# Common entanglement witnesses and their characteristics 

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#### Abstract

We investigate the issue of finding common entanglement witness for certain class of states and extend this study to the case of Schmidt number witnesses. We also introduce the notion of common decomposable and non-decomposable witness operators which is specially useful for constructing a common witness where one of the entangled states is with a positive partial transpose. Our approach is illustrated with the help of suitable examples of qutrit systems.


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## 1 Introduction

Entanglement lies at the core of quantum information theory. Although entanglement is a vital resource for processes like teleportation, dense coding, quantum key distribution and quantum computation [1, 2,3,4], its detection is a hard task. For low dimensional $(2 \otimes 2$ and $2 \otimes 3)$ states there exists a simple necessary and sufficient condition for separability [56, which is based on the fact that separable states have a positive partial transpose (PPT). For higher dimensional systems all states with negative partial transpose (NPT) are entangled but there are entangled states which have a positive partial transpose. This paradoxical behaviour of quantum entanglement in higher dimensions makes it difficult to lay down a single necessary and sufficient condition for its detection.

A major breakthrough in this direction is provided in the form of entanglement witnesses (EW) [6, 7]. An outcome of the celebrated Hahn-Banach theorem in functional analysis, entanglement witnesses are

[^0]hermitian operators with at least one negative eigenvalue. Entanglement witness provides a necessary and sufficient condition to detect entanglement. More specifically a given state is entangled if and only if there is an EW that detects it [6]. Such a property makes EW an useful tool in the experimental detection of entanglement. Though it is difficult to construct an EW that detects an unknown entangled state, several methods have been suggested in the literature pertaining to several classes of entangled states 7819 . The notion of an entanglement witness was further extended to Schmidt number witness, which detects the Schmidt number of quantum states [1011].
An interesting line of study concerning entanglement witnesses is to find a common EW for different entangled states. It was proved by Wu and Guo [12] that for a given pair of entangled states $\rho_{1}$ and $\rho_{2}$, a common EW exists if and only if $\lambda \rho_{1}+(1-\lambda) \rho_{2}$ is an entangled state $\forall \lambda \in[0,1]$. They thus arrived at a sufficient condition for entanglement for pairs of entangled states. Construction of a common entanglement witness for two entangled states not only detects them but also any state which is a convex combination of the two. Thereby one can detect a large class of entangled states if one is able to find a common EW satisfying the above criterion.
In the present work our motivation is to propose some methods to construct common EW for certain classes of states making use of the above condition of existence. We first propose some characteristics of common Schmidt number witnesses based on the analysis of common entanglement witnesses, providing suitable examples for our propositions. We then suggest schemes for finding common EW for various categories of states based on their spectral characteristics. The distinction between a common decomposable witness operator and a non-decomposable one is of relevance in the probe for finding common EW. A decomposable operator is unable to detect a PPTES (positive partially transposed entangled state), whereas a non-decomposable witness can successfully detect a PPTES. This distinction propagates to a common witness. Precisely, if one of the entangled states in a convex combination is a PPTES, then the common witness is non-decomposable. Our analysis makes use of some decomposable and nondecomposable witnesses including one which we had proposed earlier [14]. We illustrate our results through various appropriate examples from qutrit systems.
The paper is organized as follows. We begin with a statement of certain relevant definitions and results in section 2 , which are useful for the later analysis. In section 3 , we propose and study some features of common Schmidt number witnesses. Next, in section 4 we suggest methods to detect a combined pair of entangled states and construct the common EW for them. We then provide explicit examples demonstrating our methods for finding common entanglement witnesses in section 5 . In section 6 , we distinguish between a common decomposable and a non-decomposable witness operator citing examples. We conclude with a brief summary of our results in section 7 .

## 2 Prerequisites

We begin with a brief summary of a few useful definitions and results.
Definition-1: The kernel of a given density matrix $\rho \in B\left(H_{A} \otimes H_{B}\right)$ is defined as

$$
\begin{equation*}
\operatorname{ker}(\rho)=\left\{|x\rangle \in H_{A}: \rho|x\rangle=0\right\} \tag{1}
\end{equation*}
$$

Definition-2: A PPT entangled state $\delta$ is called an edge state if for any $\varepsilon>0$ and any product vector $|e, f\rangle, \delta^{\prime}=\delta-\varepsilon|e, f\rangle\langle e, f|$ is not a PPT state [8].

Definition-3: A hermitian operator $W$ is said to be an entanglement witness operator iff

> (i) $\operatorname{Tr}(W \sigma) \geq 0 \quad \forall$ separable state $\sigma$ and
> (ii) $\operatorname{Tr}(W \rho)<0$, for at least one entangled state $\rho$.

