabilities z_i^j satisfying $\sum_{i=1}^{d_j} z_i^j = 1$. \square

The introduced function $\tilde{S}(X, Y)$ is employed to quantify a distance between two collections. One may be curious about the reason why we have introduced the quantity $s(x_i^j, y_i^j)$ to define this function. Indeed, it is more common to use the function

$r(x_i^j, y_i^j) = -x_i^j \log_2(x_i^j/y_i^j)$ instead of $s(x_i^j, y_i^j)$ to quantify a discrepancy between x_i^j and y_i^j . For example, the relative entropy employs r . However, r can only be used under the condition $x_i^j \neq 0 \leftrightarrow y_i^j \neq 0$. In the theory we are constructing in this contribution, this condition does not hold.

Now we define the new measure of nonclassical correlation on the basis of Proposition 1.

Definition 4 Suppose there are m distinct eigenvalues, η_1, \dots, η_m , for a bipartite state ρ^{AB} , i.e., $\rho^{\text{AB}} = \sum_{j=1}^m \tilde{\rho}^{\eta_j}$. Let us write the dimension of the η_j eigenspace as d^{η_j} . For $\tilde{\rho}^{\eta_j}$, consider the eigenvalues $\lambda_i^{j, \text{A}}$ of the reduced matrix $\text{Tr}_{\text{B}} \tilde{\rho}^{\eta_j}$. Let d_j^{A} be the rank of $\text{Tr}_{\text{B}} \tilde{\rho}^{\eta_j}$. Consider the collections

$$X^{\text{A}}(\rho^{\text{AB}}) = \{ \{ \lambda_i^{1, \text{A}} \}_{i=1}^{d_1^{\text{A}}}, \{ \lambda_i^{2, \text{A}} \}_{i=1}^{d_2^{\text{A}}}, \dots, \{ \lambda_i^{m, \text{A}} \}_{i=1}^{d_m^{\text{A}}} \}$$

and

$$Y^{\text{A}}(\rho^{\text{AB}}) = \{ \{ \text{nim}_{\eta_1}(\lambda_i^{1, \text{A}}) \}_{i=1}^{d_1^{\text{A}}}, \{ \text{nim}_{\eta_2}(\lambda_i^{2, \text{A}}) \}_{i=1}^{d_2^{\text{A}}}, \dots, \{ \text{nim}_{\eta_m}(\lambda_i^{m, \text{A}}) \}_{i=1}^{d_m^{\text{A}}} \}$$

as functions of ρ^{AB} . The measure of nonclassical correlation from the view of the subsystem A is defined as

$$\begin{aligned} M^{\text{A}}(\rho^{\text{AB}}) &= \tilde{S}[X^{\text{A}}(\rho^{\text{AB}}), Y^{\text{A}}(\rho^{\text{AB}})] \\ &= - \sum_{j=1}^m \sum_{i=1}^{d_j^{\text{A}}} \left| \lambda_i^{j, \text{A}} - \text{nim}_{\eta_j}(\lambda_i^{j, \text{A}}) \right| \log_2 \frac{\lambda_i^{j, \text{A}}}{\eta_j d^{\eta_j}}. \end{aligned} \tag{1}$$

In the same way, $M^{\text{B}}(\rho^{\text{AB}})$ is defined:

$$M^{\text{B}}(\rho^{\text{AB}}) = \tilde{S}[X^{\text{B}}(\rho^{\text{AB}}), Y^{\text{B}}(\rho^{\text{AB}})],$$

where

$$X^B(\rho^{AB}) = \{ \{ \lambda_i^{1,B} \}_{i=1}^{d_1^B}, \{ \lambda_i^{2,B} \}_{i=1}^{d_2^B}, \dots, \{ \lambda_i^{m,B} \}_{i=1}^{d_m^B} \}$$

and

$$Y^B(\rho^{AB}) = \{ \{ \text{nim}_{\eta_1}(\lambda_i^{1,B}) \}_{i=1}^{d_1^B}, \{ \text{nim}_{\eta_2}(\lambda_i^{2,B}) \}_{i=1}^{d_2^B}, \dots, \{ \text{nim}_{\eta_m}(\lambda_i^{m,B}) \}_{i=1}^{d_m^B} \}$$

with $\lambda_i^{j,B}$ the eigenvalues of the reduced matrix $\text{Tr}_A \tilde{\rho}^{\eta_j}$ and d_j^B the rank of $\text{Tr}_A \tilde{\rho}^{\eta_j}$.

The new measure of nonclassical correlation is defined by their average

$$M(\rho^{AB}) = \frac{1}{2} [M^A(\rho^{AB}) + M^B(\rho^{AB})].$$

Let us list the basic properties of this measure.