Definition-4: A witness operator is said to be decomposable if it can be expressed in the form

$$
\begin{equation*}
D=P+Q^{T_{A}} \tag{3}
\end{equation*}
$$

where $P$ and $Q$ are positive semi-definite operators. Non-decomposable operators are those which cannot be written as in (3).
Result-1: A non-decomposable witness that can detect an edge state $\delta$ is of the form $P+Q^{T_{A}}-\varepsilon I$, where $P$ is a projector on $\operatorname{ker}(\delta)$ and $Q$ a projector on $\operatorname{ker}\left(\delta^{T_{A}}\right)$ and $0<\varepsilon \leq \varepsilon_{0}=i n f_{|e, f\rangle}\langle e, f| P+Q^{T_{A}}|e, f\rangle(|e, f\rangle$ is a product vector) [8].

Result-2: Another non-decomposable witness operator can be expressed in the form as 14:

$$
\begin{equation*}
W_{1}=Q^{T_{A}}-k(I-P), \tag{4}
\end{equation*}
$$

where $0<k \leq k_{0}$ and

$$
\begin{equation*}
k_{0}=\min \frac{\operatorname{Tr}\left(Q^{T_{A}} \sigma\right)}{\operatorname{Tr}((I-P) \sigma)} \tag{5}
\end{equation*}
$$

The minimum is taken over all separable states $\sigma$. An extension of these in tripartite systems is the witness

$$
\begin{equation*}
W_{t r i}=Q^{T_{X}}-k_{0}(I-P), \quad X=1,2,3 \tag{6}
\end{equation*}
$$

$P=\operatorname{Projector}$ on $\operatorname{Ker}\left(\delta_{t r i}\right)$ and $Q=\operatorname{Projector}$ on $\operatorname{Ker}\left(\delta_{t r i}^{T X}\right)$, where $T_{X}$ denotes the transpose taken with respect to any one of the subsystems. As before, we define

$$
\begin{equation*}
k_{0}=\min \frac{\operatorname{Tr}\left(Q^{T_{X}} \sigma\right)}{\operatorname{Tr}((I-P) \sigma)} \tag{7}
\end{equation*}
$$

where the minimum is taken over all separable states $\sigma$.

Result-3: Given a state $\rho$ whose partial transposition is negative, then one witness that can detect $\rho$ is

$$
\begin{equation*}
W=\left(\left|e_{-}\right\rangle\left\langle e_{-}\right|\right)^{T_{A}} \tag{8}
\end{equation*}
$$

where $\left|e_{-}\right\rangle$is an eigenvector corresponding to a negative eigenvalue of $\rho^{T_{A}}$.
Result-4: For a given pair of entangled states $\rho_{1}$ and $\rho_{2}$ an EW, $W^{\prime}$ common to both of them exists if and only if for $0 \leq \lambda \leq 1, \lambda \rho_{1}+(1-\lambda) \rho_{2}$ is an entangled state [12].

## 3 Common Schmidt number Witness

Consider the space $H^{N} \otimes H^{M}$, with $N<M$. Define $S_{k}$ to be the set of states whose Schmidt number is $\leq k$. Thus, $S_{1}$ is the set of separable states and the different states share the relation $S_{1} \subset S_{2} \subset S_{3} \ldots \subset$ $S_{k} . . \subset S_{N}$ and are convex 1011 .
A $k$ Schmidt witness $(k S W), W^{S}$ is defined as [10]

$$
\begin{gather*}
\operatorname{Tr}\left(W^{S} \sigma\right) \geq 0, \quad \forall \sigma \in S_{k-1}  \tag{9}\\
\operatorname{Tr}\left(W^{S} \rho\right)<0 \text { for at least one } \rho \in S_{k} \tag{10}
\end{gather*}
$$

A well-known example of a $k S W$ is $I-\frac{m}{k-1} P[10]$ where $m$ and $k$ respectively denote the dimension and Schmidt number and $P$ is a projector on $\frac{1}{\sqrt{m}} \Sigma_{i=0}^{m-1}|i i\rangle$.
Proposition-I: Suppose $\rho_{1}$ and $\rho_{2}$ are Schmidt number $k$ states. If there exists a common $k \mathrm{SW}$ for $\rho_{1}$ and $\rho_{2}$, then the Schmidt number of their convex combination will also be $k$. In other words the Schmidt number of $\lambda \rho_{1}+(1-\lambda) \rho_{2}$ is also $k(\lambda \in[0,1])$.
Proof: Since $\rho_{1}, \rho_{2}$ are in $S_{k}$ and $S_{k}$ is convex, $\lambda \rho_{1}+(1-\lambda) \rho_{2}$ cannot have a Schmidt number $>k$. Now, let $W^{S}$ be the common $k S W$ for $\rho_{1}$ and $\rho_{2}$. As a result

$$
\begin{equation*}
\operatorname{Tr}\left(W^{S}\left(\lambda \rho_{1}+(1-\lambda) \rho_{2}\right)\right)=\lambda \operatorname{Tr}\left(W^{S} \rho_{1}\right)+(1-\lambda) \operatorname{Tr}\left(W^{S} \rho_{2}\right)<0 \tag{11}
\end{equation*}
$$

since $\operatorname{Tr}\left(W^{S} \rho_{1}\right)<0, \operatorname{Tr}\left(W^{S} \rho_{2}\right)<0$. Thus the Schmidt number of $\lambda \rho_{1}+(1-\lambda) \rho_{2}$ is also $k$.

Proposition-II: Suppose $\delta_{1}$ and $\delta_{2}$ are two states with Schmidt number ( SN ) $k_{1}$ and $k_{2}$ respectively where $k_{1}>k_{2}$. Then a common witness $W_{k}$, if it exists, will be of class $k$, where $k=\min \left(k_{1}, k_{2}\right)$.
Proof: It follows from the definition of Schmidt number witness that there exists a $k_{1} S W, W_{k_{1}}$ for which $\operatorname{Tr}\left(W_{k_{1}} \delta_{1}\right)<0$, but $\operatorname{Tr}\left(W_{k_{1}} \delta_{2}\right) \geq 0$. Therefore a common witness if it exists should be of class $k$ where $k=\min \left(k_{1}, k_{2}\right)$.

## Example-I: Convex combination of two pure SN 3 states

Consider the states $\left|\Phi_{1}\right\rangle=a|00\rangle+b|11\rangle+\sqrt{1-a^{2}-b^{2}}|22\rangle$ and $\left|\Phi_{2}\right\rangle=x|00\rangle+y|11\rangle+\sqrt{1-x^{2}-y^{2}}|22\rangle$. A $3 S W$ of the form $W^{S 3}=I-\frac{3}{2} P$ detects both states for many ranges of $a, b, c, x, y, z$ (one such range
is $0.25 \leq a \leq 0.65,0.25 \leq b \leq 0.65,0.25 \leq x \leq 0.65,0.25 \leq y \leq 0.65)$. Therefore, for those ranges, $W^{S 3}$ is a common witness for the states $\left|\Phi_{1}\right\rangle$ and $\left|\Phi_{2}\right\rangle$ and thus their convex combination will have SN 3 . ( $P$ is a projector on $\frac{1}{\sqrt{3}} \Sigma_{i=0}^{2}|i i\rangle$ )

## Example-II: Convex combination of a pure SN 3 state and a pure SN 2 state

Consider now the state $\left|\Phi_{1}\right\rangle=a|00\rangle+b|11\rangle+\sqrt{1-a^{2}-b^{2}}|22\rangle$ and $|\chi\rangle=t|00\rangle+\sqrt{1-t^{2}}|11\rangle$. Here a $2 S W$ of the form $W^{S 2}=I-3 P$ detects both of them whereas the previous $3 S W$ fails to qualify as a common witness.

## Example-III: Convex combination of a mixed state and a pure SN 2 state

Consider the two-qutrit isotropic state $\Omega=\alpha P+\frac{1-\alpha}{9} I$ with $\left(-\frac{1}{8} \leq \alpha \leq 1\right)$. The $2 S W$, $W^{S 2}$ detects it $\forall 1 \geq \alpha>\frac{1}{4}$, which is exactly the range for which the isotropic state is entangled. As a result, the $2 S W$ detects $\lambda \Omega+(1-\lambda)|\chi\rangle\langle\chi|(\lambda \in[0,1])$.

## 4 Methods to construct common entanglement witness

Case-I: Let us consider that the two states described by the density operators $\rho_{1}$ and $\rho_{2}$ be negative partial transpose (NPT) states. Let us further assume that the two sets $S_{1}$ and $S_{2}$ consist of the set of all eigenvectors of $\rho_{1}^{T_{A}}$ and $\rho_{2}^{T_{A}}$ corresponding to their negative eigenvalues. In set builder notation, $S_{1}$ and $S_{2}$ can be expressed as $S_{1}=\left\{|x\rangle: \rho_{1}^{T_{A}}|x\rangle=\lambda_{-}|x\rangle, \lambda_{-}\right.$is a negative eigenvalue of $\left.\rho_{1}^{T_{A}}\right\}$ and $S_{2}=\left\{|y\rangle: \rho_{2}^{T_{A}}|y\rangle=\alpha_{-}|y\rangle, \alpha_{-}\right.$is a negative eigenvalue of $\left.\rho_{2}^{T_{A}}\right\}$. Now we propose the following theorem: Theorem 1: If $S_{1} \cap S_{2} \neq \phi$, then there exists a common witness detecting not only $\rho_{1}$ and $\rho_{2}$ both but also all the states lying on the straight line joining $\rho_{1}$ and $\rho_{2}$.
Proof: Let $S_{1} \cap S_{2} \neq \phi$. Then there exists a non-zero vector $|\eta\rangle \in S_{1} \cap S_{2}$. Let $W=(|\eta\rangle\langle\eta|)^{T_{A}}$. This gives