Proposition 3 *The measure M vanishes for any (bipartite) state having a product eigenbasis.*

Proof By Proposition 1, for any state having a product eigenbasis, $\lambda_i^{j,A} = \text{nim}_{\eta_j}(\lambda_i^{j,A})$ and $\lambda_i^{j,B} = \text{nim}_{\eta_j}(\lambda_i^{j,B})$ hold. This proves the proposition. \square

Proposition 4 *For a state $\rho^{AA'}$ of an $N \times N$ -dimensional bipartite system, $M(\rho^{AA'}) \leq \log_2 N$.*

Proof Under this condition, we have $\max_j d_j^{A(B)} \leq N$. In addition, generally $\text{nim}_y(x) \leq 2x$ for $0 \leq x, y$. Thus by Proposition 2, this proposition holds. \square

There are some properties of the measure for the pure states.

Proposition 5 (i) *The measure M vanishes for a pure state if and only if the state is a product state. Thus M never vanishes for an entangled pure state.*

(ii) *For a pure state $|\phi\rangle\langle\phi|$, M is bounded above by the entropy of entanglement, $S_{\text{vN}}(\text{Tr}_B |\phi\rangle\langle\phi|)$, where $S_{\text{vN}}(\sigma) = -\sum_{k=1}^d \lambda_k \log_2 \lambda_k$ is the von Neumann entropy for a general d -dimensional density matrix σ with the eigenvalues λ_k .*

(iii) *The measure M reduces to the entropy of entanglement for a pure state with the Schmidt coefficients $\leq 1/\sqrt{2}$.*

Proof (i) Consider the Schmidt decomposition of a pure state vector $|\phi\rangle$, namely, $|\phi\rangle = \sum_{k=1}^R \sqrt{c_k} |a_k\rangle |b_k\rangle$ with $\sqrt{c_k}$'s the Schmidt coefficients, R the Schmidt rank, and $|a_k\rangle$'s and $|b_k\rangle$'s being the eigenvectors of the reduced density matrix $\text{Tr}_B |\phi\rangle\langle\phi|$ and those of $\text{Tr}_A |\phi\rangle\langle\phi|$, respectively, corresponding to the nonzero eigenvalues c_k (c_k 's are common for the reduced density matrices). We have only to consider the value 1 for η in the calculation of M for a pure state. Then, obviously, M vanishes if

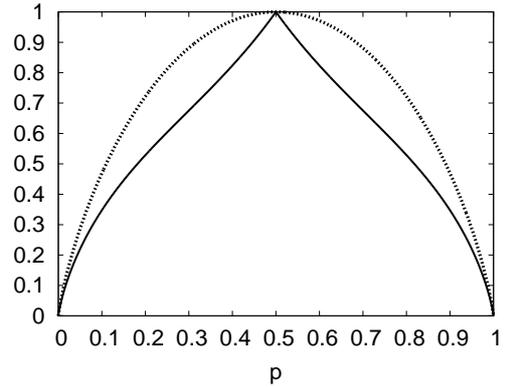


Fig. 1 Plots of $M(|\phi_p\rangle\langle\phi_p|)$ [the lower (solid) curve] and the entropy of entanglement $S_{\text{vN}}(\text{Tr}_B |\phi_p\rangle\langle\phi_p|)$ [the upper (dotted) curve] as functions of p .

and only if the Schmidt rank is 1.

(ii) Consider the real numbers $0 \leq x, y \leq 1$. The relation $\text{nim}_y(x) \leq 2x$ holds because 0 is closer or equally close to x than $2x$. Therefore, $-|x - \text{nim}_y(x)| \log_2 x \leq -x \log_2 x$. In addition, for a general pure state $|\phi\rangle\langle\phi|$, we have only to consider the value 1 for η . Therefore, $M(|\phi\rangle\langle\phi|) = M^A(|\phi\rangle\langle\phi|) = M^B(|\phi\rangle\langle\phi|) \leq S_{\text{vN}}(\text{Tr}_B |\phi\rangle\langle\phi|)$.

(iii) Recall again that, for a pure state, we have only to consider the value 1 for η . In case we have a pure state vector $|\phi'\rangle$ whose Schmidt coefficients are $\leq 1/\sqrt{2}$, the definition obviously reduces to $M(|\phi'\rangle\langle\phi'|) = M^A(|\phi'\rangle\langle\phi'|) = M^B(|\phi'\rangle\langle\phi'|) = S_{\text{vN}}(\text{Tr}_B |\phi'\rangle\langle\phi'|)$. \square

For the typical two-qubit entangled state $|\phi_p\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$, $M(|\phi_p\rangle\langle\phi_p|)$ behaves as illustrated in Fig. 1. It should be noted that, for general states, M is not an entanglement measure but a measure of nonclassical correlation. Because of the above properties, M can be seen as an entanglement measure in case the states are limited to pure states.

As important properties, for general states, the measure M satisfies local-unitary invariance and it takes its maximum value for the generalized Bell states as we will see in Section 3.4.