$$
\begin{equation*}
\operatorname{Tr}\left(W \rho_{1}\right)=\operatorname{Tr}\left((|\eta\rangle\langle\eta|)^{T_{A}} \rho_{1}\right)=\operatorname{Tr}\left((|\eta\rangle\langle\eta|) \rho_{1}^{T_{A}}\right)<0 \tag{12}
\end{equation*}
$$

With similar justifications,

$$
\begin{equation*}
\operatorname{Tr}\left(W \rho_{2}\right)<0 \tag{13}
\end{equation*}
$$

If now we consider $\rho=\lambda \rho_{1}+(1-\lambda) \rho_{2}, \lambda \in[0,1]$, then $\operatorname{Tr}(W \rho)<0$. Hence the theorem.
Case-II: Let $\delta_{1}$ and $\delta_{2}$ be two edge states. We know that a witness operator of the form $W_{\text {edge }}=$ $P+Q^{T_{A}}-\varepsilon I$ can detect an edge state $\delta$ if $P$ is a projector on $\operatorname{ker}(\delta)$ and $Q$ a projector on $\operatorname{ker}\left(\delta^{T_{A}}\right)$ and $0<\varepsilon \leq \varepsilon_{0}=\inf f_{|e, f\rangle}\langle e, f| P+Q^{T_{A}}|e, f\rangle$ where $|e, f\rangle$ is a product vector [8]. Thus we propose:
Theorem 2: $W_{\text {edge }}$ can detect both $\delta_{1}$ and $\delta_{2}$ if $\operatorname{dim}\left(\operatorname{ker}\left(\delta_{1}\right) \cap \operatorname{ker}\left(\delta_{2}\right)\right)>0 \operatorname{ordim}\left(\operatorname{ker}\left(\delta_{1}^{T_{A}}\right) \cap \operatorname{ker}\left(\delta_{2}^{T_{A}}\right)\right)>$ 0 .
Proof: Let $\operatorname{dim}\left(\operatorname{ker}\left(\delta_{1}\right) \cap \operatorname{ker}\left(\delta_{2}\right)\right)>0$, i.e., there exists at least one non-zero eigenvector $|a\rangle \in$
$\operatorname{ker}\left(\delta_{1}\right) \cap \operatorname{ker}\left(\delta_{2}\right)$. We assume $P=|a\rangle\langle a|$.
Further, let $\operatorname{dim}\left(\operatorname{ker}\left(\delta_{1}^{T_{A}}\right) \cap \operatorname{ker}\left(\delta_{2}^{T_{A}}\right)\right)>0$. We take $|b\rangle \in \operatorname{ker}\left(\delta_{1}^{T_{A}}\right) \cap \operatorname{ker}\left(\delta_{2}^{T_{A}}\right)$. Assume $Q=(|b\rangle\langle b|)^{T_{A}}$.
On taking $W_{\text {edge }}=P+Q^{T_{A}}-\varepsilon I$ with the above mentioned definition of $\varepsilon$, we obtain

$$
\begin{equation*}
\operatorname{Tr}\left(W_{\text {edge }} \delta_{1}\right)<0 \quad \text { and } \quad \operatorname{Tr}\left(W_{\text {edge }} \delta_{2}\right)<0 \tag{14}
\end{equation*}
$$

Consequently, $W_{\text {edge }}$ detects $\delta=\lambda \delta_{1}+(1-\lambda) \delta_{2}$ for $0 \leq \lambda \leq 1$ since

$$
\begin{equation*}
\operatorname{Tr}\left(W_{\text {edge }} \delta\right)=\operatorname{Tr}\left(W_{\text {edge }}\left(\lambda \delta_{1}+(1-\lambda) \delta_{2}\right)\right)<0 \tag{15}
\end{equation*}
$$