The improvement in complexity is significant as we have already mentioned: the measure is computed by using the eigenvalues of at most $2d^A d^B$ reduced matrices. The total complexity is dominated by the complexity of diagonalizing the original density matrix, $O(d^{A^3} d^{B^3})$, which is larger than the complexity of tracing out a subsystem for each truncated density matrix. In the exceptional cases where $d^A > d^{B^2}$ or $d^B > d^{A^2}$, the complexity of diagonalizing all the reduced matrices, $O[\max(d^{A^4} d^B, d^A d^{B^4})]$, becomes the largest cost. Thus the total complexity, namely, the number of the floating-

point operations in total to compute the measure, is

$$O[\max(d^A{}^3 d^B{}^3, d^A{}^4 d^B, d^A d^B{}^4)].$$

In the following examples, one may recognize the simpleness of the process to compute M .

Examples Consider the two-qubit state

$$\varsigma = \frac{1}{2}(|00\rangle\langle 00| + |1+\rangle\langle 1+|) \quad (2)$$

with $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. This has no product eigenbasis while it is separable. The nonzero eigenvalue of ς is $1/2$ with the multiplicity two. The eigenvalue of $\text{Tr}_B \varsigma$ is $1/2$ with the multiplicity two; this leads to $M^A(\varsigma) = \tilde{S}(\{\{1/2, 1/2\}, \{1/2, 1/2\}\}) = 0$. The eigenvalues of $\text{Tr}_A \varsigma$ are $(2 \pm \sqrt{2})/4$ for which the nearest integer multiples of $1/2$ are 0 and 1; this leads to $M^B(\varsigma) = \tilde{S}(\{\{(2 - \sqrt{2})/4, (2 + \sqrt{2})/4\}, \{0, 1\}\}) \simeq 0.439$. Therefore $M(\varsigma) = M^B(\varsigma)/2 \simeq 0.220$.

In view of the problem of detecting nonclassical correlation, it is easy for this example since one of the reduced density matrices is nondegenerate (see the discussion in Section 4). The next example is not an easy case in this sense.

Consider the state in a 4×4 dimensional system,

$$\zeta = \frac{1}{4}(|00\rangle\langle 00| + |2+\rangle\langle 2+| + |2+\rangle\langle 2+| + |33\rangle\langle 33|).$$

Let $U_\zeta = U_\zeta^A \otimes U_\zeta^B$ be some local unitary operation. Suppose that $\zeta' = U_\zeta \zeta U_\zeta^\dagger$ is given and M is computed for it. The global and local eigenvalues are unchanged but the matrix form may be complicated at a glance due to U_ζ . We have the same eigenvalues for $\text{Tr}_B \zeta'$ and $\text{Tr}_A \zeta'$. The eigenvalues of $\text{Tr}_B \zeta'$ are $1/4$ with multiplicity two and $(2 \pm \sqrt{2})/8$. This leads to $M(\zeta') = M^A(\zeta') = M^B(\zeta') = \tilde{S}(\{\{(2 - \sqrt{2})/8, (2 + \sqrt{2})/8, 1/4, 1/4\}, \{0, 1/2, 1/4, 1/4\}\}) \simeq 0.366$.

The examples we have seen are fixed states. In case there are parameters, it is rather complicated to write M in terms of them because it is influenced by the degeneracy of eigenvalues and it involves the function $\text{nim}_y(x)$. As M is tractable, one seldom has a motivation to decompose M analytically. One should keep M as a routine in a computational program, which might be a way to avoid the complication. This is in contrast to intractable measures, for which finding an analytical solution for a particular form of density matrices is highly motivated. As for quantum discord, which is in general intractable to compute, analytical solutions are known for particular forms of density matrices. Luo [14] gave a general solution for the two-qubit density matrices in the form $\kappa = (I \otimes I + \sum_{j=x,y,z} c_j \sigma_j \otimes \sigma_j)/4$,

where σ_j 's are Pauli matrices and c_j 's are real parameters. Ali *et al.* [1] very recently gave a general solution for the two-qubit density matrices with only diagonal and anti-diagonal elements.

The density matrix κ is special in the sense that its eigenvectors are the Bell basis vectors. With this fact and its eigenvalues $(1 - c_x - c_y - c_z)/4$, $(1 - c_x + c_y + c_z)/4$, $(1 + c_x - c_y + c_z)/4$, and $(1 + c_x + c_y - c_z)/4$, it is straightforward albeit complicated to write $M(\kappa)$ in terms of the parameters c_j .

3.2 Imperfectness in the detection range

We have achieved a reduction in the complexity by introducing the measure M . We have shown that several properties are satisfied by M in Propositions 3, 4, and 5. As a desirable property for a measure of nonclassical correlation, it never vanishes for entangled pure states. However, our main concern is to use M for general states which are mostly mixed states. As a matter of fact, the measure does not have a perfect detection range as is expected from the fact that it does not test all the local bases unlike other expensive measures like quantum discord (with both-side test) or zero-way quantum deficit. For example, the measure M vanishes for the two-qubit state

$$\sigma = \frac{1}{6}(|00\rangle\langle 00| + 2|01\rangle\langle 01| + 3|1+\rangle\langle 1+|). \quad (3)$$

Nevertheless, this state has no product eigenbasis because $|0\rangle\langle 0|$ and $|+\rangle\langle +|$ cannot be diagonalized in the same basis.

Consequently, what one can claim is that a state for which the measure does not vanish is in the outside of the set B of the states for which the measure M vanishes, and hence in the outside of the set C of the states having a product eigenbasis, as illustrated in Fig. 2. Note that the set B includes some inseparable states while C does not. For example, the measure M vanishes for the state τ , which is represented as a density matrix acting on the (3×3) -dimensional Hilbert space:

$$\tau = \frac{1}{3}(|\phi\rangle^{\text{AB}}\langle \phi| + |\psi\rangle^{\text{AB}}\langle \psi| + |\xi\rangle^{\text{AB}}\langle \xi|) \quad (4)$$

with

$$\begin{aligned} |\phi\rangle^{\text{AB}} &= \frac{|0\rangle^{\text{A}}|1\rangle^{\text{B}} + |1\rangle^{\text{A}}|0\rangle^{\text{B}}}{\sqrt{2}}, \\ |\psi\rangle^{\text{AB}} &= \frac{|1\rangle^{\text{A}}|2\rangle^{\text{B}} + |2\rangle^{\text{A}}|1\rangle^{\text{B}}}{\sqrt{2}}, \\ |\xi\rangle^{\text{AB}} &= \frac{|2\rangle^{\text{A}}|0\rangle^{\text{B}} + |0\rangle^{\text{A}}|2\rangle^{\text{B}}}{\sqrt{2}}. \end{aligned}$$

This is because the nonzero eigenvalue of τ is $1/3$ with the multiplicity three and the eigenvalue of $\text{Tr}_B \tau =$

$\text{Tr}_A \tau$ is also $1/3$ with the multiplicity three. The state τ is inseparable because its partial transposition $(I \otimes T)\tau$ (here, T is the transposition map) has the eigenvalues $-1/6$, $1/6$, and $1/3$ with multiplicities two, six, and one, respectively. Thus it has been found that M cannot detect nonclassical correlation of this negative-partial-transpose (NPT) state. This example however does not

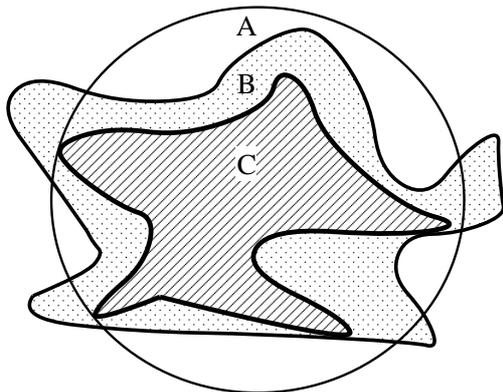


Fig. 2 The hierarchy of the quantum states. A: The convex set of the separable states. B: The nonconvex set of the states for which the measure M vanishes. C: the nonconvex set of the states having a product eigenbasis ($C \subset A \cap B$). Internal geometric structures of each set are not depicted, which should wait for future investigations.

weaken the measure M very much in light of the benefit of computational tractability. In addition, regarding M as a detection tool, there is a way to extend the detection range simply, which we will discuss later in Section 4.

3.3 Relative detection ability

One can compare the measures M and G in their detection ability using the state σ given in Eq. (3) and another particular state for a couple of qubits.

The measure M vanishes for σ while G does not vanish. We have $G(\sigma) = H(1/3) - H[(6 - \sqrt{10})/12] \simeq 0.129$ where $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function.

On the other hand, M does not vanish for the state $\sigma' = |\phi\rangle\langle\phi|/2 + (|01\rangle\langle 01| + |10\rangle\langle 10|)/4$ with $|\phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, while G vanishes. We have $M(\sigma') = M^A(\sigma') = M^B(\sigma') = \tilde{S}(\{\{1/4, 1/4\}, \{1/4, 1/4\}\}, \{\{0, 0\}, \{1/4, 1/4\}\}) = 1/2$.

Therefore, M is neither stronger nor weaker than G in detecting nonclassical correlations.

3.4 Axioms satisfied by the measure

As we have described in Section 1, there are axioms that are desired to be satisfied by a measure of nonclassical correlation. Here, we examine the measure M in this regard.

First, it is easy to verify that M vanishes for any state having a product eigenbasis as we have already shown in Proposition 3.

Second, we show that M is invariant under the local unitary operations $\mathcal{U}^A \otimes \mathcal{U}^B : \rho \mapsto (U^A \otimes U^B)\rho(U^A \otimes U^B)^\dagger$. This is easily verified from the fact that (i) $\text{Tr}_B \tilde{\rho}^\eta$ and $\text{Tr}_B(\mathcal{U}^A \otimes \mathcal{U}^B)(\tilde{\rho}^\eta)$ have the same eigenvalues, and similarly, (ii) $\text{Tr}_A \tilde{\rho}^\eta$ and $\text{Tr}_A(\mathcal{U}^A \otimes \mathcal{U}^B)(\tilde{\rho}^\eta)$ have the same eigenvalues.