Thus $W_{\text {edge }}$ is a common witness for $\delta_{1}$ and $\delta_{2}$ and detects any convex combination of $\delta_{1}$ and $\delta_{2}$.
Case-III: Let $\delta_{t r i}^{1}$ and $\delta_{t r i}^{2}$ be two tripartite edge states. Using (6) we have the following theorem:
Theorem 3: The witness $W_{t r i}$ can detect both the tripartite edge states $\delta_{t r i}^{1}$ and $\delta_{t r i}^{2}$ if $\operatorname{dim}\left(\operatorname{ker}\left(\delta_{t r i}^{1}\right) \cap\right.$ $\left.\operatorname{ker}\left(\delta_{t r i}^{2}\right)\right)>0$ or $\operatorname{dim}\left(\operatorname{ker}\left(\left(\delta_{t r i}^{1}\right)^{T_{X}}\right) \cap \operatorname{ker}\left(\left(\delta_{t r i}^{2}\right)^{T_{X}}\right)\right)>0$. Here $T_{X}$ represents the transposition with respect to any one of the subsystems.
Proof: Proof is similar to Theorem 2.

## 5 Examples from Qutrit systems

Here, we exemplify the methods to find common entanglement witnesses as laid down in section 4 for the different classifications.
Example 1: Let us consider the following states in $C^{3} \otimes C^{3}: \rho_{1}=\left|\psi_{1}\right\rangle\left\langle\psi_{1}\right|$ and $\rho_{2}=\left|\psi_{2}\right\rangle\left\langle\psi_{2}\right|$, where $\left|\psi_{1}\right\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ and $\left|\psi_{2}\right\rangle=\frac{1}{\sqrt{3}}(|00\rangle+|11\rangle+|22\rangle)$. On observation we find an eigenvector $\left|e_{-}\right\rangle=|01\rangle-|10\rangle$ common to $\rho_{1}^{T_{A}}$ and $\rho_{2}^{T_{A}}$ corresponding to their respective negative eigenvalues. On defining $U=\left|e_{-}\right\rangle\left\langle e_{-}\right|$and $W=U^{T_{A}}$, we obtain $\operatorname{Tr}\left(W \rho_{1}\right)<0$ and $\operatorname{Tr}\left(W \rho_{2}\right)<0$. Therefore, $W$ is a common witness to the entanglement in $\rho_{1}$ and $\rho_{2}$. Hence we can conclude that $\rho=\lambda \rho_{1}+(1-\lambda) \rho_{2}$ is entangled for all $\lambda \in[0,1]$ and can be detected by $W$.
Example 2: The following family of edge states in $C^{2} \otimes C^{4}$ was introduced in [13].

$$
\tau(b, s)=\frac{1}{2\left(2+b+b^{-1}\right)}\left(\begin{array}{cccccccc}
b & 0 & 0 & 0 & 0 & -1 & 0 & 0  \tag{16}\\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & b^{-1} & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & s & 0 & 0 & 0 \\
0 & 0 & 0 & s & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & b^{-1} & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & b
\end{array}\right)
$$

$$
(\tau(b, s))^{T_{A}}=\frac{1}{2\left(2+b+b^{-1}\right)}\left(\begin{array}{cccccccc}
b & 0 & 0 & 0 & 0 & 0 & 0 & s  \tag{17}\\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & b^{-1} & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & b^{-1} & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
s & 0 & 0 & 0 & 0 & 0 & 0 & b
\end{array}\right)
$$

where $0<b<1$ and $|s|<b$. We consider $\delta_{1}=\tau(0.4,0)$ and $\delta_{2}=\tau(0.5,0)$. It is observed that the eigenvector $|01\rangle+|12\rangle \in \operatorname{ker}\left(\delta_{1}\right) \cap \operatorname{ker}\left(\delta_{2}\right)$. Further, the eigenvector $|03\rangle+|12\rangle$ and $|01\rangle+|10\rangle$ lies in $\operatorname{ker}\left(\delta_{1}^{T_{A}}\right) \cap \operatorname{ker}\left(\delta_{2}^{T_{A}}\right)$. Taking the projectors as defined in Theorem (2) and $\varepsilon$ as in Result-1, we obtain the witness

$$
W_{\text {edge }}=\left(\begin{array}{cccccccc}
-\varepsilon & 0 & 0 & 0 & 0 & 1 & 0 & 0  \tag{18}\\
0 & -\varepsilon+2 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -\varepsilon & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -\varepsilon+1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\varepsilon+1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -\varepsilon & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -\varepsilon+2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -\varepsilon
\end{array}\right)
$$

This gives $\operatorname{Tr}\left(W_{\text {edge }} \delta_{1}\right)<0$ and $\operatorname{Tr}\left(W_{\text {edge }} \delta_{2}\right)<0$. Thus, $W_{\text {edge }}$ is a common witness and also detects the class of states $\delta=\lambda \delta_{1}+(1-\lambda) \delta_{2}, 0 \leq \lambda \leq 1$.