Third, it is desirable that a measure takes its maximum value for certain maximally entangled states as we have mentioned in Section 1. The dimensions d^A and d^B of subsystems A and B are set to N to consider the maximum value. Then, by Proposition 4, $M(\rho^{AB}) \leq \log_2 N$ holds. It is now easy to show that M takes its maximum value for the generalized Bell state $|\Psi_{\text{Bell}}\rangle\langle\Psi_{\text{Bell}}|$ with

$$|\Psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N |i\rangle^A |i\rangle^B.$$

This is a straightforward calculation.

It has been shown that the basic axioms are satisfied. Let us now mention that, in addition to these axioms, some mathematical properties, such as convexity and additivity properties are often investigated for a measure. For a measure in the present context, convexity should not hold because the set of the classically correlated states is a nonconvex set. The additivity properties are next investigated.

3.5 Investigation on additivity properties

The measure M has neither the subadditivity property nor the superadditivity property as we prove below. It is also shown not to be weakly additive.

Here, let us denote a splitting of a system considered for the measure M by a subscript $\cdot|\cdot$.

Proposition 6 *Neither the subadditivity*

$$M_{AC|BD}(\rho^{AB} \otimes \sigma^{CD}) \leq M_{A|B}(\rho^{AB}) + M_{C|D}(\sigma^{CD})$$

nor the superadditivity

$$M_{AC|BD}(\rho^{AB} \otimes \sigma^{CD}) \geq M_{A|B}(\rho^{AB}) + M_{C|D}(\sigma^{CD})$$

holds. In addition, the weak additivity

$$M_{AA\dots|BB\dots}(\rho^{AB \otimes m}) = m M_{A|B}(\rho^{AB})$$

does not hold.

Proof (i) First we prove that subadditivity does not hold. Consider the state $\xi = \sigma^{AB} \otimes \sigma^{CD}$ with the state σ defined by Eq. (3). We have already found $M_{A|B}(\sigma^{AB}) = 0$. Now we calculate $M_{AC|BD}(\xi)$. The state ξ has the eigenvalues $e = 0, 1/36, 1/18, 1/12, 1/9, 1/6$, and $1/4$. Let us write the truncated density matrix down to the e -eigenspace as $\tilde{\xi}^e$. We have $\tilde{\xi}^{1/12} = (|001+\rangle\langle 001+| + |1+00\rangle\langle 1+00|)/12$. This leads to $\text{Tr}_{AC}\tilde{\xi}^{1/12} = (|0+\rangle\langle 0+| + |+0\rangle\langle +0|)/12$ whose eigenvalues are $1/24, 1/8$, and 0 with multiplicity two. Similarly, $\tilde{\xi}^{1/6} = (|011+\rangle\langle 011+| + |1+01\rangle\langle 1+01|)/6$ and $\text{Tr}_{AC}\tilde{\xi}^{1/6}$ has the eigenvalues $1/12, 1/4$, and 0 with multiplicity two. For other $\tilde{\xi}^e$'s, $\text{Tr}_{AC}\tilde{\xi}^e$ has the eigenvalues equal to e . Therefore, $M_{AC|BD}^{\text{BD}}(\xi) =$

$\tilde{S}(\{\{1/24, 1/8\}, \{1/12, 1/4\}\}, \{\{0, 1/12\}, \{0, 1/6\}\}) \simeq 0.302$. In addition, it is easy to find $M_{AC|BD}^{\text{AC}}(\xi) = 0$ because $\text{Tr}_{BD}\tilde{\xi}^e$ has the eigenvalues equal to integer multiples of e for every e . Consequently, $M_{AC|BD}(\xi) = 0.151$, which is larger than $M_{A|B}(\sigma^{AB}) + M_{C|D}(\sigma^{CD}) = 0$. This is a counterexample to subadditivity.

(ii) Second we prove that superadditivity does not hold. Consider the state $\xi' = \sigma''^{AB} \otimes \sigma''^{CD}$ with $\sigma'' = (1/4)|\phi\rangle\langle\phi| + (3/8)(|01\rangle\langle 01| + |10\rangle\langle 10|)$ where $|\phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Let us first calculate $M_{A|B}(\sigma''^{AB})$. We have $\tilde{\sigma}''^{1/4} = (1/4)|\phi\rangle\langle\phi|$ and $\tilde{\sigma}''^{3/8} = (3/8)(|01\rangle\langle 01| + |10\rangle\langle 10|)$. The eigenvalues of $\text{Tr}_B\tilde{\sigma}''^{1/4}$ are $1/8$ with multiplicity two and those of $\text{Tr}_B\tilde{\sigma}''^{3/8}$ are $3/8$ with multiplicity two. Because of the symmetry of the state, this leads to $M_{A|B}(\sigma''^{AB}) = M_{A|B}^A(\sigma''^{AB}) = M_{A|B}^B(\sigma''^{AB}) = -2 \times (1/8) \log_2[(1/8)/(1/4)] = 1/4$. Let us second calculate $M_{AC|BD}(\xi')$. We have $\tilde{\xi}'^{1/16} = (1/16)(|\phi\rangle\langle\phi|)^{\otimes 2}$, $\tilde{\xi}'^{3/32} = (3/32)[|\phi\rangle\langle\phi| \otimes (|01\rangle\langle 01| + |10\rangle\langle 10|) + (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes |\phi\rangle\langle\phi|]$, and $\tilde{\xi}'^{9/64} = (9/64)(|01\rangle\langle 01| + |10\rangle\langle 10|)^{\otimes 2}$. The eigenvalues of $\text{Tr}_{BD}\tilde{\xi}'^e$ are $1/64$ for $e = 1/16, 3/16$ for $e = 3/32$, and $9/64$ for $e = 9/64$, with multiplicity four for each e . Because of the symmetry of the state, this leads to $M_{AC|BD}(\xi') = M_{AC|BD}^{\text{AC}}(\xi') = M_{AC|BD}^{\text{BD}}(\xi') = -4 \times (1/64) \log_2[(1/64)/(1/16)] = 1/8$, which is less than $M_{A|B}(\sigma''^{AB}) + M_{C|D}(\sigma''^{CD}) = 1/2$. This is a counterexample to superadditivity.

(iii) The above counterexamples shown in (i) and (ii) are also counterexamples to weak additivity. \square

4 Discussions

The main aim of introducing the measure M has been the computational tractability. The problem of quantification is harder than detection as certain axioms should be satisfied as we discuss later in this section. In view of a detection problem rather than quantification, indeed, it is known to be possible to decide

whether a given density matrix has a product eigenbasis within polynomial time for some special cases: (i) In case there are only nondegenerate eigenvalues for a given density matrix, it has a product eigenbasis if and only if the Schmidt ranks of the eigenvectors $|v\rangle^{AB}$ are all one, i.e., $|v\rangle^{AB} = |a\rangle^A|b\rangle^B$ and, for each subsystem, the Schmidt vectors ($|a\rangle$'s for A and $|b\rangle$'s for B) are either orthogonal or equal to each other, neglecting the global phase difference. (ii) In case the reduced density matrices of a given density matrix have no degeneracy, the local eigenbases are uniquely determined neglecting the global phase factors. Then, the density matrix has a product eigenbasis if and only if the product of the local eigenbases is the eigenbasis of the total system. (iii) The case only one of the reduced density matrices, say, one for the subsystem B, is nondegenerate is also an easy case. In this case, for the reduced density matrix, we have the eigenvectors $|v_j\rangle^B$ that are unique neglecting their global phase factors. Then, first we test if the density matrix ρ^{AB} is equal to $\sum_j |v_j\rangle^B \langle v_j|^B \rho^{AB} |v_j\rangle^B \otimes |v_j\rangle^B \langle v_j|^B$. If this is false, then ρ^{AB} has no product eigenbasis. If true, then ρ^{AB} has a product eigenbasis if and only if $|v_j\rangle^B \langle v_j|^B$'s are commutative to each other. (iv) There is a property very recently mentioned [7]: the commutation relations $[\rho^{AB}, \text{Tr}_B \rho^{AB} \otimes I^B] = [\rho^{AB}, I^A \otimes \text{Tr}_A \rho^{AB}] = 0$ hold if ρ^{AB} has a product eigenbasis. Any state that does not satisfy these relations has no product eigenbasis while the converse does not hold in general. For example, this detection method does not work for the generalized Bell states.

Regarding measures as detection tools, combinations of imperfect measures, that are neither stronger nor weaker to each other, makes a tool with a larger detection range. One may utilize the measures M , G , and some entanglement measure to produce a detection tool easily, which is nonvanishing for entangled states detectable by the entanglement measure.

Some existing measures, namely, the measurement-induced disturbance [8, 15] and its variants, are computable within polynomial time in case the dephasing basis is uniquely determined, namely in the case (ii), while they are not otherwise. The measure M is, in contrast, always computable within polynomial time.

As is expected for a tractable measure, M is imperfect in its detection range. The measure has been found to vanish not only for the state (3) but for the NPT state (4). It has been controversial as to which extent computational tractability has the higher priority, facing the trade-off with the detection range. As for entanglement measures, negativity and the logarithmic negativity [21] are commonly used although they are not perfect in the detection range. The basic axiom of

monotonicity has been a ground of argument for entanglement measures. Thus there must be a demand of measures of nonclassical correlation whose detection ranges may be imperfect as long as certain axioms are satisfied. As we have described in Section 1, there are possible axioms for nonclassical correlation measures: (i) a measure should vanish for any classically correlated state, (ii) a measure should be invariant under local unitary transformations, and (iii) a measure should have the maximum value for certain maximally entangled states. These are all satisfied by M as described in Section 3.4.

For a more quantitative discussion, one may prefer to have the volume of the detection range not very smaller than the volume of the set of the nonclassically correlated states. This has not been studied and it is an open problem in the present stage. We need to start from understanding the geometric structure of the sets of the classically/nonclassically correlated states. Quite recently, a very simplified discussion on the geometric structure is reported in Ref. [7].