Example 3: We consider the following class of tripartite edge states as proposed in [15]:

$$
\delta_{t r i}(a, b, c)=\frac{1}{n}\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1  \tag{19}\\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

where $n=2+a+b+c+1 / a+1 / b+1 / c$ and the basis is taken in the order $|000\rangle,|001\rangle,|010\rangle,|011\rangle$, $|100\rangle,|101\rangle,|110\rangle,|111\rangle$. The partial transpose with respect to system $C$ is given by

$$
\delta_{t r i}^{T_{C}}(a, b, c)=\frac{1}{n}\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{20}\\
0 & a & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Next we take the edge states $\delta_{t r i}^{1}=\delta_{t r i}(1,1,1)$ and $\delta_{t r i}^{2}=\delta_{t r i}(1,2,2)$. It is observed that $|111\rangle-|000\rangle \in$ $\operatorname{ker}\left(\delta_{t r i}^{1}\right) \cap \operatorname{ker}\left(\delta_{t r i}^{2}\right)$ and $|110\rangle-|001\rangle \in \operatorname{ker}\left(\left(\delta_{t r i}^{1}\right)^{T_{C}}\right) \cap \operatorname{ker}\left(\left(\delta_{t r i}^{2}\right)^{T_{C}}\right)$. Now, taking the projectors and $k$ as defined in Result 2 the witness is

$$
W_{t r i}=\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -k-1  \tag{21}\\
0 & 1-k & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -k & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -k & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -k & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-k & 0 \\
-k-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

It is found that $W_{t r i}$ detects both $\delta_{t r i}^{1}$ and $\delta_{t r i}^{2}$, thus detecting the states $\delta_{t r i}^{12}=\lambda \delta_{t r i}^{1}+(1-\lambda) \delta_{t r i}^{2}, \forall \lambda \in$ $[0,1]$.

## 6 Common decomposable and non-decomposable witness operators

Central to the idea of the detection of a PPTES is a non-decomposable witness which can successfully identify a PPTES in contrast to a decomposable witness which fails in this purpose. If we are given two states described by the density operators $\Delta_{1}$ and $\Delta_{2}$ then we can construct a witness operator common not only to the states $\Delta_{1}$ and $\Delta_{2}$ but also to the states lying on the straight line joining $\Delta_{1}$ and $\Delta_{2}$. Naturally, the next question as to whether the common witness operator is decomposable or non-decomposable. The answer lies in the nature of the states $\Delta_{1}$ and $\Delta_{2}$. The decomposable or nondecomposable nature of the common witness operator depends on the PPT or NPT nature of the states
$\Delta_{1}$ and $\Delta_{2}$. Let us suppose that $\Delta_{1}$ and $\Delta_{2}$ are two entangled states. Now if we consider the convex combination of $\Delta_{1}$ and $\Delta_{2}$, i.e., $\Delta=\lambda \Delta_{1}+(1-\lambda) \Delta_{2}, \quad 0 \leq \lambda \leq 1$, then the common decomposable witness operator and common non-decomposable witness operator can be seen as:

Common decomposable witness operator: If both $\Delta_{1}$ and $\Delta_{2}$ are NPT then a decomposable operator is enough to qualify as a common witness.

Common non-decomposable witness operator: If either $\Delta_{1}$ or $\Delta_{2}$ or both are PPT then the common witness operator is non-decomposable.

Note that if $\Delta_{1}$ is PPT and $\Delta_{2}$ is NPT, or vice-versa, then the state $\Delta$ may be NPT and it may be detected by a decomposable witness operator, but such a witness operator will not be common to $\Delta_{1}$ and $\Delta_{2}$, because either $\Delta_{1}$ or $\Delta_{2}$ is PPT, and a PPT entangled state cannot be detected by a decomposable witness operator. Let us understand the above defined common decomposable and common non-decomposable witness operators by considering the following two cases: (i) convex combination of a class of PPT mixed entangled state and a class of NPT pure entangled state and (ii) convex combination of a class of PPT mixed entangled state and a class of NPT mixed entangled state.

Case-I: Convex combination of a class of PPT mixed entangled state and a class of NPT pure entangled state

Let us consider a class of PPT mixed entangled state [16]

$$
\begin{equation*}
\rho_{1}^{e}=\frac{2}{7}\left|\psi^{+}\right\rangle\left\langle\psi^{+}\right|+\frac{\alpha}{7} \varrho_{+}+\frac{5-\alpha}{7} \varrho_{-}, \quad 3<\alpha \leq 4 \tag{22}
\end{equation*}
$$

where $\left|\psi^{+}\right\rangle=\frac{1}{\sqrt{3}}(|00\rangle+|11\rangle+|22\rangle), \varrho_{+}=\frac{1}{3}(|01\rangle\langle 01|+|12\rangle\langle 12|+|20\rangle\langle 20|)$ and $\varrho_{-}=\frac{1}{3}(|10\rangle\langle 10|+|21\rangle\langle 21|+$ $|02\rangle\langle 02|)$. Further let us consider a pure entangled state which is described by the density operator

$$
\begin{equation*}
\rho_{2}^{e}=\beta|00\rangle\langle 00|+\beta \sqrt{1-\beta^{2}}|00\rangle\langle 11|+\beta \sqrt{1-\beta^{2}}|11\rangle\langle 00|+\left(1-\beta^{2}\right)|11\rangle\langle 11| \tag{23}
\end{equation*}
$$

The convex combination of the above two states can be described by the density operator

$$
\begin{equation*}
\rho^{e}=\lambda \rho_{1}^{e}+(1-\lambda) \rho_{2}^{e}, 0 \leq \lambda \leq 1 \tag{24}
\end{equation*}
$$

Enumerating the eigenvalues of the partial transpose of the state (24) it is observed that the state has the following characterization:

| Sl. No. | $\lambda$ | $\alpha$ | $\beta$ | Nature of $\rho^{e}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $0 \leq \lambda<0.75$ | $3<\alpha \leq 3.9$ | $0.07<\beta \leq 0.99$ | $\left(\rho^{e}\right)^{T_{B}}<0$ |
| 2 | $0.75 \leq \lambda \leq 1$ | $3<\alpha \leq 3.9$ | $0 \leq \beta \leq 0.01$ | $\left(\rho^{e}\right)^{T_{B}} \geq 0$ |

Since the state $\rho^{e}$ is free entangled for the range of three parameters $0 \leq \lambda<0.75,3<\alpha \leq 3.9$, $0.07<\beta \leq 0.99$, so a decomposable witness operator is sufficient to detect it and it is given by

$$
\begin{equation*}
W^{d}=(|\chi\rangle\langle\chi|)^{T_{B}} \tag{25}
\end{equation*}
$$

where $|\chi\rangle$ is an eigenvector corresponding to a negative eigenvalue of the state $\left(\rho^{e}\right)^{T_{B}}$. The witness operator $W^{d}$ detects $\rho^{e}$ as well as the state $\rho_{2}^{e}$, but it fails to detect $\rho_{1}^{e}$, as $\rho_{1}^{e}$ is PPT and $W^{d}$ is decomposable. So, in this case we are not able to construct a common decomposable witness operator. However, using Result-1, we can construct a non-decomposable witness operator in the form

$$
\begin{equation*}
W^{n d}=|\phi\rangle\langle\phi|-\varepsilon I \tag{26}
\end{equation*}
$$

where $|\phi\rangle \in \operatorname{ker}\left(\rho^{e}\right)$. With this selection, we obtain

$$
\begin{equation*}
\operatorname{Tr}\left(W^{n d} \rho^{e}\right)=-\varepsilon<0, \quad \operatorname{Tr}\left(W^{n d} \rho_{1}^{e}\right)=-\varepsilon<0, \quad \operatorname{Tr}\left(W^{n d} \rho_{2}^{e}\right)=-\varepsilon<0 \tag{27}
\end{equation*}
$$

The above non-decomposable witness operator $W^{n d}$ not only detects $\rho_{2}^{e}$ but also detects $\rho_{1}^{e}$, and thus, it is a common non-decomposable witness operator. Let us now consider the case when $\left(\rho^{e}\right)^{T_{B}} \geq 0$ for $0.75 \leq \lambda \leq 1,3<\alpha \leq 3.9,0 \leq \beta \leq 0.01$. As $\beta \rightarrow 0$, the state $\rho_{2}^{e}$ approaches the separable projector $|11\rangle\langle 11|$. Consequently, the convex combination of $\rho_{1}^{e}$ and $\rho_{2}^{e}$ is PPT. Thus in this scenario, we can conclude that either all the states lying on the straight line joining $\rho_{1}^{e}$ and the projector $|11\rangle\langle 11|$ are separable, or we are incapable of detecting the most weak bound entangled state.