As an additional topic, there will be a particular behavior when an evolution process is evaluated by M . The value of M has an abrupt change when multiple eigenvalues of the density matrix gradually change and cross with each other. At the crossing point, the value of M most probably has discontinuity. This is because its component values are computed inside each eigenspace. Thus M has an abrupt change when multiple eigenspaces are admixed in a process. This behavior corresponds to a physical event of eigenvalue crossing, but can be seen as an unstable behavior. It is controversial if this is counted as a drawback as a measure.

Let us turn into a rather conceptual problem. It is often of general interest to find a physical interpretation of a measure to justify its quantification. Here we suggest an interpretation of the definition of the measure M . It involves a transmission of a system from a dealer to players and guess works by the players on local information.

Consider a thermal state $\rho_{\text{th}} = e^{-\beta H}$ with a system Hamiltonian H and the reciprocal temperature β . It is possible to have a transmission line (namely, a waveguide or a resonator) in resonance with an eigenfrequency ν of H under the same temperature. An eigenvalue η_ν of ρ_{th} can be written as $\eta_\nu = e^{-\beta\nu}$. This transmission line acts as a channel that projects a density matrix of the system onto an eigenspace of H corresponding to the eigenfrequency ν . Let us mention that one need not to have a transmission line directly acting on the system in case this is difficult for some physical setup. Consider the case where the system is a molecular spin system placed in a magnetic resonance spec-

trometer, for example. One may attach band-pass filters to its probe circuits so that only the frequency band around the signal frequency corresponding to ν is captured. This virtually behaves similarly as the channel.

Suppose a dealer has a bipartite system AB in the thermal state $\rho^{\text{AB}} = \rho_{\text{th}}$ and sends it through the channel. The surviving state in the channel is $\tilde{\rho}^{\eta_\nu}$ (in the notation used in Definition 1). At the other side of the channel, Alice takes the subsystem A and Bob takes the subsystem B. These players try to guess their local eigenvalues of the reduced matrices $\text{Tr}_B \tilde{\rho}^{\eta_\nu}$ and $\text{Tr}_A \tilde{\rho}^{\eta_\nu}$, respectively. A dealer may answer to the query as to whether or not a value is equal to a local eigenvalue of the subsystem indicated in the query. Unlike the scenario for the measure G described in Section 2, suppose players want to use a linear-time strategy. A natural strategy is to guess a local eigenvalue as an integer multiple of η_ν . They can easily find the value of $\eta_\nu = e^{-\beta\nu}$ from the resonance frequency ν and the temperature of the transmission line. Because a local eigenvalue does not exceed $\eta_\nu d^{\eta_\nu}$, the maximum number of queries they can try is $1 + d^{\eta_\nu}$ for one eigenvalue. The quantity $M^{\text{A(B)}}(\rho^{\text{AB}})$ consists of the components

$$- \sum_{i=1}^{d_\nu^{\text{A(B)}}} \left| \lambda_i^{\nu, \text{A(B)}} - \text{nim}_{\eta_\nu}(\lambda_i^{\nu, \text{A(B)}}) \right| \log_2 \frac{\lambda_i^{\nu, \text{A(B)}}}{\eta_\nu d^{\eta_\nu}}$$

found in Eq. (1) (we read j therein as ν here). Each component is a discrepancy between the set of the true local eigenvalues and that of their nearest guessed values for Alice (Bob) in this strategy for ν with the reduction factor $\log_2(\eta_\nu d^{\eta_\nu}) < 0$ (recall that $\text{Tr} \tilde{\rho}^{\eta_\nu} = \eta_\nu d^{\eta_\nu}$) corresponding to players' knowledge on which channel is used.

In this way, the definition of M has been interpreted in view of a physical process. The interpretation is, however, not applicable for general states other than the thermal state. Although the system Hamiltonian is H , the quantum state ρ^{AB} can be changed from the thermal state by a unitary operation. Then the correspondence between the spectrum of ρ^{AB} and that of H is broken. It is to be hoped that a different protocol-based interpretation will be found for M for general states. For the time being, the validity of M as a measure for general states relies on those axioms that we have discussed.

5 Summary

We have proposed an unconventional measure of nonclassical correlation by using truncations of a density matrix, on the basis of Proposition 1. The mathematical properties of the measure have been investigated. It

is invariant under local unitary operations and it takes the maximum value for the generalized Bell states while it is imperfect in the detection range and it has no additivity property. It is usable for a practical evaluation of quantum states because it is calculated within polynomial time in the dimension of a density matrix.

Acknowledgements The authors are thankful to Karol Życzkowski for helpful comments on a geometric structure related to Fig. 2. They are also thankful to Todd Brun for a comment on the definition of the measure M . A.S. and M.N. are supported by the “Open Research Center” Project for Private Universities: matching fund subsidy from MEXT. R.R. is supported by the FIRST program of JSPS. A.S. is supported by the Grant-in-Aid for Scientific Research from JSPS (Grant No. 21800077). R.R. and M.N. have been supported by the Grants-in-Aid for Scientific Research from JSPS (Grant Nos. 1907329 and 19540422, respectively).

References

1. Ali, M., Rau, A.R.P., Alber, G.: Quantum discord for two-qubit x states. *Phys. Rev. A* **81**, 042105 (2010)
2. Bengtsson, I., Życzkowski, K.: *Geometry of Quantum States: An Introduction to Quantum Entanglement*. Cambridge University Press, Cambridge (2006)
3. Bennett, C.H., DiVincenzo, D.P., Fuchs, C.A., Mor, T., Rains, E., Shor, P.W., Smolin, J.A., Wootters, W.K.: Quantum nonlocality without entanglement. *Phys. Rev. A* **59**, 1070–1091 (1999)
4. Bylicka, B., Chruściński, D.: Witnessing quantum discord in $2 \times N$ systems. *Phys. Rev. A* **81**, 062102 (2010)
5. Datta, A., Gharibian, S.: Signatures of nonclassicality in mixed-state quantum computation. *Phys. Rev. A* **79**, 042325 (2009)
6. Davies, E.: Information and quantum measurement. *IEEE Trans. Inf. Theory* **24**, 596–599 (1978)
7. Ferraro, A., Aolita, L., Cavalcanti, D., Cucchietti, F.M., Acín, A.: Almost all quantum states have nonclassical correlations. *Phys. Rev. A* **81**, 052318 (2010)
8. Groisman, B., Kenigsberg, D., Mor, T.: “quantumness” versus “classicality” of quantum states. *arXiv:quant-ph/0703103* (2007)
9. Groisman, B., Popescu, S., Winter, A.: Quantum, classical, and total amount of correlations in a quantum state. *Phys. Rev. A* **72**, 032317 (2005)
10. Henderson, L., Vedral, V.: Classical, quantum and total correlations. *J. Phys. A: Math. Gen.* **34**, 6899–6905 (2001)
11. Horodecki, M., Horodecki, P., Horodecki, R.: Separability of mixed states: necessary and sufficient conditions. *Phys. Lett. A* **223**, 1–8 (1996)
12. Horodecki, M., Horodecki, P., Horodecki, R., Oppenheim, J., Sen(De), A., Sen, U., Synak-Radtke, B.: Local versus nonlocal information in quantum-information theory: Formalism and phenomena. *Phys. Rev. A* **71**, 062307 (2005)
13. Li, N., Luo, S.: Classical states versus separable states. *Phys. Rev. A* **78**, 024303 (2008)
14. Luo, S.: Quantum discord for two-qubit systems. *Phys. Rev. A* **77**, 042303 (2008)
15. Luo, S.: Using measurement-induced disturbance to characterize correlations as classical or quantum. *Phys. Rev. A* **77**, 022301 (2008)
16. Ollivier, H., Zurek, W.H.: Quantum discord: A measure of the quantumness of correlations. *Phys. Rev. Lett.* **88**, 017901 (2001)
17. Oppenheim, J., Horodecki, M., Horodecki, P., Horodecki, R.: Thermodynamical approach to quantifying quantum correlations. *Phys. Rev. Lett.* **89**, 180402 (2002)
18. Pankowski, L., Synak-Radtke, B.: Can quantum correlations be completely quantum? *J. Phys. A: Math. Theor.* **41**, 075308 (2008)
19. Peres, A.: Separability criterion for density matrices. *Phys. Rev. Lett.* **77**, 1413–1415 (1996)
20. Piani, M., Horodecki, P., Horodecki, R.: No-local-broadcasting theorem for multipartite quantum correlations. *Phys. Rev. Lett.* **100**, 090502 (2008)
21. Plenio, M.B.: Logarithmic negativity: A full entanglement monotone that is not convex. *Phys. Rev. Lett.* **95**, 090503 (2005)
22. Plenio, M.B., Virmani, S.: An introduction to entanglement measures. *Quantum Inf. Comput.* **Vol. 7**, 1–51 (2007)
23. Rahimi, R., SaiToh, A.: Single-experiment-detectable nonclassical correlation witness. *Phys. Rev. A* **82**, 022314 (2010)
24. SaiToh, A., Rahimi, R., Nakahara, M.: Evaluating measures of nonclassical correlation in a multipartite quantum system. *Int. J. Quant. Inf.* **6** (Supp. 1), 787–793 (2008)
25. SaiToh, A., Rahimi, R., Nakahara, M.: Mathematical framework for detection and quantification of nonclassical correlation. *arXiv:0802.2263* (quant-ph) (2008)
26. SaiToh, A., Rahimi, R., Nakahara, M.: Nonclassical correlation in a multipartite quantum system: Two measures and evaluation. *Phys. Rev. A* **77**, 052101 (2008)
27. Werner, R.F.: Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Phys. Rev. A* **40**, 4277–4281 (1989)