Case-II: Convex combination of a class of PPT mixed entangled state and a class of NPT mixed entangled state

Let us consider a class of PPT entangled mixed state and a class of NPT entangled mixed state which are described by the density operators

$$
\begin{equation*}
\Upsilon_{1}=\frac{2}{7}\left|\psi^{+}\right\rangle\left\langle\psi^{+}\right|+\frac{\alpha}{7} \varrho_{+}+\frac{5-\alpha}{7} \varrho_{-} \quad(3<\alpha \leq 4) \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
\Upsilon_{2}=\frac{2}{7}\left|\psi^{+}\right\rangle\left\langle\psi^{+}\right|+\frac{\gamma}{7} \varrho_{+}+\frac{5-\gamma}{7} \varrho_{-} \quad(4<\gamma \leq 5) \tag{29}
\end{equation*}
$$

respectively. The convex combination of the states $\Upsilon_{1}$ and $\Upsilon_{2}$ is given by

$$
\begin{equation*}
\Upsilon=\lambda \Upsilon_{1}+(1-\lambda) \Upsilon_{2} \quad(0 \leq \lambda \leq 1) \tag{30}
\end{equation*}
$$

The nature of the resultant state described by the density operator $\Upsilon$ depends on the values of the mixing parameter $\lambda$ and the other two parameters $\alpha$ and $\gamma$, as is given in the table below:

| Sl. No. | $\alpha$ | $\gamma$ | $\lambda$ | Nature of $\Upsilon$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $3<\alpha \leq 4$ | $4<\gamma \leq 5$ | $0 \leq \lambda<\frac{\gamma-4}{\gamma-\alpha}$ | $\Upsilon^{T_{B}}<0$ |
| 2 | $3<\alpha<4$ | $4<\gamma \leq 5$ | $\frac{\gamma-4}{\gamma-\alpha} \leq \lambda \leq 1$ | $\Upsilon^{T_{B}} \geq 0$ |
| 3 | $\alpha=4$ | $4<\gamma \leq 5$ | $\lambda=1$ | $\Upsilon^{T_{B}} \geq 0$ |

The state $\Upsilon$ is NPT for the range of parameters $3<\alpha \leq 4,4<\gamma \leq 5,0 \leq \lambda<\frac{\gamma-4}{\gamma-\alpha}$, and in this case the common witness operator is a non-decomposable witness which detects $\Upsilon, \Upsilon_{1}$ and $\Upsilon_{2}$,
whereas a decomposable witness fails to detect all the three simultaneously. However, in the remaining two cases where the ranges of three parameters are given by $3<\alpha<4,4<\gamma \leq 5, \frac{\gamma-4}{\gamma-\alpha} \leq \lambda \leq 1$ and $\alpha=4,4<\gamma \leq 5, \lambda=1$, we find that the vectors $\left|v_{1}\right\rangle=|11\rangle-|00\rangle \in \operatorname{ker}\left(\Upsilon_{1}\right) \cap \operatorname{ker}\left(\Upsilon_{2}\right)$ and $\left|v_{2}\right\rangle=|22\rangle-|00\rangle \in \operatorname{ker}\left(\Upsilon_{1}\right) \cap \operatorname{ker}\left(\Upsilon_{2}\right)$. In this scenario, a non-decomposable witness operator can be constructed, which detects both $\Upsilon_{1}, \Upsilon_{2}$ and hence $\Upsilon$. Such a non-decomposable witness operator is of the form

$$
\begin{equation*}
\Gamma=P-\varepsilon I \tag{31}
\end{equation*}
$$

where $P=\left|v_{1}\right\rangle\left\langle v_{1}\right|+\left|v_{2}\right\rangle\left\langle v_{2}\right|$, and $0<\varepsilon \leq \varepsilon_{0}=i n f_{|e, f\rangle}\langle e, f| P|e, f\rangle$.

## 7 Conclusions

To summarize, in this work we have investigated the conditions for the existence of common Schmidt number and entanglement witnesses, and proposed methods for the construction of common witness operators. Common entanglement witnesses for pairs of entangled states enable us to detect a large class of entangled states, viz., when a common witness exists for two states, it enables us to detect all states lying on the line segment joining the two. Certain characteristics of the states help us to construct the common witnesses which we have discussed here. We have considered a few interesting examples of states presented earlier in the literature in the context of entanglement witnesses, and these illustrations from qutrit systems buttress our claim of suggesting schemes for finding common witnesses.
Our study shows that the nature of the common witness is significantly dictated by the positivity of the transpose of the two states. Specifically, a decomposable witness can never qualify to be a common witness if one the states is PPTES. Thus, we demarcate between a common decomposable and a nondecomposable witness. In our analysis of common Schmidt number witnesses we find that if the two states are both of SN $k$ and a common SN witness exists for them, then the convex combination will be of SN $k$. We conclude by noting that an interesting question for further study could be to find whether the converse of the above statement is true.

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